Harnack Inequalities for some Lévy Processes

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Abstract

In this paper we prove Harnack inequality for nonnegative functions which are harmonic with respect to random walks in $\mathbb{R}^d$. We give several examples when the scale invariant Harnack inequality does not hold. For any $\alpha \in (0,2)$ we also prove the Harnack inequality for nonnegative harmonic functions with respect to a symmetric Lévy process in $\mathbb{R}^d$ with a Lévy density given by $c|x|^{-d-\alpha}1_{\{|x|\le 1\}} + j(|x|)1_{\{|x|>1\}}$, where $0 \le j(r) \le cr^{-d-\alpha}$, $\forall r > 1$, for some constant c. Finally, we establish the Harnack inequality for nonnegative harmonic functions with respect to a subordinate Brownian motion with subordinator with Laplace exponent $\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$, $\lambda > 0$, where ℓ is a slowly varying function at infinity and $\alpha \in (0,2)$.

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1 Introduction

Lévy processes became very important class of processes in theory as well as in applications. Recently they have been studied very intensively. There are many important results concerning these processes and among them is also the Harnack inequality for nonnegative harmonic functions (see [1],[3],[6],[7],[8] and [22]). Harnack inequality is very important in the study of harmonic functions, in particular for proving regularity of solutions of some boundary value problems.

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Let $X = (X_t, \mathbb{P}_x)$ be a Lévy process on $\mathbb{R}^d$. A nonnegative function $h\colon \mathbb{R}^d \to [0,\infty)$ is harmonic in an open subset $D \subset \mathbb{R}^d$ with respect to X if for any open subset B of D such that $B \subset \overline{B} \subset D$,

$$h(x) = \mathbb{E}_x[h(X_{\tau_B})1_{\{\tau_B<\infty\}}], \quad \text{for all } x \in B,$$

where $\tau_B = \inf\{t > 0\colon X_t \notin B\}$ is the first exit time from B of the process X. Let $B(a,r) := \{x \in \mathbb{R}^d\colon |x-a| < r\}$ be an open ball with radius $r > 0$ and center $a \in \mathbb{R}^d$. We say that the Harnack inequality holds for X if there is $r_0 > 0$ with the property that for all $r \le r_0$ there exists a constant $C > 0$, which depends only on r, such that for any $a \in \mathbb{R}^d$ and any nonnegative function h which is harmonic in $B(a,r)$ with respect to X,

$$h(x) \le C\, h(y), \quad \text{for all } x, y \in B(a, r/2). \tag{1.1}$$

Then, by the standard chain argument, one can easily show that for any open subset $D \subset \mathbb{R}^d$ and any compact subset $K \subset D$ there exists a constant $C > 0$, depending only on D and K, such that

$$\sup_{x\in K} h(x) \le C \inf_{y\in K} h(y),$$

for any nonnegative function h which is harmonic in D with respect to X.

Generally, the constant C in (1.1) depends on $r > 0$. One would like to investigate when C does not depend on $r > 0$ if we take r small enough. To be more precise, we say that the scale invariant Harnack inequality holds if there exist $r_0 > 0$ and a constant $C = C(r_0) > 0$ such that for any $r \le r_0$, any $a \in \mathbb{R}^d$ and any nonnegative function h on $\mathbb{R}^d$ which is harmonic in $B(a,r)$ with respect to X,

$$h(x) \le Ch(y), \quad \text{for all } x, y \in B(a, r/2).$$

When C depends on r one says that the weak Harnack inequality holds.

In [22] Harnack inequality was proved for some classes of Lévy processes on $\mathbb{R}^d$, but compound Poisson case was not considered. Harmonic functions of a compound Poison process are the same as of the corresponding random walk. Harnack inequality for random walks on $\mathbb{R}^d$ with steps that are continuous random variables has not yet been considered. It was proved for some random walks on $\mathbb{Z}^d$ (see [2], [20]) and on more general graphs (see [13], [15]).

In Section 2 we investigate Harnack inequality for a random walk $X = (X_n\colon n \ge 0)$ with steps that are continuous random variables with density function given by $p(x) = j(|x|)$, for a decreasing function $j\colon (0,\infty) \to [0,\infty)$. We prove that the weak Harnack inequality always holds, but that the scale invariant one may fail.

For example, when $d = 1$, it turns out that the scale invariant Harnack inequality holds for a random walk with the steps that are exponentially distributed. On the other side, when the steps of a random walk are normally distributed, only the weak Harnack inequality holds. More generally, if

$$j(r) = Ae^{-r^\gamma},\ r > 0, \tag{1.2}$$

where $\gamma > 0$ and $A > 0$ is a normalizing constant, we prove that the scale invariant Harnack inequality holds for $\gamma \le 1$, while for $\gamma > 1$ only the weak Harnack inequality holds. In the latter case we give a counterexample which shows that the constant in Harnack inequality depends on $r > 0$.

Suppose that for some constant $L > 0$ the following is true:

$$j(r) \le L\, j(2r), \quad \text{for } 0 < r \le 1, \tag{1.3}$$

$$j(r) \le L\, j(r+1), \quad \text{for } r \ge \frac{1}{2}. \tag{1.4}$$

In this case, we prove that the scale invariant Harnack inequality holds. One can check that for j as in (1.2) the condition (1.4) is satisfied for $\gamma \le 1$, but it is not satisfied for $\gamma > 1$. In particular, the condition (1.4) is satisfied for random walks with exponentially distributed steps, and it is not satisfied for random walks with normally distributed steps. We give example of j not satisfying condition (1.3) such that the scale invariant Harnack inequality does not hold. Therefore, if j does not satisfy (1.3) or (1.4), the scale invariant Harnack inequality need not hold.

In Section 3 we consider a Lévy process on $\mathbb{R}^d$ with the Lévy density given by

$$c\, |x|^{-d-\alpha}\, 1_{\{|x| \le 1\}} + j(|x|)\, 1_{\{|x|>1\}}, \tag{1.5}$$

where $c > 0$ is a constant, $\alpha \in (0, 2)$ and $j\colon (1, \infty) \to [0, \infty)$ is a nonnegative decreasing function such that

$$j(r) \le c r^{-d-\alpha}, \quad \text{for all } r > 1. \tag{1.6}$$

The Lévy density (1.5) for small $|x|$ coincides with the Lévy density of the rotationally invariant α-stable process. When $j \equiv 0$ we get a truncated α-stable process, which was considered in [18]. In that paper the (scale invariant) Harnack inequality was proved. We can use similar techique to compare Green functions for small balls for the process with Lévy density (1.5) and the rotationally invariant α-stable process (cf. Proposition 3.1). Apart from the condition (1.6) we suppose only that j is a decreasing function. This case was not covered in [1],[3], [7] and [22]. In [11] the authors remark that using the Meyer's construction method heat kernel estimates can be obtained, which allow them to prove the parabolic Harnack inequality. Our technique is much simpler and does not involve heat kernel estimates.

We would like to point out the difference between the random walk case (i.e. the case of the finite Lévy measure) and the case of the infinite Lévy measure. In the latter case we obtain the scale invariant Harnack inequality regardless of the condition (1.4).

In Section 4 we consider Harnack inequality for nonnegative harmonic functions of subordinate Brownian motion, where the Laplace exponent ϕ of the corresponding subordinator is given by

$$\phi(\lambda) = \lambda^{\alpha/2} \ell(\lambda),\ \lambda > 0. \tag{1.7}$$

Here $\alpha \in (0, 2)$ and ℓ is a continuous and slowly varying function at infinity, that is,

$$\lim_{x\to\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1, \quad \text{for all } \lambda > 0.$$

The Lévy density of the subordinate Brownian motion will be of the form $j(|x|)$, for some decreasing function j. We prove the Harnack inequality under some conditions on

ℓ. In [23] Harnack inequality was proved in the case of $\ell \equiv b$, for some constant $b > 0$, but with an additional condition on behavior of $j(r)$ for large r. Harnack inequality for ϕ given in (1.7) was proved in [19] under assumption on the behavior of the tail of the corresponding Lévy measure. In our case we do not need such conditions.

We introduce notation we shall often use in the sequel. If f and g are functions, we write $f \sim g$ if the quotient f/g converges to 1. For $D \subset \mathbb{R}^d$ we define the diameter of D by $\operatorname{diam} D = \sup\{|x-y| \colon x, y \in D\}$. If $A, B \subset \mathbb{R}^d$, the set $A - B$ is defined by

$$A - B = \{a - b : a \in A,\, b \in B\}.$$

The volume of the unit ball in $\mathbb{R}^d$ is denoted by ν_d and the area of the unit sphere in $\mathbb{R}^d$ is denoted by σ_d. The Gamma function is defined by

$$\Gamma(x) := \int_0^\infty t^{x-1}\, e^{-t}\, dt, \quad \text{for } x > 0.$$

In every section we denote constants that we use troughout the section by $C_1, C_2, \ldots$ and $R_0, R_1, \ldots$. Sometimes we use constants $c_1, c_2, \ldots$ in the proofs and each of these lower case constants is relevant only for the proof containing it. For $a, b \in \mathbb{R}$ we define $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

2 Random Walks

Let $X = (X_n, \mathbb{P}_x)$ be a Markov chain on $\mathbb{R}^d$ with transition kernel $P(x, dy)$. Assume that $P(x, dy)$ is absolutely continuous with respect to the Lebesgue measure, that is,

$$P(x, dy) = p(y - x)\, dy,$$

where p is a nonnegative function on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} p(x)\, dx = 1$. For an open set $A \subset \mathbb{R}^d$ the first exit time of X from A is defined by

$$\tau_A = \inf\{n \geq 0 \colon X_n \notin A\}.$$

Proposition 2.1 There exists $R_0 > 0$ such that for any open subset $D \subset \mathbb{R}^d$ with $\operatorname{diam} D \leq R_0$:

(i) $\mathbb{P}_x(\tau_D < \infty) = 1$;

(ii) for every $x \in D$ and every Borel subset $F \subset D^c$

$$\mathbb{P}_x(X_{\tau_D} \in F) = \int_F K_D(x, z)\, dz, \tag{2.1}$$

where

$$K_D(x, z) = p(z - x) + \sum_{n=1}^{\infty} \Phi_n(x, z) \tag{2.2}$$

and

$$\Phi_n(x, z) = \int_D \ldots \int_D \int_D p(x_1 - x) \ldots p(x_n - x_{n-1}) p(z - x_n)\, dx_n\, dx_{n-1} \ldots dx_1,\ n \geq 1.$$

Proof. Take $R_0 > 0$ such that $\mathbb{P}_0(X_1 \in B(0, 2R_0)) < 1$. Let $D \subset \mathbb{R}^d$ be an open subset with $\operatorname{diam} D \le R_0$. Since $D - D \subset B(0, 2R_0)$, it follows that $\theta := \mathbb{P}_0(X_1 \in D - D) < 1$ and thus

$$\begin{aligned}\mathbb{P}_x(\tau_D > n) &= \mathbb{P}_x(X_1 \in D, \dots, X_n \in D) \\ &= \int_D \dots \int_D p(x_1 - x)p(x_2 - x_1)\dots p(x_n - x_{n-1})\, dx_n \dots dx_1 \le \theta^n,\end{aligned}$$

for any $n \in \mathbb{N}$ and any $x \in D$. Therefore,

$$\mathbb{P}_x(\tau_D < \infty) = 1, \quad \text{for all } x \in D.$$

Let $F \subset D^c$ be a Borel subset and $x \in D$. Then we have

$$\begin{aligned}&\mathbb{P}_x(X_{\tau_D} \in F) = \sum_{n=0}^{\infty} \mathbb{P}_x(X_{\tau_D} \in F,\, \tau_D = n+1) \\ &= \mathbb{P}_x(X_1 \in F) + \sum_{n=1}^{\infty} \mathbb{P}_x(X_1 \in D, \dots, X_n \in D, X_{n+1} \in F) \\ &= \int_F p(z-x)dz + \sum_{n=1}^{\infty} \int_F \int_D \cdots \int_D p(x_1 - x) \cdots p(x_n - x_{n-1}) p(z - x_n)\, dx_n \dots\, dx_1\, dz.\end{aligned}$$

□

The function $K_D(\cdot, \cdot)$ in Proposition 2.1 is called the Poisson kernel for D with respect to X.

