Unavoidable collections of balls for censored stable processes

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Abstract

We study avoidability of collections of balls in bounded $C^{1,1}$ opens sets for censored α-stable processes, $\alpha \in (1,2)$. The results are analog to the ones obtained for Brownian motion in S. J. Gardiner, M. Ghergu, *Champagne subregions of the unit ball with unavoidable bubbles*, Ann. Acad. Sci. Fenn. Math. **35** (2010) 321-329. On the way we derive a Wiener-Aikawa-type criterion for minimal thinness with respect to the censored stable processes.

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1 Introduction

Let A be a Borel subset of the unit ball $B(0,1) \subset \mathbb{R}^d$, $d \geq 2$, not containing the origin. Then A is said to be unavoidable if Brownian motion starting from the origin almost surely hits A before hitting the boundary ∂B. More precisely, let $X = (X_t, \mathbb{P}_x)$ denote a standard Brownian motion in $\mathbb{R}^d$ and let $T_C = \inf\{t > 0 : X_t \in C\}$ be the hitting time of a Borel set $C \subset \mathbb{R}^d$. Then $A \subset B$ is *unavoidable* if $\mathbb{P}_0(T_A < T_{\partial B(0,1)}) = 1$ and *avoidable* if $\mathbb{P}_0(T_A < T_{\partial B(0,1)}) < 1$.

Problem of avoidability for sets A that are unions of balls has been studied recently. Let $\{\overline{B}(x_n, r_n)\}_{n\geq 1}$ be a collection of pairwise disjoint closed balls contained in $B(0,1)$ satisfying $|x_n| \to 1$ and $\sup\limits_{n\geq 1} \frac{r_n}{1-|x_n|} < 1$. Define $A := \cup_{n=1}^{\infty} \overline{B}(x_n, r_n)$. The domain $B(0,1) \setminus A$ is often called the champagne region and the balls are called bubbles. Avoidability of balls in the unit disc in $\mathbb{R}^2$ was studied in [3, 18], and in higher dimensions in [9]. Those results were extended in [12].

The aim of this paper is to prove analogous results for a class of jump processes. Note that if Brownian motion is replaced by the rotationally invariant α-stable process, then any collection of balls in the unit ball is avoidable since the process jumps out of the unit ball with positive probability before hitting A.

The natural choice for the jump process replacing Brownian motion in avoidability problems in $B(0,1)$ is the censored α-stable process with $\alpha \in (1,2)$. Roughly, this process is constructed from the symmetric α-stable process by suppressing the jumps landing outside of the ball and continuing at the place where the suppressed jump has occurred. Such a process is transient (for $\alpha \in (1,2)$) and converges to the boundary at its lifetime. In particular, with probability one it cannot be killed while inside the state space – if this were possible every subset of B would be avoidable. The censored stable processes were rigorously constructed and studied in [6]. Fine properties of their potential theory in bounded $C^{1,1}$ open sets was further developed in [7] (for more detail see Section 2). These results will allow us to replace the unit ball by a bounded $C^{1,1}$ open set.

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be a bounded $C^{1,1}$ open set, denote by $\delta_D(x)$ the distance of $x \in D$ to the boundary ∂D, and let $Y = (Y_t, \mathbb{P}_x)$ be the censored α-stable process in D with $\alpha \in (1,2)$. The lifetime of Y will be denoted by ζ. By [6, Theorem 1.1] it holds that $Y_{\zeta-} \in \partial D$.

The concept of avoidability for the process Y and the set D is defined analogously to the case of the Brownian motion and the unit ball. To be more precise, let $\{\overline{B}(x_n, r_n)$ be a collection of pairwise disjoint closed balls in D such that $\delta_D(x_n) \to 0$ and $\sup\limits_{n\geq 1} \frac{r_n}{\delta_D(x_n)} < 1/2$, and let $A = \cup_{n\geq 1}\overline{B}(x_n, r_n)$. The open set $D \setminus A$ will be called a champagne subregion of D, and as before the balls are called bubbles. Fix a point $x_0 \in D$ such that $x_0 \notin A$, and let $T_A = \inf\{t > 0 : Y_t \in A\}$. We will say that A is avoidable if $\mathbb{P}_{x_0}(T_A < \zeta) < 1$, and unavoidable otherwise. The first main result of this paper is the analog of [12, Theorem 1].

Theorem 1.1 *Let $D \setminus A$ be the champagne subregion of a bounded $C^{1,1}$ open subset D.*

(a) If A is unavoidable, then

$$\sum_{n\geq 1} \frac{\delta_D(x_n)^{2\alpha-2}}{|x_n - z|^{d+\alpha-2}} r_n^{d-\alpha} = \infty \quad \textit{for } \sigma - a.e.\ z \in \partial D\,. \tag{1.1}$$

(b) Conversely, if (1.1) and the separation condition

$$\inf_{j\neq k} \frac{|x_j - x_k|}{r_k^{1-\alpha/d} \delta_D(x_k)^{\alpha/d}} > 0 \tag{1.2}$$

hold, then A is unavoidable.

Here σ denotes the surface measure on ∂D.

The analog of [12, Theorem 2] seems to make sense only in the unit ball $B = B(0,1)$. It concerns radii r_n which are of the form $r_n = (1 - |x_n|)\phi(|x_n|)$ where $\phi : [0,1) \to (0,1)$ is decreasing. For $a \in (0,1)$ let

$$N_a(x) = \sharp[B(x, a(1 - |x|)) \cap \{x_n : n \in \mathbb{N}\}]$$

be the number of centers in the ball $B(x, a(1 - |x|)$, and let $M : [0,1) \to [1,\infty)$ be an increasing function satisfying

$$M(1 - \tfrac{t}{2}) \leq cM(1 - t) \quad \text{for all } t \in (0,1)\,. \tag{1.3}$$

Theorem 1.2 *Let $\phi : [0,1) \to (0,1)$ be a decreasing function and $M : [0,1) \to [1,\infty)$ be a function satisfying (1.3). Let $B \setminus A$ be a champagne subregion of the ball B such that $r_n = (1 - |x_n|)\phi(|x_n|)$.*

(a) If A is unavoidable and there are constants $a \in (0,1)$ and $b > 0$ such that $N_a(x) \leq bM(|x|)$ for all $x \in B$, then

$$\int_0^1 \frac{\phi(t)^{d-\alpha} M(t)}{1-t}\, dt = \infty\,. \tag{1.4}$$

(b) Conversely, if (1.4) *holds together with the separation condition*

$$\inf_{m\neq n} \frac{|x_m - x_n|}{\phi(|x_n|)^{1-\alpha/d}(1-|x_n|)} > 0\,, \tag{1.5}$$

and there are constants $a \in (0,1)$ and $b \geq 0$ such that $N_a(x) \geq bM(|x|)$ for all $x \in B$, then A is unavoidable.

