Harnack inequality and Hölder regularity estimates for a Lévy process with small jumps of high intensity

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Abstract

We consider a Lévy process in $\mathbb{R}^d$ ($d \geq 3$) with the characteristic exponent

$$\Phi(\xi) = \frac{|\xi|^2}{\ln(1+|\xi|^2)} - 1.$$

The scale invariant Harnack inequality and apriori estimates of harmonic functions in Hölder spaces are proved.

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1 Introduction

In the theory of differential equations Harnack inequalities and regularity properties of harmonic functions play important role. Many local and non-local operators can

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be considered as infinitesimal generators of some Markov processes. In this sense the potential theory has a probabilistic counterpart.

Recently, probabilistic methods turned out to be very successful (cf. [BL02]) in some steps in the proofs of the Harnack inequality and regularity estimates. Extensions to certain classes of Lévy processes were obtained in [SV04, BS05, RSV06, ŠSV06, KS07, Mim10]. More general jump processes were treated in [BK05a, BK05b, CK03].

Let $X = (X_t, \mathbb{P}_x)_{t\geq 0,\ x\in\mathbb{R}^d}$ be a Lévy process in $\mathbb{R}^d$. We say that a function $h\colon \mathbb{R}^d \to [0,\infty)$ is *harmonic* in an open set $D \subset \mathbb{R}^d$ (with respect to the X) if for any open set $B \subset D$ such that $B \subset \overline{B} \subset D$ the following is true

$$h(x) = \mathbb{E}_x[h(X_{\tau_B})1_{\{\tau_B<\infty\}}] \ \ \forall\ x \in B,$$

where $\tau_D = \inf\{t > 0 : X_t \notin B\}$ is the *first exit time* from B. Denote by $B(x, r)$ the (open) ball in $\mathbb{R}^d$ with center $x \in \mathbb{R}^d$ and radius $r > 0$.

The *scale invariant Harnack inequality* holds for the process X if there exists a constant $c > 0$ and $r_0 > 0$ such that for any $x_0 \in \mathbb{R}^d$, $r \in (0, r_0)$ and any non-negative function h on $\mathbb{R}^d$ which is harmonic in $B(x_0, r)$ the following inequality holds

$$h(x) \leq c\, h(y) \ \ \forall\ x, y \in B(x_0, r/2).$$

It is worth mentioning that this type of Harnack inequality does not imply Hölder continuity directly via Moser's method of oscillation reduction. The relation of this two properties is currently investigated (cf. [Kas]).

If $X = (X_t, \mathbb{P}_x)_{t\geq 0,\ x\in\mathbb{R}^d}$ is a purely discontinuous Lévy process, then

$$\mathbb{E}_x[e^{i\xi\cdot(X_t-X_0)}] = e^{-t\Phi(\xi)}$$

with

$$\Phi(\xi) = \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\xi\cdot y} - 1 - i\xi\cdot y1_{\{|y|<1\}}\right)\Pi(dy). \tag{1.1}$$

The measure Π in (1.1) is known as the *Lévy measure*. It is a measure on $\mathbb{R}^d \setminus \{0\}$ which satisfies the following integrability condition

$$\int_{\mathbb{R}^d\setminus\{0\}} (1 \wedge |y|^2)\Pi(dy). \tag{1.2}$$

The aim of this paper is to prove the scale invariant Harnack inequality and Hölder regularity estimates of harmonic functions for purely discontinuous Lévy process in $\mathbb{R}^d$ ($d \geq 3$) with the characteristic exponent

$$\Phi(\xi) = \frac{|\xi|^2}{\ln(1 + |\xi|^2)} - 1.$$

We prove in Proposition 2.1 that Φ is of the form (1.1) with $\Pi(dy) = j(|y|)\, dy$, where $j\colon (0,\infty) \to [0,\infty)$ is a decreasing function satisfying

$$j(r) \sim \frac{\Gamma(d/2-1)}{\pi^{d/2}} \cdot \frac{1}{r^{d+2}\left(\ln\frac{1}{r}\right)^2} \quad \text{as} \quad r \to 0. \tag{1.3}$$

Using asymptotics given in (1.3) to verify condition (1.2) it is easy to see that the corresponding function is on the edge of integrability. This means that the process X has small jumps of very high intensity.

The rotationally invariant α-stable process, $\alpha \in (0,2)$, is an example of a purely discontinuous Lévy process with the Lévy measure of the form

$$\frac{\alpha 2^{\alpha-1}\Gamma((d+\alpha)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)} \cdot \frac{1}{|y|^{d+\alpha}}\, dy.$$

In particular, (1.3) implies that small jumps of X have higher intensity than the corresponding small jumps of any rotationally invariant α-stable process.

The potential operator G defined by

$$Gf(x) = \mathbb{E}_x \left[\int_0^\infty f(X_t)\, dt \right]$$

has a density (with respect to the Lebesgue measure) $G(x,y)$ called the *Green function*. In Proposition 2.3 we obtain the following asymptotics

$$G(x,y) \sim \frac{\Gamma(d/2-1)}{2\pi^{d/2}} |x-y|^{2-d} \ln \frac{1}{|x-y|} \quad \text{as} \quad |x-y| \to 0. \tag{1.4}$$

The Green functions of the rotationally invariant α-stable process and the Brownian motion are given, respectively, by

$$G^{(\alpha)}(x,y) = \frac{2^\alpha \pi^{d/2}\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)} |x-y|^{\alpha-d} \qquad G^{(2)}(x,y) = \frac{4\pi^{d/2}}{\Gamma(n/2-1)} |x-y|^{2-d}.$$

This suggests that the process X is in some sense closer to the Brownian motion than any stable process.

In this paper we assume that $d \geq 3$. The first main result is the scale invariant Harnack inequality.

Theorem 1.1 (Harnack inequality) There exist $R > 0$ and $L_1 > 0$ such that for any $x_0 \in \mathbb{R}^d$ and $r \in (0,R)$ and any non-negative bounded function h on $\mathbb{R}^d$ which is harmonic with respect to X in $B(x_0, 6r)$,

$$h(x) \leq L_1\, h(y) \quad \text{for all } x, y \in B(x_0, r).$$

□

The following result shows that harmonic functions locally satisfy uniform Hölder estimates.