Remark 2.2 Let $D \subset \mathbb{R}^d$ be an open subset such that $\operatorname{diam} D \le R_0$ and let $h\colon D^c \to [0, \infty)$ be a Borel function. Using Proposition 2.1 we see that the following is true:

$$\mathbb{E}_x[h(X_{\tau_D})] = \int_{D^c} K_D(x, z) h(z)\, dz, \quad \text{for all } x \in D.$$

Definition 2.3 Let $D \subset \mathbb{R}^d$ be an open subset. A nonnegative Borel function $h\colon \mathbb{R}^d \to [0, \infty)$ is *harmonic* in D with respect to X if for any bounded open subset $B \subset \overline{B} \subset D$,

$$h(x) = \mathbb{E}_x[h(X_{\tau_B}) 1_{\{\tau_B < \infty\}}], \quad \text{for all } x \in B.$$

A Borel function $h\colon \mathbb{R}^d \to \mathbb{R}$ is *regular harmonic* in D with respect to X if

$$h(x) = \mathbb{E}_x[h(X_{\tau_D}) 1_{\{\tau_D < \infty\}}], \quad \text{for all } x \in D. \tag{2.3}$$

Remark 2.4 (i) Using the strong Markov property we can check that a regular harmonic function in D is harmonic in D.

(ii) When $\operatorname{diam} D \le R_0$, it follows from Proposition 2.1 (i) that we do not need terms $1_{\{\tau_B < \infty\}}$ and $1_{\{\tau_D < \infty\}}$ in Definition 2.3.

Using Remark 2.2 we can easily check that the following proposition holds.

Proposition 2.5 Let $D \subset \mathbb{R}^d$ be an open set with $\operatorname{diam} D \le R_0$. If h is a nonnegative function on $\mathbb{R}^d$ which is regular harmonic in D with respect to X, then

$$h(x) = \int_{D^c} K_D(x,z)h(z)\,dz, \quad \text{for all } x \in D.$$

□

In this section we suppose that p is of the form

$$p(x) = j(|x|), \; x \in \mathbb{R}^d, \, x \neq 0,$$

where $j\colon (0,\infty) \to [0,\infty)$ is a decreasing function.

Remark 2.6 In the rest of this section, the same proofs will work for a function $j\colon (0,\infty) \to [0,\infty)$ such that

$$j(t) \le M j(s), \quad \text{for all } 0 < s < t,$$

for some constant $M > 0$.

The following lemma will be useful in many places.

Lemma 2.7 Let $g\colon (0,\infty) \to [0,\infty)$ be a decreasing function. For any $a \in \mathbb{R}^d$ and $r > 0$ the following inequality holds

$$g(|z-x|) \le 32^d \nu_d^{-1} r^{-d} \int_{B(a,r)} g(|z-u|)\,du, \quad \text{for all } x \in B(a,3r/4), \; z \in B(a,2r)^c. \tag{2.4}$$

Proof. Let $x \in B(a,3r/4)$. For every $z \in B(a,2r)^c$ there exists $x_z \in B(a,r)$ such that

$$B(x_z, r/32) \subseteq \{u \in B(a,r)\colon |z-u| \le |z-x|\}.$$

Since g is decreasing, it follows that

$$\begin{aligned}\int_{B(a,r)} g(|z-u|)du &\ge \int_{B(x_z,r/32)} g(|z-u|)du \\ &\ge |B(x_z,r/32)|\, g(|z-x|) = \nu_d \,(r/32)^d \, g(|z-x|).\end{aligned}$$

□

Proposition 2.8 There exist $R_1 \in (0,R_0]$ and a constant $C_1 = C_1(d) > 0$ such that for any $a \in \mathbb{R}^d$ and any $r \le R_1$,

$$K_{B(a,r)}(x,z) \le j(|z-x|) + C_1 r^{-d}, \quad \text{for all } x \in B(a,r/2), \; z \in B(a,3r) \setminus B(a,r). \tag{2.5}$$

Proof. Let $a \in \mathbb{R}^d$. Take $R_1 \in (0,R_0]$ such that

$$\int_{B(0,4R_1)} j(|u|)\,du \le 1/2$$

and $j(4R_1) > 0$. Let $r \le R_1$, $w \in B(a,r/2)$, $z \in B(a,3r) \setminus B(a,r)$ and $n \ge 1$. By the triangle inequality we have

$$\begin{aligned}
\Phi_n(w,z) &= \int_{B(a,r)\times\ldots\times B(a,r)} j(|x_1-w|)\ldots j(|z-x_n|)\,dx_1\ldots\,dx_n \\
&\le \int_{B(a,r)\times\ldots\times B(a,r)\cap\{|x_1-w|\ge\frac{|z-w|}{n+1}\}} j(|x_1-w|)\ldots j(|z-x_n|)\,dx_1\ldots\,dx_n \\
&+ \sum_{k=2}^{n}\int_{B(a,r)\times\ldots\times B(a,r)\cap\{|x_k-x_{k-1}|\ge\frac{|z-w|}{n+1}\}} j(|x_1-w|)\ldots j(|z-x_n|)\,dx_1\ldots\,dx_n \\
&+ \int_{B(a,r)\times\ldots\times B(a,r)\cap\{|z-x_n|\ge\frac{|z-w|}{n+1}\}} j(|x_1-w|)\ldots j(|z-x_n|)\,dx_1\ldots\,dx_n \\
&\le (n+1) j\left(\frac{|z-w|}{n+1}\right)\left(\int_{B(0,4r)} j(|u|)\,du\right)^n \le (n+1)2^{-n} j\left(\frac{r}{2(n+1)}\right).
\end{aligned}$$

Therefore,

$$\Phi_n(w,z) \le (n+1)2^{-n} j\left(\frac{r}{2(n+1)}\right) \quad \text{for } n\ge 1,\ w\in B(a,r),\ z\in B(a,3r)\setminus B(a,r). \tag{2.6}$$

On the other hand, using monotonicity of j, we have

$$\begin{aligned}
1 &\ge \int_{B(0,r/4)} j(|u|)\,du = \sigma_d \int_0^{\frac{r}{4}} j(s)s^{d-1}\,ds = \sigma_d \sum_{n=1}^{\infty}\int_{\frac{r}{2(n+2)}}^{\frac{r}{2(n+1)}} j(s)s^{d-1}\,ds \\
&\ge \sigma_d\, 2^{-d}\, r^d \sum_{n=1}^{\infty} \frac{j\left(\frac{r}{2(n+1)}\right)}{(n+2)^{d+1}} \ge \sigma_d\, 2^{-d}\, r^d \sum_{n=1}^{\infty}(n+1)2^{-n} j\left(\frac{r}{2(n+1)}\right)\frac{2^n}{(n+2)^{d+2}} \\
&\ge c_1\sigma_d\, 2^{-d}\, r^d \sum_{n=1}^{\infty}(n+1)2^{-n} j\left(\frac{r}{2(n+1)}\right),
\end{aligned}$$

where $c_1 = c_1(d) > 0$ is a constant such that $\frac{2^n}{(n+2)^{d+1}} \ge c_1$ for all $n \ge 1$. Hence, by the last display and (2.6) we get

$$\sum_{n=1}^{\infty}\Phi_n(w,z) \le c_1^{-1}\sigma_d^{-1}2^d r^{-d}. \tag{2.7}$$

Using Proposition 2.1 and (2.7) it follows that for $x \in B(a,r/2)$, $z \in B(a,3r)\setminus B(a,r/2)$,

$$K_{B_r}(x,z) \le j(|z-x|) + c_1^{-1}\sigma_d^{-1}2^d r^{-d}.$$

Therefore, we may take $C_1 = c_1^{-1}\sigma_d^{-1}2^d$. □

Proposition 2.9 There exists a constant $C_2 = C_2(d) > 0$ such that for any $a \in \mathbb{R}^d$ and any $r \le R_1$,

$$K_{B(a,r)}(x,z) \le j(|z-x|) + C_2 r^{-d}\int_{B(a,3r/2)} j(|z-u|)\,du, \tag{2.8}$$

for all $x \in B(a, r/2)$, $z \in B(a, 3r)^c$ and

$$K_{B(a,r)}(x, z) \geq j(|z - x|) + j(3r/2) \int_{B(a,r)} j(|z - u|)\, du, \tag{2.9}$$

for all $x \in B(a, r/2)$, $z \in B(a, r)^c$.

Proof. It follows from the choice of R_1 in the proof of the Proposition 2.8 that

$$\int_{B(0,2R_1)} j(|u|)\, du \leq 1/2. \tag{2.10}$$

Let $a \in \mathbb{R}^d$, $r \leq R_1$, $x \in B(a, r/2)$ and $z \in B(a, 3r)^c$. Using Lemma 2.7 and (2.10) we have

$$\begin{aligned}
\Phi_n(x, z) &= \int_{B(a,r)} \dots \int_{B(a,r)} \int_{B(a,r)} j(|x_1 - x|) \dots j(|x_n - x_{n-1}|)\, j(|z - x_n|)\, dx_n\, dx_{n-1} \dots dx_1 \\
&\leq 32^d\, \nu_d^{-1} (3r/2)^{-d} \left(\int_{B(0,2r)} j(|w|)\, dw \right)^n \int_{B(a,3r/2)} j(|z - u|)\, du \\
&\leq (64/3)^d\, \nu_d^{-1} r^{-d} 2^{-n} \int_{B(a,3r/2)} j(|z - u|)\, du.
\end{aligned}$$

It follows from Proposition 2.1 and the last display that

$$K_{B(a,r)}(x, z) \leq j(|z - x|) + (64/3)^d\, \nu_d^{-1} r^{-d} \int_{B(a,3r/2)} j(|z - u|)\, du.$$

Hence, we may take $C_2 = (64/3)^d\, \nu_d^{-1}$. It follows from Proposition 2.1 that for $x \in B(a, r/2)$ and $z \in B(a, r)^c$,

$$K_{B(a,r)}(x, z) \geq j(|z - x|) + \Phi_1(x, z) \geq j(|z - x|) + j(3r/2) \int_{B_r} j(|z - x_1|)\, dx_1.$$

□

Now we can prove the weak Harnack inequality.

Proposition 2.10 There exists a constant $C_3 = C_3(d) > 0$ such that for any $a \in \mathbb{R}^d$ and any $r \leq R_1$,

$$h(x) \leq 2\, \frac{j(r/2) + C_3 r^{-d}}{j(7r/2)} h(y), \quad \text{for all } x, y \in B(a, r/2),$$

for any nonnegative function h on $\mathbb{R}^d$ which is harmonic in $B(a, 2r)$ with respect to X.