By comparing the statements of Theorems 1.1 and 1.2 with [12, Theorems 1 an 2.2] one sees that they are identical except for parameter α replacing 2. In proving Theorems 1.1 and 1.2 we will closely follow the ideas from [12]. In order to implement those ideas we had to develop certain potential-theoretic results for censored stable processes which are standard in case of Brownian motion (or classical potential theory). The first such result is quasi-additivity of capacity related to the censored stable process, see Proposition 3.1. Here we follow the exposition from [2, Part II, Section 7] and only indicate the necessary changes mostly related to the construction of a comparable measure. The second necessary ingredient is a Wiener-Aikawa-type condition for minimal thinness of a set near the boundary point, see Propositions 4.2 and 4.4. The third result is a modification of a Aikawa-Borichev-type quasi-additivity of capacity from [1, Theorem 3], see Proposition 5.2. With these results at hand we prove Theorems 1.1 and 1.2 in Sections 6 and 7. Scaling plays a significant role in the proof of Theorem 1.1. We note that in case of censored stable process scaling is more delicate than in case of Brownian motion due to the fact that both the process Y and all related potential-theoretic properties are confined to the state space D.

We end this introduction with a few remarks on other works about avoidability of sets for jump processes. In a recent preprint [13] the authors study the question of smallness of unavoidable sets in the context of balayage spaces. Their examples include Brownian motion, symmetric α-stable processes in $\mathbb{R}^d$, $\alpha \in (0,2)$, and censored α-stable processes in bounded $C^{1,1}$-open sets, $\alpha \in (1,2)$. In the latter case their results do not overlap with ours – the goal in [13] is to construct unavoidable collections of balls, or more generally unavoidable sets, having certain smallness properties, while Theorems 1.1 and 1.2 give sufficient and necessary conditions for a given collection of balls to be unavoidable. Another work that treats unavoidable collections of balls in $\mathbb{R}^d$ for certain class of isotropic Lévy processes is [17] with results more in the spirit of the current paper.

Notation and conventions about constants: The constants C_G, C_M, C_H, C that we introduce in the next section, as well as the constant C_1 introduced in Section 6, stay fixed throughout the paper. The other constants, denoted by lowercase letters $c_1, c_2, \dots$, appear only locally in the paper and their numbering starts afresh in each section. Throughout the paper we use the notation $f(r) \asymp g(r)$ as $r \to a$ to denote that $f(r)/g(r)$ stays between two positive constants as $r \to a$.

2 Preliminaries about censored stable processes

Let $Y = (Y_t, \mathbb{P}_x)$ be a censored α-stable process, $\alpha \in (1,2)$, in a bounded $C^{1,1}$ open set $D \subset \mathbb{R}^d$. Such process has been studied in [6, 7, 8, 14]. We will list several properties proved in those papers.

We first note that it follows from [6] that Y is a transient Hunt process with finite lifetime ζ, and $Y_{\zeta-} \in \partial D$. The Dirichlet form $(\mathcal{E}^D, \mathcal{F}^D)$ of Y is given by

$$\mathcal{E}^D(u,v) = \frac{1}{2}\mathcal{A}(d,-\alpha) \int_D \int_D \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\, dy\, dx\,, \quad u,v \in C_c^\infty(D)\,, \tag{2.1}$$

where $\mathcal{F}^D$ is the closure of $C_c^\infty(D)$ under $\mathcal{E}_1^D = \mathcal{E}^D + \langle \cdot, \cdot \rangle_{L^2(D)}$. Here $\mathcal{A}(d,-\alpha)$ is an explicit, but unimportant constant.

The following Hardy's inequality is proved in [8, Corollary 2.4]: There exists $c = c(D,\alpha) > 0$ such that

$$\mathcal{E}^D(u,u) \ge c \int_D \frac{u(x)^2}{\delta_D(x)^\alpha}\, dx\,, \quad u \in \mathcal{F}^D\,. \tag{2.2}$$

Let $G^D(x,y)$ denote the Green function of Y. The existence and sharp two-sided estimates for G^D are proved in [7, Theorem 1.1]: There exists $C_G = C_G(D,\alpha) \ge 1$ such that

$$\begin{aligned} C_G^{-1} & \left(1 \wedge \frac{\delta_D(x)^{\alpha-1}}{|x-y|^{\alpha-1}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha-1}}{|x-y|^{\alpha-1}}\right) |x-y|^{\alpha-d} \le G^D(x,y) \\ \le \;\; & C_G \left(1 \wedge \frac{\delta_D(x)^{\alpha-1}}{|x-y|^{\alpha-1}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha-1}}{|x-y|^{\alpha-1}}\right) |x-y|^{\alpha-d}\,, \qquad x,y \in D\,. \end{aligned} \tag{2.3}$$

The constant $C_G(D,\alpha)$ can be chosen to be domain translation and dilation invariant.

Let $x_0 \in D$ and define the Martin kernel based at x_0 by

$$M^D(x,y) = \frac{G^D(x,y)}{G^D(x_0,y)}\,, \quad x,y \in D\,.$$

It is proved in [7, Theorem 1.2] that for each $Q \in \partial D$ there exists the limit $M^D(x,Q) := \lim_{y\to Q} M^D(x,y)$ and M^D is jointly continuous on $D\times\partial D$. Further, there exists $C_M = C_M(x_0,D,\alpha)$ such that

$$C_M^{-1} \frac{\delta_D(x)^{\alpha-1}}{|x-z|^{d+\alpha-2}} \le M^D(x,z) \le C_M \frac{\delta_D(x)^{\alpha-1}}{|x-z|^{d+\alpha-2}}\,, \quad x \in D, z \in \partial D\,. \tag{2.4}$$

Consequently, the Martin boundary $\partial_M D$ of D and the minimal Martin boundary $\partial_m D$ (with respect to Y) can be both identified with the Euclidean boundary ∂D.