Theorem 1.2 (Hölder continuity) There exists $R' > 0$, $\beta > 0$ and $L_2 > 0$ such that for all $a \in \mathbb{R}^d$, $r \in (0, R')$ and any bounded function h on $\mathbb{R}^d$ which is harmonic in $B(x_0, r)$ we have

$$|h(x) - h(y)| \leq L_2 \|h\|_\infty r^{-\beta} |x-y|^\beta \quad \text{for all} \quad x, y \in B(x_0, r/3).$$

$\square$

The main ingredient in the proofs of the Harnack inequality and regularity of harmonic functions is a Krylov-Safonov type estimate. More precisely, we prove that there is a constant $c > 0$ and $r_0 > 0$ such that for any $r \in (0, r_0)$, $x_0 \in \mathbb{R}^d$ and closed set $A \subset B(x_0, r)$ the following estimate holds

$$\mathbb{P}_y(T_A < \tau_{B(x_0,r)}) \geq c \frac{\mathrm{Cap}(A)}{\mathrm{Cap}(B(x_0,r))} \quad \forall\, y \in B(x_0, r/2) \tag{1.5}$$

(cf. Proposition 2.7). Here Cap denotes the 0-order capacity (cf. Section 2) and $T_A = \inf\{t > 0 \colon X_t \in A\}$.

This kind of estimate for jump processes appeared first in [BL02] with the Lebesgue measure instead of capacity. In [SV04] it was extended to some Markov processes. If we apply Lemma 3.4 from [SV04] to our case, together with (1.3) we obtain an estimate

$$\mathbb{P}_y(T_A < \tau_{B(x_0,r)}) \geq \frac{c}{\ln \frac{1}{r}} \frac{|A|}{|B(x_0,r)|} \quad \forall\, y \in B(x_0, r/2), \tag{1.6}$$

which is not scale invariant.

Here we follow the ideas from [ŠSV06] and [RSV06] and use the capacity instead of the Lebesgue measure to prove the scale invariant Harnack inequality. Furthermore, we use the same idea with capacities instead of the Lebesgue measure to prove that harmonic functions are Hölder continuous.

For $x \in \mathbb{R}^d$ and $r > 0$ by $B(x_0, r)$ we denote the open ball in $\mathbb{R}^d$ with radius r and center a. We say that $f(x) \sim g(x)$ as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

2 Preliminaries and preparatory results

A function $\phi\colon (0,\infty) \to (0,\infty)$ is called a *Bernstein function* if $\phi \in C^\infty((0,\infty))$ and

$$(-1)^{n-1}\phi^{(n)}(\lambda) \geq 0 \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \lambda > 0.$$

We say that $\phi\colon (0,\infty) \to (0,\infty)$ is a *completely monotone function* if $\phi \in C^\infty((0,\infty))$ and

$$(-1)^{n}\phi^{(n)}(\lambda) \geq 0 \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\} \quad \text{and} \quad \lambda > 0.$$

A *subordinator* $S = (S_t)_{t\geq 0}$ is a Lévy process taking values in $[0,\infty)$ and starting at 0. The Laplace transform of S_t is given by

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \ \lambda > 0,$$

where ϕ is called the *Laplace exponent* and it is of the form (cf. page 72 in [Ber96])

$$\phi(\lambda) = d\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\mu(dt). \tag{2.1}$$

Here $d \geq 0$ and μ is a measure on $(0, \infty)$ (called a *Lévy measure*) satisfying

$$\int_{(0,\infty)} (1 \wedge t)\mu(dt) < \infty.$$

Using [SSV10, Theorem 3.2] we conclude that the Laplace exponent ϕ of S is a Bernstein function. Conversely, if ϕ is a Bernstein function such that $\lim_{\lambda \to 0+} \phi(\lambda) = 0$, then there exists a subordinator S with the Laplace exponent ϕ (cf. [Ber96, Theorem I.1]).

We say that $f\colon (0, \infty) \to (0, \infty)$ is a *complete Bernstein function* if it has representation (2.1) such that the Lévy measure has a completely monotone density (with respect to the Lebesgue measure). It follows from [SSV10, Proposition 7.1] that $f^*\colon (0, \infty) \to (0, \infty)$ defined by

$$f^*(\lambda) = \frac{\lambda}{f(\lambda)}$$

is also a complete Bernstein function.

The *potential measure* U of the subordinator S is defined by

$$U(A) = \mathbb{E}\left[\int_0^\infty 1_{\{S_t \in A\}}\, dt\right],\ A \subset [0, \infty).$$

The Laplace transform of U is then

$$\mathcal{L}U(\lambda) = \int_{(0,\infty)} e^{-\lambda t}\, U(dt) = \frac{1}{\phi(\lambda)},\ \lambda > 0. \tag{2.2}$$

Let $B = (B_t, \mathbb{P}_x)_{t \geq 0, x \in \mathbb{R}^d}$ be a Brownian motion in $\mathbb{R}^d$ independent of the subordinator S. The process $X = (X_t, \mathbb{P}_x)_{t \geq 0,\ x \in \mathbb{R}^d}$ defined by

$$X_t = B_{S_t},\ t \geq 0$$

is called the *subordinate Brownian motion.* By [Sat99, Theorem 30.1] we conclude that X is a Lévy process and

$$\mathbb{E}_x\left[e^{i\xi \cdot (X_t - X_0)}\right] = e^{-t\Phi(\xi)},\ \xi \in \mathbb{R}^d$$

with $\Phi(\xi) = \phi(|\xi|^2)$. Moreover, we can rewrite Φ in the following way

$$\Phi(\xi) = \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{i\xi \cdot y} - 1 - i\xi \cdot y 1_{\{|y| \leq 1\}}\right) j(|y|)\, dy$$

where $j\colon (0, \infty) \to (0, \infty)$ is given by

$$j(r) = (4\pi)^{-d/2} \int_{(0,\infty)} t^{-d/2} e^{-r^2/4t}\, \mu(dt),\ r > 0.$$

Note that j is a non-increasing function.

From now on we denote by $S = (S_t)_{t\geq 0}$ the subordinator with the Laplace exponent

$$\phi(\lambda) = \frac{\lambda}{\ln(1+\lambda)} - 1$$

and by X the correspoding subordinate Brownian motion.

Note that

$$\ell(\lambda) = \ln(1+\lambda) = \int_0^\infty (1 - e^{-\lambda t})\, t^{-1} e^{-t}\, dt$$

and thus ℓ is a complete Bernstein function. Therefore, ϕ is also a complete Bernstein function and so the Lévy measure of the subordinator S has a completely monotone density $\mu(t)$.