Proof. Let $a \in \mathbb{R}^d$, $r \leq R_1$, $x, y \in B(a, r/2)$, $z_1 \in B(a, 3r) \setminus B(a, r)$ and $z_2 \in B(a, 3r)^c$. Using Proposition 2.8 and monotonicity of j it follows that

$$\begin{aligned}
K_{B(a,r)}(x, z_1) \leq j(r/2) + C_1 r^{-d} &\leq \frac{j(r/2) + C_1\, r^{-d}}{j(7r/2)} j(|z_1 - y|) \\
&\leq \frac{j(r/2) + C_1 r^{-d}}{j(7r/2)} K_{B(a,r)}(y, z_1).
\end{aligned}$$

On the other hand, it follows from Proposition 2.9 and Lemma 2.7 that

$$
\begin{aligned}
K_{B(a,r)}(x,z_2) &\le (32^d\nu_d^{-1}+C_2)r^{-d}\int_{B(a,3r/2)} j(|z_2-u|)\,du \\
&\le \frac{(32^d\nu_d^{-1}+C_2)r^{-d}}{j(9r/4)} K_{B(a,3r/2)}(y,z_2).
\end{aligned}
$$

Set $C_3 = C_1 \vee (32^d\nu_d^{-1}+C_2)$. Using monotonicity of j and the last two displays we get

$$
K_{B(a,r)}(x,z) \le \frac{j(r/2)+C_3r^{-d}}{j(7r/2)} K_{B(a,r)}(y,z), \quad \text{for all } z\in B(a,3r)\setminus B(a,r), \tag{2.11}
$$

$$
K_{B(a,r)}(x,z) \le \frac{j(r/2)+C_3r^{-d}}{j(7r/2)} K_{B(a,3r/2)}(y,z), \quad \text{for all } z\in B(a,3r)^c. \tag{2.12}
$$

Let $h\colon \mathbb{R}^d\to\mathbb{R}$ be a nonnegative function which is harmonic in $B(a,2r)$ with respect to X. Since h is regular harmonic in $B(a,r)$ and in $B(a,3r/2)$, it follows from Proposition 2.5, (2.11) and (2.12) that

$$
\begin{aligned}
h(x) &= \int_{B(a,r)^c} K_{B(a,r)}(x,z)h(z)\,dz \\
&= \int_{B(a,3r)\setminus B(a,r)} K_{B(a,r)}(x,z)h(z)\,dz + \int_{B(a,3r)^c} K_{B(a,r)}(x,z)h(z)\,dz \\
&\le \frac{j(r/2)+C_3r^{-d}}{j(7r/2)}\left(\int_{B(a,3r)\setminus B(a,r)} K_{B(a,r)}(y,z)h(z)\,dz + \int_{B(a,3r)^c} K_{B(a,3r/2)}(y,z)h(z)\,dz\right) \\
&\le \frac{j(r/2)+C_3r^{-d}}{j(7r/2)}\left(\int_{B(a,r)^c} K_{B(a,r)}(y,z)h(z)\,dz + \int_{B(a,3r/2)^c} K_{B(a,3r/2)}(y,z)h(z)\,dz\right) \\
&= 2\,\frac{j(r/2)+C_3r^{-d}}{j(7r/2)} h(y).
\end{aligned}
$$

□

Remark 2.11 If j is bounded by K, one can moreover prove an estimate for the Poisson kernel. More precisely, there exists $R_2\in(0,R_0]$ and a constant $C_4=C_4(d,K,R_2)>0$ such that for any $a\in\mathbb{R}^d$ and any $r\le R_2$

$$
j(|z-x|)+C_4^{-1}\int_{B(a,r)} j(|z-u|)\,du \le K_{B(a,r)}(x,z) \le j(|z-x|)+C_4\int_{B(a,r)} j(|z-u|)\,du, \tag{2.13}
$$

for all $x\in B(a,r/2)$, $z\in B(a,r)^c$. Using estimate (2.13) it follows that there exists a constant $C_5=C_5(d,K,R_2)>0$ such that for any $a\in\mathbb{R}^d$ and any $r\le R_2$,

$$
K_{B(a,r)}(x,z) \le C_5\, r^{-d}\, K_{B(a,r)}(y,z), \quad \text{for all } x,y\in B(a,r/2),\ z\in B(a,r)^c, \tag{2.14}
$$

which, by Proposition 2.5, implies that

$$
h(x) \le C_5\, r^{-d}\, h(y), \quad \text{for all } x,y\in B(a,r/2), \tag{2.15}
$$

for any nonnegative function h which is harmonic in $B(a,2r)$ with respect to X. In this case one can also get (2.15) from Proposition 2.10. This remark shows that, in the case of bounded j, it is possible to get the inequality (2.14) between the Poisson kernels, which is not possible in the general case.

So far we have showed that the weak Harnack inequality holds. Next we would like to see when the scale invariant Harnack inequality holds.

Theorem 2.12 Suppose that (1.3) and (1.4) hold. There exist $R_3 \in (0, R_1]$ and a constant $C_6 = C_6(d, L) > 0$ such that for any $a \in \mathbb{R}^d$ and $r \le R_3$,

$$K_{B(a,r)}(x,z) \le C_6\, K_{B(a,r)}(y,z), \quad \text{for all } x,y \in B(a,r/2),\ z \in B(a,r)^c.$$

In particular, the scale invariant Harnack inequality holds, that is, for any $a \in \mathbb{R}^d$, any $r \le R_3$ and any nonnegative function h which is harmonic in $B(a,2r)$,

$$h(x) \le C_6\, h(y), \quad \text{for all } x,y \in B(a,r/2).$$

Remark 2.13 The condition (1.3) is automatically satisfied for bounded j such that $j(2) > 0$. Indeed, if $K > 0$ is a constant such that $j(s) \le K$, for all $s > 0$, then

$$j(s) \le K \le \frac{K}{j(2)} j(2s), \quad \text{for all } s \le 1.$$

Proof. Let $a \in \mathbb{R}^d$. Take $R_3 \le R_1 \wedge \frac{1}{4}$ such that

$$2L \int_{B(0,4R_3)} j(|v|)\, dv \le 1/2. \tag{2.16}$$

Let $r \le R_3$ and $x,y \in B(a,r/2)$. If $z \in B(x,1/2) \cap B(a,r)^c$, then

$$|z-y| \le |z-x| + r \le 3|z-x|.$$

Hence, by monotonicity of j and (1.3) we have

$$j(|z-y|) \ge j(3|z-x|) \ge L^{-2} j(|z-x|). \tag{2.17}$$

On the other hand, if $z \in B(x,1/2)^c$, it follows from (1.4) that

$$j(|z-y|) \ge j(|z-x|+1) \ge L^{-1} j(|z-x|), \tag{2.18}$$

since

$$|z-y| \le |z-x| + 1.$$

Therefore, it follows from (2.17) and (2.18) that

$$j(|z-x|) \le L^2 j(|z-y|), \quad \text{for all } x,y \in B(a,r/2),\ z \in B(a,r)^c, \tag{2.19}$$

since $L \ge 1$.

Let $z \in B(a,3r) \setminus B(a,r)$ and $u \in B(a,r)$. By the triangle inequality it follows that

$$
\begin{aligned}
&\int_{B(a,r)} j(|v-u|)\, j(|z-v|)\, dv \le \\
\le& \int_{B(a,r)\cap\{|v-u|\ge \frac{|z-u|}{2}\}} j(|v-u|) j(|z-v|)\, dv + \int_{B(a,r)\cap\{|z-v|\ge \frac{|z-u|}{2}\}} j(|v-u|) j(|z-v|)\, dv \\
\le& \int_{B(a,r)\cap\{|v-u|\ge \frac{|z-u|}{2}\}} j\left(\frac{|z-u|}{2}\right) j(|z-v|)\, dv + \int_{B(a,r)\cap\{|z-v|\ge \frac{|z-u|}{2}\}} j(|v-u|) j\left(\frac{|z-u|}{2}\right) dv \\
\le& L\, j(|z-u|) \left(\int_{B(a,r)\cap\{|v-u|\ge \frac{|z-u|}{2}\}} j(|z-v|)\, dv + \int_{B(a,r)\cap\{|z-v|\ge \frac{|z-u|}{2}\}} j(|v-u|)\, dv \right) \\
\le& 2L\, j(|z-u|) \int_{B(0,4r)} j(|v|)\, dv,
\end{aligned}
$$

where we have used monotonicity of j in the third and (1.3) in the fourth line. From the last display and (2.16) we have

$$
\int_{B_r} j(|v-u|)\, j(|z-v|)\, dv \le \frac{j(|z-u|)}{2}, \quad \text{for all } u \in B(a,r),\ z \in B(a,3r) \setminus B(a,r)
$$

and hence by iteration

$$
\begin{aligned}
\Phi_n(x,z) &= \int_{B_r} \dots \int_{B_r} \int_{B_r} j(|x_1 - x|) \dots j(|x_n - x_{n-1}|) j(|z - x_n|)\, dx_n\, dx_{n-1} \dots dx_1 \\
&\le 2^{-n} j(|z-x|), \quad \text{for all } n \ge 1,\ x \in B(a,r/2),\ z \in B(a,3r) \setminus B(a,r).
\end{aligned}
$$

It follows from Proposition 2.1 that

$$
j(|z-x|) \le K_{B(a,r)}(x,z) \le \frac{3}{2} j(|z-x|),\ \forall x \in B(a,r/2),\ z \in B(a,3r) \setminus B(a,r)
$$

and thus by (1.3) we have

$$
\frac{K_{B(a,r)}(x,z)}{K_{B(a,r)}(y,z)} \le \frac{3}{2} \frac{j(r/2)}{j(7r/2)} \le \frac{3}{2} L^3, \quad \text{for all } x, y \in B(a,r/2),\ z \in B(a,3r) \setminus B(a,r).
$$

Let $x, y \in B(a,r/2)$ and $z \in B(a,3r)^c$. Proposition 2.9 and (2.19) imply

$$
\begin{aligned}
K_{B(a,r)}(x,z) &\le j(|z-x|) + C_2 r^{-d} \int_{B(a,3r/2)} j(|z-u|)\, du \le L^2 (1 + C_2\, (3/2)^d\, \nu_d) j(|z-y|) \\
&\le L^2 (1 + C_2\, (3/2)^d\, \nu_d) K_{B_r}(y,z).
\end{aligned}
$$

Finally, we can take $C_6 = \frac{3}{2} L^3 \vee L^2 (1 + C_2\, (3/2)^d\, \nu_d)$. □

Now we consider a few examples which show that the scale invariant Harnack inequality does not always hold.

Example 2.14 Suppose that j satisfies the following conditions:

(i) There exists $K > 0$ such that $j(s) \le K$, for all $s > 0$;

(ii) There exists $s_0 > 0$ such that $j(s_0) > 0$ and $j(s) = 0$, for all $s > s_0$.

We will show that the scale invariant Harnack inequality does not hold in this case.

