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Abstract

This paper describes two models for simulating flow in naturally fractured petroleum reservoirs, one for single phase flow of a fluid of constant compressibility, and the other for two-phase, incompressible, immiscible flow. Both models are based on the dual porosity concept. In each model the flow in an individual matrix block is simulated using the standard equations describing flow in unfractured media, and the matrix/fracture interaction is based on the imposition of proper boundary conditions on the surface of the block. The models are presented in an easily parallelizable form.

Introduction

Double porosity models of flow through a naturally fractured petroleum reservoir were first described by Barenblatt, Zheltov, and Kochina[1] and Warren and Root[2]; their models were for single phase flow under the assumption of quasi-steady state flow in the matrix blocks. Kazemi[3] and de Swaan O.[4] considered the fully unsteady model. In this paper a somewhat more general single phase model will be considered, along with a model for two-phase, immiscible flow; Thomas *et al*[5] have studied a different double porosity model of this problem.

The fractured reservoir Ω will be idealized as a porous medium having a regular geometric pattern of fractures separating the medium into matrix blocks Ω_i. The diameter of each Ω_i is supposed small in comparison to that of Ω. The fracture system and each matrix block will be considered to be distinct, coupled porous media. The flow in each matrix block will be treated in a standard manner, based on a proper form of Darcy's law and conservation of mass. Similarly, the flow in the fractures will be modelled through the same physics, except that a (distributed) source term is induced by the flow between the blocks and the fractures. No direct flow between blocks (i.e. without passing through the fractures) will be permitted; thus, each matrix block interacts, through proper boundary conditions, with the surrounding fractures, but with no other block. For convenience in the models formulated below, it will also be assumed that the blocks are not directly affected by external sources or sinks.

The Single Phase Model

In addition to the assumptions described above, assume that the single phase fluid is of constant compressibility; i.e.,

$$R^{-1}dR = c\,dP, \quad \rho^{-1}d\rho = c\,dp, \tag{1}$$

c a positive constant, in the entire system. (Capital letters generally denote fracture quantities, small letters the corresponding matrix quantities.) Gravitational terms will be linearized:

References at end of paper.

$$R(x,t)^2 = \{R_0(x) + [R(x,t) - R_0(x)]\}^2 \approx R_0(x)[2R(x,t) - R_0(x)], \quad (2)$$

where $R_0(x)$ is a fixed, conveniently chosen reference density, such as the initial density distribution. (This function can be updated from time to time in a simulation; this possibility will be ignored in the following discussion.)

The boundary condition relating the flow in a block to that in the surrounding fractures is based on the assumption, inherent in the concept of a double porosity model, that the blocks are quite small with respect to any practical spatial discretization parameter for the simulation of the flow in the fractures. Assume that

$$\rho(x,t) = R(x_i,t), \qquad x\epsilon\partial\Omega_i,\ 0\le t\le T, \quad (3)$$

where x_i is the centroid of Ω_i. Similarly, assume that

$$\rho(x,0) = R(x_i,0) = R^0(x_i), \qquad x\epsilon\Omega_i, \quad (4)$$

where R^0 is the initial (fracture) density.

The boundary condition is based on having three families of parallel surfaces to represent the fractures. If, for instance, the reservoir Ω were a horizontal slab fractured by two families of parallel, vertical planes, a different boundary condition would be required to include properly the effect of gravity; only the case covered by (3) will be considered here.

The partial differential equation for the single phase flow in a block Ω_i is given by

$$\phi\frac{\partial\rho}{\partial t} - \nabla\cdot\left[\frac{k}{\mu c}\nabla\rho\right] = 0, \qquad x\epsilon\Omega_i,\ 0<t\le T; \quad (5)$$

note that, in line with the boundary condition (3), gravitational effects have been omitted on the block. Again, this omission would need to be addressed for other fracture geometries. The matrix block Ω_i transmits through its surface a flow of fluid given by

$$-\int_{\partial\Omega_i}\frac{k}{\mu c}\nabla\rho\cdot\nu_{\Omega_i}d\sigma = -\int_{\Omega_i}\phi\frac{\partial\rho}{\partial t}\,dx. \quad (6)$$

Average this function over Ω_i to give the source function

$$Q(x,t) = -\frac{1}{|\Omega_i|}\int_{\Omega_i}\phi\frac{\partial\rho}{\partial t}\,dx, \quad x\epsilon\Omega_i,\ 0<t\le T. \quad (7)$$

The flow in the fractures can be described by the differential equation

$$\Phi\frac{\partial R}{\partial t} - \nabla\cdot\left[\frac{K}{\mu c}\nabla R - 2R_0 g\frac{K}{\mu}\nabla Z[R - {}^1\!/_2 R_0]\right] = f_e + Q, \qquad x\epsilon\Omega,\ 0<t\le T. \quad (8)$$

where $Z=Z(x)$ is the vertical coordinate and f_e is the imposed external mass flow rate. For simplicity, the boundary condition on $\partial\Omega$ will be taken to be "no flow:"

$$\left[\frac{K}{\mu c}\nabla R - 2R_0 g\frac{K}{\mu}\nabla Z[R - {}^1\!/_2 R_0]\right]\cdot\nu_\Omega = 0, \qquad x\epsilon\partial\Omega,\ 0<t\le T. \quad (9)$$

