ANALYSIS OF JUMP PROCESSES WITH NONDEGENERATE JUMPING KERNELS

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Abstract. We prove regularity estimates for functions which are harmonic with respect to certain jump processes. The aim of this article is to extend the method of Bass-Levin[BL02] and Bogdan-Sztonyk[BS05] to more general processes. Furthermore, we establish a new version of the Harnack inequality that implies regularity estimates for corresponding harmonic functions.

1. Introduction

Let $\alpha \in (0,2)$. We define a non-local operator $\mathcal{L}$ by

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d\setminus\{0\}} (f(x+h) - f(x) - \langle \nabla f(x), h\rangle \mathbb{1}_{\{|h|\leq 1\}}) n(x,h)\, dh, \tag{1.1}$$

for $f \in C_b^2(\mathbb{R}^d)$. Here $n\colon \mathbb{R}^d \times \big(\mathbb{R}^d \setminus \{0\}\big) \to [0,\infty)$ is a measurable function with

$$c_1 |h|^{-d-\alpha} \leq n(x,h) \leq c_2 |h|^{-d-\alpha} \tag{1.2}$$

for every $h \in \mathbb{R}^d \setminus \{0\}$, any $x \in \mathbb{R}^d$ and fixed positive reals $c_1 < c_2$. Note that $n(x,h) = |h|^{-d-\alpha}$ for every h implies $\mathcal{L}f = -c(\alpha)(-\Delta)^{\alpha/2} f$ with some appropriate constant $c(\alpha)$.

In [BL02] it is shown that harmonic functions with respect to $\mathcal{L}$ satisfy a Harnack inequality in the following sense: There is a constant $c_3 \geq 1$ such that for every ball B_R the following implication holds:

$$f \geq 0 \text{ in } \mathbb{R}^d,\ f \text{ harmonic in } B_R \quad \Rightarrow \quad \forall\, x,y \in B_{R/2} : f(x) \leq c_3 f(y)\,.$$

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In [BL02] it is also shown that harmonic functions with respect to $\mathcal{L}$ satisfy the following a-priori estimate: There are constants $\beta \in (0,1)$, $c_4 \geq 1$ such that for every ball B_R the following implication holds:

$$f \text{ harmonic in } B_R \quad \Rightarrow \quad \|f\|_{C^\beta(\overline{B_{R/2}})} \leq c_4 \|f\|_\infty \,.$$

This result and its proof recently generated several research activities, see the short discussion below. Our aim is to prove similar results under weaker assumptions on the kernel n.

Let us be more precise. We consider kernels $n\colon \mathbb{R}^d \times \big(\mathbb{R}^d \setminus \{0\}\big) \to [0,\infty)$ that satisfy for every $x, h \in \mathbb{R}^d$, $h \neq 0$

$$n(x,h) = n(x,-h) \tag{1.3}$$

and

$$k_1\big(\tfrac{h}{|h|}\big) j(|h|) \leq n(x,h) \leq k_2\big(\tfrac{h}{|h|}\big) j(|h|) \tag{1.4}$$

where $k_1, k_2\colon S^{d-1} \to [0,\infty)$ are measurable bounded symmetric functions on the unit sphere satisfying the following conditions: There are $\delta > 0, N \in \mathbb{N}, \varepsilon_1, \ldots, \varepsilon_N > 0$ and $\eta_1, \ldots, \eta_N \in S^{d-1}$ such that for $S_i = S^{d-1} \cap \big(B(\eta_i, \varepsilon_i) \cup B(-\eta_i, \varepsilon_i)\big)$

$$k_2(\xi) \geq k_1(\xi) \geq \delta \;\text{ if } \xi \in \bigcup_{i=1}^N S_i \quad \text{and} \quad k_2(\xi) = k_1(\xi) = 0 \text{ otherwise.} \tag{1.5}$$

Let $j : (0,\infty) \to [0,\infty)$ be a function such that $\int_{\mathbb{R}^d} (|z|^2 \wedge 1)\, j(|z|)\, dz$ is finite. We assume further:

(J1) There exists $\alpha \in (0,2)$ and a function $\ell\colon (0,2) \to (0,\infty)$ which is slowly varying at 0 (i.e. $\lim\limits_{r \to 0+} \dfrac{\ell(\lambda r)}{\ell(r)} = 1$ for any $\lambda > 0$) and bounded away from 0 and ∞ on every compact interval such that

$$j(t) = \frac{\ell(t)}{t^{d+\alpha}} \;\text{ for every } 0 < t \leq 1 \,.$$

(J2) There is a constant $\kappa \geq 1$ such that

$$j(t) \leq \kappa j(s) \;\text{ whenever }\; 1 \leq s \leq t \,.$$

In order to establish regularity estimates we need an additional weak assumption.

(J3) There is $\sigma > 0$ such that

$$\limsup_{R\to\infty} R^\sigma \int\limits_{|z|>R} j(|z|)\,dz \le 1\,.$$

If this condition holds, then one can always choose $\sigma \in (0,\alpha)$.

Remark 1.1. *The symmetry assumption (1.3) is used only in Proposition 2.4 and can be dispensed with if $\alpha \in (0,1)$.*

Example 1: If a kernel n satisfies condition (1.2), then it also satisfies (J1)-(J3). Choose $N = 1$, $\varepsilon_1 = 4$, i.e. $S_1 = S^{d-1}$, $k_1 \equiv \delta = c_1$, $k_2 \equiv c_2$, $j(s) = s^{-d-\alpha}$ in (1.4), $\ell \equiv 1$ in (J1), $\kappa = 1$ in (J2) and $\sigma \in (0,\alpha)$ arbitrarily in (J3). In general, (J1)-(J3) hold for jumping kernels corresponding to stable processes, stable-like processes and truncated versions. Sums of such jumping kernels can be considered, too.

Example 2: Let $N \in \mathbb{N}$, $\eta_1,\dots,\eta_N \in S^{d-1}$ and $\varepsilon_1,\dots,\varepsilon_N$ be positive real numbers such that the sets $S_i = S^{d-1} \cap \big(B(\eta_i,\varepsilon_i) \cup B(-\eta_i,\varepsilon_i)\big)$ are pairwise disjoint for $i = 1,\dots,N$. Set $B = \bigcup\limits_{i=1}^{N} S_i$. Let $k_1 = \delta \mathbb{1}_B$ for some $\delta > 0$ and $k_2 = ck_1$ for some $c > 1$. Let $j(s) = s^{-d-\alpha}$ for $s > 0$. Then our assumptions are satisfied if (1.4) and (1.3) hold true. For the particular choice where $x \mapsto n(x,h)$ is constant (case of Lévy process), this class of examples is treated in [BS05, p.148], where it is shown that for $N = \infty$ the Harnack inequality fails.

Given a linear operator $\mathcal{L}$ as in (1.1) we assume that there exists a strong Markov process $X = (X_t, \mathbb{P}^x)$ with paths that are right-continuous with left limits such that the process

$$\left\{ f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\,ds \right\}_{t\ge 0}$$

is a $\mathbb{P}^x$-martingale for all $x \in \mathbb{R}^d$ and $f \in C_b^2(\mathbb{R}^d)$. We say that a bounded function $f : \mathbb{R}^d \to \mathbb{R}$ is harmonic with respect to $\mathcal{L}$ in an open set Ω if $f\big(X_{\min(t,\tau_{\Omega'})}\big)$ is a right-continuous martingale for every open $\Omega' \subset \mathbb{R}^d$ with $\overline{\Omega'} \subset \Omega$.

We can prove the following version of the Harnack inequality.

Theorem 1.2. *Assume (J1) and (J2). There exist constants $c_1, c_2 \ge 1$ such that for every $x_0 \in \mathbb{R}^d$, $r \in (0,\frac{1}{4})$ and every bounded function $f\colon \mathbb{R}^d \to \mathbb{R}$ which is non-negative in $B(x_0,4r)$ and harmonic in $B(x_0,4r)$ the following estimate holds*

$$f(x) \le c_1 f(y) + c_2 \left(\frac{r^\alpha}{\ell(r)}\right) \sup_{v\in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) n(v, z-v)\,dz$$

for all $x, y \in B(x_0, r)$.

Remark 1.3. *If f is, in addition, non-negative in all of $\mathbb{R}^d$, then the classical version of the Harnack inequality follows, i.e. for all $x, y \in B(x_0, r)$:*

$$f(x) \le c_1 f(y)\,.$$

As a corollary to the Harnack inequality we obtain the following regularity result.