Take $x_r = (r/4, 0 \dots, 0)$, $y = (0, \dots, 0)$, $p_r = (s_0 + r/8, 0 \dots, 0)$ and define functions $h_r(x) = \mathbb{E}_x[1_{B(p_r, r/8)}(X_{\tau_{B(0,r)}})]$, where $0 < r \le R_2$. Each function h_r is regular harmonic in $B(0, r)$ with respect to X. It follows from Proposition 2.5 and (2.13) that

$$\frac{h_r(x_r)}{h_r(y)} = \frac{\int_{B(p_r,r/8)} K_{B(0,r)}(x_r, z)\,dz}{\int_{B(p_r,r/8)} K_{B(0,r)}(y, z)\,dz} \ge \frac{\int_{B(p_r,r/8)} j(|z - x_r|)\,dz}{\int_{B(p_r,r/8)} (j(|z|) + C_4 \int_{B(0,r)} j(|z-u|)\,du)\,dz}$$
$$\ge \frac{j(s_0)}{C_4\, K\, \nu_d\, r^d} \to \infty, \text{ as } r \to 0.$$

△

It is easy to see that non-degenerate j satisfying the assumptions of the Proposition 2.12 have support $(0, \infty)$, that is, $j(s) > 0$, for all $s > 0$. Example 2.14 shows that if j has a bounded support, then the scale invariant Harnack inequality need not hold. In other words, the full support is necessary but not sufficient.

Remark 2.15 It can be showed by using (2.13) that for $d = 1$ and

$$j(s) = A(1 - s^\alpha) 1_{(0,1)}(s), \ s > 0,$$

where $\alpha > 0$ and $A > 0$ is the constant such that $\int_0^\infty j(s)\,ds = 1$, the scale invariant Harnack inequality does not hold. This case was not covered by Example 2.14.

Example 2.16 Let

$$j(s) = \begin{cases} j_1(s), & 0 < s < 1 \\ e^{-s^\gamma}, & s \ge 1 \end{cases}, \tag{2.20}$$

where $\gamma > 1$ and $j_1 \colon (0,1) \to [0, \infty)$ is a decreasing function such that $j_1(1) \ge e^{-1}$ and $\displaystyle\int_{\mathbb{R}^d} j(|u|)\,du = 1$.

Set $B_r := B(0, r)$ for $r > 0$. Let $r \le R_1$ and $z \in B^c_{1+r}$. Define

$$\Psi(z; r) := \sum_{n=1}^\infty \int_{B_r} \int_{B_r} \dots \int_{B_r} j(|x_1|) j(|x_2 - x_1|) \dots j(|x_n - x_{n-1}|) e^{|z|^\gamma - |z - x_n|^\gamma} dx_n \dots dx_2\, dx_1$$
$$= e^{|z|^\gamma} \sum_{n=1}^\infty \Phi_n(0, z).$$

It follows from Proposition 2.1 that

$$\frac{K_{B_r}(x, z)}{K_{B_r}(0, z)} \ge \frac{j(|z - x|)}{j(|z|) + \sum_{n=1}^\infty \Phi_n(0, z)} = \frac{e^{|z|^\gamma - |z - x|^\gamma}}{1 + \Psi(z; r)}, \tag{2.21}$$

where $x \in B_{r/2}$.

First we need to show that $\Psi(\cdot;r)\colon B^c_{1+r}\to\mathbb{R}$ is continuous for $r\le R_1$. Recall that

$$\int_{B_{3R_1}} j(|u|)\,du \le 1/2.$$

Let $r\le R_1$, $R>1+r$ and $\varepsilon>0$. Then there exists $n_0\in\mathbb{N}$ such that for all $z\in B_R\backslash B_{1+r}$,

$$|\Psi(z;r)-e^{|z|^\gamma}\sum_{n=1}^{n_0}\Phi_n(0,z)|\le e^{R^\gamma}\sum_{n=n_0+1}^{\infty}\left(\int_{B_{2r}} j(|u|)\,du\right)^n \le e^{R^\gamma}\sum_{n=n_0+1}^{\infty}2^{-n}<\varepsilon. \tag{2.22}$$

Let $z\in B_R\setminus B_{1+r}$. By the dominated convergence theorem there exists $\delta>0$ such that for $w\in B_R\setminus B_{1+r}$, $|z-w|<\delta$ and $1\le n\le n_0$ we have

$$|e^{|z|^\gamma}\Phi_n(0,z)-e^{|w|^\gamma}\Phi_n(0,w)|<\frac{\varepsilon}{n_0}. \tag{2.23}$$

Therefore, by (2.22) and (2.23) we have

$$|\Psi(z;r)-\Psi(w;r)|<3\varepsilon \quad \text{for } w\in B_R\setminus B_{1+r},\ |z-w|<\delta.$$

Since $R>1+r$ was arbitrary, it follows that $\Psi(\cdot;r)\colon B^c_{1+r}\to\mathbb{R}$ is continuous.

Let $r\le R_2\wedge 1$ and $a=r/(2\sqrt{d})$. Then $[0,a]^d\subset B_r$. The mean value theorem implies that for $u=(u_1,\ldots,u_d)\in B_r$ with $u_1,\ldots,u_d\ge 0$ and $z=(s,\ldots,s)\in B^c_{1+r}$ with $s\ge 1+r$, we have

$$|z|^\gamma-|z-u|^\gamma=\sum_{i=1}^{d}\gamma(s-\vartheta u_i)|z-\vartheta u|^{\gamma-2}u_i\ge \gamma 2^{1-\gamma}s^{\gamma-1}\sum_{i=1}^{d}u_i, \tag{2.24}$$

where $\vartheta\in(0,1)$. Therefore, for $z_n=(1+n,\ldots,1+n)$, we obtain

$$\begin{aligned}\Psi(z_n;r)&\ge\int_{B_r} e^{|z_n|^\gamma-|z_n-u|^\gamma}\,du\ge\int_0^a\cdots\int_0^a \exp\{\gamma\,2^{1-\gamma}n^{\gamma-1}\sum_{i=1}^{d}u_i\}\,du_1\ldots du_n\\ &=\left(\int_0^a \exp\{\gamma\,2^{1-\gamma}n^{\gamma-1}t\}\,dt\right)^d=\left(\frac{\exp\{\gamma 2^{-\gamma}d^{-1/2}rn^{\gamma-1}\}-1}{\gamma\,2^{1-\gamma}n^{\gamma-1}}\right)^d\to\infty,\end{aligned}$$

as $n\to\infty$. On the other hand,

$$\Psi((1+r,\ldots,1+r);r)\le e^{d^{\gamma/2}(1+r)^\gamma}\sum_{n=1}^{\infty}2^{-n}=e^{d^{\gamma/2}(1+r)^\gamma}\le e^{d^{\gamma/2}(1+R_1)^\gamma}.$$

Therefore, by continuity of $\Psi(\cdot;r)$, there exists $z_r=(s_r,\ldots,s_r)\in B^c_{1+r}$ such that $\Psi(z_r;r)=2e^{d^{\gamma/2}(1+R_1)^\gamma}$. We claim that there exists a sequence (r_n) such that $\lim_n r_n=0$ and $\lim_n |z_{r_n}|^{\gamma-1}r_n=\infty$. Otherwise, there would exist a constant $c_1>0$ such that $|z_r|^{\gamma-1}r\le c_1$ for all r small enough. By the mean value theorem we would have

$$\begin{aligned}\Psi(z_r;r)&\le\exp\{\gamma\,d2^{\gamma-1}|z_r|^{\gamma-1}r\}\sum_{n=1}^{\infty}\left(\int_{B_{2r}} j(|u|)\,du\right)^n\\ &\le 2e^{\gamma\,d2^{\gamma-1}c_1}\int_{B_{2r}} j(|u|)\,du<2e^{d^{\gamma/2}(1+R_1)^\gamma},\end{aligned}$$

for r small enough which is in contradiction with the choice of z_r. Hence, for $x_r = (r/4, \dots, r/4)$, using (2.21) and (2.24) as before, we obtain

$$\frac{K_{B_{r_n}}(x_{r_n}, z_{r_n})}{K_{B_r}(0, z_{r_n})} \geq \frac{\exp\{\gamma\, d\, 2^{-1-\gamma}|z_{r_n}|^{\gamma-1} r_n\}}{1 + 2e^{d^{\gamma/2}(1+R_1)^\gamma}} \to \infty \text{ as } n \to \infty.$$

Therefore, the scale invariant Harnack inequality does not hold. △

Remark 2.17 Let

$$j(s) = A\, e^{-s^\gamma},\ s > 0,$$

where $\gamma > 0$ and $A > 0$ is a constant such that $\int_{\mathbb{R}^d} j(|u|)\, du = 1$. It follows from Example 2.16 that the scale invariant Harnack inequality does not hold for $\gamma > 1$. We remark that the condition (1.4) in Theorem 2.12 is not satisfied. On the other hand, if $\gamma \leq 1$, it is easy to check that the conditions (1.3) and (1.4) in Theorem 2.12 are satisfied and therefore the scale invariant Harnack inequality holds.

Example 2.18 Let

$$j(s) = A(e^{-s} 1_{[1,\infty)}(s) + \sum_{n=1}^{\infty} a_n 1_{[3^{-n}, 3^{-(n-1)})}(s)), \quad \text{for } s > 0,$$

where $A > 0$ is a constant and (a_n) is a sequence defined by

$$a_1 = 1$$
$$a_{n+1} = \begin{cases} n a_n, & n = 3^k - 1, \quad \text{for some } k \in \mathbb{N} \\ a_n, & \text{otherwise.} \end{cases}$$

Set $B_r = B(0, r)$ for $r > 0$. First we show that $j(|\cdot|) \in L^p(\mathbb{R}^d)$, for any $p \geq 1$:

$$\begin{aligned}
\int_{\mathbb{R}^d} j(|x|)^p\, dx &= Ae^{-1} + A\sigma_d \int_0^1 j(s)^p\, s^{d-1}\, ds = Ae^{-1} + A\sigma_d \sum_{n=1}^{\infty} a_n^p \int_{3^{-n}}^{3^{-(n-1)}} s^{d-1}\, ds \\
&\leq Ae^{-1} + 3^d A\sigma_d d^{-1} \sum_{n=1}^{\infty} a_n^p\, 3^{-nd} = Ae^{-1} + 3^d A\sigma_d d^{-1} \sum_{k=1}^{\infty} \sum_{n=3^{k-1}}^{3^k - 1} a_n^p 3^{-nd} \\
&\leq Ae^{-1} + 3^d A\sigma_d d^{-1}(1 - 3^{-d})^{-1} \sum_{k=1}^{\infty} a_{3^{k-1}}^p 3^{-3^{k-1} d} =: A_p < \infty,
\end{aligned}$$

by the ratio test. Therefore, we can choose $A > 0$ such that $\int_{\mathbb{R}^d} j(|u|)\, du = 1$. Let $x \in B_r$ and $z \in B_r^c$. It follows from the Cauchy-Schwarz inequality that for $u, v \in B_r$

$$\int_{B_r} j(|w-u|)\, j(|v-w|)\, dw \leq \left[\int_{B_r} j(|w-u|)^2\, dw\right]^{1/2} \left[\int_{B_r} j(|v-w|)^2\, dw\right]^{1/2} \leq A_2$$

and hence, by Proposition 2.1, we get

$$\begin{aligned}
K_{B_r}(0, z) &\leq j(|z-x|) + A_2 + \\
&\quad + A_2 \sum_{n=2}^{\infty} \int_{B_r} \dots \int_{B_r} j(|x_1 - x|) \dots j(|x_{n-1} - x_{n-2}|)\, dx_{n-1} \dots dx_1.
\end{aligned}$$

If $r \le R_1$, it follows

$$K_{B_r}(x,z) \le j(|z-x|) + A_2 + A_2 \sum_{n=1}^{\infty} \left(\int_{B_{2r}} j(|u|)\, du \right)^n \le .j(|z-x|) + 2A_2 \qquad (2.25)$$

Define $x_n = (3^{-3^n}, 0, \dots, 0)$, $y_n = (-3^{-3^n}, 0, \dots, 0)$, $A_n = B((3^{-3^n+1}, 0, \dots, 0), 3^{-3^n-1})$ and $r_n = 2 \cdot 3^{-3^n}$ for $n \ge 1$. Then for $w \in A_n$ we have

$$|w - x_n| \le \frac{7}{3} 3^{-3^n}$$
$$|w - y_n| \ge \frac{11}{9} 3^{-3^n+1}$$

and so

$$\frac{\mathbb{P}_{x_n}(X_{\tau_{B_{r_n}}} \in A_n)}{\mathbb{P}_{y_n}(X_{\tau_{B_{r_n}}} \in A_n)} = \frac{\int_{A_n} K_{B_r}(x_n, z)\, dz}{\int_{A_n} K_{B_r}(y_n, z)\, dz}$$
$$\ge \frac{j(\frac{7}{3} 3^{-3^n})}{j(\frac{11}{9} 3^{-3^n+1}) + 2A_2} = \frac{a_{3^n}}{a_{3^n-1} + 2A_2} = \frac{a_{3^n-1}}{a_{3^n-1} + 2A_2}(3^n - 1) \to \infty$$

as $n \to \infty$. Thus, the scale invariant Harnack inequality does not hold. △

Remark 2.19 It is easy to see that the conditions (1.3) and (1.4) are not satisfied in Examples 2.16 and 2.18, respectively. Therefore, if j does not satisfy (1.3) or (1.4) the scale invariant Harnack inequality need not hold.