Arbogast[6] has studied the mathematical aspects of the model defined by (3) through (9) in a slightly modified form that has no effect whatsoever on the numerical model approximating it. Under the assumption that the data functions f_e and R^0 and the reference density function R_0 are smooth, he has shown that the coupled differential system is well-posed; i.e., there exists a unique solution to the system and this solution depends continuously on the data. The continuous dependence is expressed explicitly for bounds in Sobolev spaces of R and ρ in terms of norms on f_e, R^0, and R_0. The effect of the matrix source function Q is determined to be one of stabilization; it induces a bound on the spatial block averages of $\partial R/\partial t$ in the Sobolev space $H^{-1/4}(0,T)$, which adds mildly to the usual bound for $\partial R/\partial t$ in $L^2(0,T;L^2(\Omega))$ in terms of the same norms on f_e, R^0, and R_0 for a standard, nonfractured reservoir. In the case that each Ω_i is a rectangular parallelepiped, Arbogast showed that the Warren-Root model is essentially equivalent to taking a term like the first term in the series representation of the source function Q.

A Finite Element Approximation of the Single Phase Model

Define the bilinear forms $B(u_1,u_2)$ on $H^1(\Omega)$ and $b_i(u_1,u_2)$ on $H^1(\Omega_i)$ by

$$B(u_1,u_2) = \left[\frac{K}{\mu c}\nabla u_1,\ \nabla u_2\right] = \int_\Omega\frac{K}{\mu c}\nabla u_1\cdot\nabla u_2\,dx \quad (10)$$

and

$$b_i(u_1,u_2) = \left[\frac{k}{\mu c}\nabla u_1, \nabla u_2\right]_i = \int_{\Omega_i} \frac{k}{\mu c}\nabla u_1\cdot\nabla u_2\, dx \qquad (11)$$

Then, the weak form of equations (3) through (9) appropriate for Galerkin approximation is given by seeking maps $R:[0,T]\rightarrow H^1(\Omega)$ and $\rho:[0,T]\rightarrow H^1(\cup_i\Omega_i)$ such that, if $\Gamma=2R_0gK\mu^{-1}\nabla Z$,

$$\left[\phi\frac{\partial R}{\partial t}, w\right] + B(R, w) - (\Gamma R, \nabla w) = (f_e, w) + (Q, w) - {}^1/_2(\Gamma R_0, \nabla w), \quad w\in H^1(\Omega), \qquad (12)$$

and, for all i,

$$\left[\phi\frac{\partial \rho}{\partial t}, z\right]_i + b_i(\rho, z) = 0, \quad z\in H^1{}_0(\Omega_i), \qquad (13)$$

with R and ρ remaining subject to the initial conditions (4) and the boundary conditions (3) and (9).

Let $\mathcal{M}_H\subset H^1(\Omega)$ and, for each i, $\mathcal{N}_{i,h_i}\subset H^1{}_0(\Omega_i)$ be standard C^0 finite element spaces such that

$$\inf_{w\in\mathcal{M}_H} \|u - w\|_{H^1(\Omega)} \le C\|u\|_{H^\Theta(\Omega)}H^{\Theta-1}, \qquad (14)$$

$$\inf_{z\in\mathcal{N}_{i,h_i}} \|u - z\|_{H^1(\Omega_i)} \le C\|u\|_{H^{\theta_i}(\Omega_i)}h_i^{\theta_i-1}, \qquad (15)$$

where Θ and θ_i are integers greater than or equal to two and C is independent of u; the parameters H and h_i represent the diameters of the polygons which partition Ω and Ω_i, respectively.

The approximations at time $t_f{}^n$ for R and ρ will be denoted by $R_H{}^n$ and $\rho_i{}^n$, $x\in\Omega_i$, respectively. Consider first the approximation of ρ between times $t_f{}^{n-1}$ and $t_f{}^n = t_f{}^{n-1}+\Delta t_f$. Let $\Delta t_m=\Delta t_f/N$, where N is a positive integer; N=1 is a feasible choice. In order to be able to discretize (13) implicitly, it is useful to split the approximation of ρ on Ω_i into two parts, one representing the flow in Ω_i over $(t_f{}^{n-1},t_f{}^n]$ caused by the density distribution at time $t_f{}^{n-1}$ and the other the flow caused by changing the boundary values from $R_H{}^{n-1}$ to $R_H{}^n$ over the time interval. Let

$$\rho_i{}^{n-1}\in\{R_H{}^{n-1}(x_i) + z : z\in\mathcal{N}_{i,h_i}\} = \mathcal{N}_{i,h_i}{}^{n-1} \qquad (16)$$

be the approximation to ρ on Ω_i at time $t_f{}^{n-1}$. Determine $\rho 1_{i,n}{}^{\ell}\in\mathcal{N}_{i,h_i}{}^{n-1}$ such that, for $\ell=1,\ldots,N$,

$$\left[\phi\frac{\rho 1_{i,n}{}^{\ell} - \rho 1_{i,n}{}^{\ell-1}}{\Delta t_m}, z\right]_i + b_i(\rho 1_{i,n}{}^{\ell}, z) = 0, \quad z\in\mathcal{N}_{i,h_i}, \qquad (17)$$

$$\rho 1_{i,n}{}^{0} = \rho_i{}^{n-1}, \qquad (18)$$

and then compute

$$Q1_i{}^n = -\frac{1}{|\Omega_i|}\int_{\Omega_i}\phi\frac{\rho 1_{i,n}{}^{N} - \rho_i{}^{n-1}}{\Delta t_f}\, dx. \qquad (19)$$