Theorem 1.4. *Assume (J1), (J2) and (J3). Then there exist $\beta \in (0,1)$, $c_3, c_4 \ge 1$ such that for every $x_0 \in \mathbb{R}^d$, every $R \in (0,1)$, every function $f : \mathbb{R}^d \to \mathbb{R}$ which is harmonic in $B(x_0, R)$ and every $\rho \in (0, R/2)$*

$$\sup_{x,y \in B(x_0,\rho)} |f(x) - f(y)| \le c_3 \|f\|_\infty (\rho/R)^\beta\,, \tag{1.6}$$

$$\text{in particular} \qquad \|f\|_{C^\beta(\overline{B(x_0,R/2)})} \le c_4 \|f\|_\infty\,. \tag{1.7}$$

Let us comment on the differences between our results and those of [BL02]:

(1) We can treat kernels $n(x,h)$ for which the quantity

$$\inf_{x\in\mathbb{R}^d} \liminf_{r\to 0+} \frac{|\{h \in B(0,r); n(x,h) = 0\}|}{|B(0,r)|}$$

is arbitrarily close to 1, e.g. $n(x,h)$ as in (1.9).
(2) For fixed $x \in \mathbb{R}^d$, upper and lower bounds for $n(x,h)$ may not allow for scaling.
(3) Large jumps of the process might not be comparable, i.e. the quantity

$$\sup \left\{ \frac{n(x,h_1)}{n(y,h_2)}; |x-y| \le 1, |h_1 - h_2| \le 1, |h_2| + |h_1| \ge 2 \right\}$$

might be infinite.
(4) We establish a new version of the Harnack inequality and derive a-priori Hölder regularity estimate as a consequence. In a different setting, this procedure was recently established in [Kas].

The constants in the main results of our work and [BL02] depend on α. It would be desirable to adopt the technique further such that results would be robust for $\alpha \to 2$. Under an assumption like (1.2), this has been acheived with analytic techniques in [Sil06] and [Kas].

Comparing our results to the local theory of second order partial differential equations, a natural question arises: Which is a natural class of kernels n such that similar results hold true?

We call a kernel n of the above type nondegenerate if there is a function $N : (0,1) \to (0,\infty)$ with $\lim\limits_{\rho\to 0+} N(\rho) = +\infty$ and $\lambda, \Lambda > 0$ such that for every $\rho \in (0,1)$ and $x \in \mathbb{R}^d$ the symmetric matrix $[A^\rho_{ij}(x)]_{i,j=1}^d$ defined by

$$A^\rho_{ij}(x) = N(\rho) \int\limits_{\{0<|h|\le\rho\}} h_i h_j n(x,h)\, dh\,.$$

satisfies for every $\xi \in \mathbb{R}^d$

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{d} A^\rho_{i,j}(x)\xi_i\xi_j \le \Lambda|\xi|^2\,. \tag{1.8}$$

If n depends only on h and $N(\rho) = \rho^{\alpha-2}$, then this condition implies that the corresponding Lévy process has a smooth density, see [Pic96]. Note that condition (1.2) implies the nondegeneracy condition (1.8) with $N(\rho) = \rho^{\alpha-2}$ but is not necessary, just consider the example

$$n(x,h) = |h|^{-d-\alpha}\mathbb{1}_{\{|h_1|\ge 0.99|h|\}}\,. \tag{1.9}$$

Note that (1.8) holds under our assumptions.

Let us comment on other articles that generalize the results of [BL02]. Note that we do not include works on nonlocal Dirichlet forms. [SV04] gives conditions on Lévy processes and more general Markov jump processes such that the theory of [BL02] is applicable. In [BK05a] the theory is extended to the variable order case and to situations where the lower and upper bound in (1.2) behave differently for $|h| \to 0$. In these cases, regularity of harmonic functions does not hold. Regularity is established in [BK05b] for variable order cases under additional assumptions. Fine potential theoretic results are obtained in [BSS02, BS05] for stable processes. The case of Lévy processes with truncated stable Lévy densities is covered in [KS07] and generalized in [Mim10]. As mentioned above there is an independent approach with analytic methods developed in [Sil06, CS09] covering linear and fully nonlinear integro-differential operators.

Notation: For two functions f and g we write $f(t) \sim g(t)$ if $f(t)/g(t) \to 1$. For $A \subset \mathbb{R}^d$ open or closed τ_A denotes the first exit time of the Markov process under consideration. T_A denotes the the first hitting time of the set A.

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2. Some probabilistic estimates

In this section we prove useful auxiliary results. We follow closely the ideas of [BL02]. However, we need to provide several computations because of the appearance of a slowly varying function in (J1). The proofs of Proposition 2.7 and Proposition 2.9 are significantly different from their counterparts in [BL02].

The following proposition will be used often in obtaining probabilistic estimates.

Proposition 2.1. *Let $A, B \subset \mathbb{R}^d$ be disjoint Borel sets. Then for every bounded stopping time T*

$$\mathbb{E}^x\left[\sum_{s\le T} \mathbb{1}_{\{X_{s-}\in A, X_s\in B\}}\right] = \mathbb{E}^x\left[\int_0^T \int_B \mathbb{1}_A(X_s)\, n(X_s, u - X_s)\, du\right]$$

for every $x \in \mathbb{R}^d$.

Proof. By [BL02, Proposition 2.3] it follows that the process

$$\left\{\sum_{s\le t} \mathbb{1}_{\{X_{s-}\in A, X_s\in B\}} - \int_0^t \int_B \mathbb{1}_A(X_s)\, n(X_s, u - X_s)\, du\right\}_{t\ge 0}$$

is a $\mathbb{P}^x$-martingale. Therefore the result follows by the optional stopping theorem. □

The following result, taken from the theory of regular variation, will be repeatedly used throughout the paper.

Proposition 2.2. *Assume that $\ell\colon (0,2) \to (0,\infty)$ varies slowly at 0 and let $\beta_1 > -1$ and $\beta_2 > 1$. Then the following is true:*

(i) $\displaystyle\int_0^r u^{\beta_1}\ell(u)\, du \sim \frac{r^{1+\beta_1}}{1+\beta_1}\ell(r)$ *as $r \to 0+$,*

(ii) $\displaystyle\int_r^1 u^{-\beta_2}\ell(u)\, du \sim \frac{r^{1-\beta_2}}{\beta_2 - 1}\ell(r)$ *as $r \to 0+$.*

Proof. By a change of variables and using [BGT87, Proposition 1.5.10] we obtain

$$\int_0^r u^{\beta_1}\ell(u)\, du = \int_{r^{-1}}^\infty u^{-\beta_1-2}\ell(u^{-1})\, du \sim \frac{r^{1+\beta_1}\ell(r)}{1+\beta_1},$$

since $u \mapsto \ell(u^{-1})$ varies slowly at infinity. This proves (i). Similarly, with the help of [BGT87, Proposition 1.5.8] we obtain (ii). □

Remark 2.3. *Using* [BGT87, Theorem 1.5.4] *we conclude that for a function* $\ell\colon (0,2) \to (0,\infty)$ *that varies slowly at* 0 *there exists a non-increasing function* $\phi\colon (0,2) \to (0,\infty)$ *such that*

$$\lim_{r\to 0+} \frac{r^{-d-\alpha}\ell(r)}{\phi(r)} = 1.$$

Before proving our main probabilistic estimates, note that (1.5) implies that there exists $\vartheta \in (0,\pi/2]$ such that for every $i \in \{1,\dots,N\}$

$$n(x,h) \geq \delta\, j(|h|) \quad \text{for all } \ h \in \mathbb{R}^d,\ h \neq 0,\ \frac{|\langle h,\eta_i\rangle|}{|h|} \geq \cos\vartheta. \tag{2.1}$$

2.1. **Exit time estimates.**

Proposition 2.4. *There exists a constant* $C_1 > 0$ *such that for every* $x_0 \in \mathbb{R}^d$, $r \in (0,1)$ *and* $t > 0$

$$\mathbb{P}^{x_0}(\tau_{B(x_0,r)} \leq t) \leq C_1 t \frac{\ell(r)}{r^\alpha}.$$

Proof. Again, we closely follow the ideas in [BL02]. Let $x_0 \in \mathbb{R}^d$, $r \in (0,1)$ and let $f \in C^2(\mathbb{R}^d)$ be a positive function such that

$$f(x) = \begin{cases} |x-x_0|, & |x-x_0| \leq \frac{r}{2} \\ r^2, & |x-x_0| \geq r \end{cases}$$

and

$$|f(x)| \leq c_1 r^2, \quad \left|\frac{\partial f}{\partial x_i}(x)\right| \leq c_1 r \quad \text{and} \quad \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right| \leq c_1,$$

for some constant $c_1 > 0$.

Let $x \in B(x_0,r)$. We estimate $\mathcal{L}f(x)$ in a few steps.