Let U be an open subset of D. A function $u : D \to [0,\infty)$ is harmonic in U with respect to Y if for every open set V such that $V \subset \overline{V} \subset U$ it holds that

$$u(x) = \mathbb{E}_x[u(Y_{\tau_V})] \quad \text{for all } x \in V\,.$$

Here $\tau_V = \inf\{t > 0 : Y_t \notin V\}$ is the exit time of Y from V. The function u is regular harmonic in U if

$$u(x) = \mathbb{E}_x[u(Y_{\tau_U})] \quad \text{for all } x \in U\,.$$

It is well known that regular harmonic functions are harmonic. Moreover, for $y \in D$, $x \mapsto G^D(x,y)$ is harmonic in $D\setminus\{y\}$, and for every $\epsilon > 0$ regular harmonic in $D\setminus\overline{B}(y,\epsilon)$. Further, by [6, Theorem

3.2], harmonic functions satisfy Harnack inequality. More precisely, we can deduce from that result that there exists a constant $C_H = C_H(d,\alpha) > 0$ such that for every ball $B(x,r) \subset D$ and every nonnegative function u on D which is harmonic in $B(x,r)$,

$$\sup_{y\in B(x,r/2)} u(y) \le C_H \inf_{y\in B(x,r/2)} u(y)\,. \tag{2.5}$$

Let Cap^D denote the capacity with respect to Y. It is proved in [7, (3.10)] that there exists a constant $C = C(D,\alpha) \ge 1$ such that for every ball $B(x,r) \subset D$ satisfying $B(x,2r) \subset D$,

$$C^{-1} r^{d-\alpha} \le \mathrm{Cap}^D B(x,r) \le C r^{d-\alpha}\,. \tag{2.6}$$

We look now at some scaling properties related to Y. For $r > 0$, let $r^{-1}D := \{x \in \mathbb{R}^d : rx \in D\}$. By [7, Remark 2.3], $\{r^{-1}Y_{r^\alpha t}, \mathbb{P}_x\}$ has the same distribution as the censored stable process in $r^{-1}D$ started at the point $r^{-1}x$ and

$$G^{r^{-1}D}(x,y) = r^{d-\alpha} G^D(rx,ry)\,, \quad x,y \in r^{-1}D\,.$$

A simple computation using (2.1) gives that

$$\mathcal{E}^{r^{-1}D}(u,u) = r^{-d+\alpha}\mathcal{E}^D(\widehat{u},\widehat{u})\,, \quad u \in \mathcal{F}^{r^{-1}D}\,,$$

where $\widehat{u}(x) = u(r^{-1}x)$, $x \in D$. Therefore

$$\begin{aligned} \mathrm{Cap}^{r^{-1}D}(A) &= \inf\{\mathcal{E}^{r^{-1}D}(u,u) : u \in \mathcal{F}^{r^{-1}D}, u \ge 1 \text{ a.e. on } A\} \\ &= r^{-d+\alpha}\inf\{\mathcal{E}^D(\widehat{u},\widehat{u}) : \widehat{u} \in \mathcal{F}^D,\ \widehat{u} \ge 1 \text{ a.e. on } rA\} \\ &= r^{-d+\alpha}\mathrm{Cap}^D(rA)\,. \end{aligned} \tag{2.7}$$

Finally, let $\mathcal{W}$ denote the family of all excessive functions with respect to Y. It is proved in [13, Corollary 6.4], that $(D,\mathcal{W})$ is a balayage space in the sense of [5]. This will allow us to freely use results from [5]. Recall that for $u \in \mathcal{W}$ and $B \subset D$, $R_u^B = \inf\{w \in \mathcal{W} : w \ge v \text{ on } B\}$ is the reduced function of u onto B, while its lower-semicontinuous regularization $\widehat{R}_u^B \in \mathcal{W}$ is called the balayage of u onto B. If $T_B = \inf\{t > 0 : Y_t \in B\}$, then $\widehat{R}_u^B(x) = \mathbb{E}_x u(X_{T_B})$ giving the probabilistic interpretation of the balayage, see [5, VI.4].

3 Quasi-additivity of capacity

The goal of this section is to prove that Cap^D is quasi-additive with respect to a Whitney decomposition of D.

Let $\{Q_j\}_{j\ge1}$ be the Whitney decomposition of D. For each Q_j let Q_j^* denote the double of Q_j and let x_j denote the center of Q_j. Then $\{Q_j, Q_j^*\}$ is a quasi-disjoint decomposition of D, cf. [2, pp. 146-147]. A kernel $k : D \times D \to [0, +\infty]$ is said to satisfy the Harnack property with respect to $\{Q_j, Q_j^*\}$, cf. [2, p. 147], if

$$k(x, y) \asymp k(x', y) \text{ for all } x, x' \in Q_j \text{ and all } y \in D \setminus Q_j^*,$$

for all cubes Q_j (with constants not depending on the cube). One way to get such kernels is as follows. Suppose that $u : D \to [0, \infty)$ is a function satisfying the scale invariant Harnack inequality of the form

$$\sup_{Q_j} u \le c_1 \inf_{Q_j} u \quad \text{for all } Q_j,$$

where c_1 does not depend on Q_j. Typical u's are the constant function $u \equiv 1$ and $u = g$ where $g(x) := G^D(x, x_0) \wedge 1$, and $x_0 \in D$ fixed (see (2.3)). Define the kernel $k : D \times D \to [0, \infty]$ by

$$k(x, y) := \frac{G^D(x, y)}{u(x)u(y)}, \quad x, y \in D,$$

Note that $x \mapsto G^D(x, y)$ is regular harmonic in Q_j for every $y \in D \setminus Q_j^*$. Hence by the scale invariant Harnack inequality (2.5) and the assumption on u, we see that k satisfies the Harnack property with respect to $\{Q_j, Q_j^*\}$. For a measure λ on D let $\lambda_u(dy) := \lambda(dy)/u(y)$. Then

$$K\lambda(x) := \int_D k(x, y)\, \lambda(dy) = \int_D \frac{G^D(x, y)}{u(x)u(y)} \lambda(dy) = \frac{1}{u(x)} \int_D G^D(x, y) \frac{\lambda(dy)}{u(y)} = \frac{1}{u(x)} G^D \lambda_u(dy).$$

We define the capacity with respect to the kernel k as follows:

$$\mathrm{Cap}_u(E) := \inf\{\|\lambda\| : K\lambda \ge 1 \text{ on } E\}.$$

For a compact set $F \subset D$, consider the balayage $\widehat{R}_u^F$. Being a potential, $\widehat{R}_u^F = G^D \lambda_u^F$ for a measure λ_u^F supported in F. Define the energy of F (with respect to u) as

$$\gamma_u(F) := \int_D \int_D G^D(x, y) \lambda_u^F(dx)\, \lambda_u^F(dy) = \int_D G^D \lambda_u^F(x)\, \lambda_u^F(dx) = \mathcal{E}_D(G^D \lambda_u^F, G^D \lambda_u^F).$$

This definition of energy is in the usual way extended first to open, and then to Borel subsets of D. By using the dual definition of capacity

$$\mathrm{Cap}_u(F) = \sup\{\mu(F) :\, \mu(D \setminus F) = 0, K\mu \le 1\}, \tag{3.1}$$

for compact subsets $F \subset D$, see e.g. [11, Théorème 1.1], it is standard to show that

$$\gamma_u(E) = \mathrm{Cap}_u(E), \quad E \subset D.$$

Note that in case $u \equiv 1$, $\gamma_1(E) = \mathrm{Cap}_1(E) = \mathrm{Cap}^D(E)$.