Next, let $\rho 2_i{}^{\ell}\in\{\ell\Delta t_m/\Delta t_f + z : z\in\mathcal{N}_{i,h_i}\} = \{\ell/N + z : z\in\mathcal{N}_{i,h_i}\}$ for $\ell=1,\ldots,N$ satisfy

$$\left[\phi\frac{\rho 2_i{}^{\ell} - \rho 2_i{}^{\ell-1}}{\Delta t_m}, z\right]_i + b_i(\rho 2_i{}^{\ell}, z) = 0, \quad z\in\mathcal{N}_{i,h_i}, \qquad (20)$$

$$\rho 2_i{}^{0} = 0, \qquad (21)$$

and compute

$$G_i{}^n = |\Omega_i|^{-1}\int_{\Omega_i}\phi(\rho 2_i{}^{N} - \rho 2_i{}^{0})\, dx = |\Omega_i|^{-1}\int_{\Omega_i}\phi\rho 2_i{}^{N}\, dx. \qquad (22)$$

Then, the matrix source function Q on Ω_i at time $t_f{}^n$ is approximated by

$$Q(x,t_f{}^n) \approx Q1_i{}^n - G_i\frac{R_H{}^n(x_i) - R_H{}^{n-1}(x_i)}{\Delta t_f}. \qquad (23)$$

Now, approximate R at time $t_f{}^n$ by $R_H{}^n\in\mathcal{M}_H$ such that

$$\left[\phi\frac{R_H{}^n - R_H{}^{n-1}}{\Delta t_f}, w\right] + \sum_i G_i(1, w)_i\frac{R_H{}^n(x_i) - R_H{}^{n-1}(x_i)}{\Delta t_f} + B(R_H{}^n, w) - (\Gamma R_H{}^n, \nabla w) = (f_e(t_f{}^n), w) + \sum_i Q1_i{}^n(1,w)_i - {}^1/_2(\Gamma R_0, \nabla w), \quad w\in\mathcal{M}_H. \qquad (24)$$

The stabilizing effect of the flow from the matrix

blocks into the fractures is readily apparent in (24), since $G_j>0$.

The finite element algorithm can be summarized as follows. The constants G_j, which must be computed for blocks associated with quadrature points in the fracture calculation, can be evaluated in a pre-processor. Then, the initial condition R^0 must be approximated by $R_H^0 \in \mathcal{M}_H$, and the initial values ρ_j^0 are determined from it. Given R_H^{n-1} and ρ_j^{n-1}, the general time step consists of three parts. First, the collection of functions $\rho 1_{j,n}^N$ must be found by solving (17), (18) for each relevant j, and the corresponding values $Q1_j^n$ computed. Second, (24) must be solved for R_H^n. Finally, ρ_j^n must be evaluated:

$$\rho_j^n = \rho 1_{j,n}^N + (R_H^n(x_j) - R_H^{n-1}(x_j))\rho 2_j^N, \quad (25)$$

so that R_H and ρ_j, all j, have been updated and the time step completed.

The blocks can be treated simultaneously; i.e., in parallel. Then, only the single number $Q1_j^n$ must be transmitted from the calculation on the block Ω_j to the fracture calculation. Next, the calculation for the density R_H^n in the fractures takes place, after which the single number $R_H^n(x_j)$ must be returned to each block to permit the update (25) and the determination of $\mathcal{M}_{j,h_j}^n$. The algorithm can be implemented very efficiently on a computer system having one quite fast node, such as a vector computer, tied to a collection of less expensive nodes, such as reasonable workstations. Note that the number of parameters associated with the fracture calculation (i.e. $\dim \mathcal{M}_H$) should be expected to be much larger than the number associated with an individual block (i.e. $\dim \mathcal{M}_{j,h_j}$). The blocks can be assigned to the workstations, a few to each station. (More sophisticatedly, a fraction to the fast node and the rest portioned out to the slower nodes.) The fracture calculation would be made by the fast node. Since so little information is passed between the fracture calculation and the block calculations, only a modest bus capacity is required.

The convergence of the approximate solution R_H and ρ_j, all j, to the solution R and ρ follows easily from the argument of Arbogast[6] for a Crank-Nicolson version of the algorithm; Arbogast restricted his attention to N=1, but no significant change in his argument is needed to treat the case considered here.

A finite difference procedure can be constructed with the same concepts as used in the finite element case. The same parallel features will occur in the algorithm.