First

$$\begin{aligned} &\int_{B(x_0,r)} \big(f(x+h) - f(x) - \langle \nabla f(x), h\rangle \mathbb{1}_{\{|h|\leq 1\}}\big)\, n(x,h)\, dh \\ &\leq c_2 \int_{B(x_0,r)} |h|^2 n(x,h)\, dh \leq c_2 \int_{B(x_0,r)} |h|^{2-d-\alpha}\ell(|h|)\, dh \\ &\leq c_3 r^{2-\alpha}\ell(r), \end{aligned}$$

where in the last line we have used Proposition 2.2 (i). Similarly, by Proposition 2.2 (ii) on $B(x_0,r)^c$ we get

$$\int_{B(x_0,r)^c} (f(x+h)-f(x))\, n(x,h)\, dh \le \|f\|_\infty \int_{B(x_0,r)^c} n(x,h)\, dh$$
$$\le \|f\|_\infty \left(\int_{B(x_0,1)\setminus B(x_0,r)} |h|^{-d-\alpha}\ell(|h|)\, dh + \int_{B(x_0,1)^c} n(x,h)\, dh \right)$$
$$\le c_1 r^2 \left(c_4 r^{-\alpha}\ell(r) + c_5\right) \le c_6 r^{2-\alpha}\ell(r).$$

In the last inequality we have used the fact that $\lim_{r\to 0+} r^{-\alpha}\ell(r) = \infty$ (cf. [BGT87, Proposition 1.3.6 (v)]). Finally, by symmetry of the kernel, we have

$$\int_{B(x_0,1)\setminus B(x_0,r)} \langle h, \nabla f(x)\rangle n(x,h)\, dh = 0. \tag{2.2}$$

Therefore, by preceding estimates, we conclude that there is a constant $c_7 > 0$ such that for all $x \in \mathbb{R}^d$ and $r \in (0,1)$

$$\mathcal{L} f(x) \le c_7 r^{2-\alpha}\ell(r). \tag{2.3}$$

It follows from the optional stopping theorem that

$$\mathbb{E}^{x_0} f(X_{t\wedge\tau_{B(x_0,r)}}) - f(x_0) = \mathbb{E}^{x_0}\int_0^{t\wedge\tau_{B(x_0,r)}} \mathcal{L}f(X_s)\, ds \le c_7 t r^{2-\alpha}\ell(r), \quad t>0. \tag{2.4}$$

On $\{\tau_{B(x_0,r)} \le t\}$ one has $X_{t\wedge\tau_{B(x_0,r)}} \notin B(x_0,r)$ and so $f(X_{t\wedge\tau_{B(x_0,r)}}) \ge r^2$. Then (2.4) gives

$$\mathbb{P}^{x_0}(\tau_{B(x_0,r)} \le t) \le c_7 t\, r^{-\alpha}\ell(r).$$

□

Proposition 2.5. *There exists a constant $C_2 > 0$ such that for every $r \in (0,1)$ and $x_0 \in \mathbb{R}^d$*

$$\inf_{y\in B(x_0,r/2)} \mathbb{E}^y \tau_{B(x_0,r)} \ge C_2 \frac{r^\alpha}{\ell(r)}.$$

Proof. Let $r \in (0,1)$, $x_0 \in \mathbb{R}^d$ and $y \in B(x_0,r/2)$. Using Proposition 2.4 we obtain

$$\mathbb{P}^y(\tau_{B(x_0,r)} \le t) \le \mathbb{P}^y(\tau_{B(y,r/2)} \le t) \le C_1\, t\, r^{-\alpha}\ell(r) \quad \text{for} \quad t>0.$$

Let

$$t_0 = \frac{r^\alpha}{2C_1\ell(r)}.$$

Then

$$\mathbb{E}^y \tau_{B(x_0,r)} \ge t_0 \mathbb{P}^y(\tau_{B(x_0,r)} \ge t_0) \ge \frac{r^\alpha}{2C_1\ell(r)}.$$

□

Proposition 2.6. *There exists a constant $C_3 > 0$ such that for every $r \in (0, \frac{1}{2})$ and $x_0 \in \mathbb{R}^d$*

$$\sup_{y \in B(x_0,r)} \mathbb{E}^y \tau_{B(x_0,r)} \le C_3 \frac{r^\alpha}{\ell(r)}.$$

Proof. Let $r \in (0, \frac{1}{2})$, $x_0 \in \mathbb{R}^d$ and $y \in B(x_0, r)$. Denote by S the first time when process $(X_t)_{t \ge 0}$ has a jump larger than $2r$, i.e.

$$S = \inf\{t > 0 \colon |X_t - X_{t-}| > 2r\}.$$

Assume first that $\mathbb{P}^y(S \le \frac{r^\alpha}{\ell(r)}) \le \frac{1}{2}$. Then by Proposition 2.1

$$\begin{aligned} \mathbb{P}^y\left(S \le \tfrac{r^\alpha}{\ell(r)}\right) &= \mathbb{E}^y \left[\sum_{s \le \frac{r^\alpha}{\ell(r)} \wedge S} \mathbb{1}_{\{|X_s - X_{s-}| > 2r\}} \right] \\ &= \mathbb{E}^y \left[\int_0^{\frac{r^\alpha}{\ell(r)} \wedge S} \int_{B(0,2r)^c} n(X_s, h)\, dh\, ds \right] \end{aligned} \tag{2.5}$$

Choose arbitrary $\xi_0 \in \{\eta_1, \ldots, \eta_N\}$ and let ϑ be as in (2.1). Then

$$\begin{aligned} \int_{B(0,2r)^c} n(X_s, h)\, dh &\ge \int_{\left\{h \in \mathbb{R}^d \colon 2r \le |h| < 1, \frac{|\langle h, \xi_0 \rangle|}{|h|} \ge \cos\vartheta \right\}} n(X_s, h)\, dh \\ &\ge \delta \int_{\left\{h \in \mathbb{R}^d \colon 2r \le |h| < 1, \frac{|\langle h, \xi_0 \rangle|}{|h|} \ge \cos\vartheta \right\}} \frac{\ell(|h|)}{|h|^{d+\alpha}}\, dh \\ &\ge c_1 \int_{2r}^1 \frac{\ell(t)}{t^{1+\alpha}}\, dt \ge c_2 \frac{\ell(r)}{r^\alpha}, \end{aligned}$$

where in the last inequality we have used Proposition 2.2 (ii). Using this estimate we get from (2.5) the following estimate

$$\begin{aligned} \mathbb{P}^y\left(S \le \frac{r^\alpha}{\ell(r)}\right) &\ge c_2 \frac{\ell(r)}{r^\alpha} \mathbb{E}^y \left[\frac{r^\alpha}{\ell(r)} \wedge S \right] \\ &\ge c_2 \mathbb{P}^y \left(S > \frac{r^\alpha}{\ell(r)} \right) \ge \frac{c_2}{2}. \end{aligned}$$

Therefore, in any case the following inequality holds:

$$\mathbb{P}^y\left(S \le \frac{r^\alpha}{\ell(r)}\right) \ge \frac{1}{2} \wedge \frac{c_2}{2}.$$

Since $S \geq \tau_{B(x_0,r)}$ we conclude

$$\mathbb{P}^y\left(\tau_{B(x_0,r)} \leq \frac{r^\alpha}{\ell(r)}\right) \geq \mathbb{P}^y\left(S \leq \frac{r^\alpha}{\ell(r)}\right) \geq c_3,$$

with $c_3 = \frac{1}{2} \wedge \frac{c_2}{2}$. By the Markov property, for $m \in \mathbb{N}$ we obtain

$$\begin{aligned}\mathbb{P}^y\left(\tau_{B(x_0,r)} > (m+1)\frac{r^\alpha}{\ell(r)}\right) &\leq \mathbb{P}^y\left(\tau_{B(x_0,r)} > m\frac{r^\alpha}{\ell(r)}, \tau_{B(x_0,r)} \circ \theta_{m\frac{r^\alpha}{\ell(r)}} > \frac{r^\alpha}{\ell(r)}\right) \\ &= \mathbb{E}^y\left[\mathbb{P}^{X_{m\frac{r^\alpha}{\ell(r)}}}\left(\tau_{B(x_0,r)} > \frac{r^\alpha}{\ell(r)}\right); \tau_{B(x_0,r)} > m\frac{r^\alpha}{\ell(r)}\right] \\ &\leq (1-c_3)\mathbb{P}^y\left(\tau_{B(x_0,r)} > m\frac{r^\alpha}{\ell(r)}\right),\end{aligned}$$

where θ_s denotes the usual shift operator. By iteration we obtain

$$\mathbb{P}^y\left(\tau_{B(x_0,r)} > m\frac{r^\alpha}{\ell(r)}\right) \leq (1-c_3)^m, \ m \in \mathbb{N}.$$

Finally,

$$\begin{aligned}\mathbb{E}^y\tau_{B(x_0,r)} &\leq \frac{r^\alpha}{\ell(r)}\sum_{m=0}^\infty (m+1)\mathbb{P}^y\left(\tau_{B(x_0,r)} > m\frac{r^\alpha}{\ell(r)}\right) \\ &\leq \frac{r^\alpha}{\ell(r)}\sum_{m=0}^\infty (m+1)(1-c_3)^m \leq c_4\frac{r^\alpha}{\ell(r)}.\end{aligned}$$

□

2.2. **Krylov-Safonov type estimate.** Fix $\vartheta \in (0, \pi/2]$ such that (2.1) holds.