A Borel measure σ_u (defined on Borel subsets if D) is comparable to the capacity Cap_u with respect to $\{Q_j\}$ if there exists $c_2 > 0$ such that

$$\sigma_u(Q_j) \asymp \mathrm{Cap}_u(Q_j), \quad \text{for all } Q_j,$$
$$\sigma_u(E) \le c_2 \mathrm{Cap}_u(E), \quad \text{for all Borel } E.$$

Define

$$\sigma_u(E) := \int_E u(x)^2 \delta_D(x)^{-\alpha}\, dx\,, \quad E \subset D\,.$$

We claim that σ_u is comparable with C_u. Note that on Q_j we have $u \asymp u(x_j)$, hence $\widehat{R}_u^{Q_j} \asymp u(x_j)\widehat{R}_1^{Q_j}$, implying $G^D\lambda_u^{Q_j} \asymp u(x_j)G^D\lambda_1^{Q_j}$ (everywhere on D). Therefore,

$$\begin{aligned}
\gamma_u(Q_j) &= \int_{Q_j} G^D\lambda_u^{Q_j}(x)\,\lambda_u^{Q_j}(dx) \asymp u(x_j)\int_{Q_j} G^D\lambda_1^{Q_j}(x)\,\lambda_u^{Q_j}(dx)\\
&= u(x_j)\int_{Q_j} G^D\lambda_u^{Q_j}(x)\,\lambda_1^{Q_j}(dx) \asymp u(x_j)^2\int_{Q_j} G^D\lambda_1^{Q_j}(x)\lambda_1^{Q_j}(dx)\\
&= u(x_j)^2\mathrm{Cap}^D(Q_j)\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sigma_u(Q_j) &= \int_{Q_j} u(x)^2\delta_D(x)^{-\alpha}\,dx \asymp u(x_j)^2(\mathrm{diam}Q_j)^{-\alpha}\,|Q_j|^d\\
&\asymp u(x_j)^2(\mathrm{diam}Q_j)^{-\alpha}\,(\mathrm{diam}Q_j)^d \asymp u(x_j)^2\mathrm{Cap}^D(Q_j)\,,
\end{aligned}$$

where the last asymptotic equality follows from (2.6). Thus, $\gamma_u(Q_j) \asymp \sigma_u(Q_j)$. Further, for any Borel $E \subset D$ and compact $F \subset E$, by using (2.2), we have

$$\begin{aligned}
\gamma_u(E) &\geq \gamma_u(F) = \mathcal{E}(G^D\lambda_u^F, G^D\lambda_u^F) \geq c_3\int_D (G^D\lambda_u^F)(x)^2\delta_D(x)^{-\alpha}\,dx\\
&\geq c_3\int_F (G^D\lambda_u^F)(x)^2\delta_D(x)^{-\alpha}\,dx = c_3\int_F u(x)^2\delta_D(x)^{-\alpha}\,dx = c_3\sigma_u(F)\,.
\end{aligned}$$

This proves that $\gamma_u(E) \geq c_3\sigma_u(E)$.

Now we can invoke [2, Theorem 7.1.3] and conclude that $\gamma_u = \mathrm{Cap}_u$ is quasi-additive with respect to $\{Q_j\}$.

Proposition 3.1 *The Green energy γ_u is quasi-additive with respect to $\{Q_j\}$:*

$$\gamma_u(E) \asymp \sum_{j\geq 1}\gamma_u(E\cap Q_j)\,.$$

4 Minimal thinness

In this section we prove a Wiener-Aikawa-type conditions for minimal thinness of a set near the boundary point.

Recall that $M^D(x,z)$ denotes the Martin kernel at z (based at $x_0 \in D$). The Martin boundary $\partial_M D$ and the minimal Martin boundary $\partial_m D$ of D (with respect to Y) are identified with its Euclidean boundary ∂D. Recall that a set $E \subset D$ is said to be minimally thin at $z \in \partial_m D$ if

$\widehat{R}^E_{M^D(\cdot,z)} \neq M^D(\cdot, z)$, cf. [10]. It is known, see e.g. [16], that every excessive function u of Y can be uniquely represented as

$$u(x) = G^D\mu(x) + M^D\nu(x) = \int_D G^D(x,y)\,\mu(dy) + \int_{\partial D} M^D(x,z)\,\nu(dz)\,.$$

The function $M^D\nu$ is the greatest harmonic minorant of u.

By following the proof of [4, Theorem 9.2.6] and using [10, Lemma 2.7] instead of [4, Lemma 9.2.2(c)] one obtains the next characterizations of minimal thinness.

Proposition 4.1 *Let $A \subset D$. The following are equivalent:*
(a) A is minimally thin at $z \in \partial_m D$;
(b) There exists an excessive function $u = G^D\mu + M^D\nu$ such that

$$\liminf_{x\to z, x\in A} \frac{u(x)}{M^D(x,z)} > \nu(\{z\})\,; \tag{4.1}$$

(c) There exists a potential $u = G^D\mu$ such that

$$\liminf_{x\to z, x\in A} \frac{u(x)}{M^D(x,z)} = +\infty\,. \tag{4.2}$$

Proof. We sketch the proof following the proof of [4, Theorem 9.2.6]. Clearly, (c) implies (b). Assume that (b) holds. Then there exists a Martin topology neighborhood W of z and $a > \nu(\{z\})$ such that $u \geq aM^D(\cdot, z)$ on $A \cap W$. If $\widehat{R}^{A\cap W}_{M^D(\cdot,z)} = M^D(\cdot, z)$, then $u \geq \widehat{R}^{A\cap W}_u \geq aM^D(\cdot, z)$ everywhere. Thus $u - aM^D(\cdot, z)$ is excessive, hence $u - aM^D(\cdot, z) = G^D\mu + M^D\widetilde{\nu}$ for a (non-negative) measure $\widetilde{\nu}$ on ∂D. On the other hand, $u - aM^D(\cdot, z) = G^D\mu + M^D\nu_{|\partial D\setminus\{z\}} + (\nu(\{z\}) - a)M^D(\cdot, z)$. This implies that $\widetilde{\nu} = \nu_{|\partial D\setminus\{z\}} + (\nu(\{z\}) - a)\delta_z$ yielding $\widetilde{\nu}(z) = \nu(\{z\}) - a < 0$, which is a contradiction. Hence $\widehat{R}^{A\cap W}_{M^D(\cdot,z)} \neq M^D(\cdot, z)$, i.e., A is minimally thin at z. Thus (b) implies (a).