The Two-Phase, Immiscible Model

This model is intended to simulate an incompressible waterflood in the fractured reservoir Ω, again employing a dual porosity model. The model will be formulated for a horizontal, linear flood here in order to meet the length constraint for the paper. Gravity and more space variables can be treated by essentially the same techniques. In the fractures, let $S=S_w$, $P=P_w$, $P_c=P_o-P_w$, $\Lambda_\theta = KK_{r\theta}/\mu_\theta$ (θ=o,w), and $\Lambda=\Lambda_w+\Lambda_o$. With Q_w denoting the (usually negative) water source term resulting from imbibition into the matrix blocks and Q_o the corresponding oil source term, the differential equations describing the flow in the fractures (external source terms are omitted here; again, they could be added with little complication) are the usual

$$\Phi\frac{\partial S}{\partial t} - \frac{\partial}{\partial x}\left[\Lambda_w\frac{\partial P}{\partial x}\right] = Q_w, \quad (26)$$

$$-\Phi\frac{\partial S}{\partial t} - \frac{\partial}{\partial x}\left[\Lambda_o\frac{\partial P_o}{\partial x}\right] = Q_o, \quad (27)$$

for $0<x<L=|\Omega|$ and $t>0$. If (26) and (27) are added, the pressure equation

$$-\frac{\partial}{\partial x}\left[\Lambda\frac{\partial P}{\partial x} + \Lambda_o P_c'\frac{\partial S}{\partial x}\right] = 0 \quad (28)$$

results, as incompressibility requires that $Q_w+Q_o=0$. Initial values S(x,0) must be specified, along with boundary conditions for $t>0$. Assume injection of water at x=0 at a specified volumetric rate. Then, for $t>0$,

$$-\Lambda_w\frac{\partial P}{\partial x}(0,t) = f(t) \geq 0, \quad -\Lambda_o\frac{\partial P_o}{\partial x}(0,t) = 0. \quad (29)$$

At x=L, the flow out of the domain splits proportionally to the mobilities of the phases:

$$-\Lambda_w\frac{\partial P}{\partial x}(L,t) = \frac{\Lambda_w}{\Lambda}f(t), \quad -\Lambda_o\frac{\partial P_o}{\partial x}(L,t) = \frac{\Lambda_o}{\Lambda}f(t). \quad (30)$$

These conditions amount to the assumption that viscous forces dominate capillary forces at the outflow face. The initial pressure is determined by (28) and the boundary conditions (29) and (30).

The flow in an individual matrix block is governed by the standard equations

$$\phi\frac{\partial s}{\partial t} - \frac{\partial}{\partial x}\left[\lambda_w\frac{\partial p}{\partial x}\right] = 0, \tag{31}$$

$$-\frac{\partial}{\partial x}\left[\lambda\frac{\partial p}{\partial x} + \lambda_o p_c'\frac{\partial s}{\partial x}\right] = 0, \tag{32}$$

for $x\epsilon\Omega_i$ and $t>0$. It will be assumed that each block is sufficiently small with respect to Ω that imbibition dominates viscous forces on the block. Thus, the pressure change in the fractures across a block will be ignored. The first consequence of this assumption is that the initial condition in the block will be taken to be

$$p(x,0) = P(x_i,0),\ p_o(x,0) = P_o(x_i,0),\ x\epsilon\Omega_i; \tag{33}$$

thus,

$$p_c(s(x,0)) = P_c(S(x_i,0)), \qquad x\epsilon\Omega_i. \tag{34}$$

(The function p_c can vary from block to block, but this possibility will not be considered here.) A related consequence is that the boundary conditions on the block are given by

$$p(x,t) = P(x_i,t),\ p_o(x,t) = P_o(x_i,t),\ x\epsilon\partial\Omega_i, \tag{35}$$

so that

$$p_c(s(x,t)) = P_c(S(x_i,t)), \qquad x\epsilon\partial\Omega_i. \tag{36}$$

The matrix/fracture interaction in differential form is quite similar to that for the single phase problem. The term Q_w can be evaluated easily on Ω_i as

$$Q_w = -\frac{1}{|\Omega_i|}\int_{\Omega_i}\phi\frac{\partial s}{\partial t}\,dx, \qquad x\epsilon\Omega_i. \tag{37}$$

A Finite Difference Approximation for the Two-Phase, Immiscible Model

The finite difference procedure in an individual block should be appropriately designed to treat an imbibition-dominated flood. In the appendix, it is shown that, if

$$\lambda_o(s_o) \sim \alpha(s_o - s_{ro})^\delta, \quad \lambda_w(s_o) \sim \beta, \quad p_c'(s_o) \sim \gamma, \tag{38}$$

for s_o slightly larger than s_{ro} and if the water saturation in the face at $y=0$ is set to $1-s_{ro}$ and the capillary pressure external to the face set to zero, then the solution for $s_o(y,t)$ has the asymptotic form

$$s_o(y,t) \sim \left[\frac{\delta+1}{\alpha\gamma}q_w(0,t)\right]^{1/(\delta+1)} y^{1/(\delta+1)}. \tag{39}$$

In order that the differences of the values of s between neighboring mesh points be reasonably nearly equal, a mesh point distribution proportional to $y^{\delta+1}$ near $y=0$ is indicated.