Proposition 2.7. *Let $\lambda \in \left(0, \frac{\sin\vartheta}{8}\right]$. There exists a constant $C_4 = C_4(\lambda) > 0$ such that for every $x_0 \in \mathbb{R}^d$, $r \in (0, \frac{1}{2})$, closed set $A \subset B(x_0, \lambda r)$ and $x \in B(x_0, \lambda r)$,*

$$\mathbb{P}^x(T_A < \tau_{B(x_0,r)}) \geq C_4\frac{|A|}{|B(x_0,r)|}.$$

Proof. Choose arbitrary $\xi_0 \in \{\eta_1, \dots, \eta_N\}$ and set $\tilde{x}_0 = x_0 - \frac{r}{2}\xi_0$. The idea is to choose $\lambda \in (0, \frac{1}{8}]$ such that

$$\frac{|\langle u-v, \xi_0\rangle|}{|u-v|} \geq \cos\vartheta \tag{2.6}$$

for all $u \in B(x_0, 2\lambda r)$, $v \in B(\tilde{x}_0, 2\lambda r)$. Since for every $u \in B(x_0, 2\lambda r)$ and $v \in B(\tilde{x}_0, 2\lambda r)$

$$\frac{|\langle u-v, \xi_0\rangle|}{|u-v|} \geq \frac{\sqrt{(\frac{r}{4})^2 - (2\lambda r)^2}}{\frac{r}{4}} = \sqrt{1-(8\lambda)^2}.$$

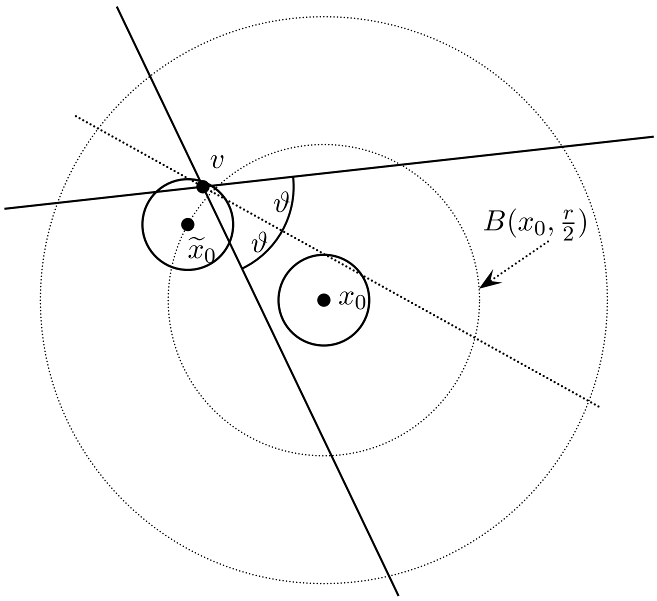

FIGURE 1. The choice of $\widetilde{x}_0$ and λ.

it is enough to choose $\lambda \in (0, \frac{1}{8}]$ such that

$$\sqrt{1-(8\lambda)^2} \geq \cos\vartheta,$$

or, more explicitly,

$$\lambda \leq \frac{\sin\vartheta}{8}.$$

For $s > 0$ we denote $B(x_0, s)$ and $B(\tilde{x}_0, s)$ by B_s and $\tilde{B}_s$. Let $r \in (0,1)$, $\lambda \in (0, \frac{\sin\vartheta}{8}]$, $x \in B_{\lambda r}$ and let $A \subset B_{\lambda r}$ be a closed subset. The strong Markov property now implies

$$\begin{aligned} \mathbb{P}^x(T_A < \tau_{B_r}) &\geq \mathbb{P}^x\left(X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}, X_{\tau_{\tilde{B}_{2\lambda r}}} \circ \theta_{\tau_{B_{2\lambda r}}} \in A\right) \\ &= \mathbb{E}^x\left[\mathbb{P}^{X_{\tau_{B_{2\lambda r}}}}(X_{\tau_{\tilde{B}_{2\lambda r}}} \in A); X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}\right]. \end{aligned} \tag{2.7}$$

For every $y \in \tilde{B}_{\lambda r}$ and $t > 0$ Proposition 2.1 and (2.6) yield

$$\begin{aligned} &\mathbb{P}^y(X_{\tau_{\tilde{B}_{2\lambda r}} \wedge t} \in A) = \mathbb{E}^y\left[\sum_{s \leq \tau_{\tilde{B}_{2\lambda r}} \wedge t} \mathbb{1}_{\{X_{s-} \neq X_s, X_s \in A\}}\right] \\ &\quad = \mathbb{E}^y\left[\int_0^{\tau_{\tilde{B}_{2\lambda r}} \wedge t} \int_A n(X_s, z - X_s)\, dz\, ds\right] \geq \delta\, \mathbb{E}^y\left[\int_0^{\tau_{\tilde{B}_{2\lambda r}} \wedge t} \int_A \frac{\ell(|z - X_s|)}{|z - X_s|^{d+\alpha}}\, dz\, ds\right]. \end{aligned}$$

Letting $t \to \infty$ and using the monotone convergence theorem we deduce

$$\mathbb{P}^y(X_{\tau_{\tilde{B}_{2\lambda r}}} \in A) \geq \delta\, \mathbb{E}^y \left[\int_0^{\tau_{\tilde{B}_{2\lambda r}}} \int_A \frac{\ell(|z - X_s|)}{|z - X_s|^{d+\alpha}}\, dz\, ds \right].$$

Since $|z - X_s| \leq r/2 + 4\lambda r \leq r$, by Remark 2.3 we conclude

$$\begin{aligned} \mathbb{P}^y(X_{\tau_{\tilde{B}_{2\lambda r}}} \in A) &\geq c_1 \frac{\ell(r)}{r^{d+\alpha}} |A| \mathbb{E}^y \tau_{\tilde{B}_{2\lambda r}} \\ &\geq c_2 \ell(r) \frac{|A|}{|B_r|} r^{-\alpha} \mathbb{E}^y \tau_{\tilde{B}_{2\lambda r}}. \end{aligned}$$

Using Proposition 2.5 we deduce

$$\mathbb{P}^y(X_{\tau_{\tilde{B}_{2\lambda r}}} \in A) \geq c_3 \frac{\ell(r)}{\ell(2\lambda r)} \lambda^\alpha \frac{|A|}{|B_r|}. \tag{2.8}$$

Since ℓ varies slowly at 0 we finally obtain

$$\mathbb{P}^y(X_{\tau_{\tilde{B}_{2\lambda r}}} \in A) \geq c_4 \frac{|A|}{|B_r|} \quad \text{for all} \quad y \in \tilde{B}_{\lambda r}, \tag{2.9}$$

for some constant $c_4 = c_4(\lambda) > 0$. By symmetry and (2.9) we deduce

$$\mathbb{P}^x(X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}) \geq c_4 \frac{|\tilde{B}_{\lambda r}|}{|\tilde{B}_r|} \quad \text{for all} \quad x \in B_{\lambda r}. \tag{2.10}$$

Finally, by (2.7), (2.9) and (2.10) we get

$$\mathbb{P}^x(T_A < \tau_{B_r}) \geq c_4^2 \lambda^d \frac{|A|}{|B_r|}.$$

□

2.3. **Restricted Harnack inequality.** The aim of this subsection is to establish a Harnack inequality for a restricted class of harmonic functions.

The following lemma can be proved similarly as [Mim10, Lemma 2.7].

Lemma 2.8. *Let $g\colon (0,\infty) \to [0,\infty)$ be a function satisfying*

$$g(s) \leq c g(t) \quad \text{for all} \quad 0 < t \leq s,$$

for some constant $c > 0$. There is a constant $c' > 0$ such that for any $x_0 \in \mathbb{R}^d$ and $r > 0$ we have

$$g(|z - x|) \leq c' r^{-d} \int_{B(x_0, r)} g(|z - u|)\, du,$$

for all $x \in B(x_0, r/2)$ and $z \in B(x_0, 2r)^c$.