Let $T = (T_t)_{t\geq 0}$ be the subordinator with the Laplace exponent ℓ. This process is known as the *gamma subordinator*. Since the Lévy measure of T has a completely monotone density, we deduce that the Lévy measure of S has also a completely monotone density, which we denote by $\mu(t)$. It follows from [SSV10, Corollary 10.7 and Corrolary 10.8] that V has a non-increasing density $v(t)$ and that the following is true

$$v(t) = 1 + \int_t^\infty \mu(s)\, ds, \ \ t > 0. \tag{2.3}$$

By [ŠSV06, Theorem 2.2] we get the following asymptotic behavior of v

$$v(t) \sim t^{-1} \left(\ln \frac{1}{t} \right)^{-2} \quad \text{as } t \to 0+ . \tag{2.4}$$

Now we can prove the asymptotic behavior of the jumping function J. The proof of the following proposition is basically the proof of [ŠSV06, Lemma 3.1] but with the use of Potter's theorem (cf. [BGT87, Theorem 1.5.6 (ii)]), which was also done in [KSV, Lemma 5.1].

Recall that the Lévy measure of the subordinate Brownian motion X has a density $J(y) = j(|y|)$.

Proposition 2.1 The following asymptotic behavior of the function j holds

$$j(r) \sim \frac{4\Gamma(\frac{d}{2} - 1)}{\pi^{d/2}} \cdot \frac{1}{r^{d+2} \left(\ln \frac{1}{r^2} \right)^2} \quad \text{as} \quad r \to 0+ .$$

Proof. Using (2.3) and (2.4) we get

$$\int_t^\infty \mu(s)\, ds \sim \frac{1}{t(\ln t)^2}, \quad \text{as} \quad t \to 0+$$

and thus by the Karamata's monotone density theorem (see [BGT87, Theorem 1.7.2]) we have

$$\mu(t) \sim \frac{1}{t^2 (\ln t)^2} \quad \text{as} \quad t \to 0+ . \tag{2.5}$$

By change of variable we get

$$\begin{aligned} j(r) &= (4\pi)^{-d/2} \int_0^\infty t^{-d/2} e^{-\frac{r^2}{4t}} \mu(t)\, dt \\ &= 4^{-1}\pi^{-d/2} r^{-d+2} \int_0^\infty t^{d/2-2} e^{-t} \mu\left(\frac{r^2}{4t}\right) dt \\ &= 4^{-1}\pi^{-d/2} r^{-d+2} \mu(r^2) \int_0^\infty t^{d/2-2} e^{-t} \frac{\mu\left(\frac{|h|^2}{4t}\right)}{\mu(r^2)}\, dt. \end{aligned} \tag{2.6}$$

By Potter's theorem (cf. Theorem 1.5.6 (iii) in [BGT87]) we see that there is a constant $c_1 > 0$ such that

$$\frac{\mu\left(\frac{r^2}{4t}\right)}{\mu(r^2)} \le c_1 (t^{2-1/2} \vee t^{2+1/2}) \quad \text{for all } t > 0 \text{ and } r > 0.$$

Therefore we can apply the dominated convergence theorem in (2.6) and so we obtain

$$\lim_{r\to 0} \frac{j(r)}{4^{-1}\pi^{-d/2} r^{-d+2} \mu(r^2)} = \Gamma(d/2-1).$$

□

If $d \ge 3$, then by [Sat99, Corollary 37.6] and Proposition 2.1 we conclude that X is transient. Thus we can define a measure $G(x, \cdot)$ in the following way

$$G(x, A) = \mathbb{E}_x \left[\int_0^\infty 1_{\{X_t \in A\}}\, dt \right] = \int_0^\infty \mathbb{P}_x(X_t \in A)\, dt. \tag{2.7}$$

By [Sat99, Proposition 28.1] we conclude that X has a transition density $p(t, x, y)$ and therefore we see that a measure defined by (2.7) has a density which we denote by

$$G(x, y) = \int_0^\infty p(t, x, y)\, dt$$

and call the *Green function* of X. Using [Sat99, Theorem 30.1] we see that $G(x, y) = g(|x - y|)$, with

$$g(r) = (4\pi)^{-d/2} \int_{(0,\infty)} t^{-d/2} e^{-r^2/4t}\, U(dt), \tag{2.8}$$

and U is the potential measure of the subordinator S. Combinig Corollaries 10.7 and 10.8 in [SSV10] we see that U has a completely monotone density $u(t)$.

Lemma 2.2 We have the following asymptotics of u

$$u(t) \sim \ln \frac{1}{t} \quad \text{as} \quad t \to 0+, \qquad\qquad u(t) \to 2 \quad \text{as} \quad t \to \infty.$$

Proof. We can readily check that

$$\phi(\lambda) \sim \frac{\lambda}{2} \quad \text{as} \quad \lambda \to 0+, \qquad\qquad \phi(\lambda) \sim \frac{\lambda}{\ln \lambda} \quad \text{as} \quad \lambda \to \infty,$$

and thus by (2.2) we deduce

$$\mathcal{L}U(\lambda) \sim \frac{2}{\lambda} \quad \text{as} \quad \lambda \to 0+, \qquad\qquad \mathcal{L}U(\lambda) \sim \frac{\ln \lambda}{\lambda} \quad \text{as} \quad \lambda \to \infty.$$

By the Karamata's Tauberian theorem (cf. [BGT87, Theorem 1.7.1]) we conclude that

$$U(t) \sim 2t \quad \text{as} \quad t \to \infty, \qquad\qquad U(t) \sim t \ln \frac{1}{t} \quad \text{as} \quad t \to 0+\,.$$

Finally, using Karamata's monotone density theorem (cf. [BGT87, Theorem 1.7.2]) we get

$$u(t) \sim 2 \quad \text{as} \quad t \to \infty, \qquad\qquad u(t) \sim \ln \frac{1}{t} \quad \text{as} \quad t \to 0+\,.$$

□

Proposition 2.3 The following is true

$$g(r) \sim \frac{\Gamma(d/2-1)}{2\pi^{d/2}} r^{2-d} \ln \frac{1}{r} \quad \text{as} \quad r \to 0+, \qquad g(r) \sim \frac{\Gamma(d/2-1)}{2\pi^{d/2}} r^{2-d} \quad \text{as} \quad r \to \infty.$$

Proof. Using (2.8) and changing variable we have

$$\begin{aligned} g(r) &= 4^{-1}\pi^{-d/2} r^{-d+2} \int_0^\infty t^{d/2-2} e^{-t} u\left(\frac{r^2}{4t}\right) dt \\ &= 4^{-1}\pi^{-d/2} r^{-d+2} u(r^2) \int_0^\infty t^{d/2-2} e^{-t} \frac{u\left(\frac{r^2}{4t}\right)}{u(r^2)}\, dt. \end{aligned} \tag{2.9}$$

From Potter's theorem (cf. [BGT87, Theorem 1.5.6 (ii)]) we deduce that there is a constant $c_1 > 0$ such that

$$\frac{u\left(\frac{r^2}{4t}\right)}{u(r^2)} \le c_1 (t^{1/2} \vee t^{-1/2}) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d,\ x \neq 0.$$

Therefore we can use the dominated convergence theorem in (2.9) and thus

$$\lim_{|x|\to 0} \frac{g(r)}{4^{-1}\pi^{-d/2} r^{-d+2} u(r^2)} = \Gamma(d/2-1).$$

Using Proposition 2.2 we obtain the required asymptotics. □

We will need the following technical lemma later.