Let the local coordinates for Ω_i be denoted by y, $0\le y\le L_i=|\Omega_i|$, and let $\{y_{i,j} : j=0,\dots,J_i\}$ be a partition of Ω_i, with $h_{i,j}=y_{i,j}-y_{i,j-1}$. Discretize (31) and (32) in the following manner. For $1\le j\le J_i-1$ and $\ell=1,\dots,N$, let

$$\phi_{i,j}\frac{s_n{}^{\ell}{}_{i,j} - s_n{}^{\ell-1}{}_{i,j}}{\Delta t_m} - \frac{2}{h_{i,j+1}+h_{i,j}}\left[\lambda_w{}^{n-1}{}_{i,j+1/2}\frac{p_n{}^{\ell}{}_{i,j+1} - p_n{}^{\ell}{}_{i,j}}{h_{i,j+1}} - \lambda_w{}^{n-1}{}_{i,j-1/2}\frac{p_n{}^{\ell}{}_{i,j} - p_n{}^{\ell}{}_{i,j-1}}{h_{i,j}}\right] = 0, \tag{40}$$

$$\lambda^{n-1}{}_{i,j+1/2}\frac{p_n{}^{\ell}{}_{i,j+1} - p_n{}^{\ell}{}_{i,j}}{h_{i,j+1}} - \lambda^{n-1}{}_{i,j-1/2}\frac{p_n{}^{\ell}{}_{i,j} - p_n{}^{\ell}{}_{i,j-1}}{h_{i,j}} + (\lambda_o p_c')^{n-1}{}_{i,j+1/2}\frac{s_n{}^{\ell}{}_{i,j+1} - s_n{}^{\ell}{}_{i,j}}{h_{i,j+1}} - (\lambda_o p_c')^{n-1}{}_{i,j-1/2}\frac{s_n{}^{\ell}{}_{i,j} - s_n{}^{\ell}{}_{i,j-1}}{h_{i,j}} = 0. \tag{41}$$

The boundary conditions (35) and (36) can be applied as follows:

$$p_n{}^{\ell}{}_{i,0} = p_n{}^{\ell}{}_{i,J_i} = P^{n-1}{}_i + \ell N^{-1}(P^n{}_i - P^{n-1}{}_i), \tag{42}$$

$$s_n{}^0{}_{i,0} = s_n{}^0{}_{i,J_i} = p_c{}^{-1}(P_c(S^{n-1}{}_i)), \tag{43}$$

$$s_n{}^{\ell}{}_{i,0} = s_n{}^{\ell}{}_{i,J_i} = s_n{}^0{}_{i,0} + \frac{\ell P_c'(S^{n-1}{}_i)}{N p_c'(s_n{}^0{}_{i,0})}(S^n{}_i - S^{n-1}{}_i), \tag{44}$$

for $\ell=1,\dots,N$. The initial condition for the saturation in the block Ω_i is given by

$$s_n{}^0{}_{i,j} = s^{n-1}{}_{i,j} = s_{n-1}{}^N{}_{i,j}, \qquad 1<j<J_i. \tag{45}$$

Note that in general, (43) implies that $s_n{}^0{}_{i,0}\neq s^{n-1}{}_{i,0}$ and $s_n{}^0{}_{i,J_i}\neq s^{n-1}{}_{i,J_i}$; (43) restores the capillary equilibrium that is slightly lost by the linearization in (44).

The coefficients λ and $\lambda_o p_c'$ should be evaluated differently. Since $\lambda=\lambda_w+\lambda_o$ is a smooth function of s and is bounded away from zero, it suffices to take

$$\lambda^{n-1}{}_{i,j+1/2} = \lambda(1/2[s^{n-1}{}_{i,j} + s^{n-1}{}_{i,j+1}]). \tag{46}$$

However, since $\lambda_o p_c'$ vanishes at $s=1-s_{ro}$, it is better to use a harmonic average of its values. So, let

$$(\lambda_o p_c')^{n-1}{}_{i,j-1/2} = \frac{h_{i,j}}{\int_{y_{i,j-1}}^{y_{i,j}} \frac{1}{(\lambda_o p_c')(s^{n-1}{}_i(y))}\,dy}. \tag{47}$$

Evaluate the integral approximately by interpreting $s^{n-1}{}_i(y)$ as the linear interpolant of $s^{n-1}{}_{i,j-1}$ and $s^{n-1}{}_{i,j}$ and then using a two-point Gauss quadrature rule. The evaluation of $\lambda_w{}^{n-1}{}_{i,j+1/2}$ can be made analogously to that of $\lambda^{n-1}{}_{i,j+1/2}$. Note that the coefficients on Ω_i are being held fixed over the fracture time step $(t_f{}^{n-1}, t_f{}^n)$.

The calculation for $s_n{}^N{}_{i,j}$ can be decoupled from that for $S^n{}_i$ in a fashion corresponding to the splitting (17),(18) and (20),(21) for the solution in the block Ω_i in the single phase problem. Let $s_n{}^\ell{}_{i,j}=s1_n{}^\ell{}_{i,j}+s2_n{}^\ell{}_{i,j}$, where $s1_n$ reflects the effect of the conditions existing in the block at time $t_f{}^{n-1}$ and $s2_n$ the effect of changing the saturation in the surrounding fractures; note that changing the pressure in the fractures has no effect on the saturation in the block, as no flow results from a uniform change in the pressure. Then, for $j=1,\dots,J_i-1$ and $\ell=1,\dots,N$,

$$\phi_{i,j}\frac{s1_n{}^\ell{}_{i,j} - s1_n{}^{\ell-1}{}_{i,j}}{\Delta t_m}$$

$$- \frac{2}{h_{i,j+1}+h_{i,j}}\left[\lambda_w{}^{n-1}{}_{i,j+1/2}\frac{p1_n{}^\ell{}_{i,j+1} - p1_n{}^\ell{}_{i,j}}{h_{i,j+1}} - \lambda_w{}^{n-1}{}_{i,j-1/2}\frac{p1_n{}^\ell{}_{i,j} - p1_n{}^\ell{}_{i,j-1}}{h_{i,j}}\right] = 0, \tag{48}$$