Suppose that (a) holds. By [10, Lemma 2.7], there exists an open subset $U \subset \mathbb{R}^d$ such that $A \subset U$, and U is minimally thin at z. By the analog of [4, Theorem 9.2.5], there is a decreasing sequence $(W_n)_{n\geq 1}$ of Martin topology open neighborhoods of z shrinking to z and such that $\widehat{R}^{U\cap W_n}_{M^D(\cdot,z)}(x_0) \leq 2^{-n}$. Let $u_1 := \sum_{n=1}^\infty \widehat{R}^{U\cap W_n}_{M^D(\cdot,z)}$. Then u_1 is a sum of potentials, hence a potential itself since $u_1(x_0) < \infty$. Further, $\widehat{R}^{U\cap W_n}_{M^D(\cdot,z)} = M^D(\cdot, z)$ on the open set $U \cap W_n$. Therefore, $u_1(x)/M^D(x,z) \to \infty$ as $x \to z$, $x \in U$. Thus (c) holds. □

Let D be a $C^{1,1}$ open set. Fix $x_0 \in D$ and let M^D be the Martin kernel of D based at x_0. The following proposition is an analog of [5, Proposition V. 4.15]. A similar result is proved in [15] in case of isotropic Lévy processes satisfying certain conditions.

Proposition 4.2 *Let $E \subset D$ and let $z \in \partial D$. Define*

$$E_n = E \cap \{x \in D:\ 2^{-n-1} \leq |x - z| < 2^{-n}\}\,, \quad n \geq 1\,.$$

Then E is minimally thin at z if and only if $\sum_{n=1}^\infty R^{E_n}_{M^D(\cdot,z)}(x_0) < \infty$.

Proof. Assume that $\sum_{n=1}^\infty R^{E_n}_{M^D(\cdot,z)}(x_0)<\infty$. Then there exists $n_0\in\mathbb{N}$ such that

$$\sum_{n=n_0}^\infty R^{E_n}_{M^D(\cdot,z)}(x_0)<\frac{1}{2}M^D(x_0,z)\,.$$

Let $B=B(z,2^{-n_0})$. Then $A:=B\cap E\subset \cup_{n=n_0}^\infty E_n$. Therefore,

$$R^{A}_{M^D(\cdot,z)}(x_0)\le \sum_{n=n_0}^\infty R^{E_n}_{M^D(\cdot,z)}(x_0)<\frac{1}{2}M^D(x_0,z)\,,$$

implying $\widehat{R}^{A}_{M^D(\cdot,z)}(x_0)<\frac12 M^D(x_0,z)$. Hence, there exists an excessive function u such that $u\ge M^D(\cdot,z)$ on A and $u(x_0)<\frac12 M^D(x_0,z)$. By the Riesz decomposition, $u=G^D\mu+M^D\nu$. Thus, $M^D\nu(x_0)=\int_{\partial_M D}M^D(x_0,z)\,\nu(dz)<\frac12 M^D(x_0,z)$ implying that $\nu(\{z\})<\frac12$. Therefore,

$$\liminf_{x\to z, x\in A}\frac{u(x)}{M^D(x,z)}\ge 1>\frac12>\nu(\{z\})\,.$$

By Proposition 4.1, A is minimally thin at z. Clearly, E is also minimally thin at z.

Conversely, suppose that E is minimally thin at z. By Proposition 4.1, there exists a potential u such that

$$\liminf_{x\to z, x\in E}\frac{u(x)}{M^D(x,z)}=+\infty\,.$$

Let $c_1=\max\{C_G,C_M\}$ where C_G and C_M are constants from (2.3) and (2.4), respectively. Without loss of generality, we may assume that

$$u(x_0)\le\frac{1}{2c_2}\,,\quad \text{where } c_2:=8c_1^4\left(\frac83\right)^d\,. \tag{4.3}$$

There exists $n_1\in\mathbb{N}$ such that $u(x)>M^D(x,z)$ for all $x\in E\cap B(z,2^{-n_1})$. Thus, $E\subset \overline{B}(z,2^{-n_1})^c\cup\{u>M^D(\cdot,z)\}$. For $n\ge n_1$ define

$$G_n=\left\{x\in D:\ 2^{-n-1}<|x-z|<2^{-n}, u(x)>M^D(x,z)\right\}\qquad\text{and}\qquad G=\bigcup_{n=n_1}^\infty G_n\,.$$

Let $x\in E_n$. Since $|x-z|\le 2^{-n_1}$, we have that $u(x)>M^D(x,z)$ and thus $x\in G_n$. This shows that $E_n\subset G_n$, $n\ge n_1$. Therefore, it suffices to show that $\sum_{n=n_1}^\infty R^{G_n}_{M^D(\cdot,z)}(x_0)<\infty$. Since $u>M^D(\cdot,z)$ on G, it follows that $R^{G}_{M^D(\cdot,z)}(x_0)\le u(x_0)$.

Let $i\in\{1,2,3\}$. For every $n\in\mathbb{N}$ let

$$U_n=G_{n_1+3n+i}\,.$$

Since $i\in\{1,2,3\}$ is arbitrary, it suffices to show that $\sum_{n=1}^\infty R^{U_n}_{M^D(\cdot,z)}(x_0)<\infty$. Let $U=\bigcup_{n=1}^\infty U_n$. Then $U\subset G$ and thus $R^{U}_{M^D(\cdot,z)}(x_0)\le u(x_0)$. Note that since U is open, $\widehat{R}^{U}_{M^D(\cdot,z)}=R^{U}_{M^D(\cdot,z)}$ (see

[5, p.205]). Since u is a potential, the same holds for $\widehat{R}^U_{M^D(\cdot,z)}$, hence there exists a measure μ such that $R^U_{M^D(\cdot,z)} = G^D\mu$. Moreover, since $R^U_{M^D(\cdot,z)}$ is harmonic on $\overline{U}^c$ (cf. [5, III.2.5]), $\mu(U^c) = 0$. Let $\mu_n := \mu_{|\overline{U}_n}$. Since $\overline{U}_n$ are pairwise disjoint,

$$\mu = \sum_{n=1} \mu_n \qquad \text{and} \qquad G^D\mu = \sum_{n=1}^{\infty} G^D\mu_n \,.$$

Fix $n \in \mathbb{N}$ and consider $l \in \mathbb{N}$, $x \in U_n$, $y \in \overline{U}_l$. Then

$$|x-y| \geq \frac{3}{4}|x-z|\,, \qquad x \in U_n, y \in \overline{U_l}\,, l \neq n\,.$$