Lemma 2.4 (a) Let $f\colon (0,1) \to \mathbb{R}$ be defined by

$$f(t) = \frac{t^{d-2}}{\ln \frac{1}{t}}.$$

Then f is stricly increasing and

$$f^{-1}(t) \sim (d-2)^{-1/(d-2)}\, t^{1/(d-2)} \left(\ln \frac{1}{t}\right)^{1/(d-2)} \quad \text{as} \quad t \to 0+\,.$$

(b) The following is true

$$\int_0^r s^{d-1} g(s)\, ds \sim \frac{\Gamma(d/2-1)}{4\pi^{d/2}} r^2 \ln \frac{1}{r} \quad \text{as} \quad r \to 0+\,.$$

Proof.

(a) It is easy to see that f is a strictly increasing function. Define $h\colon (0,1) \to \mathbb{R}$ by

$$h(t) = t^{1/(d-2)} \left(\ln \frac{1}{t}\right)^{1/(d-2)}.$$

Then

$$\begin{aligned}
\lim_{t\to 0+} \frac{f^{-1}(t)}{h(t)} &= \lim_{t\to 0+} \frac{f^{-1}(f(t))}{h(f(t))} = \lim_{t\to 0+} \frac{t}{\frac{t}{(\ln \frac{1}{t})^{\frac{1}{d-2}}} \left(\frac{t^{d-2}}{\ln \frac{1}{t}}\right)^{\frac{1}{d-2}}} \\
&= \lim_{t\to 0+} \left(\frac{\ln \frac{1}{t}}{(d-2)\ln \frac{1}{t} + \ln \ln \frac{1}{t}}\right)^{\frac{1}{d-2}} = (d-2)^{-\frac{1}{d-2}}.
\end{aligned}$$

(b) By applying Karamata's theorem (cf. [BGT87, Proposition 1.5.8]) we get

$$\begin{aligned}
\int_0^r s^{d-1} g(s)\, ds &\sim \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \int_0^r s \ln \frac{1}{s^2}\, ds \\
&\sim \frac{\Gamma(d/2-1)}{4\pi^{d/2}} r^2 \ln \frac{1}{r} \quad \text{as} \quad r \to 0+\,.
\end{aligned}$$

□

Let $D \subset \mathbb{R}^d$ be an open set. We define the *killed process* $X^D = (X_t^D)_{t\geq 0}$ by killing process X upon exiting set D, i.e.

$$X_t^D = \begin{cases} X_t, & t < \tau_D \\ \partial, & t \geq \tau_D. \end{cases}$$

Here ∂ is an extra point adjoined to D. In this case the killed process also has a transition density and it is given (cf. proof of Theorem 2.4 in [CZ01]) by

$$p_D(t,x,y) = p(t,x,y) - \mathbb{E}_x\left[p(t-\tau_D, X_{\tau_D}, y); \tau_D < t\right]. \tag{2.10}$$

The Green function of X^D also exists and it is given by

$$G_D(x,y) = \int_0^\infty p_D(t,x,y)\,dt = G(x,y) - \mathbb{E}_x\left[G(X_{\tau_D},y)\right] \quad \text{for } x,y \in D. \tag{2.11}$$

Since $\mathbb{P}_x(X_{\tau_{B(x_0,r)}} \in \partial B(x_0,r)) = 0$ (cf. [Szt00]) for $x_0 \in \mathbb{R}^d$ and $r > 0$, it follows from Theorem 1 in [IW62] that for any non-negative function $h\colon \mathbb{R}^d \to [0,\infty)$ we have

$$\mathbb{E}_x[h(X_{\tau_{B(x_0,r)}})] = \int_{\overline{B(x_0,r)}^c} \int_{B(x_0,r)} G_{B(x_0,r)}(x,y) j(|z-y|) h(z)\,dy\,dz. \tag{2.12}$$

If we define *Poisson kernel* $K_{B(x_0,r)}\colon B(x_0,r) \times \overline{B(x_0,r)}^c \to [0,\infty)$ by

$$K_{B(x_0,r)}(x,z) = \int_{B(x_0,r)} G_{B(x_0,r)}(x,y) j(|z-y|)\,dy \quad \text{for } x \in B(x_0,r),\ z \in \overline{B(x_0,r)}^c,$$

from (2.12) we get

$$\mathbb{E}_x[h(X_{\tau_{B(x_0,r)}})] = \int_{\overline{B(x_0,r)}^c} K_{B(x_0,r)}(x,z) h(z)\,dz. \tag{2.13}$$

Proposition 2.5 There exists $R_0 \in (0,1)$ and a constant $C_1 > 0$ such that for any $r \le R_0$ and $x_0 \in \mathbb{R}^d$,

$$G_{B(x_0,4r)}(x,y) \ge C_1 r^{2-d} \ln\frac{1}{r} \quad \text{for all } x,y \in B(x_0,r). \tag{2.14}$$

Proof. Choose $0 < c_1 < 1 < c_2$ such that

$$c_1^2 \left(\frac{1}{2}\right)^{d-2} - c_2^2 \left(\frac{1}{3}\right)^{d-2} > 0.$$

Using Proposition 2.3 we can choose $R_0 \in (0,1)$ such that for $r \le 3R_0$ we have

$$c_1 c_3 r^{2-d} \ln\frac{1}{r} \le g(r) \le c_2 c_3 r^{2-d} \ln\frac{1}{r}, \quad c_1 \le \frac{\ln\frac{1}{2r}}{\ln\frac{1}{r}} \le c_2, \quad c_1 \le \frac{\ln\frac{1}{3r}}{\ln\frac{1}{r}} \le c_2, \tag{2.15}$$

where $c_3 = \frac{\Gamma(d/2-1)}{2\pi^{d/2}}$. Let $r \le R_0$, $x_0 \in \mathbb{R}^d$ and $x,y \in B(x_0,r)$. By (2.15) and monotonicity of g we get

$$\begin{aligned}
G_{B(x_0,4r)}(x,y) &= G(x,y) - \mathbb{E}_x[G(Y_{\tau_{B(x_0,4r)}},y)] = g(|x-y|) - \mathbb{E}_x[g(|Y_{\tau_{B(x_0,4r)}} - y|)] \\
&\ge g(2r) - g(3r) \ge c_3 \left(c_1 (2r)^{2-d} \ln\frac{1}{2r} - c_2\,(3r)^{2-d} \ln\frac{1}{3r} \right) \\
&= c_3 r^{2-d} \ln\frac{1}{r} \left(c_1 \left(\frac{1}{2}\right)^{d-2} \frac{\ln\frac{1}{2r}}{\ln\frac{1}{r}} - c_2 \left(\frac{1}{3}\right)^{d-2} \frac{\ln\frac{1}{3r}}{\ln\frac{1}{r}} \right) \\
&\ge c_3 r^{2-d} \ln\frac{1}{r} \frac{c_1^2 \left(\frac{1}{2}\right)^{d-2} - c_2^2 \left(\frac{1}{3}\right)^{d-2}}{c_1\, c_2}.
\end{aligned}$$