Proposition 2.9. *There is a constant $\lambda_0 \in (0, \frac{1}{16})$ so that for every $\lambda \in (0, \lambda_0]$ there exists a constant $C_5 = C_5(\lambda) \geq 1$ such that for all $x_0 \in \mathbb{R}^d$, $r \in (0, \frac{1}{2})$ and $x, y \in B(x_0, \lambda r)$*

$$\mathbb{E}^x[H(X_{\tau_{B(x_0,\lambda r)}})] \leq C_5 \mathbb{E}^y[H(X_{\tau_{B(x_0,r)}})],$$

for every non-negative function $H\colon \mathbb{R}^d \to [0, \infty)$ supported in $B(x_0, 3r/2)^c$.

Proof. Let $x_0 \in \mathbb{R}^d$, $r \in (0, \frac{1}{2})$ and let $x, y \in B(x_0, \lambda r)$, where $\lambda \in (0, \lambda_0)$ and $\lambda_0 \in (0, \frac{1}{16})$ is chosen later. λ_0 will depend only on constants in our main assumptions. Take $z \in B(x_0, 3r/2)^c$. There are only two cases.

Case 1: There exists $u_0 \in B(x_0, \lambda r)$ so that $n(u_0, z - u_0) > 0$.

Case 2: $n(u, z - u) = 0$ for all $u \in B(x_0, \lambda r)$.

We consider Case 1. By (1.4) and (1.5) there exist $\xi' \in \{\pm\eta_1, \ldots, \pm\eta_N\}$ and $\vartheta' \in (0, \frac{\pi}{2}]$ with

$$\frac{\langle z - u_0, \xi' \rangle}{|z - u_0|} \geq \cos \vartheta'.$$

Note that ξ', ϑ' depend on u_0, z, x_0 and r but $\vartheta' \geq \vartheta$ uniformly with ϑ as in (2.1).

Set $\tilde{x_0} = x_0 - \frac{r}{2}\xi'$ and take $\lambda_0 \leq \frac{\sin\vartheta}{16}$. Let $B_s := B(x_0, s)$ and $\tilde{B}_s := B(\tilde{x}_0, s)$. As in (2.6), for $\lambda \leq \lambda_0$ we have

$$\frac{|\langle u - v, \xi' \rangle|}{|u - v|} \geq \cos \xi' \ \text{ for all } \ u \in B_{2\lambda r},\ v \in \tilde{B}_{2\lambda r}\,.$$

Choose $\tilde{z_0} \in \partial B_{r/2}$ so that the following conditions hold:

$$\begin{aligned} |z - w| &\leq |z - u| && \text{for all } \ u \in B_{2\lambda r},\ w \in B(\tilde{z_0}, \tfrac{\lambda r}{4})\,, \\ \frac{\langle w - v, \xi' \rangle}{|w - v|} &\geq \cos \vartheta' && \text{for all } \ v \in \tilde{B}_{2\lambda r},\ w \in B(\tilde{z_0}, \tfrac{\lambda r}{4})\,, \\ \frac{\langle z - w, \xi' \rangle}{|z - w|} &\geq \cos \vartheta' && \text{for all } \ w \in B(\tilde{z_0}, \tfrac{\lambda r}{4})\,. \end{aligned} \tag{2.11}$$

In the appendix we briefly explain the geometric argument behind the choice of $\tilde{z_0} \in \partial B_{r/2}$.

Let $B'_s = B(\tilde{z}_0, s)$. By the strong Markov property,

$$\begin{aligned}
\mathbb{E}^y\left[\int_0^{\tau_{B_r}} n(X_s, z-X_s)\, ds\right] &\geq \mathbb{E}^y\left[\int_{\tau_{B_{2\lambda r}}}^{\tau_{B_r}} n(X_s, z-X_s)\, ds; X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}\right] \\
&= \mathbb{E}^y\left[\left\{\int_0^{\tau_{B_r}} n(X_s, z-X_s)\, ds\right\} \circ \theta_{\tau_{B_{2\lambda r}}}; X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}\right] \\
&= \mathbb{E}^y\left[\mathbb{E}^{X_{\tau_{B_{2\lambda r}}}}\left[\int_0^{\tau_{B_r}} n(X_s, z-X_s)\, ds\right]; X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}\right]. \qquad (2.12)
\end{aligned}$$

Similarly, for $v \in \tilde{B}_{\lambda r}$ we have

$$\mathbb{E}^v\left[\int_0^{\tau_{B_r}} n(X_s, z-X_s)\, ds\right] \geq \mathbb{E}^v\left[\mathbb{E}^{X_{\tau_{\tilde{B}_{2\lambda r}}}}\left[\int_0^{\tau_{B_r}} n(X_s, z-X_s)\, ds\right]; X_{\tau_{\tilde{B}_{2\lambda r}}} \in B'_{\frac{\lambda r}{8}}\right]. \qquad (2.13)$$

Let $w \in B'_{\frac{\lambda r}{8}}$. Then (J1), (J2), Proposition 2.5 and (2.11) yield

$$\begin{aligned}
\mathbb{E}^w\left[\int_0^{\tau_{B_r}} n(X_s, z-X_s)\, ds\right] &\geq \mathbb{E}^w\left[\int_0^{\tau_{B'_{\frac{\lambda r}{4}}}} n(X_s, z-X_s)\, ds\right] \\
&\geq c_1 \mathbb{E}^w\left[\int_0^{\tau_{B'_{\frac{\lambda r}{4}}}} j(|z-X_s|)\, ds\right] \geq c_2 \mathbb{E}^w \tau_{B'_{\frac{\lambda r}{4}}} (4\lambda r)^{-d} \int_{B_{4\lambda r}} j(|z-u|)\, du \\
&\geq c_3 \lambda^{\alpha-d} \frac{r^{\alpha-d}}{\ell(\frac{\lambda r}{4})} \int_{B_{4\lambda r}} j(|z-u|)\, du\,. \qquad (2.14)
\end{aligned}$$

Combining (2.12), (2.13) and (2.14) we obtain

$$\begin{aligned}
&\mathbb{E}^y\left[\int_0^{\tau_{B_r}} n(X_s, z-X_s)\, ds\right] \\
&\geq c_3 \lambda^{\alpha-d} \frac{r^{\alpha-d}}{\ell(\frac{\lambda r}{4})} \int_{B_{4\lambda r}} j(|z-u|)\, du\, \mathbb{E}^y\left[\mathbb{P}^{X_{\tau_{B_{2\lambda r}}}}(X_{\tau_{\tilde{B}_{2\lambda r}}} \in B'_{\frac{\lambda r}{8}}); X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}\right].
\end{aligned}$$

Similarly as in the proof of Proposition 2.7 we obtain, for some $c_4 = c_4(\lambda) > 0$

$$\mathbb{P}^v(X_{\tau_{\tilde{B}_{2\lambda r}}} \in B'_{\frac{\lambda r}{8}}) \geq c_4 \quad \text{for all} \quad v \in \tilde{B}_{\lambda r}$$

and

$$\mathbb{P}^u(X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}) \geq c_4 \quad \text{for all} \quad u \in B_{\lambda r}\,.$$

Therefore,

$$\mathbb{E}^y\left[\int_0^{\tau_{B_r}} n(X_s, z-X_s)\,ds\right] \geq c_5 \frac{r^{\alpha-d}}{\ell(\frac{\lambda r}{4})}\int_{B_{4\lambda r}} j(|z-u|)\,du\,. \tag{2.15}$$

On the other hand, by Proposition 2.6 and Lemma 2.8,

$$\begin{aligned}\mathbb{E}^x\left[\int_0^{\tau_{B_{\lambda r}}} n(X_s, z-X_s)\,ds\right] &\leq c_6\mathbb{E}^x\left[\int_0^{\tau_{B_{\lambda r}}} j(|z-X_s|)\,ds\right]\\ &\leq c_7\mathbb{E}^x\tau_{B_{\lambda r}}(4r)^{-d}\int_{B_{4\lambda r}} j(|z-u|)\,du\\ &\leq c_8\frac{r^{\alpha-d}}{\ell(2\lambda r)}\int_{B_{4\lambda r}} j(|z-u|)\,du. \end{aligned} \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$\mathbb{E}^x\left[\int_0^{\tau_{B_{\lambda r}}} n(X_s, z-X_s)\,ds\right] \leq c_9\mathbb{E}^y\left[\int_0^{\tau_{B_r}} n(X_s, z-X_s)\,ds\right]. \tag{2.17}$$