$$\lambda^{n-1}{}_{i,j+1/2}\frac{p1_n{}^\ell{}_{i,j+1} - p1_n{}^\ell{}_{i,j}}{h_{i,j+1}} - \lambda^{n-1}{}_{i,j-1/2}\frac{p1_n{}^\ell{}_{i,j} - p1_n{}^\ell{}_{i,j-1}}{h_{i,j}} + (\lambda_o p_c')^{n-1}{}_{i,j+1/2}\frac{s1_n{}^\ell{}_{i,j+1} - s1_n{}^\ell{}_{i,j}}{h_{i,j+1}} - (\lambda_o p_c')^{n-1}{}_{i,j-1/2}\frac{s1_n{}^\ell{}_{i,j} - s1_n{}^\ell{}_{i,j-1}}{h_{i,j}} = 0, \tag{49}$$

with

$$p1_n{}^\ell{}_{i,0} = p1_n{}^\ell{}_{i,J_i} = p^{n-1}{}_i \text{ (or zero)}, \tag{50}$$

$$s1_n{}^\ell{}_{i,0} = s1_n{}^\ell{}_{i,J_i} = s_n{}^0{}_{i,0}, \tag{51}$$

$$s1_n{}^0{}_{i,j} = s^{n-1}{}_{i,j}, \qquad 1<j<J_i, \tag{52}$$

and

$$p2_n{}^\ell{}_{i,j} = \frac{P_c'(S^{n-1}{}_i)}{p_c'(s_n{}^0{}_{i,0})}(S^n{}_i - S^{n-1}{}_i)\,q_n{}^\ell{}_{i,j}, \tag{53}$$

$$s2_n{}^\ell{}_{i,j} = \frac{P_c'(S^{n-1}{}_i)}{p_c'(s_n{}^0{}_{i,0})}(S^n{}_i - S^{n-1}{}_i)\,r_n{}^\ell{}_{i,j}, \tag{54}$$

where

$$\phi_{i,j}\frac{r_n{}^\ell{}_{i,j} - r_n{}^{\ell-1}{}_{i,j}}{\Delta t_m} - \frac{2}{h_{i,j+1}+h_{i,j}}\left[\lambda_w{}^{n-1}{}_{i,j+1/2}\frac{q_n{}^\ell{}_{i,j+1} - q_n{}^\ell{}_{i,j}}{h_{i,j+1}} - \lambda_w{}^{n-1}{}_{i,j-1/2}\frac{q_n{}^\ell{}_{i,j} - q_n{}^\ell{}_{i,j-1}}{h_{i,j}}\right] = 0, \tag{55}$$

$$\lambda^{n-1}{}_{i,j+1/2}\frac{q_n{}^\ell{}_{i,j+1} - q_n{}^\ell{}_{i,j}}{h_{i,j+1}} - \lambda^{n-1}{}_{i,j-1/2}\frac{q_n{}^\ell{}_{i,j} - q_n{}^\ell{}_{i,j-1}}{h_{i,j}} + (\lambda_o p_c')^{n-1}{}_{i,j+1/2}\frac{r_n{}^\ell{}_{i,j+1} - r_n{}^\ell{}_{i,j}}{h_{i,j+1}}$$

$$-(\lambda_0 p_c')^{n-1}{}_{i,j-1/2}\frac{r_n{}^{\ell}{}_{i,j} - r_n{}^{\ell}{}_{i,j-1}}{h_{i,j}} = 0, \qquad (56)$$

with

$$q_n{}^{\ell}{}_{i,0} = q_n{}^{\ell}{}_{i,J_i} = 0, \qquad (57)$$

$$r_n{}^{\ell}{}_{i,0} = r_n{}^{\ell}{}_{i,J_i} = \ell N^{-1}, \qquad (58)$$

$$r_n{}^{0}{}_{i,j} = 0, \qquad 1<j<J_i. \qquad (59)$$

Note that q and r must be recalculated each fracture time step, since the coefficients change.

Next, evaluate the matrix/fracture interaction:

$$Q1^n_i = -\frac{1}{2\Delta t_f L_i}\sum_{j=1}^{J_i}\phi_{i,j}\{s1_n{}^{N}{}_{i,j-1} + s1_n{}^{N}{}_{i,j} - s1_n{}^{0}{}_{i,j-1} - s1_n{}^{0}{}_{i,j}\}h_{i,j}, \qquad (60)$$

$$G^n_i = \frac{1}{2L_i}\sum_{j=1}^{J_i}\phi_{i,j}\{r_n{}^{N}{}_{i,j-1} + r_n{}^{N}{}_{i,j}\}h_{i,j}, \qquad (61)$$

$$Q_W{}^n_i = Q1^n_i - G^n_i\frac{P_c'(S^{n-1}{}_i)\,(S^n_i - S^{n-1}{}_i)}{p_c'(s_n{}^{0}{}_{i,0})\;\Delta t_f}. \qquad (62)$$