Hence we may take

$$C_1 = c_3 \frac{c_1^2 \left(\frac{1}{2}\right)^{d-2} - c_2^2 \left(\frac{1}{3}\right)^{d-2}}{c_1\, c_2} > 0.$$

□

Proposition 2.6 There exist $R_1 \in (0, R_0]$ and a constant $C_2 > 0$ such that for any $r \le R_1$ and $x_0 \in \mathbb{R}^d$,

$$K_{B(x_0,r)}(x,z) \le C_2\, K_{B(x_0,4r)}(y,z) \quad \text{for all } x,y \in B(x_0, r/2),\ z \in B(x_0,4r)^c. \qquad (2.16)$$

Proof. Take $r \le R_0$, $x_0 \in \mathbb{R}^d$, $x, y \in B(x_0, r/2)$ and $z \in B(a, 3r)^c$. Using Proposition 2.5 we get

$$\begin{aligned} K_{B(x_0,4r)}(y,z) &= \int_{B(x_0,4r)} G_{B(x_0,4r)}(y,u)\, j(|z-u|)\, du \ge \int_{B(x_0,r)} G_{B(x_0,4r)}(y,u)\, j(|z-u|)\, du \\ &\ge C_1 r^{2-d} \ln \frac{1}{r} \int_{B(x_0,r)} j(|z-u|)\, du. \qquad (2.17) \end{aligned}$$

On the other side, using Lemma 2.7 in [Mim10] applied to j, Proposition 2.3 and Lemma 2.4 (b) we see that there exists $R_1 \in (0, R_0]$ and constants $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned} K_{B(x_0,r)}(x,z) = & \int_{B(x_0,3r/4)} G_{B(x_0,r)}(x,v) j(|z-v|)\, dv \\ & + \int_{B(a,r)\setminus B(x_0,3r/4)} G_{B(x_0,r)}(x,v) j(|z-v|)\, dv \\ \le & c_1 r^{-d} \int_{B(x_0,r)} j(|z-u|)\, du \int_{B(a,3r/4)} g(|x-v|)\, dv \\ & + c_2 r^{2-d} \ln \frac{1}{r} \int_{B(x_0,r)\setminus B(x_0,3r/4)} j(|z-u|)\, du \\ \le & c_3\, r^{2-d} \ln \frac{1}{r} \int_{B(x_0,r)} j(|z-u|)\, du + c_2 r^{2-d} \ln \frac{1}{r} \int_{B(x_0,r)} j(|z-u|)\, du \\ = & (c_2 + c_3) r^{2-d} \ln \frac{1}{r} \int_{B(x_0,r)} j(|z-u|)\, du, \end{aligned}$$

where in the first term in the last inequality we have used

$$\int_{B(x_0,3r/4)} g(|x-v|)\, dv \le \int_{B(x,2r)} g(|x-v|)\, dv \le c_1' r^2 \ln \frac{1}{r}.$$

Therefore

$$K_{B(x_0,r)}(x,z) \le \frac{c_2 + c_3}{C_1} K_{B(x_0,4r)}(y,z).$$

□

For a measure ρ on $\mathbb{R}^d$ we define its *potential* by

$$G\rho(x) = \int_{\mathbb{R}^d} G(x,y)\,\rho(dx).$$

Denote by Cap the (0-order) *capacity* with respect to X (cf. Section II.2 in [Ber96]). It is proved in Corollary II.8 in [Ber96] that for any compact set $K \subset \mathbb{R}^d$ there exists a measure ρ_K, called the *equilibrium measure*, which is supported by K and satisfies

$$G\rho_K(x) = \mathbb{P}_x(T_K < \infty) \quad \text{for a.e. } x \in \mathbb{R}^d.$$

Moreover, the following is true

$$\frac{1}{\mathrm{Cap}(K)} = \inf\left\{\int_{\mathbb{R}^d} G\rho(x)\,\rho(dx) \colon \rho \text{ is a probability measure supported by } K\right\}$$

and the infimum is attained at the equilibrium measure ρ_K. If we combine Lemma 2.4 (b) and Proposition 5.3 in [ŠSV06] we conclude that there exists constants $C_3, C_4 > 0$ such that

$$C_3 \frac{r^{d-2}}{\ln\frac{1}{r}} \le \mathrm{Cap}\left(\overline{B(x_0,r)}\right) \le C_4 \frac{r^{d-2}}{\ln\frac{1}{r}} \quad \text{for } x_0 \in \mathbb{R}^d,\ 0 < r \le 1. \tag{2.18}$$

Now we can prove a Krylov-Safonov-type estimate.

Proposition 2.7 There exists a constant $C_5 > 0$ such that for any $x_0 \in \mathbb{R}^d$, $r \le R_2$, closed subset A of $B(x_0, 2r)$ and $y \in B(x_0, r)$,

$$\mathbb{P}_y(T_A < \tau_{B(x_0,4r)}) \ge C_5 \frac{\mathrm{Cap}(A)}{\mathrm{Cap}\left(\overline{B(x_0,r)}\right)}.$$