Define $\mu_n' = \mu - \mu_n$ and let $x \in U_n$. By using (2.3) and (2.4),

$$\begin{aligned}
G^D\mu_n'(x) &= \int_D G^D(x,y)\,\mu_n'(dy) = M^D(x,z)\int_D \frac{G^D(x,y)}{M^D(x,z)}\,\mu_n'(dy) \qquad (4.4)\\
&\leq c_1^2 M^D(x,z)\int_D \frac{|x-y|^{\alpha-d}\frac{\delta_D(x)^{\alpha-1}}{|x-y|^{\alpha-1}}\frac{\delta_D(y)^{\alpha-1}}{|x-y|^{\alpha-1}}}{\frac{\delta_D(x)^{\alpha-1}}{|x-z|^{d+\alpha-2}}}\,\mu_n'(dy)\\
&= c_1^2 M^D(x,z)\int_D \left(\frac{|x-z|}{|x-y|}\right)^{d+\alpha-2} \delta_D(y)^{\alpha-1}\,\mu_n'(dy)\\
&\leq c_1^2\left(\frac{4}{3}\right)^d M^D(x,z)\int_D \delta_D(y)^{\alpha-1}\,\mu_n'(dy)\,.
\end{aligned}$$

We now compare $\delta_D(y)^{\alpha-1}$ with $G^D(x_0,y)$. By choosing n_1 even larger, we can assume that $\frac{1}{2}\delta_D(x_0) \leq |x_0-y|$, $\delta_D(y) \leq |x_0-y|$ and $\frac{1}{2}|x_0-z| \leq |x_0-y| \leq 2|x_0-z|$. Therefore,

$$1 \wedge \frac{\delta_D(y)^{\alpha-1}}{|x_0-y|^{\alpha-1}} = \frac{\delta_D(y)^{\alpha-1}}{|x_0-y|^{\alpha-1}} \qquad \text{and} \qquad 1 \wedge \frac{\delta_D(x_0)^{\alpha-1}}{|x_0-y|^{\alpha-1}} \geq 2^{1-\alpha}\frac{\delta_D(x_0)^{\alpha-1}}{|x_0-y|^{\alpha-1}}\,.$$

Hence,

$$\begin{aligned}
G^D(x_0,y) &\geq c_1^{-1}|x_0-y|^{\alpha-d}\left(1 \wedge \frac{\delta_D(x_0)^{\alpha-1}}{|x_0-y|^{\alpha-1}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha-1}}{|x_0-y|^{\alpha-1}}\right)\\
&\geq c_1^{-1}|x_0-y|^{\alpha-d}2^{1-\alpha}\frac{\delta_D(x_0)^{\alpha-1}}{|x_0-y|^{\alpha-1}}\frac{\delta_D(y)^{\alpha-1}}{|x_0-y|^{\alpha-1}}\\
&= c_1^{-1}2^{1-\alpha}|x_0-y|^{-d-\alpha+2}\,\delta_D(x_0)^{\alpha-1}\delta_D(y)^{\alpha-1}\\
&\geq c_1^{-1}2^{1-\alpha}2^{-d-\alpha+2}|x_0-z|^{-d-\alpha+2}\delta_D(x_0)^{\alpha-1}\delta_D(y)^{\alpha-1}\\
&\geq c_1^{-2}2^{3-2\alpha-d}M^D(x_0,z)\delta_D(y)^{\alpha-1} \geq c_1^{-2}2^{-1-d}\delta_D(y)^{\alpha-1}\,. \qquad (4.5)
\end{aligned}$$

Thus we have that

$$\begin{aligned}
G^D\mu_n'(x) &\leq c_1^2\left(\frac{4}{3}\right)^d M^D(x,z)\int_D c_1^2 2^{d+3}G^D(x_0,y)\,\mu_n'(dy) \qquad (4.6)\\
&= c_2 M^D(x,z)G^D\mu_n'(x_0) \leq c_2 M^D(x,z)G^D\mu(x_0)\\
&\leq c_2 M^D(x,z)u(x_0) \leq \frac{1}{2}M^D(x,z)\,.
\end{aligned}$$

Since $G^D\mu_n + G^D\mu_n' = G^D\mu = R^U_{M^D(\cdot,z)} = M^D(\cdot,z)$ on U, it follows that $G^D\mu_n = M^D(\cdot,z) - G^D\mu_n' \geq M^D(\cdot,z) - \frac{1}{2}M^D(\cdot,z) = \frac{1}{2}M^D(\cdot,z)$ on U_n. This implies that $G^D\mu_n \geq \frac{1}{2}R^{U_n}_{M^D(\cdot,z)}$. Finally,

$$\sum_{n=1}^{\infty} R^{U_n}_{M^D(\cdot,z)}(x_0) \leq 2\sum_{n=1}^{\infty} G^D\mu_n(x_0) = 2G^D\mu(x_0) < \infty\,.$$

□

Suppose that $\frac{1}{2}\delta_D(x_0) \leq |x_0 - y| \leq 2\delta_D(x_0)$, $\delta_D(y) \leq |x_0 - y|$ and $|x_0 - y| \geq \frac{1}{2}|x_0 - z|$. It is shown in the proof of Proposition 4.2, see (4.5), that $G^D(y,x_0) \geq c_3^{-1}\delta_D(y)^{\alpha-1}$ (with an explicit constant $c_3 > 1$). In the same way (even easier) it follows that $G^D(y,x_0) \leq c_3\delta_D(y)^{\alpha-1}$. Let $g(y) := G^D(y,x_0)\wedge 1$. It follows from the above discussion that $g(y) \asymp \delta_D(y)^{\alpha-1}$. Since

$$M^D(y,z) \asymp \frac{\delta_D(y)^{\alpha-1}}{|y-z|^{d+\alpha-2}}\,,$$

we see that there exists $c_4 > 1$ such that

$$c_4^{-1}\frac{g(x)}{|x-z|^{d+\alpha-2}} \leq M^D(x,z) \leq c_4\frac{g(x)}{|x-z|^{d+\alpha-2}}\,, \quad x \in D\,. \tag{4.7}$$

In particular, if $E_n = E \cap \{x \in D : 2^{-n-1} \leq |x-z| < 2^{-n}\}$, then

$$c_5^{-1}2^{n(d+\alpha-2)}g(x) \leq M^D(x,z) \leq c_5 2^{n(d+\alpha-2)}g(x)\,, \quad x \in E_n\,.$$

This implies that

$$c_5^{-1}2^{n(d+\alpha-2)}R_g^{E_n} \leq R^{E_n}_{M^D(\cdot,z)} \leq c_5 2^{n(d+\alpha-2)}R_g^{E_n}\,.$$

In particular,

$$\sum_{n=1}^{\infty} R^{E_n}_{M^D(\cdot,z)}(x_0) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} 2^{n(d+\alpha-2)}R_g^{E_n}(x_0) < \infty\,. \tag{4.8}$$

Note that $\widehat{R}_g^{E_n}$ is a potential, hence there exists a measure λ_n (supported by $\overline{E}_n$) such that $\widehat{R}_g^{E_n} = G^D\lambda_n$. Also, $\widehat{R}_g^{E_n} = g = G^D(\cdot,x_0)$ on E_n (except a polar set, and at least for large n), hence

$$\begin{aligned}\widehat{R}_g^{E_n}(x_0) &= G^D\lambda_n(x_0) = \int_{\overline{E}_n} G^D(x_0,y)\,\lambda_n(dy) = \int_{\overline{E}_n} g(y)\,\lambda_n(dy)\\ &= \int_{\overline{E}_n} \widehat{R}_g^{E_n}(y)\,\lambda_n(dy) = \int_D\int_D G^D(x,y)\,\lambda_n(dy)\,\lambda_n(dx) = \gamma_g(E_n)\,,\end{aligned}$$

the Green energy of E_n with respect to g. We conclude from (4.8) that

$$\sum_{n=1}^{\infty} R^{E_n}_{M^D(\cdot,z)}(x_0) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} 2^{n(d+\alpha-2)}\gamma_g(E_n) < \infty\,. \tag{4.9}$$

Thus we have proved the following Wiener-type criterion for minimal thinness.