3 Lévy processes

Let $Y = (Y_t, \mathbb{P}_x)$ be a pure jump Lévy process, that is, Y is a Lévy process such that

$$\mathbb{E}_x[e^{i\xi\cdot(Y_t - Y_0)}] = e^{-t\Psi(\xi)}, \ \xi \in \mathbb{R}^d, \ t \ge 0,$$

where the characteristic exponent Ψ of Y is given by

$$\Psi(\xi) = \int_{\mathbb{R}^d} (1 - e^{i\xi\cdot y} + i\xi \cdot y 1_{\{|y|\le 1\}})\nu(dy).$$

Here ν is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$, $\int_{\mathbb{R}^d}(1 \wedge |y|^2)\nu(dy) < \infty$. Suppose that ν is of the form $\nu(dy) = J^Y(y)\, dy$ with

$$J^Y(y) = \mathcal{A}(d,\alpha)|y|^{-d-\alpha} 1_{\{|y|\le 1\}} + j(|y|) 1_{\{|y|>1\}}, \ y \in \mathbb{R}^d, \ y \ne 0, \qquad (3.1)$$

where $\alpha \in (0,2)$, $\mathcal{A}(d,\alpha) = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{d+\alpha}{2})}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}$ and $j\colon (1,\infty) \to \mathbb{R}$ is a function satisfying

$$0 \le j(s) \le \mathcal{A}(d,\alpha) s^{-d-\alpha}, \quad \text{for all } s > 1. \qquad (3.2)$$

We call J^Y the Lévy density of Y. Using symmetry we can check that

$$\Psi(\xi) = \mathcal{A}(d,\alpha) \int_{\{|y|<1\}} \frac{1 - \cos(\xi \cdot y)}{|y|^{d+\alpha}}\, dy + \int_{\{|y|\ge 1\}} (1 - \cos(\xi \cdot y)) j(|y|)\, dy.$$

Since

$$\int_{\{|y|<1\}} \frac{1-\cos(\xi\cdot y)}{|y|^{d+\alpha}}\,dy = |\xi|^\alpha \int_{\{|y|<|\xi|\}} \frac{1-\cos(\frac{\xi\cdot y}{|\xi|})}{|y|^{d+\alpha}}\,dy,$$

we see that there exist constants $M_1, M_2 > 0$ such that

$$M_1^{-1}|\xi|^2 \le \Psi(\xi) \le M_1(|\xi|^2+1), \quad \text{for } \xi\in\mathbb{R}^d,\ |\xi|\le 1/2 \tag{3.3}$$

and

$$M_2^{-1}|\xi|^\alpha \le \Psi(\xi) \le M_2|\xi|^\alpha, \quad \text{for } \xi\in\mathbb{R}^d,\ |\xi|\ge 1/2. \tag{3.4}$$

It follows from (3.4) and [21, Propositon 28.1] that Y_t has smooth density $p^Y(t,x,y)$. In the rest of this section we assume that $d\ge 3$. Using Chung-Fuchs type criterion [21, Corollary 37.6] and (3.3) we conclude that Y is transient, so we can define the Green function of Y by

$$G^Y(x,y) = \int_0^\infty p^Y(t,x,y)\,dt, \quad x,y\in\mathbb{R}^d,\ x\ne y.$$

For an open set $D\subset\mathbb{R}^d$ we define the first exit time from D of the process Y by

$$\tau_D^Y = \inf\{t>0\colon Y_t\notin D\}.$$

Denote by Y^D the process obtained by killing process Y upon leaving D, that is,

$$Y_t^D = \begin{cases} Y_t, & t<\tau_D^Y \\ \partial, & \text{otherwise,}\end{cases}$$

where ∂ is an extra point adjoined to D. Following the beginning of the proof of [12, Theorem 2.4] we see that Y_t^D has density $p_D^Y(t,x,y)$ given by

$$p_D^Y(t,x,y) = p^Y(t,x,y) - \mathbb{E}_x[p^Y(t-\tau_D^Y, Y_{\tau_D^Y}, y); \tau_D^Y < t], \quad x,y\in D.$$

It follows that the Green function G_D^Y of Y^D is given by

$$G_D^Y(x,y) = \int_0^\infty p_D^Y(t,x,y)\,dt, \quad x,y\in D.$$

Let $X=(X_t,\mathbb{P}_x)$ be a rotationally invariant α-stable process in $\mathbb{R}^d$, that is, X is a Lévy process such that

$$\mathbb{E}_x[e^{i\xi\cdot(X_t-X_0)}] = e^{-t|\xi|^\alpha},\ \xi\in\mathbb{R}^d,\ t\ge 0.$$

It is known that the Lévy density of X is

$$J^X(y) = \mathcal{A}(d,\alpha)|y|^{-d-\alpha},\ y\in\mathbb{R}^d,\ y\ne 0.$$

Therefore, X is a special case of the process Y considered at the beginning of this section.

It follows from [14, Example 1.4.1] that the Dirichlet form $(\mathcal{E}^Y, \mathcal{F}^Y)$ corresponding to the process Y is given by

$$\mathcal{E}^Y(u,v) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(x)-u(y))(v(x)-v(y))J^Y(x-y)\,dx\,dy$$
$$\mathcal{F}^Y = \{u \in L^2(\mathbb{R}^d) \colon \mathcal{E}^Y(u,u) < \infty\}.$$

We can rewrite $(\mathcal{E}^Y, \mathcal{F}^Y)$ as

$$\mathcal{E}^Y(u,v) = \int_{\mathbb{R}^d} \hat{u}(\xi)\bar{\hat{v}}(\xi)\Psi(\xi)\,d\xi \tag{3.5}$$
$$\mathcal{F}^Y = \{u \in L^2(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2\Psi(\xi)\,d\xi < \infty\}, \tag{3.6}$$

where $\hat{u}(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i\xi\cdot y}u(y)\,dy$ is the Fourier transform of u. Define

$$\mathcal{E}_1^Y(u,v) = \mathcal{E}^Y(u,v) + (u,v)_{L^2(\mathbb{R}^d)},\ u,v \in \mathcal{F}^Y.$$

In particular, the Dirichlet form of the rotationally invariant α-stable process X is

$$\mathcal{E}^X(u,v) = \frac{1}{2}\mathcal{A}(d,\alpha)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\,dx\,dy$$
$$\mathcal{F}^X = \{u \in L^2(\mathbb{R}^d) \colon \mathcal{E}^X(u,u) < \infty\}.$$

Alternatively, one has

$$\mathcal{E}^X(u,v) = \int_{\mathbb{R}^d} \hat{u}(\xi)\bar{\hat{v}}(\xi)|\xi|^\alpha\,d\xi \tag{3.7}$$
$$\mathcal{F}^X = \{u \in L^2(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2|\xi|^\alpha\,d\xi < \infty\}, \tag{3.8}$$

Using (3.3) and (3.4) we can check that $\mathcal{F}^X = \mathcal{F}^Y$. In the sequel we shall denote $\mathcal{F}^X$ and $\mathcal{F}^Y$ by $\mathcal{F}$. It follows from (3.3), (3.4), (3.5) and (3.7) that there exists a constant $M_3 > 0$ such that

$$M_3^{-1}\mathcal{E}_1^X(u,u) \le \mathcal{E}_1^Y(u,u) \le M_3\mathcal{E}_1^X(u,u), \quad \text{for all } u \in \mathcal{F}.$$

Using this estimate we see that a set has zero capacity with respect to $(\mathcal{E}^X, \mathcal{F})$ if and only if it has zero capacity with respect to $(\mathcal{E}^Y, \mathcal{F})$ (see [14, Chapter 2]). We say that a statement holds quasi-everywhere (q.e.) on a subset A of $\mathbb{R}^d$ if there exists a subset N of zero capacity such that the statement is true on $A \setminus N$. Therefore a statement holds q.e. with respect to X if and only if it holds q.e. with respect to Y.

It follows from [14, Theorem 4.4.2] that the Dirichlet form corresponding to the killed process Y^D is $(\mathcal{E}^Y, \mathcal{F}_D)$, where

$$\mathcal{F}_D = \{u \in \mathcal{F} \colon u = 0 \text{ q.e. on } D^c\}.$$

If $u, v \in \mathcal{F}_D$, then we can rewrite $\mathcal{E}^Y(u,v)$ as

$$\mathcal{E}^Y(u,v) = \frac{1}{2}\int_D\int_D (u(x)-u(y))(v(x)-v(y))J^Y(x-y)\,dxdy + \int_D u(x)v(x)\kappa_D^Y(x)\,dx,$$

where

$$\kappa_D^Y(x) = \int_{D^c} J^Y(x-y)\,dy.$$

The Dirichlet form corresponding to the killed rotationally invariant α-stable process X^D is $(\mathcal{E}^X, \mathcal{F}_D)$. For $u, v \in \mathcal{F}_D$ we have

$$\mathcal{E}^X(u,v) = \frac{1}{2}\mathcal{A}(d,\alpha)\int_D\int_D (u(x)-u(y))(v(x)-v(y))\frac{dxdy}{|x-y|^{d+\alpha}} + \int_D u(x)v(x)\kappa_D^X(x)\,dx,$$

where

$$\kappa_D^X(x) = \mathcal{A}(d,\alpha)\int_{D^c}\frac{dy}{|x-y|^{d+\alpha}}.$$

Let $D \subset \mathbb{R}^d$ be an open subset such that $\operatorname{diam} D \le \frac{1}{2}$. Consider the following semigroup

$$P_t^D f(x) = \mathbb{E}_x[e^{\int_0^t q_D(X_s^D)\,ds} f(X_t^D)],$$

where $q_D(x) = \kappa_D^X(x) - \kappa_D^Y(x)$. It follows from (3.2) that $q_D \ge 0$. The Dirichlet form corresponding to the semigroup $\{P_t^D\}$ is

$$\mathcal{E}^X(u,v) - \int_D u(x)v(x)q_D(x)\,dx = \mathcal{E}^Y(u,v),$$

since $J^Y(y) = J^X(y)$ for $|y| \le 1$ (see [14, Lemma 4.6.7]). Therefore, $\{P_t^D\}$ is the semigroup of the process Y^D.
In order to prove the scale invariant Harnack inequality, we compare Green functions for small balls with respect to the processes X and Y.