The finite difference equations in the fractures can be constructed as follows. Partition (here uniformly, for simplicity of notation) Ω: $H_i=H=L/M$; $x_i=iH$, $i=0,\ldots,M$. For $i=1,\ldots,M-1$, let

$$\left[\Phi_i + G^n_i\frac{P_c'(S^{n-1}{}_i)}{p_c'(s_n{}^{0}{}_{i,0})}\right]\frac{S^n_i - S^{n-1}{}_i}{\Delta t_f} - H^{-2}[\Lambda_W{}^{n-1}{}_{i+1/2}(P^n_{i+1} - P^n_i) - \Lambda_W{}^{n-1}{}_{i-1/2}(P^n_i - P^n_{i-1})] = Q1^n_i, \qquad (63)$$

$$\Lambda^{n-1}{}_{i+1/2}(P^n_{i+1} - P^n_i) - \Lambda^{n-1}{}_{i-1/2}(P^n_i - P^n_{i-1}) + (\Lambda_0 P_c')^{n-1}{}_{i+1/2}(S^n_{i+1} - S^n_i) - (\Lambda_0 P_c')^{n-1}{}_{i-1/2}(S^n_i - S^n_{i-1}) = 0. \qquad (64)$$

The boundary conditions (29) and (30) can be incorporated in a discretization of (26) and (27) for $i=0$:

$$\left[\Phi_0 + G^n_0\frac{P_c'(S^{n-1}{}_0)}{p_c'(s_n{}^{0}{}_{0,0})}\right]\frac{S^n_0 - S^{n-1}{}_0}{\Delta t_f} - 2\Lambda_W{}^{n-1}{}_{1/2}H^{-2}(P^n_1 - P^n_0) = 2H^{-1}f(t_f{}^n) + Q1^n_0, \qquad (65)$$

$$\left[\Phi_0 + G^n_0\frac{P_c'(S^{n-1}{}_0)}{p_c'(s_n{}^{0}{}_{0,0})}\right]\frac{S^n_0 - S^{n-1}{}_0}{\Delta t_f} + 2\Lambda_0{}^{n-1}{}_{1/2}H^{-2}[(P^n_1 - P^n_0) + P_c'{}^{n-1}{}_{1/2}(S^n_1 - S^n_0)] = Q1^n_0, \qquad (66)$$

The boundary conditions at $x=L$ lead to the equations

$$\left[\Phi_M + G^n_M\frac{P_c'(S^{n-1}{}_M)}{p_c'(s_n{}^{0}{}_{M,0})}\right]\frac{S^n_M - S^{n-1}{}_M}{\Delta t_f} + 2\Lambda_W{}^{n-1}{}_{M-1/2}H^{-2}(P^n_M - P^n_{M-1}) = -\frac{2\Lambda_W{}^{n-1}{}_M}{H\Lambda^{n-1}{}_M}f(t_f{}^n) + Q1^n_M, \qquad (67)$$

and

$$\left[\Phi_M + G^n_M\frac{P_c'(S^{n-1}{}_M)}{p_c'(s_n{}^{0}{}_{M,0})}\right]\frac{S^n_M - S^{n-1}{}_M}{\Delta t_f} - 2\Lambda_0{}^{n-1}{}_{M-1/2}H^{-2}[(P^n_M - P^n_{M-1}) + P_c'{}^{n-1}{}_{M-1/2}(S^n_M - S^n_{M-1})] = \frac{2\Lambda_0{}^{n-1}{}_M}{H\Lambda^{n-1}{}_M}f(t_f{}^n) + Q1^n_M, \qquad (68)$$

Note that the coefficient functions Λ, Λ_W, Λ_0, and P_c' must be evaluated for $S=S^{n-1}{}_M$, as well as for values corresponding to the midpoints of the intervals, where it suffices to take $S = {}^1/_2(S^{n-1}{}_i + S^{n-1}{}_{i+1})$ at $x_{i+1/2}$.

After solving for S^n_i and P^n_i, the saturations and pressures of the blocks can be updated:

$$s^n_{i,j} = s1_n{}^{N}{}_{i,j} + \frac{P_c'(S^{n-1}{}_i)}{p_c'(s_n{}^{0}{}_{i,0})}r_n{}^{N}{}_{i,j}(S^n_i - S^{n-1}{}_i), \qquad (69)$$

$$p^n_{i,0} = p^n_{i,J_i} = P^n_i. \qquad (70)$$

The sequence of calculations above completes a time step in the fractures. The remarks on parallelization for the single phase, fractured problem apply in like fashion to this problem.

Nomenclature

Symbols separated by a semicolon refer to fracture and matrix quantities, respectively. Generally, capitals denote fracture quantities, while small letters denote the corresponding matrix quantities.