Next, we consider Case 2, i.e. $n(u, z-u) = 0$ for all $u \in B(x_0, \lambda r)$. Also in this case, assertion (2.17) holds true, because

$$\begin{aligned}\mathbb{E}^y\left[\int_0^{\tau_{B_r}} n(X_s, z-X_s)\,ds\right] &\geq 0\,,\\ \mathbb{E}^x\left[\int_0^{\tau_{B_{\lambda r}}} n(X_s, z-X_s)\,ds\right] &= 0\,.\end{aligned} \tag{2.18}$$

We have shown that (2.17) always holds. It is enough to prove the proposition for $H = \mathbb{1}_A$, where $A \subset B(x_0, 3r/2)^c$. We conclude from Proposition 2.1 and (2.17) that

$$\begin{aligned}\mathbb{P}^y(X_{\tau_{B_r}} \in A) &= \int_A \mathbb{E}^y\left[\int_0^{\tau_{B_r}} n(X_s, z-X_s)\,ds\right]dz\\ &\geq c_9^{-1}\int_A \mathbb{E}^x\left[\int_0^{\tau_{B_{\lambda r}}} n(X_s, z-X_s)\,ds\right]dz\\ &= c_9^{-1}\mathbb{P}^x(X_{\tau_{B_{\lambda r}}} \in A).\end{aligned}$$

□

3. Harnack inequality

In this section we prove Theorem 1.2.

Proof of Theorem 1.2. Since f is non-negative in $B(x_0, 4r)$, we may assume that $\inf_{x\in B(x_0,r)} f(x)$ is positive. If not, we would prove the claim for $f_\varepsilon = f + \varepsilon$ and then consider $\varepsilon \to 0+$. By taking a constant multiple of f we may further assume $\inf_{x\in B(x_0,r)} f(x) = \frac{1}{2}$.

Choose $u \in B(x_0, r)$ such that $f(u) \le 1$. By Proposition 2.6 and using properties of slowly varying functions we can find a constant $c_1 > 0$ such that for all $u, v \in \mathbb{R}^d$ and $s \in (0, r]$

$$\mathbb{E}^u \tau_{B(v,2s)} \le c_1 \frac{s^\alpha}{\ell(s)} \quad \text{and} \quad \mathbb{E}^u \tau_{B(v,s)} \le c_1 \frac{r^\alpha}{\ell(r)}. \tag{3.1}$$

From Proposition 2.7 we deduce that there is a constant $c_2 > 0$ and $\lambda \in (0, \frac{\sin\vartheta}{16}]$ such that for all $A \subset B(x_0, 2\lambda r)$ and $y \in B(x_0, 2\lambda r)$

$$\mathbb{P}^y(T_A < \tau_{B(x_0,2r)}) \ge c_2 \frac{|A|}{|B(x_0, 2r)|}. \tag{3.2}$$

Similarly, by Proposition 2.7 we see that there exists a constant $c_3 \in (0, 1)$ such that for every $x \in \mathbb{R}^d$, $s < r$ and $C \subset B(x, \lambda s)$ with $|C|/|B(x, \lambda s)| \ge \frac{1}{3}$

$$\mathbb{P}^x(T_C < \tau_{B(x,s)}) \ge c_3.$$

The idea of the proof is to show that f is bounded from the above in $B(x_0, r)$ by

$$c_4 \left(1 + \frac{r^\alpha}{\ell(r)} \sup_{v\in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) n(v, z - v)\, dz \right),$$

for some constant $c_4 > 0$ that does not depend on f. This will be proved by contradiction.

Define

$$\eta = \frac{c_3}{3} \quad \text{and} \quad \zeta = \frac{\eta}{2C_5}, \tag{3.3}$$

where C_5 is taken from Proposition 2.9.

Assume that there exists $x \in B(x_0, \frac{3r}{2})$ such that $f(x) = K$ for some

$$K > \max\left\{ \frac{K_0}{\zeta}, \frac{2 \cdot 8^d \lambda^{-d} K_0}{c_2 \zeta} \right\},$$

where

$$K_0 = 1 + c_1 \frac{r^\alpha}{\ell(r)} \sup_{v\in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) n(v, z-v)\, dz. \tag{3.4}$$

Let $s = \left(\frac{2K_0}{c_2\zeta K}\right)^{1/d} 2\lambda^{-1} r$. Then $s < \frac{r}{4}$ and

$$|B(x,\lambda s)| = \frac{2K_0}{c_2\zeta\, K} |B(x_0, 2r)| \,.$$

Set $B_s := B(x,s)$ and $\tau_s := \tau_{B(x,s)}$. Let A be a compact subset of

$$A' = \{w\in B(x,\lambda s)\colon f(w)\geq \zeta K\}.$$

By the optional stopping theorem, (3.1), (3.2) and Proposition 2.1

$$\begin{aligned}
1 &\geq f(u) = \mathbb{E}^u[f(X_{T_A\wedge\tau_{B(x_0,2r)}})]\\
&\geq \mathbb{E}^u[f(X_{T_A\wedge\tau_{B(x_0,2r)}}); T_A<\tau_{B(x_0,2r)}] - \mathbb{E}^u[f^-(X_{T_A\wedge\tau_{B(x_0,2r)}}); T_A>\tau_{B(x_0,2r)}]\\
&\geq \zeta K\, \mathbb{P}^u(T_A<\tau_{B(x_0,2r)}) - \mathbb{E}^u[f^-(X_{\tau_{B(x_0,2r)}})]\\
&= \zeta K\, \mathbb{P}^u(T_A<\tau_{B(x_0,2r)}) - \mathbb{E}^u\left[\int_0^{\tau_{B(x_0,2r)}}\int_{B(x_0,4r)^c} f^-(z)n(X_t, z-X_t)\, dz\, dt\right]\\
&\geq c_2\,\zeta K \frac{|A|}{|B(x_0,2r)|} - c_1 \tfrac{r^\alpha}{\ell(r)} \sup_{v\in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z)n(v,z-v)\, dz.
\end{aligned}$$

Using (3.4) we obtain

$$\begin{aligned}
&\frac{|A|}{|B(x,\lambda s)|} \leq\\
&\leq \left(1 + c_1\frac{r^\alpha}{\ell(r)} \sup_{v\in B(x_0,2r)}\int_{B(x_0,4r)^c} f^-(z)n(v,z-v)\, dz\right) \frac{|B(x_0,2r)|}{c_2\zeta K |B(x,\lambda s)|}\\
&= \frac{K_0}{c_2\zeta K}\frac{|B(x_0,2r)|}{|B(x,\lambda s)|} = \tfrac{1}{2},
\end{aligned}$$

which implies

$$\frac{|A'|}{|B(x,\lambda s)|} \leq \frac{1}{2}.$$

Let $C\subset B(x,\lambda s)\setminus A'$ be a compact subset such that

$$\frac{|C|}{|B(x,\lambda s)|} \geq \frac{1}{3}. \tag{3.5}$$

Let $H = f^+\,\mathbb{1}_{B^c_{3s/2}}$. Assume that

$$\mathbb{E}^x[H(X_{\tau_{\lambda s}})] > \eta K. \tag{3.6}$$

Then for any $y \in B(x, \lambda s)$ we have

$$\begin{aligned} f(y) &= \mathbb{E}^y f(X_{\tau_s}) = \mathbb{E}^y f^+(X_{\tau_s}) - \mathbb{E}^y f^-(X_{\tau_s}) \\ &= \mathbb{E}^y f^+(X_{\tau_s}) - \mathbb{E}^y[f^-(X_{\tau_s}); X_{\tau_s} \notin B(x_0, 4r)] \\ &\geq \mathbb{E}^y[f^+(X_{\tau_s}); X_{\tau_s} \notin B_{3s/2}] - \mathbb{E}^y[f^-(X_{\tau_s}); X_{\tau_s} \notin B(x_0, 4r)]. \end{aligned}$$

Applying Proposition 2.9 to H it follows

$$\begin{aligned} f(y) \geq C_5^{-1} & \mathbb{E}^x[f^+(X_{\tau_{\lambda s}}); X_{\tau_{\lambda s}} \notin B_{3s/2}] \\ & - c_1 \frac{r^\alpha}{\ell(r)} \sup_{v \in B(x_0, 2r)} \int_{B(x_0, 4r)^c} f^-(z) n(v, z - v)\, dz. \end{aligned}$$

Combining the last display with the assumption (3.6) and the definition of ζ in (3.3) gives

$$f(y) \geq C_5^{-1} \eta K - K_0 = \zeta K \left(2 - \tfrac{K_0}{\zeta K}\right) \geq \zeta K \quad \text{for all} \quad y \in B(x, \lambda s),$$

which is a contradiction to (3.5). Therefore $\mathbb{E}^x[H(X_{\tau_{\lambda s}})] \geq \eta K$.