Proposition 3.1 There exists $R_0 \in (0, 1/4]$ such that for any $a \in \mathbb{R}^d$ and $r \le R_0$,

$$G^X_{B(a,r)}(x,y) \le G^Y_{B(a,r)}(x,y) \le 2G^X_{B(a,r)}(x,y), \quad \text{for all } x, y \in B(a,r). \tag{3.9}$$

□

This was done in [18, Proposition 3.2]. Since $q_D \ge 0$, the proof is almost the same and thus we omit it here.

Let $a \in \mathbb{R}^d$ and $r > 0$. It is proved in [24] that $\mathbb{P}_x(Y_{\tau_{B(a,r)}} \in \partial B(a,r)) = 0$, for all $x \in B(a,r)$ and hence it follows from [16, Theorem 1] that for a nonnegative Borel function f on $\mathbb{R}^d$ the following formula holds

$$\mathbb{E}_x[f(Y_{\tau^Y_{B(a,r)}})] = \int_{\overline{B(a,r)}^c}\int_{B(a,r)} G^Y_{B(a,r)}(x,u)J^Y(z-u)f(z)\,du\,dz,\ x \in B(a,r).$$

If we define

$$K^Y_{B(a,r)}(x,z) = \int_{B(a,r)} G^Y_{B(a,r)}(x,u)J^Y(z-u)\,dy,\ x \in D,\ z \in \overline{B(a,r)}^c, \tag{3.10}$$

then

$$\mathbb{E}_x[f(Y_{\tau_{B(a,r)}})] = \int_{\overline{B(a,r)}^c} K^Y_{B(a,r)}(x,z)f(z)\,dz,\ x \in B(a,r). \tag{3.11}$$

$K^Y_{B(a,r)}(x,z)$ is called the Poisson kernel of $B(a,r)$ with respect to Y.

Proposition 3.2 There exists a constant $C_1 = C_1(d, \alpha) > 0$ such that for any $a \in \mathbb{R}^d$ and $r \le R_0$

$$K^Y_{B(a,r)}(x,z) \le C_1 K^Y_{B(a,r)}(y,z), \quad \text{for all } x,y \in B(a,r/2),\ z \in B(a,2r) \setminus B(a,r). \tag{3.12}$$

Proof. Let $a \in \mathbb{R}^d$, $r \le R_0$, $x, y \in B(a, r/2)$ and $z \in B(a, 2r) \setminus B(a, r)$. Since $R_0 \le \frac{1}{4}$, it follows that

$$|z-u| \le 3r \le 3/4, \quad \text{for } u \in B(a,r),$$

and hence

$$K^Y_{B(a,r)}(x,z) = \mathcal{A}(d,\alpha) \int_{B(a,r)} \frac{G^Y_{B(a,r)}(x,u)}{|z-u|^{d+\alpha}}\, du. \tag{3.13}$$

Using Proposition 3.1 and (3.13) we get

$$K^X_{B(a,r)}(x,z) \le K^Y_{B_r}(x,z) \le 2K^X_{B(a,r)}(x,z). \tag{3.14}$$

The explicit formula for the Poisson kernel of the ball $B(a, r)$ with respect to the rotationally invariant α-stable process X is

$$K^X_{B(a,r)}(x,z) = c_1 \frac{(r^2 - |x-a|^2)^{\alpha/2}}{(|z-a|^2 - r^2)^{\alpha/2}} \frac{1}{|x-z|^d},\ x \in B(a,r),\ z \in B(a,r)^c,$$

for some constant $c_1 = c_1(d, \alpha) > 0$. It follows from this formula that there exists a constant $c_2 = c_2(d, \alpha) > 0$ such that

$$K^X_{B_r}(x,z) \le c_2 K^X_{B_r}(y,z) \tag{3.15}$$

Using (3.14) and (3.15) we get (3.12). □

Proposition 3.3 There exists a constant $C_2 = C_2(d, \alpha) > 0$ such that for any $a \in \mathbb{R}^d$ and $r \le \frac{R_0}{2}$,

$$K^Y_{B(a,r)}(x,z) \le C_2\, K^Y_{B(a,2r)}(y,z), \quad \text{for all } x,y \in B(a,r/2),\ z \in B(a,2r)^c. \tag{3.16}$$

Proof. Let $a \in \mathbb{R}^d$, $r \le \frac{R_0}{2}$, $x, y \in B(a, r/2)$ and $z \in B(a, 2r)^c$. It follows from Proposition 3.1 that

$$\int_{B(a,r)} G^X_{B(a,r)}(x,u) J^Y(z-u)\, du \le K^Y_{B(a,r)}(x,z) \le 2 \int_{B(a,r)} G^X_{B(a,r)}(x,u) J^Y(z-u)\, du. \tag{3.17}$$

Using [10, Corollary 1.3] we see that there exists a constant $c_1 = c_1(d, \alpha) > 0$ such that

$$G^X_{B(a,r)}(x,u) \le c_1 |x-u|^{\alpha-d}, \quad \text{for all } u \in B(a,r). \tag{3.18}$$

By Lemma 2.7 we have

$$J^Y(z-u) \le 32^d\, \nu_d^{-1} r^{-d} \int_{B(a,r)} J^Y(z-v)\, dv, \quad \text{for all } u \in B(a, 3r/4). \tag{3.19}$$

Using (3.17), (3.18) and (3.19) we get

$$\begin{aligned}
K^Y_{B(a,r)}(x,z) \leq & 2\int_{B(a,r)} G^X_{B(a,r)}(x,u) J^Y(z-u)\,du \\
= & 2\int_{B(a,r)\setminus B(a,3r/4)} G^X_{B(a,r)}(x,u) J^Y(z-u)\,du + \\
& + 2\int_{B(a,3r/4)} G^X_{B(a,r)}(x,u) J^Y(z-u)\,du \\
\leq & 2c_1 \int_{B(a,r)\setminus B(a,3r/4)} |u-x|^{\alpha-d} J^Y(z-u)\,du + \\
& + 2\,c_1\,32^d\,\nu_d^{-1}\,r^{-d} \int_{B(a,3r/4)} |u-x|^{\alpha-d}\,du \int_{B(a,r)} J^Y(z-v)\,dv \\
\leq & 2^{1+2(d-\alpha)}\,c_1 r^{\alpha-d} \int_{B_r\setminus B(a,3r/4)} J^Y(z-u)\,du + \\
& + 2^{1+5d+\alpha}\,c_1\,\nu_d^{-1}\,r^{-d} \int_{B(0,2r)} |u|^{\alpha-d}\,du \int_{B(a,r)} J^Y(z-v)\,dv \\
\leq & (2^{1+2(d-\alpha)}\,c_1 + 2^{1+5d+\alpha}\,c_1\,\nu_d^{-1}\alpha^{-1}\sigma_d) r^{\alpha-d} \int_{B(a,r)} J^Y(z-v)\,dv.
\end{aligned}$$

Therefore, for $c_3 = c_1(2^{1+2(d-\alpha)} + 2^{1+5d+\alpha}\,\nu_d^{-1}\alpha^{-1}\sigma_d)$ we have

$$K^Y_{B(a,r)}(x,z) \leq c_3 \int_{B(a,r)} J^Y(z-u)\,du. \tag{3.20}$$

By [10, Corollary 1.3] again, there exists a constant $c_4 = c_4(d,\alpha) > 0$ such that

$$G^X_{B(a,2r)}(x,u) \geq c_4|x-u|^{\alpha-d} \quad \text{for all } u \in B(a,r).$$

Hence, by (3.17) we have

$$\begin{aligned}
K^Y_{B(a,2r)}(y,z) \geq & \int_{B(a,2r)} G^X_{B(a,2r)}(y,u)\,J^Y(z-u)\,du \geq \int_{B(a,r)} G^X_{B(a,2r)}(y,u)\,J^Y(z-u)\,du \\
& \geq c_4 \int_{B(a,r)} |u-y|^{\alpha-d} J^Y(z-u)\,du \geq (3/2)^{\alpha-d}\,c_4 r^{\alpha-d} \int_{B(a,r)} J^Y(z-u)\,du.
\end{aligned}$$

Using the last display and (3.20) we get

$$K^Y_{B(a,r)}(x,z) \leq c_3 c_4^{-1} (3/2)^{d-\alpha}\,K^Y_{B(a,2r)}(y,z)$$

and thus we may set $C_2 = c_3 c_4^{-1}(3/2)^{d-\alpha}$. □

Theorem 3.4 (**Harnack inequality**) There exists a constant $C_3 = C_3(d,\alpha) > 0$ such that for any $a \in \mathbb{R}^d$, $r \leq \frac{R_0}{2}$ and any positive function h on $\mathbb{R}^d$ which is harmonic in $B(a,4r)$ with respect to Y it follows that

$$h(x) \leq C_3 h(y), \quad \text{for all } x, y \in B(a, r/2).$$

Proof. Let $a \in \mathbb{R}^d$ and $x, y \in B(a, r/2)$. Using (3.11), Proposition 3.2, Proposition 3.3 and fact that $\mathbb{P}_y(Y_{\tau_{B(a,2r)}} \in \partial B(a, 2r)) = 0$ we see that

$$
\begin{aligned}
h(x) = \mathbb{E}_x[h(Y_{\tau_{B(a,r)^Y}})] &= \int_{\overline{B(a,r)}^c} K^Y_{B(a,r)}(x,z)h(z)\,dz = \\
&= \int_{B(a,2r)\setminus\overline{B(a,r)}} K^Y_{B(a,r)}(x,z)h(z)\,dz + \int_{B(a,2r)^c} K^Y_{B(a,r)}(x,z)h(z)\,dz \\
&\le C_1 \int_{B(a,2r)\setminus\overline{B(a,r)}} K^Y_{B(a,r)}(y,z)h(z)\,dz + C_2 \int_{B(a,2r)^c} K^Y_{B(a,2r)}(y,z)h(z)\,dz \\
&\le C_1 \int_{\overline{B(a,r)}^c} K^Y_{B(a,r)}(y,z)h(z)\,dz + C_2 \int_{\overline{B(a,2r)}^c} K^Y_{B(a,2r)}(y,z)h(z)\,dz \\
&= C_1 h(y) + C_2 \mathbb{E}_y[h(Y_{\tau^Y_{B(a,2r)}})] = (C_1 + C_2)h(y),
\end{aligned}
$$

where the last equality follows from the harmonicity of h in $B(a, 4r)$. □

4 Subordinate Brownian motion

A Lévy process $S = (S_t : t \ge 0)$ is called a subordinator if it has a.s. increasing paths which take values in $[0, \infty)$ and $S_0 = 0$. It is convenient to use the Laplace transform of the law of S_t, which is

$$\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)},\ \lambda > 0,$$

where

$$\phi(\lambda) = d\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(dt). \tag{4.1}$$

(see [4, p. 72]). Here $d \ge 0$ and μ is σ-finite measure on $(0, \infty)$ such that $\int_0^\infty (t \wedge 1)\mu(dt) < \infty$. We call $\phi : (0, \infty) \to (0, \infty)$ the Laplace exponent, μ the Lévy measure and d the drift of the subordinator S.