Corollary 4.3 *Let $E \subset D$. Define*

$$E_n = E \cap \{x \in D :\ 2^{-n-1} \le |x-z| < 2^{-n}\}\,, \quad n \ge 1\,.$$

Then E is minimally thin at z if and only if $\sum_{n=1}^{\infty} 2^{n(d+\alpha-2)}\gamma_g(E_n) < \infty$.

Finally, we prove a version of Aikawa-type criterion for minimal thinness.

Proposition 4.4 *Let D be a bounded $C^{1,1}$ open set, $z \in \partial D$, $E \subset D$, and let x_j denote the center of Q_j. The following are equivalent:*

(a) E is minimally thin at z;

(b)

$$\sum_{j\ge 1} \mathrm{dist}(z, Q_j)^{-d-\alpha+2} g(x_j)^2 \mathrm{Cap}^D(E\cap Q_j) < \infty\,; \tag{4.10}$$

(c)

$$\sum_{j\ge 1} \frac{\mathrm{dist}(Q_j, \partial D)^{2(\alpha-1)}}{\mathrm{dist}(z, Q_j)^{d+\alpha-2}} \mathrm{Cap}^D(E\cap Q_j) < \infty\,. \tag{4.11}$$

Proof. (a)$\Leftrightarrow$ (b) By Corollary 4.3, E is minimally thin at z if and only if $\sum_{n=1}^{\infty} 2^{n(d+\alpha-2)}\gamma_g(E_n) < \infty$. Further, let $A_n = \{x \in \mathbb{R}^d : 2^{-n-1} \le |x-z| < 2^{-n}\}$ so that $E_n = E \cap A_n$. If $A_n \cap Q_j \neq \emptyset$, then $\mathrm{dist}(z, Q_j) \asymp 2^{-n}$. By quasi-additivity of γ_g,

$$\begin{aligned}
\sum_{n=1}^{\infty} 2^{n(d+\alpha-2)}\gamma_g(E_n) &\asymp \sum_{n=1}^{\infty} 2^{-n(-d-\alpha+2)} \sum_{j\ge 1} \gamma_g(E_n \cap Q_j) \\
&\asymp \sum_{j\ge 1} \sum_{n, A_n\cap Q_j \neq \emptyset} \mathrm{dist}(z, Q_j)^{-d-\alpha+2}\gamma_g(E_n\cap Q_j) \\
&= \sum_{j\ge 1} \mathrm{dist}(z, Q_j)^{-d-\alpha+2} \sum_{n, A_n\cap Q_j \neq \emptyset} \gamma_g(E_n\cap Q_j) \\
&\asymp \sum_{j\ge 1} \mathrm{dist}(z, Q_j)^{-d-\alpha+2}\gamma_g(E\cap Q_j)\,.
\end{aligned}$$

For the last line we argue as follows: One inequality is subadditivity of capacity. For another note that there exists $N \in \mathbb{N}$ such that for every Q_j, $\sum_{n, A_n\cap Q_j\neq\emptyset} 1 = \sum_n 1_{A_n\cap Q_j} \le N$. Hence, $\sum_{n, A_n\cap Q_j\neq\emptyset} \gamma_g(E\cap A_n \cap Q_j) \le \sum_{n, A_n\cap Q_j\neq\emptyset} \gamma_g(E\cap Q_j) \le N\gamma_g(E\cap Q_j)$.

Finally, we use that $\gamma_g(E\cap Q_j) \asymp g(x_j)^2 \mathrm{Cap}^D(E\cap Q_j)$ which is proved in the same way as [2, Part II, Proposition 7.3.1] to finish the proof.

(b)$\Leftrightarrow$ (c) Note that

$$g(x_j) = G^D(x_j, x_0) \wedge 1 \asymp G^D(x_j, x_0) \asymp \delta_D(x_j)^{\alpha-1} \asymp \mathrm{dist}(Q_j, \partial D)^{\alpha-1}\,.$$

hence the series (4.10) converges if and only if the series (4.11) converges. □

5 Aikawa-Borichev-type result

The purpose of this section is to prove the analog of [1, Theorem 3] for the censored stable process.

Let $B(x,r) \subset D$ such that $B(x,2r) \subset D$. Define $\eta(r;x) > 0$ as the radius such that

$$|B(x,\eta(r;x))| = \mathrm{Cap}^D B(x,r)\,,$$

and let $\eta^*(r;x) = \max\{\eta(r;x), 16r\}$. By (2.6) $C^{-1}r^{d-\alpha} \le \mathrm{Cap}^D B(x,r) \le Cr^{d-\alpha}$. Let σ_d be the volume of the unit ball. Since $|B(x,\eta(r;x))| = \sigma_d \eta(r;x)^d$, we have that $C^{-1}r^{d-\alpha} \le \sigma_d \eta(r;x)^d \le Cr^{d-\alpha}$, implying

$$C^{-1/d}\sigma_d^{-1/d} r^{1-\alpha/d} \le \eta(r;x) \le C^{1/d}\sigma_d^{-1/d} r^{1-\alpha/d}\,. \tag{5.1}$$

Define

$$\eta_l(r;x) := C^{-1/d}\sigma_d^{-1/d} r^{1-\alpha/d} \quad \text{and} \quad \eta_u(r;x) := C^{1/d}\sigma_d^{-1/d} r^{1-\alpha/d}\,.$$

We recall now that a $C^{1,1}$ open set D satisfies the following interior ball condition: There exists $R = R(D) > 0$ such that for all $x_0 \in D$ with $\delta_D(x_0) < R$ there is $z_0 \in \partial D$ so that $|x_0 - z_0| = \delta_D(x_0)$ and that $B(y_0, R) \subset D$ where $y_0 = z_0 + R(x_0 - z_0)/|x_0 - z_0|$. Note that if $a > 0$, then aD is again a $C^{1,1}$ open set and $R(aD) = aR(D)$.