Proof. Let $x_0 \in \mathbb{R}^d$, $r \le R_1$ and let $A \subset B(x_0, 2r)$ be a closed subset. We may assume that $\mathrm{Cap}(A) > 0$. Let ρ_A be the equilibrium measure of A. If $G_{B(x_0,4r)}$ is the Green function of the process X killed upon exiting from $B(x_0, 4r)$, then for $y \in B(x_0, r)$ we have

$$\begin{aligned} G_{B(x_0,4r)}\rho_A(y) &= \mathbb{E}_y[G_{B(x_0,4r)}\rho_A(Y_{T_A}); T_A < \tau_{B(x_0,4r)}] \\ &\le \sup_{z\in\mathbb{R}^d} G_{B(x_0,4r)}\rho_A(z)\mathbb{P}_y(T_A < \tau_{B(x_0,4r)}) \\ &\le \mathbb{P}_y(T_A < \tau_{B(x_0,4r)}) \end{aligned} \tag{2.19}$$

since $G_{B(x_0,4r)}\rho_A(z) \le 1$. Also, for any $y \in B(x_0, r)$ we have

$$\begin{aligned} G_{B(x_0,4r)}\nu_A(y) &= \int_{\mathbb{R}^d} G_{B(x_0,4r)}(y,z)\nu_A(dz) \ge \rho_A(\mathbb{R}^d) \inf_{z\in B(x_0,4r)} G_{B(x_0,4r)}(y,z) \\ &= \mathrm{Cap}(A) \inf_{z\in B(x_0,4r)} G_{B(x_0,4r)}(y,z). \end{aligned} \tag{2.20}$$

Using (2.19), (2.20) and Proposition 2.5 we obtain

$$\mathbb{P}_y(T_A < \tau_{B(x_0,4r)}) \geq C_1 \mathrm{Cap}(A) r^{2-d} \ln \frac{1}{r^2}.$$

By (2.18) we see that

$$\mathbb{P}_y(T_A < \tau_{B(x_0,4r)}) \geq C_1 C_3 \frac{\mathrm{Cap}(A)}{\mathrm{Cap}(\overline{B(x_0,r)})}$$

for $r \leq R_2$. □

3 Harnack inequality

Proof of Theorem 1.1. Define $f(t) = \frac{t^{d-2}}{\ln \frac{1}{t}}$. By Lemma 2.4 (a) we can choose $R \leq \frac{R_2}{9} \wedge \frac{1}{4}$ such that

$$f^{-1}(r) \leq 2c_0 r^{1/(d-2)} \left(\ln \frac{1}{r}\right)^{1/(d-2)} \quad \text{for all} \quad r \leq R \tag{3.1}$$

and for some constant $c_0 > 0$.

Let $x_0 \in \mathbb{R}^d$ and $r \leq R$. Without loss of generality we may suppose

$$\inf_{z \in B(x_0,r)} h(z) = \frac{1}{2}$$

Let $z_0 \in B(x_0,r)$ be such that $h(z_0) \leq 1$. It is enough to show that h is bounded from above by some constant independent of h. By Proposition 2.7 there exists $c_1 > 0$ such that

$$\mathbb{P}_x(T_F < \tau_{B(x,s)}) \geq c_1, \tag{3.2}$$

for any $s \in (0, R_2)$, $x \in \mathbb{R}^d$ and compact $F \subset B(x, s/3)$ such that

$$\frac{\mathrm{Cap}(F)}{\mathrm{Cap}(\overline{B(x,s/3)})} \geq \frac{1}{3}.$$

Put

$$\eta = \frac{c_1}{3}, \quad \zeta = \frac{\eta}{3} \wedge \frac{\eta}{C_2}. \tag{3.3}$$

Suppose that there exists $x \in B(x_0,r)$ such that $h(x) = K$ for

$$K > \frac{2 \cdot 6^{2d-4} c_0^{2d-4} d^2}{C_3\, C_5\, \zeta}. \tag{3.4}$$

It is possible to choose a unique $s > 0$ such that

$$f(s/3) = \frac{2\, C_4}{C_3\, C_5\, \zeta\, K} f(r),$$

since f is strictly increasing and continuous on $(0,1)$. Set $c_2 := \frac{2\,C_4}{C_3\,C_5\,\zeta\,K} < 2^{1-d}$.

Using inequality

$$p \ln \frac{1}{p} < 2\sqrt{p} \quad \text{for} \quad p \in (0,1)$$

and (3.1) we get

$$\begin{aligned}
\frac{s}{3} &\leq 2c_0 \left(c_2 f(r)\right)^{1/(d-2)} \left(\ln \frac{1}{c_2 f(r)}\right)^{1/(d-2)} \\
&\leq (2c_0)^{1+1/(d-2)} c_2^{1/(d-2)} \left(\frac{\ln \frac{1}{c_2}}{\ln \frac{1}{r}} + d - 2 - \frac{\ln\ln \frac{1}{r}}{\ln \frac{1}{r}}\right)^{1/(d-2)} r \\
&\leq (2c_0)^{1+1/(d-2)} \left(c_2 \ln \frac{1}{c_2} + (d-2)c_2\right)^{1/(d-2)} r \\
&\leq (2c_0)^{1+1/(d-2)} \left(2\sqrt{c_2} + (d-2)c_2\right)^{1/(d-2)} r \\
&\leq (2c_0)^{1+1/(d-2)} d^{1/(d-2)} c_2^{1/2(d-2)} r
\end{aligned}$$

and thus $s \leq r$ by (3.4).

By (2.18) we obtain

$$\mathrm{Cap}(\overline{B(x, s/3)}) \geq \frac{2}{C_5\,\zeta\,K} \mathrm{Cap}(\overline{B(x_0, r)}).$$

Let A be a compact subset of

$$A' = \{t \in B(x, s/3) \colon h(t) \geq \zeta K\}.$$

By the optional stopping theorem we have

$$\begin{aligned}
1 \geq h(z_0) &\geq E_{z_0}[h(X_{T_A \wedge \tau_{B(x_0,4r)}}); T_A < \tau_{B(x_0,4r)}] \\
&\geq \zeta K \mathbb{P}_{z_0}(T_A < \tau_{B(x_0,4r)}) \geq C_5 \zeta K \frac{\mathrm{Cap}(A)}{\mathrm{Cap}(\overline{B(x_0, r)})},
\end{aligned}$$

where in the last inequality we have used Proposition 2.7. Therefore,

$$\frac{\mathrm{Cap}(A)}{\mathrm{Cap}(\overline{B(x, s/3)})} = \frac{\mathrm{Cap}(A)}{\mathrm{Cap}(\overline{B(x_0, r)})} \cdot \frac{\mathrm{Cap}(\overline{B(x_0, r)})}{\mathrm{Cap}(\overline{B(x, s/3)})} \leq \frac{1}{2}$$

and thus, by the subadditivity of capacity, there exists a compact set

$$F \subset \overline{B(x, s/3)} \setminus A'$$

such that

$$\frac{\mathrm{Cap}(F)}{\mathrm{Cap}(\overline{B(x, s/3)})} \geq \frac{1}{3}. \tag{3.5}$$

Next we prove that

$$\mathbb{E}_x[h(X_{\tau_{B(x,s)}}); X_{\tau_{B(x,s)}} \notin B(x, 4s)] \leq \eta K.$$