The potential measure of the subordinator S is given by

$$U(A) = \mathbb{E} \int_0^\infty 1_{\{S_t \in A\}} dt,\ A \in \mathcal{B}([0, \infty))$$

and its Laplace transform is

$$\mathcal{L}U(\lambda) = \int_0^\infty e^{-\lambda t} U(dt) = \mathbb{E} \int_0^\infty e^{-\lambda S_t}\,dt = \int_0^\infty e^{-t\phi(\lambda)}\,dt = \frac{1}{\phi(\lambda)}.$$

A function $\phi : (0, \infty) \to (0, \infty)$ is called Bernstein function if $\phi \in C^\infty((0, \infty))$ and

$$(-1)^n \phi^{(n)} \le 0, \quad \text{for all } n \in \mathbb{N}.$$

Here $\phi^{(n)}$ denotes the n-th derivative of ϕ. It is well known (see [17, Theorem 3.9.4]) that ϕ is a Bernstein function such that $\lim_{\lambda \to 0+} \phi(\lambda) = 0$ if and only if it is of the form given by (4.1).

A function $\phi\colon (0,\infty) \to (0,\infty)$ is a complete Bernstein function if there exists a Bernstein function η such that

$$\phi(\lambda) = \lambda^2 \mathcal{L}\eta(\lambda), \quad \text{for all } \lambda > 0.$$

A subordinator whose Laplace exponent ϕ is a complete Bernstein function such that $\lim_{\lambda\to 0+} \phi(\lambda) = 0$ is called a complete subordinator.

Let $X = (X_t, \mathbb{P}_x)$ be a d-dimensional Brownian motion independent of the subordinator S. It follows from [21, Theorem 30.1] that $Y_t := X_{S_t}$ defines a Lévy process $Y = (Y_t\colon t \geq 0)$ with the characteristic exponent

$$\Phi(\xi) = \phi(|\xi|^2),\ \xi \in \mathbb{R}^d.$$

We call Y a subordinate Brownian motion. The Lévy measure of Y has density $J(x) = j(|x|),\ x \in \mathbb{R}^d \setminus \{0\}$, where

$$j(r) = (4\pi)^{-d/2} \int_0^\infty t^{-d/2} e^{-\frac{r^2}{4t}} \mu(dt),\ r > 0. \tag{4.2}$$

It is easy to see that j is a decreasing function.

Suppose that $S = (S_t\colon t \geq 0)$ is a complete subordinator with Lévy measure satisfying $\mu(0,\infty) = \infty$. It follows from [23, Theorem 2.1] that the potential measure of S has a density $u\colon (0,\infty) \to (0,\infty)$ which is a decreasing function such that $\int_0^1 u(t)\,dt < \infty$. Furthermore, suppose that the Laplace exponent of S satisfies

$$\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda),\ \lambda > 0,$$

where $\alpha \in (0,2)$ and $\ell\colon (0,\infty) \to (0,\infty)$ is a continuous slowly varying function, that is, ℓ is a continuous function satisfying

$$\lim_{\lambda\to\infty} \frac{\ell(t\lambda)}{\ell(\lambda)} = 1, \quad \text{for all } t > 0.$$

Using Karamata's Tauberian Theorem and Karamata's Monotone Density Theorem (see [5, Theorem 1.7.1, Theorem 1.7.2]) we conclude that

$$u(t) \sim \frac{1}{\Gamma(\alpha/2)} \frac{t^{\alpha/2-1}}{\ell(1/t)},\ t \to 0+\,. \tag{4.3}$$

Let Y be the corresponding subordinate process. In the sequel we will assume that Y is transient. This will be the case if

$$\int_0^a \frac{\lambda^{d/2-1}}{\phi(\lambda)}\,d\lambda < \infty, \quad \text{for some } a > 0$$

(see (3.1) in [23]), which is satisfied for $d \geq 3$. In this case there exists a Green function $G(x,y)$ which is given by $G(x,y) = g(|x-y|)$, $x,y \in \mathbb{R}^d$, $x \neq y$, where

$$g(r) = (4\pi)^{-d/2} \int_0^\infty t^{-d/2} e^{-\frac{r^2}{4t}}\, u(t)\,dt,\ r > 0. \tag{4.4}$$

It is easy to see that g is a decreasing function. In order to get estimates for the jumping kernel J and the Green function G of the process Y we need some assumptions on the slowly varying function ℓ. Hence in the rest of this section we assume the following conditions:

A1. There exist functions $h_1, h_2\colon (0,\infty) \to (0,\infty)$ and a constant $D > 0$ such that

$$\int_0^\infty t^{(d+\alpha)/2-1} e^{-t} h_1(t)\, dt < \infty, \tag{4.5}$$

$$\int_0^\infty t^{(d-\alpha)/2-1} e^{-t} h_2(t)\, dt < \infty \tag{4.6}$$

and

$$\frac{1}{h_2(t)} \le \frac{\ell(1/y)}{\ell(4t/y)} \le h_1(t), \quad \text{for all } t, y > 0,\ t > D\, y. \tag{4.7}$$

A2. If $d = 1$ or $d = 2$, there exist constants $c_0 > 0$ and $\gamma > 0$ such that

$$u(t) \sim c_0 t^{\gamma-1},\ t \to \infty.$$

In comparison with conditions in [19], we do not need additional condition on behavior of the Lévy density $\mu(t)$ for large t (see condition A4 in [19, p. 12]).

Lemma 4.1 There exist constants $C_1 = C_1(d,\alpha) > 0$ and $C_2 = C_2(d,\alpha) > 0$ such that

$$j(r) \sim C_1\, r^{-d-\alpha} \ell(1/r^2),\ r \to 0+ \tag{4.8}$$

and

$$g(r) \sim C_2 \frac{r^{\alpha-d}}{\ell(1/r^2)},\ r \to 0+\,. \tag{4.9}$$

Proof. Since

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \mu(t)\, dt = \lambda \int_0^\infty e^{-\lambda t} \mu(t,\infty)\, dt$$

we get

$$\int_0^\infty e^{-\lambda t} \mu(t,\infty)\, dt \sim \lambda^{\alpha/2-1} \ell(\lambda),\ \lambda \to \infty$$

and then, using Karamata's Tauberian Theorem (see [5, Theorem 1.7.1]),

$$V(t) \sim \frac{t^{1-\alpha/2} \ell(1/t)}{\Gamma(2-\alpha/2)},\ t \to 0+,$$

where $V(t) = \int_0^t \mu(s,\infty)\, ds$. Using Karamata's Monotone Density Theorem twice (see [5, Theorem 1.7.2]) we conclude

$$\mu(t,\infty) \sim \frac{(1-\alpha/2) t^{-\alpha/2} \ell(1/t)}{\Gamma(2-\alpha/2)},\ t \to 0+$$

and finally

$$\mu(t) \sim \frac{(1-\alpha/2)\alpha\, t^{-1-\alpha/2} \ell(1/t)}{2\Gamma(2-\alpha/2)},\ t \to 0+\,. \tag{4.10}$$

Using [23, Lemma 3.3], (4.5) and (4.10) we get

$$j(r) \sim 2^{\alpha-1} \pi^{-d/2} \alpha\, \Gamma((d+\alpha)/2) \Gamma(1-\alpha/2)^{-1}\, r^{-d-\alpha} l(1/r^2),\ r \to 0+\,.$$

Using [23, Lemma 3.3], (4.6) and (4.3) we have

$$g(r) \sim 2^{-\alpha}\pi^{-d/2}\Gamma((d-\alpha)/2)\Gamma(\alpha/2)^{-1}\,\frac{r^{\alpha-d}}{\ell(1/r^2)},\ r \to 0+.$$

□

Proposition 4.2 There exist $R_0 > 0$ and a constant $C_3 = C_3(d, \alpha, \ell) > 0$ such that for any $a \in \mathbb{R}^d$ and $r \le R_0$

$$G_{B(a,3r)}(x,y) \ge C_3 \frac{1}{r^{d-\alpha}\ell(1/r^2)}, \quad \text{for all } x \in B(a, r/2),\ y \in B(a,r). \tag{4.11}$$

Proof. Choose $0 < c_1 < 1 < c_2$ such that

$$c_1^2\left(\frac{2}{3}\right)^{d-\alpha} - c_2^2\left(\frac{1}{2}\right)^{d-\alpha} > 0.$$

Let $a \in \mathbb{R}^d$. Using Lemma 4.1 choose $R_0 > 0$ such that for $r \le 2R_0$ we have

$$\frac{c_1 C_2}{r^{d-\alpha}\ell(\frac{1}{r^2})} \le g(r) \le \frac{c_2 C_2}{r^{d-\alpha}\ell(\frac{1}{r^2})}, \quad c_1 \le \frac{\ell(\frac{4}{9}\frac{1}{r^2})}{\ell(\frac{1}{r^2})} \le c_2, \quad c_1 \le \frac{\ell(\frac{1}{4}\frac{1}{r^2})}{\ell(\frac{1}{r^2})} \le c_2. \tag{4.12}$$

Let $r \le R_0$, $x \in B(a, r/2)$ and $y \in B(a,r)$. By (4.12) and monotonicity of g we get

$$\begin{aligned}
G_{B(a,3r)}(x,y) &= G(x,y) - \mathbb{E}_x[G(Y_{\tau_{B(a,3r)}}, y)] = g(|x-y|) - \mathbb{E}_x[g(|Y_{\tau_{B(a,3r)}} - y|)] \\
&\ge g(3r/2) - g(2r) \ge C_2\Big(\frac{c_1}{\left(\frac{3r}{2}\right)^{d-\alpha}\ell(\frac{4}{9}\frac{1}{r^2})} - \frac{c_2}{(2r)^{d-\alpha}\,\ell(\frac{1}{4}\frac{1}{r^2})}\Big) \\
&= \frac{C_2}{r^{d-\alpha}}\Big(\frac{c_1\,\left(\frac{2}{3}\right)^{d-\alpha}}{\frac{\ell(\frac{4}{9}\frac{1}{r^2})}{\ell(\frac{1}{r^2})}\ell(\frac{1}{r^2})} - \frac{c_2\,\left(\frac{1}{2}\right)^{d-\alpha}}{\frac{\ell(\frac{1}{4}\frac{1}{r^2})}{\ell(\frac{1}{r^2})}\ell(\frac{1}{r^2})}\Big) \\
&\ge C_2\frac{c_1^2\left(\frac{2}{3}\right)^{d-\alpha} - c_2^2\left(\frac{1}{2}\right)^{d-\alpha}}{c_1\, c_2}\frac{1}{r^{d-\alpha}\ell(1/r^2)}.
\end{aligned}$$

Hence we may take

$$C_3 = C_2\frac{c_1^2\left(\frac{2}{3}\right)^{d-\alpha} - c_2^2\left(\frac{1}{2}\right)^{d-\alpha}}{c_1\, c_2} > 0.$$

□

Proposition 4.3 There exist $R_1 \in (0, R_0]$ and a constant $C_4 = C_4(d, \alpha, \ell) > 0$ such that for any $a \in \mathbb{R}^d$ and $r \le R_1$

$$G_{B(a,r)}(x,y) \le C_4 \frac{1}{|x-y|^{d-\alpha}\ell(1/|x-y|^2)}, \quad \text{for all } x, y \in B(a,r) \tag{4.13}$$

and

$$G_{B(a,r)}(x,y) \le C_4 \frac{1}{r^{d-\alpha}\ell(1/r^2)}, \quad \text{for all } x \in B(a,r/2),\ y \in B(a,r) \setminus B(a, 3r/4). \tag{4.14}$$