Let $M = \sup_{v \in B_{3s/2}} f(v)$. Then

$$\begin{aligned} K = f(x) &= \mathbb{E}^x[f(X_{T_C}); T_C < \tau_s] + \mathbb{E}^x[f(X_{\tau_s}); \tau_s < T_C, X_{\tau_s} \in B_{3s/2}] \\ &\quad + \mathbb{E}^x[f(X_{\tau_s}); \tau_s < T_C, X_{\tau_s} \notin B_{3s/2}] \\ &\leq \zeta K\, \mathbb{P}^x(T_C < \tau_s) + M(1 - \mathbb{P}^x(T_C < \tau_s)) + \eta K \end{aligned}$$

and thus

$$\frac{M}{K} \geq \frac{1 - \eta - \zeta\, \mathbb{P}^x(T_C < \tau_s)}{1 - \mathbb{P}^x(T_C < \tau_s)}.$$

From the last display we conclude that $M \geq K(1 + 2\beta)$ with $\beta = \frac{c_3}{6(1-c_3)} + \frac{\zeta}{2} > 0$. Thus there exists $x' \in B(x, \frac{3s}{2})$ so that $f(x') \geq K(1 + \beta)$.

Using this procedure we obtain sequences (x_n) and (s_n) such that $x_{n+1} \in B(x_n, \frac{3s_n}{2})$ and $K_n := f(x_n) \geq (1 + \beta)^{n-1} K$. Thus

$$\sum_{n=1}^{\infty} |x_{n+1} - x_i| \leq \tfrac{3}{2} \sum_{n=1}^{\infty} s_i \leq c_5 \left(\tfrac{K_0}{K}\right)^{1/d} r,$$

for some constant $c_5 > 0$.

If $K > K_0\, c_5^d$, then (x_n) is a sequence in $B(x_0, \frac{3r}{2})$ such that

$$\lim_{n \to +\infty} f(x_n) \geq \lim_{n \to +\infty} (1 + \beta)^{n-1} K_1 = \infty.$$

This is a contradiction with the boundedness of f and so $K \leq c_5^d K_0$. Thus

$$\sup_{v\in B(x_0,r)} f(v) \leq c_5^d\, K_0$$
$$= c_5^d \left(1 + \frac{r^\alpha}{\ell(r)} \sup_{v\in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) n(v, z-v)\, dz \right).$$

Now, let $x, y \in B(x_0, r)$. Then

$$f(x) \leq c_5^d \left(1 + \frac{r^\alpha}{\ell(r)} \sup_{v\in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) n(v, z-v)\, dz \right)$$
$$\leq 2c_5^d f(y) + c_5^d \frac{r^\alpha}{\ell(r)} \sup_{v\in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) n(v, z-v)\, dz.$$

The proof is complete. □

4. Regularity estimates

In this section we prove a general tool that allows to deduce regularity estimates from the version of the Harnack equality given in Theorem 1.2. This approach is developed in [Kas], see also Theorem 3 in [DK].

Theorem 4.1. *Let $m\colon \mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right) \to [0,\infty)$ be a measurable function such that $\sup\limits_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (|h|^2 \wedge 1) m(x,h)\, dh$ is finite. Assume there is a function $\gamma : (0,\infty) \to (0,\infty)$ such that for all $x, h \in \mathbb{R}^d, h \neq 0$*

$$k\left(\tfrac{h}{|h|}\right) \gamma(|h|) \leq m(x,h) \leq \gamma(|h|)\,, \tag{4.1}$$

where $k : S^{d-1} \to [0,\infty)$ is a measurable bounded symmetric function such that there is $\delta > 0$ and a non-empty open set $I \subset S^{d-1}$ with $k(\xi) \geq \delta$ for every $\xi \in I$. Furthermore, assume that

$$\limsup_{R\to\infty} R^{\sigma_1} \int\limits_{B(0,R)^c} \gamma(|u|)\, du \leq 1\,, \qquad \liminf_{r\to 0+} r^{\sigma_2} \int\limits_{B(0,r)^c} \gamma(|u|)\, du \geq 1\,, \tag{4.2}$$

with $0 < \sigma_1 \leq \sigma_2$. Let $\mathcal{L}$ be a non-local operator defined by

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d\setminus\{0\}} (f(x+h) - f(x) - \langle \nabla f(x), h\rangle\, \mathbb{1}_{\{|h|\leq 1\}}) m(x,h)\, dh \tag{4.3}$$

for $f \in C_b^2(\mathbb{R}^d)$.
Assume that harmonic functions with respect to $\mathcal{L}$ satisfy a Harnack inequality, i.e.

there exist constants $c_1, c_2 \geq 1$ such that for every $x_0 \in \mathbb{R}^d$, $r \in (0, \frac{1}{4})$ and for every bounded function $f \colon \mathbb{R}^d \to \mathbb{R}$ which is non-negative in $B(x_0, 4r)$ and harmonic in $B(x_0, 4r)$ the following Harnack inequality holds for all $x, y \in B(x_0, r)$

$$f(x) \leq c_1 f(y) + c_2 M(x_0, r) \sup_{v \in B(x_0, 2r)} \int_{B(x_0,4r)^c} f^-(z) m(v, z - v)\, dz\,, \tag{4.4}$$

where $M(x_0, r) = (\int_{B(x_0,4r)^c} m(x_0, z - x_0)\, dz)^{-1}$.
Then there exist $\beta \in (0, 1)$, $c \geq 1$ such that for every $x_0 \in \mathbb{R}^d$, every $R \in (0, 1)$, every function $f : \mathbb{R}^d \to \mathbb{R}$ which is harmonic in $B(x_0, R)$ and every $\rho \in (0, R/2)$

$$\sup_{x,y \in B(x_0,\rho)} |f(x) - f(y)| \leq c \|f\|_\infty (\rho/R)^\beta\,. \tag{4.5}$$

Remark: Conditions (4.1), (4.2), (4.3) do not imply in general that $\mathcal{L}$ satisfies a Harnack inequality, see the discussion of Example 2.

Let us illustrate this result by giving two examples.

Example 3: $m(x, h) = |h|^{-d-\alpha}$, i.e. $k \equiv 1$, $\gamma(t) = t^{-d-\alpha}$, $\sigma_1 = \sigma_2 = \alpha$. Then $\mathcal{L} = c(\alpha)\Delta^{\alpha/2}$. The Harnack inequality (4.4) then becomes

$$f(x) \leq c_1 f(y) + c_2 r^\alpha \int_{B(x_0,4r)^c} f^-(z) |z - x_0|^{-d-\alpha}\, dz\,, \tag{4.6}$$

and the theorem can be applied. Note that the function f in (4.6) might be negative outside of $B(x_0, 4r)$.

Example 4: $m(x, h) \asymp |h|^{-d-\alpha}$, i.e. $k \equiv 1$, $\gamma(t) = t^{-d-\alpha}$, $\sigma_1 = \sigma_2 = \alpha$, cf. [BL02]. The Harnack inequality can be formulated as in (4.6).

Proof of Theorem 1.4. We apply Theorem 4.1. Let $k = k_1$ as in (1.4) and $I = B_1$ as in (1.5). Set $m(x, h) = n(x, h)$, $\gamma(t) = j(t)$, $\sigma_1 = \sigma$ and $\sigma_2 = \alpha - \varepsilon$ where $\varepsilon \in (0, \alpha - \sigma)$ is arbitrary. Then the first condition in (4.2) follows from (J3). The second condition follows from

$$r^{\sigma_2} \int_r^\infty s^{d-1} j(s)\, ds = r^{\alpha-\varepsilon} \int_r^\infty s^{-1-\alpha} \ell(s)\, ds \sim (1/\alpha) r^{-\varepsilon} \ell(r) \to +\infty \text{ for } r \to 0+\,,$$

where we use Proposition 2.2 (ii). It remains to check that there is a constant $c > 0$ such that for every $x_0 \in \mathbb{R}^d$ and every $r \in (0, \frac{1}{4})$

$$\frac{r^\alpha}{\ell(r)} \leq c M(x_0, r), \quad \text{i.e.} \quad \int_{B(x_0,4r)^c} m(x_0, z - x_0)\, dz \leq c \frac{\ell(r)}{r^\alpha}\,.$$