If the latter is not true, then

$$\mathbb{E}_x[h(X_{\tau_{B(x,s)}}); X_{\tau_{B(x,s)}} \notin B(x,4s)] > \eta K$$

and by Proposition 2.6 and (2.13) for any $y \in B(x, s/3)$ we have

$$\begin{aligned} h(y) &= \mathbb{E}_y[h(X_{\tau_{B(x,4s)}})] = \mathbb{E}_y[h(X_{\tau_{B(x,4s)}}); X_{\tau_{B(x,4s)}} \notin B(x,4s)] \\ &= \int_{\overline{B(x,4s)}^c} K_{B(x,4s)}(y,z)h(z)\,dz \geq C_2^{-1} \int_{\overline{B(x,4s)}^c} K_{B(x,s)}(y,z)h(z)\,dz \\ &= C_2^{-1}\mathbb{E}_y[h(X_{\tau_{B(x,s)}}); X_{\tau_{B(x,s)}} \notin B(x,4s)] \geq C_2^{-1}\eta K \geq \zeta K, \end{aligned}$$

which is a contradiction with (3.5) and the definition of the set A'.

Set

$$M = \sup_{B(x,4s)} h.$$

We have

$$\begin{aligned} K = h(x) &= \mathbb{E}_x[h(Y_{\tau_{B(x,s)}})] \\ &= \mathbb{E}_x[h(Y_{T_F}); T_F < \tau_{B(x,s)}] + \mathbb{E}^x[h(Y_{\tau_{B(x,s)}}); \tau_{B(x,s)} < T_F, X_{\tau_{B(x,s)}} \in B(x,4s)] \\ &+ \mathbb{E}_x[h(Y_{\tau_{B(x,s)}}); \tau_{B(x,s)} < T_F, X_{\tau_{B(x,s)}} \notin B(x,4s)] \\ &\leq \zeta K \mathbb{P}_x(\tau_{B(x,s)} < T_F) + M\mathbb{P}_x(\tau_{B(x,s)} < T_F) + \eta K \\ &= \zeta K \mathbb{P}_x(\tau_{B(x,s)} < T_F) + M(1 - \mathbb{P}_x(\tau_{B(x,s)} < T_F)) + \eta K \end{aligned}$$

and thus

$$\frac{M}{K} \geq \frac{1 - \eta - \zeta \mathbb{P}^x(\tau_{B(x,s)} < T_F)}{1 - \mathbb{P}^x(\tau_{B(x,s)} < T_F)} \geq 1 + 2\beta,$$

for some $\beta > 0$. It follows that there exists $x' \in B(x,4s)$ such that $h(x') \geq K(1+\beta)$. Repeating this procedure, we get a sequence (x_n) such that $h(x_n) \geq K(1+\beta)^{n-1}$ and $|x_{n+1} - x_n| \leq \left(12 \cdot (2c_0)^{1+1/(d-2)} d^{1/(d-2)} c_2^{1/2(d-2)}\right)^n r$ and so

$$\sum_{n=1}^{\infty} |x_{n+1} - x_n| \leq c_3 K^{-1/2(d-2)} r$$

If $K > c_3^{2d-4}$ we can find sequence $(x_n)_n$ in $B(x_0, 2r)$ such that $h(x_n) \to \infty$ which is a contradiction to h being bounded. Therefore

$$\sup_{x \in B(x_0,r)} h(x) \leq c_3^{2d-4}.$$

□

4 Regularity

Lemma 4.1 There exists a constant $C_6 > 0$ such that for all $r \in (0, 1/8)$, $s \in [4r, 1/2)$ and $a \in \mathbb{R}^d$ we have

$$\mathbb{P}_x(X_{\tau_{B(x_0,r)}} \notin B(a,s)) \le C_6 \frac{r^2 \ln \frac{1}{r}}{s^2 \left(\ln \frac{1}{s}\right)^2} \quad \text{for all} \quad x \in B(x_0, r/2).$$

Proof. Let $r \in (0, 1/8)$, $s \in [4r, 1/2)$ and $a \in \mathbb{R}^d$. Using (2.12) with $h = 1_{B(a,s)^c}$ for $x \in B(x_0, r/2)$ we have

$$\begin{aligned}
\mathbb{P}_x(X_{\tau_{B(x_0,r)}} \notin B(a,s)) &= \int_{B(x_0,r)} G_{B(x_0,r)}(x,u) \int_{B(a,s)^c} j(|z-u|)\, dz\, du \\
&\le \int_{B(x_0,r)} g(|u-x|) \int_{B(u,s/2)^c} j(|z-u|)\, dz\, du \\
&= \int_{B(0,s/2)^c} j(|z|)\, dz \cdot \int_{B(y-a,r)} g(|u|)\, du \\
&\le \int_{B(0,s/2)^c} j(|z|)\, dz \cdot \int_{B(0,2r)} g(|u|)\, du,
\end{aligned}$$

where in the second inequality we have used the facts that $B(u, s/2) \subset B(x_0, s)$ and $G_{B(x_0,r)}(y,u) \le g(|u-y|)$, while in the last inequality we have used $B(y - x_0, r) \subset B(0, 2r)$. Now the conclusion follows from Proposition 2.1 and Proposition 2.3. $\square$

Proof of Theorem 1.2. Let $r \in (0,1)$, $x_0 \in \mathbb{R}^d$ and let $h\colon \mathbb{R}^d \to [0,\infty)$ be bounded by $M > 0$ and harmonic in $B(x_0, r)$.

Let $z_0 \in B(x_0, r/3)$. Define

$$r_n = \gamma_1 4^{-n},$$

where we choose $\gamma_1 > 0$ small enough so that $B(x_0, 2r_1) \subset B(z_0, r/2)$. Set $B_n = B(z_0, r_n)$ and $\tau_n = \tau_{B_n}$ and

$$s_n = \gamma_2 b^{-n},$$

where constants $\gamma_2 > 0$ and $b > 1$ will be chosen later. Let

$$m_n = \inf_{x \in B_n} h(x) \quad \text{and} \quad M_n = \sup_{x \in B_n} h(x).$$

It is enough to prove that

$$M_k - m_k \le s_k \tag{4.1}$$

for all $k \ge k_0$ for some $k_0 \in \mathbb{N}$. We prove this by induction. Assume that (4.1) holds for $k \in \{1, 2, \ldots, n\}$.

Let $\varepsilon > 0$ and let $x, y \in B_{n+1}$ such that $h(x) \le m_{n+1} + \varepsilon$ and $h(y) \ge M_{n+1} - \varepsilon$. Our aim is to show that $h(y) - h(x) \le s_{n+1}$. Then we have

$$M_{n+1} - m_{n+1} \le s_{n+1} + 2\varepsilon$$

and since $\varepsilon > 0$ is arbitrary, we get (4.1) for $k = n + 1$.