Proof. Let $a \in \mathbb{R}^d$. Using Lemma 4.1 we can choose $R_1 \in (0, R_0]$ such that for $r \le R_1$

$$G_{B(a,r)}(x,y) \le g(|x-y|) \le 2C_2 \frac{1}{|x-y|^{d-\alpha}\ell(1/|x-y|^2)}, \quad \text{for all } x,y \in B(a,r).$$

It follows that for $x \in B(a, r/2)$ and $y \in B_r \setminus B(a, 3r/4)$ we get

$$G_{B(a,r)}(x,y) \le g(|x-y|) \le g(r/4) \le 2C_2 \frac{4^{d-\alpha}}{r^{d-\alpha}\ell(16/r^2)} \le \frac{2C_2\, c_1\, 4^{d-\alpha}}{r^{d-\alpha}\ell(1/r^2)},$$

where $c_1 > 0$ is a constant such that

$$\frac{\ell(1/s^2)}{\ell(16/s^2)} \le c_1, \quad \text{for all } s \le r_0.$$

□

Let $a \in \mathbb{R}^d$ and $r > 0$. It is proved in [24] that $\mathbb{P}_x(Y_{\tau_{B(a,r)}} \in \partial B(a,r)) = 0$ and so it follows from [16, Theorem 1] that for a nonnegative Borel function f on $\mathbb{R}^d$ the following formula holds

$$\mathbb{E}_x[f(Y_{\tau^Y_{B(a,r)}})] = \int_{B(a,r)} \int_{\overline{B(a,r)}^c} G^Y_{B(a,r)}(x,u) J^Y(z-u) f(z)\, dz\, du, \ x \in B(a,r).$$

If we define the Poisson kernel for the $B(a,r)$ with respect to Y by

$$K_{B(a,r)}(x,z) = \int_{B_r} G_{B_r}(x,u) J(z-u)\, du, \quad \text{for } x \in B(a,r),\ z \in \overline{B(a,r)}^c,$$

then for any nonnegative Borel function f on $\mathbb{R}^d$ we have

$$\mathbb{E}_x[f(Y_{\tau^Y_{B(a,r)}})] = \int_{\overline{B(a,r)}^c} K_{B(a,r)}(x,z) f(z)\, dz, \quad \text{for all } x \in B(a,r).$$

Proposition 4.4 There exist $R_2 \in (0, R_1]$ and a constant $C_5 = C_5(d, \alpha, \ell) > 0$ such that for any $a \in \mathbb{R}^d$ and $r \le R_2$,

$$K_{B(a,r)}(x,z) \le C_5\, K_{B(a,3r)}(y,z) \ \text{ for all } x,y \in B(a,r/2),\ z \in B(a,3r)^c. \tag{4.15}$$

Proof. Let $a \in \mathbb{R}^d$. Take $r \le R_0$, $x, y \in B(a, r/2)$ and $z \in B(a,3r)^c$. Using Proposition 4.2 we get

$$\begin{aligned} K_{B(a,3r)}(y,z) &= \int_{B(a,3r)} G_{B(a,3r)}(y,u)\, j(|z-u|)\, du \ge \int_{B(a,r)} G_{B(a,3r)}(y,u)\, j(|z-u|)\, du \\ &\ge \frac{C_3}{r^{d-\alpha}\ell(1/r^2)} \int_{B(a,r)} j(|z-u|)\, du. \end{aligned} \tag{4.16}$$

On the other side, using Proposition 4.3 and Lemma 2.7 it follows that

$$\begin{aligned}
K_{B(a,r)}(x,z) &= \int_{B(a,3r/4)} G_{B(a,r)}(x,v) j(|z-v|)\,dv + \int_{B(a,r)\setminus B(a,3r/4)} G_{B(a,r)}(x,v) j(|z-v|)\,dv \\
&\le C_4\, 32^d \nu_d^{-1} r^{-d} \int_{B(a,r)} j(|z-u|)\,du \int_{B(a,3r/4)} \frac{dv}{|x-v|^{d-\alpha}\ell(1/|x-v|^2)} + \\
&\quad + \frac{C_4}{r^{d-\alpha}\ell(1/r^2)} \int_{B(a,r)\setminus B(a,3r/4)} j(|z-u|)\,du \\
&\le C_4\, 32^d\, \nu_d^{-1}\, \sigma_d\, r^{-d} \int_{B(a,r)} j(|z-u|)\,du \int_0^{2r} \frac{s^{\alpha-1}\,ds}{\ell(1/s^2)} + \\
&\quad + \frac{C_4}{r^{d-\alpha}\ell(1/r^2)} \int_{B(a,r)} j(|z-u|)\,du
\end{aligned}$$

Using [5, Proposition 1.5.10] we have

$$\int_0^{2r} \frac{s^{\alpha-1}\,ds}{\ell(1/s^2)} \sim \frac{1}{\alpha}\,\frac{(2r)^\alpha}{\ell(1/(2r)^2)},\ \ r\to 0+$$

and hence we see that there exist $r_2 \in (0, R_1]$ such that for any $r \le R_2$ we have

$$\int_0^{2r} \frac{s^{\alpha-1}\,ds}{\ell(1/s^2)} \le 2\,\frac{2^\alpha}{\alpha}\,\frac{r^\alpha}{\ell(1/r^2)},$$

since ℓ is slowly varying. Using the last display we can continue estimating $K_{B(a,r)}(x,z)$ by

$$K_{B(a,r)}(x,z) \le \frac{C_4(2^{\alpha+1}\alpha^{-1}\,32^d\,\nu_d^{-1}\,\sigma_d + 1)}{r^{d-\alpha} j(1/r^2)} \int_{B_r} j(|z-u|)\,du. \tag{4.17}$$

Finally, by (4.16) and (4.17)

$$K_{B_r}(x,z) \le C_4 C_3^{-1}(2^{\alpha+1}\alpha^{-1}\,32^d\,\nu_d^{-1}\,\sigma_d + 1)\,K_{B_{3r}}(y,z).$$

□

To prove the Harnack inequality we need Krylov-Safonov-type estimates. Define

$$\eta(r) = r^{-2}\int_0^r s^{d+1} j(s)\,ds + \int_r^\infty s^{d-1} j(s)\,ds,\ \ r>0. \tag{4.18}$$

Lemma 4.5 There exists $R_3 \in (0, R_2]$ and a constant $C_6 = C_6(d,\alpha) > 0$ such that

$$\eta(r) \le C_6\, r^{-\alpha}\ell(1/r^2), \quad \text{for all } r \le R_3.$$

Proof. For $r > 0$ define $\eta_1(r) = r^{-2}\int_0^r s^{d+1} j(s)\,ds$ and $\eta_2(r) = \int_r^\infty s^{d-1} j(s)\,ds$. It follows from Lemma 4.1, [5, Proposition 1.5.10] and [5, Proposition 1.5.8] that there exists $r_0 \in (0, R_2]$ such that for $r \le r_0$ we have $c_1 = \int_{r_0}^{\infty} s^{d-1} j(s)\, ds < \infty$ and

$$\eta_1(r) \le 2C_1 r^{-2} \int_0^r s^{1-\alpha} \ell(1/s^2)\, ds \le 4C_1(2-\alpha)^{-1} r^{-\alpha} \ell(1/r^2),$$
$$\eta_2(r) = \int_r^{r_0} s^{d-1} j(s)\, ds + \int_{r_0}^{\infty} s^{d-1} j(s)\, ds \le 2C_1 \int_r^{r_0} s^{-1-\alpha} \ell(1/s^2)\, ds + c_1$$
$$\le 4C_1 \alpha^{-1} r^{-\alpha} \ell(1/r^2) + c_1.$$

Therefore,

$$\eta(r) \le (4C_1(2-\alpha)^{-1} + 4C_1\alpha^{-1}) r^{-\alpha} \ell(1/r^2) + c_1$$

and for

$$c_2 = (4C_1(2-\alpha)^{-1} + 4C_1\alpha^{-1}) \vee c_1$$

we get

$$\eta(r) \le c_1(r^{-\alpha}\ell(1/r^2) + 1).$$

Since $r^{-\alpha}\ell(1/r^2)$ is dominant term, there exist $R_3 \in (0, r_0]$ and $C_6 \ge c_1$ such that

$$\eta(r) \le C_6 r^{-\alpha} \ell(1/r^2), \quad \text{for all } r \le R_3.$$

□

It follows from [22, Lemma 3.4] that there exists a constant $C_7 = C_7(d, \alpha) > 0$ such that for every $a \in \mathbb{R}^d$, $r \in (0, 1)$, $A \subseteq B(a, r)$ and $x \in B(a, 2r)$

$$\mathbb{P}_x(T_A < \tau_{B(a,3r)}) \ge C_7 \frac{r^d j(4r)}{\eta(r)} \frac{|A|}{|B(a,r)|}, \tag{4.19}$$

where the first hitting time of the set A by the process X is defined by

$$T_A = \inf\{t > 0 \colon X_t \in A\}.$$

Now we can prove Krylov-Safonov estimate.

Proposition 4.6 There exist $R_4 \in (0, R_3]$ and a constant $C_8 = C_8(d, \alpha) > 0$ such that for every $a \in \mathbb{R}^d$, $r \le r_3$, $A \subseteq B(a, r)$ and $x \in B(a, 2r)$

$$\mathbb{P}_x(T_A < \tau_{B(a,3r)}) \ge C_8 \frac{|A|}{|B(a,r)|}. \tag{4.20}$$

Proof. It follows from [5, Theorem 1.5.4], Lemma 4.1 and Lemma 4.5 that there exists a constant $c_1 > 0$ and $R_4 \in (0, R_3]$ such that for any $r \le R_4$

$$\frac{r^d j(4r)}{\eta(r)} \ge 2^{-1} C_1^{-1} C_6^{-1}.$$

Therefore, by (4.19) it follows

$$\mathbb{P}_x(T_A < \tau_{B(a,3r)}) \ge 2^{-1} C_1^{-1} C_6^{-1} C_7 \frac{|A|}{|B(a,r)|},$$

for any $r \le R_4$. □

Using Proposition 4.4 and Proposition 4.6 we see that for any $a \in \mathbb{R}^d$ and $r \le R_4$:

(i) $\mathbb{E}_x H(X_{\tau_{B(a,r)}}) \le C_5 \mathbb{E}_y H(X_{\tau_{B(a,3r)}})$, for all $x, y \in B(a, r/2)$ and any nonnegative Borel function H on $\mathbb{R}^d$ with support in $B(a,3r)^c$;

(ii) $\mathbb{P}_x(T_A < \tau_{B(a,3r)}) \ge C_8 \frac{|A|}{|B(a,r)|}$, for all $A \subseteq B(a,r)$ and $x \in B(a,2r)$.

Thus, we can slightly modify the proof of the [22, Theorem 2.2] (see also [3, Theorem 3.6]) and get the Harnack inequality.

Theorem 4.7 (**Harnack inequality**) There exists a constant $C_9 = C_9(d, \alpha, \ell) > 0$ such that for any $a \in \mathbb{R}^d$ and $r \le R_4 \wedge \frac{1}{4}$ and any function h which is nonnegative, bounded on $\mathbb{R}^d$ and harmonic in $B(a, 16r)$ with respect to X, we have

$$h(x) \le C_9 h(y) \quad \text{for all } x, y \in B(a, r).$$

□

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