$B(u_1,u_2)$; $= ((K/\mu c)\nabla u_1, \nabla u_2))$

$b_i(u_1,u_2)$ $= ((k/\mu c)\nabla u_1, \nabla u_2))_i$

C positive constant

c compressibility

$d\sigma$ surface differential

f volumetric waterflood rate

f_θ external source/sink

G_i scaled effect on matrix source function due to changes on $\partial\Omega_i$

g gravitational constant

H; $h_i, h_{i,j}$ spatial discretization parameter

J_i number of intervals partitioning L_i

K; k permeability

$K_{r\theta}$; $k_{r\theta}$ relative permeabilities

L; L_i length of linear reservoir; linear block

M $= L/H$, number of intervals partitioning L

N $= \Delta t_f/\Delta t_m$, number of block time steps per fracture time step

P; p pressure [$= P_w$; p_w in immiscible flow]

P_θ; p_θ pressure

P_c; p_c capilary pressure, oil minus water

$p1_n$ matrix pressure due to conditions on the block at time t_f^{n-1}

$p2_n$ matrix pressure due to effect of saturation change on $\partial\Omega_i$

Q, Q_θ matrix source function

$Q1^n$ matrix source due to conditions on the block at time t_f^{n-1}

q scaled pressure due to effect of saturation change on $\partial\Omega_i$

q_θ $= -\lambda_\theta(\partial p_\theta/\partial x)$, linear flow through $\partial\Omega_i$

R density in the fractures

R^0 initial density in the fractures

R_0 reference density

R_H finite element approximant of R

r scaled saturation due to effect of saturation change on $\partial\Omega_i$

S; s $= S_w$; s_w

S_θ; s_θ saturation

s_{ro} residual oil saturation

$s1_n$ saturation due to conditions on the block at time t_f^{n-1}

$s2_n$ saturation due to effect of saturation change on $\partial\Omega_i$

T maximal time of interest

t time

t_f^n $= n\Delta t_f$, n *th* time level

u dummy function

w test function

x space coordinate

x_i centroid of i *th* block, also i *th* fracture mesh location

y local space coordinate on a matrix block

$y_{i,j}$ i *th* block's j *th* mesh location

Z horizontal coordinate

z test function

$\mathcal{M}_H$ fracture's Galerkin approximation space

$\mathcal{M}_{i,h_i}$ i *th* block's Galerkin approximation space

$\mathcal{M}_{i,h_i}^n$ $= \mathcal{M}_{i,h_i} + R_H^n$, $\mathcal{M}_{i,h_i}$ with imposed boundary condition

α asymptotic constant

β asymptotic constant

Γ $= 2R_0gK\mu^{-1}\nabla Z$

γ asymptotic constant

Δt_f;Δt_m time discretization parameter

δ asymptotic constant

$\partial\Omega$; $\partial\Omega_i$ boundary of Ω; of Ω_i

Θ; θ_i order of approximation of $\mathcal{M}_H$; $\mathcal{M}_{i,h_i}$.

Λ; λ $= \Lambda_w + \Lambda_o$; $\lambda_w + \lambda_o$

Λ_θ; λ_θ $= KK_{r\theta}/\mu_\theta$; $kk_{r\theta}/\mu_\theta$

μ, μ_θ viscosity

ν_Ω, ν_{Ω_i} outer unit normal to $\partial\Omega$; to $\partial\Omega_i$

ρ density in the blocks

ρ_i finite element approximant of ρ on Ω_i

$\rho 1_{i,n}$ matrix density due to conditions on the block at time t_f^{n-1}

$\rho 2_{i,n}$ scaled matrix density due to effect of density change on $\partial\Omega_i$

Φ; ϕ porosity

Φ_i; $\phi_{i,j}$ $= \Phi(x_i)$; $\phi(y_{i,j})$

Ω; Ω_i reservoir; i *th* matrix block

(u_1,u_2); $= \int_\Omega u_1u_2\,dx$

$(u_1,u_2)_i$ $= \int_{\Omega_i} u_1u_2\,dx$

$|\Omega|$;$|\Omega_i|$ volume (or length) of Ω; of Ω_i

subscripts

f	fracture
i	i *th* matrix block, also i *th* fracture mesh location or interval
$i+1/2$	an evaluation at midpoint of fracture mesh
i,j	j *th* matrix mesh location or interval on i *th* block
$j+1/2$	an evaluation at midpoint of block mesh
m	matrix
n	last fracture time t_f^{n-1} [see superscript ℓ]
o	oil
w	water
θ	o or w

superscripts

ℓ	block time level from last fracture time level [see subscript n]
n	fracture time level

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Appendix

Asymptotic Behavior for the Imbibition Problem

We consider the behavior of the saturation near an imbibition face at y=0 when water saturation at that face is set at $s_w=1-s_{ro}$. Normalize so that $s_{ro}=0$. Assume that for s_o small

$$\lambda_o(s_o) \sim \alpha s_o^{\delta}, \ \delta>0, \tag{71}$$

$$\lambda_w(s_o) \sim \beta, \tag{72}$$

$$p_c'(s_o) \sim \gamma. \tag{73}$$

Then, the total flow $q_w+q_o=0$ can be expressed as

$$(\beta + \alpha s_o^{\delta})\frac{\partial p_w}{\partial y} + \alpha\gamma s_o^{\delta}\frac{\partial s_o}{\partial y} = 0. \tag{74}$$

So,

$$-\beta\frac{\partial p_w}{\partial y} = q_w \sim \frac{\alpha\gamma s_o^{\delta}}{1+\alpha\beta^{-1}s_o^{\delta}}\frac{\partial s_o}{\partial y} \sim \alpha\gamma s_o^{\delta}\frac{\partial s_o}{\partial y} \tag{75}$$

and

$$s_o(y,t) \sim \left[\frac{\delta+1}{\alpha\gamma}q_w(0,t)y\right]^{1/(\delta+1)}. \tag{76}$$

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