This condition follows from

$$\int_{B(x_0,4r)^c} m(x_0, z-x_0)\,dz \le \int_{B(x_0,4r)^c} j(|z-x_0|)\,dz \le c_3 \frac{\ell(4r)}{(4r)^\alpha} \le c_4 \frac{\ell(r)}{r^\alpha}\,, \tag{4.7}$$

where we use Proposition 2.2 (ii) again. □

Proof of Theorem 4.1. For $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$ let ν_r^x denote the measure on $B(x_0,r)^c$ defined by

$$\nu_r^x(A) = \Big(\int\limits_A \gamma(|z-x|)\,dz \Big) \Big(\int\limits_{B(x_0,r)^c} \gamma(z-x_0)\,dz \Big)^{-1}$$

for every Borel set $A \subset B(x_0,r)^c$. With some positive constant $c_5 \ge 1$ depending on k we obtain for every bounded function $f\colon \mathbb{R}^d \to \mathbb{R}$

$$\begin{aligned} &M(x_0,r) \sup_{x\in B(x_0,r/2)} \int\limits_{B(x_0,r)^c} f^-(z) m(x,z-x)\,dz \\ &\le c_5 \Big(\int\limits_{B(x_0,r)^c} \gamma(|y-x_0|)\,dy \Big)^{-1} \sup_{x\in B(x_0,r/2)} \int\limits_{B(x_0,r)^c} f^-(z)\gamma(|z-x|)\,dz\,. \end{aligned}$$

This observation together with the main assumption of the theorem ensures that there exist constants $c_1, c_2 \ge 1$ such that for every such $x_0 \in \mathbb{R}^d$, $r \in (0,1)$ and every bounded function $f\colon \mathbb{R}^d \to \mathbb{R}$ which is non-negative in $B(x_0,r)$ and harmonic in $B(x_0,r)$ the following estimate holds

$$\sup_{B(x_0,r/4)} f \le c_1 \inf_{B(x_0,r/4)} f + c_2 \sup_{x\in B(x_0,r/2)} \int\limits_{B(x_0,r)^c} f^-(z)\nu_r^x(dz)\,. \tag{4.8}$$

We aim to apply Lemma 11 from [DK]. Note that it is not important for the application of [DK, Lemma 11] whether harmonicity is defined with respect to an operator $\mathcal{L}$ or some Dirichlet form. Assumption (4.2) implies that there are $c_6 \ge 1$ and $R_0 > 1$ such that for every $R > R_0$, $r \in (0,1)$ and $x \in B(x_0, r/2)$

$$\int\limits_{B(x_0,R)^c} \gamma(|z-x|)\,dz \le c_6 R^{-\sigma_1} \tag{4.9}$$

Moreover, there is $c_7 \ge 1$ with

$$\Big(\int\limits_{B(x_0,r)^c} \gamma(|z-x_0|)\,dz \Big)^{-1} \le c_7 r^{\sigma_2}\,. \tag{4.10}$$

Estimates (4.9) and (4.10) imply:

$$\exists c_8 \geq 1 \ \forall r \in (0,1) \ \exists j_0 \geq 1 \ \forall j \geq j_0 \ \forall x \in B(x_0, \tfrac{r}{2}) :$$
$$\nu_r^x\big(B(x_0, 2^j r)^c\big) \leq c_8 (2^j r)^{-\sigma_1} r^{\sigma_2} \leq c_8 2^{-\sigma j} .$$

Recall that we assumed $\sigma_1 \leq \sigma_2$. Note that $2^{-\sigma} < 1$ and $c_8^{1/j} \to 1$ for $j \to \infty$. We finally proved

$$\sup_{0<r<1} \limsup_{j\to\infty} (\eta_{r,j})^{1/j} < 1, \qquad \text{where } \eta_{r,j} := \sup_{x \in B(x_0, r/2)} \nu_r^x(B(x_0, 2^j r)^c) < \infty . \quad (4.11)$$

Lemma 11 from [DK] can be applied. The proof is complete. □

Appendix

We explain the geometric arguments behind the proof of Proposition 2.9

Given $\eta \in S^{d-1}$ and $\rho > 0$ we define a cone $V(\eta, \rho) \subset \mathbb{R}^d$ as follows. Set $S(\eta, \rho) = \big(B(\eta, \rho) \cup B(-\eta, \rho)\big) \cap S^{d-1}$ and $V(\eta, \rho) = \{x \in \mathbb{R}^d | x \neq 0, \frac{x}{|x|} \in S(\eta, \rho)\}$.

From now on, we keep $\eta \in S^{d-1}$ and $\rho > 0$ fixed and write V instead of $V(\eta, \rho)$. Choose $\vartheta \in (0, \frac{\pi}{2}]$ so that $\rho^2 = 2(1 - \cos\vartheta)$.

Using a simple geometric argument one can establish the following fact:

Let $\lambda \in (0, \frac{\sin\vartheta}{8})$, $x_0 \in \mathbb{R}^d$, $r \in (0,2)$, $u_0 \in B_{\lambda r}(x_0)$ and $z \in B(x_0, \frac{3r}{2})^c$. Assume $z \in u_0 + V$. Set $\widetilde{x_0} = x_0 - \frac{r}{2}\xi \in \partial B(x_0, \frac{r}{2})$ where $\xi \in \{+\eta, -\eta\}$ is chosen so that $\langle z - u_0, \xi\rangle > 0$, see Figure 2. Then the choice of λ implies

(1) $B(\tilde{x}_0, 2\lambda r) \subset \bigcap_{u \in B(x_0, 2\lambda r)} (u + V)$.

Moreover, there is $\tilde{z}_0 \in \partial B(x_0, \frac{r}{2})$ such that

(2) $B(\tilde{z}_0, \frac{\lambda r}{4}) \subset \bigcap_{v \in B(\tilde{x}_0, 2\lambda r)} (v + V)$,
(3) $z \in \bigcap_{w \in B(\tilde{z}_0, \frac{\lambda r}{4})} (w + V)$,
(4) $|z - \tilde{z}_0| < |z - x_0|$
and thus $|z - w| < |z - u|$ for all $u \in B(x_0, 4\lambda r), w \in B(\tilde{z}_0, \frac{\lambda r}{4})$.

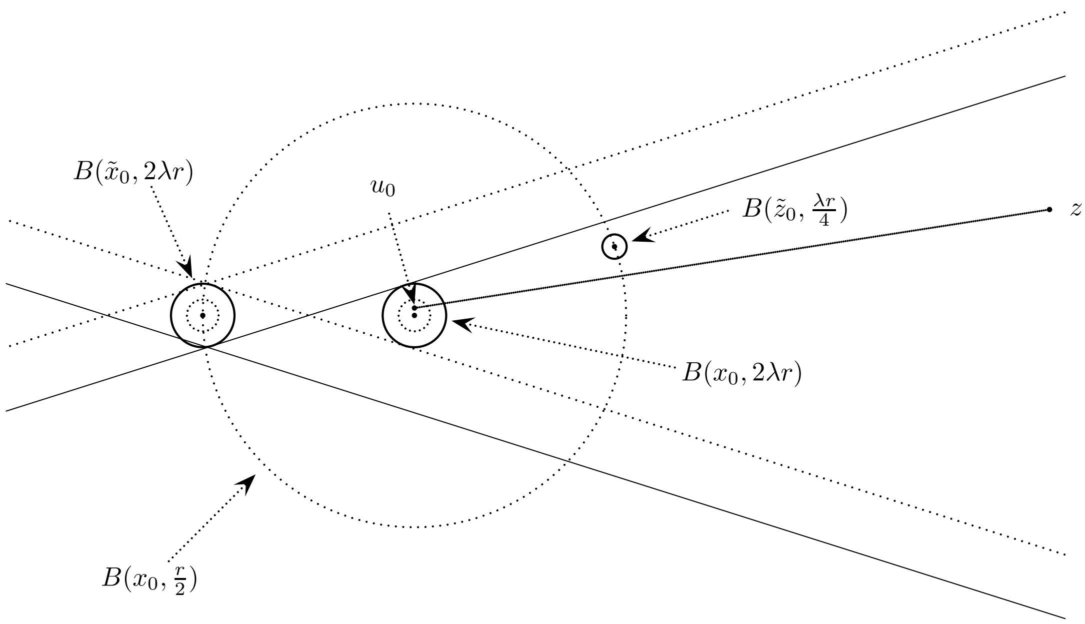

FIGURE 2. The choice of $\tilde{x}_0$ and $\tilde{z}_0$.

These conditions assure that the Markov jump process under consideration has a strictly positive probability to jump from a neighborhood of x_0 via neighborhoods of $\tilde{x}_0$ and $\tilde{z}_0$ to z. One could avoid the introduction of $\tilde{z}_0$ and let the process jump directly from the neighborhood of $\tilde{x}_0$ to z but this would result in a slightly stronger assumption than (J2).

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