Let $A = \{x \in B_n \colon h(x) \le (m_n + M_n)/2\}$ and assume that

$$\frac{\mathrm{Cap}(A)}{\mathrm{Cap}(B_n)} \ge \frac{1}{2}$$

(if this is not true, then we consider function $M - h$ and use the subadditivity of capacity). By Choquet's theorem A is capacitable and therefore there exists a compact subset $K \subset A$ such that

$$\frac{\mathrm{Cap}(K)}{\mathrm{Cap}(B_n)} \ge \frac{1}{3}.$$

By the optional stopping theorem, we have

$$\begin{aligned}
h(x) - h(y) &= \mathbb{E}_x[h(X_{\tau_n \wedge T_K}) - h(y)] \\
&= \mathbb{E}_x[h(X_{\tau_n \wedge T_K}) - h(y); T_K < \tau_n, X_{\tau_n} \in B_{n-1} \setminus B_n] \\
&\quad + \mathbb{E}_x[h(X_{\tau_n \wedge T_K}) - h(y); T_K > \tau_n, X_{\tau_n} \in B_{n-1} \setminus B_n] \\
&\quad + \sum_{i=1}^{n-2} \mathbb{E}_x[h(X_{\tau_n \wedge T_K}) - h(y); X_{\tau_n} \in B_{n-i-1} \setminus B_{n-i}] \\
&\quad + \mathbb{E}_x[h(X_{\tau_n \wedge T_K}) - h(y); X_{\tau_n} \notin B_1] \\
&\le \left(\frac{m_n + M_n}{2} - m_n \right) \mathbb{P}_x(T_K < \tau_n) + (M_{n-1} - m_{n-1}) \mathbb{P}_x(T_K > \tau_n) \\
&\quad + \sum_{i=1}^{n-2} (M_{n-i-1} - m_{n-i-1}) \mathbb{P}_x(X_{\tau_n} \notin B_{n-i-1}) + 2M \mathbb{P}_x(X_{\tau_n} \notin B_1).
\end{aligned}$$

It follows from Proposition 2.7 that there is a constant $c_1 > 0$ such that $\mathbb{P}_x(T_K < \tau_n) \ge c_1$. Using this and Lemma 4.1 we get from the previous display the following

$$h(x) - h(y) \le \frac{1}{2} s_n + s_{n-1}(1 - c_1) + C_4 \sum_{i=1}^{n-2} s_{n-i-1} \frac{r_n^2 \ln \frac{1}{r_n}}{r_{n-i}^2 \ln \frac{1}{r_{n-i}}} + 2MC_4 \frac{r_n^2 \ln \frac{1}{r_n}}{r_1^2 \ln \frac{1}{r_1}}.$$

It is easy to check that there exist constants $c_3, c_4 > 0$ such that

$$\begin{aligned}
h(x) - h(y) &\le s_{n+1} \left(\frac{b}{2} + b^2(1 - c_1) + c_3 \, n \, b^2 \sum_{i=1}^{n-2} \frac{(16/b^2)^{-i}}{n - i} + c_4 \frac{2M}{\gamma_2} b \, n \, (16/b)^{-n} \right) \\
&\le s_{n+1} \left(\frac{b}{2} + b^2(1 - c_1) + c_3 \, n \, b^2 (16/b)^{-n} \sum_{i=2}^{\infty} (16/b^2)^{-i} + c_4 \frac{2M}{\gamma_2} b \, n \, (16/b)^{-n} \right).
\end{aligned} \tag{4.2}$$

First choose $b \in (1, 2)$ such that $\frac{b}{2} + b^2(1 - c_1) < 1$ and let $\gamma_2 > 2M$. It is easy to see that the last two terms in the parenthesis in (4.2) can be made arbitrary small for n large enough. Thus there is $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ we have

$$h(x) - h(y) \le s_{n+1},$$

which was to be proved. □

References

[Ber96] J. Bertoin, *Lévy processes*, Cambridge University Press, Cambridge, 1996.

[BGT87] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Cambridge University Press, Cambridge, 1987.

[BK05a] R. F. Bass and M. Kassmann, *Harnack inequalities for non-local operators of variable order*, Trans. Amer. Math. Soc. **357** (2005), 837–850.

[BK05b] ______, *Hölder continuity of harmonic functions with respect to operators of variable order*, Comm. Partial Differential Equations **30** (2005), 1249–1259.

[BL02] R. F. Bass and D. Levin, *Harnack inequalities for jump processes*, Potential Anal. **17** (2002), 375–388.

[BS05] K. Bogdan and P. Sztonyk, *Harnack's inequality for stable Lévy processes*, Potential Anal. **22** (2005), 133–150.

[CK03] Z.-Q. Chen and T. Kumagai, *Heat kernel estimates for stable-like processes on d-sets*, Stochastic Process. Appl. **108** (2003), no. 1, 27–62.

[CZ01] K. L. Chung and Z. Zhao, *From Brownian motion to Schrödinger equation*, Springer, Berlin, 2001.

[IW62] N. Ikeda and S. Watanabe, *On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes*, J. Math. Kyoto Univ. **2** (1962), 79–95.

[Kas] M. Kassmann, *Harnack inequalities and Hölder regularity estimates for non-local operators revisited*, (preprint, 2010).

[KS07] P. Kim and R. Song, *Potential theory of truncated stable processes*, Math. Z. **256** (2007), 139–173.

[KSV] P. Kim, R. Song, and Z. Vondraček, *Two-sided Green function estimates for killed subordinate Brownian motions*, (preprint, 2010).

[Mim10] A. Mimica, *Harnack inequalities for some Lévy processes*, Potential Anal. **32** (2010), 275–303.

[RSV06] M. Rao, R. Song, and Z. Vondraček, *Green function estimates and Harnack inequality for subordinate Brownian motions*, Potential Anal. **25** (2006), 1–27.

[Sat99] K.-I. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, 1999.

[ŠSV06] H. Šikić, R. Song, and Z. Vondraček, *Potential theory of geometric stable processes*, Prob. Theory Related Fields **135** (2006), 547–575.

[SSV10] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein functions: theory and applications*, Walter de Gruyter, Berlin, 2010.

[SV04] R. Song and Z. Vondraček, *Harnack inequalities for some classes of Markov processes*, Math. Z. **246** (2004), 177–202.

[Szt00] P. Sztonyk, *On harmonic measure for Lévy processes*, Probab. Math. Statist. **20** (2000), 383–390.