Lemma 5.1 *Let $0 < 2r \le r^* < R/2$ and let $x_0 \in D$ be such that $\delta_D(x_0) < R/2$ and $B(x_0, r) \subset D$. There exists $\widetilde{x} \in D$ such that $|\widetilde{x} - x_0| = \frac{3}{4} r^*$ and for any $\theta \le \frac14$ it holds that*

$$B(\widetilde{x}, \theta r^*) \subset B(x_0, r^*) \cap D\,.$$

Moreover, if $x \notin B(x_0, r^*)$ and $\rho = \mathrm{dist}(x, B(x_0, r))$, then $B(\widetilde{x}, \theta r^*) \subset B(x, 5\rho)$.

Proof. Let $z_0 \in \partial D$ so that $|x_0 - z_0| = \delta_D(x_0)$, and let $y_0 = z_0 + R(x_0 - z_0)/|x_0 - z_0|$. Then $B(y_0, R) \subset D$. Note that $r < \delta_D(x_0)$ since $B(x_0, r) \subset D$. If $y \in B(x_0, r)$, then

$$|y - y_0| \le |y - x_0| + |x_0 - y_0| < r + |z_0 - y_0| - |z_0 - x_0| = r + R - \delta_D(x_0) < R\,,$$

so $B(x_0, r) \subset B(y_0, R)$. Further, $|x_0 - y_0| = R - \delta_D(x_0) > R - R/2 > r^*$. Let $\widetilde{x}$ be the point on the segment connecting x_0 and y_0 such that $|\widetilde{x} - x_0| = \frac34 r^*$.

Let $y \in B(\widetilde{x}, \theta r^*)$. Then

$$|x_0 - y| \le |x_0 - \widetilde{x}| + \theta r^* \le \frac34 r^* + \frac14 r^* = r^*\,,$$

and

$$|y_0 - y| \le |y_0 - \widetilde{x}| + |\widetilde{x} - y| = |y_0 - x_0| - |x_0 - \widetilde{x}| + |\widetilde{x} - y| < R - \delta_D(x_0) - \frac34 r^* + \theta r^* < R\,.$$

This shows that $B(\widetilde{x}, \theta r^*) \subset B(x_0, r^*) \cap B(y_0, R) \subset B(x_0, r^*) \cap D$.

Let $x \notin B(x_0, r^*)$; then $|x - x_0| = \rho + r \le \rho + r^*$ and $\rho \ge r^* - r > r^*/2$. If $y \in B(x_0, r^*)$, then

$$|y - x| \le |y - x_0| + |x_0 - x| < r^* + \rho + r^* \le 5\rho\,.$$

Hence $B(x_0, r^*) \subset B(x, 5\rho)$, and thus also $B(\widetilde{x}, \theta r^*) \subset B(x, 5\rho)$. □

Proposition 5.2 *Suppose that $\{B(x_j,r_j)\}_{j\geq 1}$ is a collection of balls contained in D such that $\delta_D(x_j) < R/2$ and $\eta^*(r_j;x_j) < R/2$ for all j, and the family $\{B(x_j,\eta^*(r_j;x_j))\}_{j\geq 1}$ is pairwise disjoint. If E is a Borel set contained in $\cup_{j\geq 1}B(x_j,r_j)$, then*

$$\mathrm{Cap}^D E \leq \sum_{j\geq 1} \mathrm{Cap}^D(B(x_j,r_j)\cap E) \leq c\mathrm{Cap}^D E\,,$$

where the constant $c>1$ depends on Y only.

Proof. We follow the proof of [1, Theorem 3] and indicate only necessary changes. For a finite measure ν on D, let $\|\nu\| = \nu(D)$ denote its total mass. First note that it follows from (3.1) that

$$\mathrm{Cap}^D(B(x_j,r_j)\cap E) = \sup\{\|\nu\| : \nu(D\setminus(B(x_j,r_j)\cap E)) = 0, G^D\nu \leq 1 \text{ on } D\}\,.$$

Hence, given $\epsilon>0$, there exists a measure μ_j concentrated on $B(x_j,r_j)\cap E$ such that $G^D\mu_j \leq 1$ on D and $\|\mu_j\| \geq \mathrm{Cap}^D(B(x_j,r_j)\cap E) - 2^{-j}\epsilon$. Let μ'_j be the measure on D defined by

$$d\mu'_j = \frac{\|\mu_j\|}{|B(x_j,\eta^*(r_j;x_j))|}\mathbf{1}_{B(x_j,\eta^*(r_j;x_j))\cap D}\,dx\,.$$

It is clear from the proof of [1, Theorem 3] that once we prove the inequality

$$G^D\mu_j(x) \leq cG^D\mu'_j(x), \qquad x\in B(x_j,\eta^*(r_j;x_j))^c\,, \tag{5.2}$$

where $c = c(d,\alpha)>0$, the rest of the proof follows in the exactly same way.

For simplicity, we write $r_j^* := \eta^*(r_j;x_j)$. We use Lemma 5.1 with x_j instead of x_0 and let $\widetilde{x}_j$ be the point corresponding to $\widetilde{x}$; then $|\widetilde{x}_j - x_j| = \frac{3}{4}r_j^*$ and with $\theta = 1/8$ we have

$$B(\widetilde{x}_j, 8^{-1}r_j^*) \subset B(x_j,r_j^*)\cap D\,,$$

Moreover, if $x\in B(x_j,r_j^*)^c$ and $\rho_j = \mathrm{dist}(x,B(x_j,r_j))$, then $B(\widetilde{x}_j,8^{-1}r_j^*)\subset B(x,5\rho_j)$. In the following calculation we use that for $s>0$ and $t\geq 1$, it holds that

$$1\wedge\frac{s}{t} \leq 1\wedge s \leq t\left(1\wedge\frac{s}{t}\right)\,.$$

By (2.3) we have

$$\begin{aligned}
G^D\mu'_j(x) &\geq C_G^{-1}\int_{B(x,5\rho_j)\cap B(x_j,r_j^*)\cap D}\left(1\wedge\frac{\delta_D(x)^{\alpha-1}}{|x-y|^{\alpha-1}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha-1}}{|x-y|^{\alpha-1}}\right)|x-y|^{\alpha-d}\mu'_j(dy)\\
&\geq C_G^{-1}\left(1\wedge\frac{\delta_D(x)^{\alpha-1}}{5^{\alpha-1}\rho_j^{\alpha-1}}\right)5^{\alpha-d}\rho_j^{\alpha-d}\int_{B(\widetilde{x}_j,8^{-1}r_j^*)}\left(1\wedge\frac{\delta_D(y)^{\alpha-1}}{5^{\alpha-1}\rho_j^{\alpha-1}}\right)\mu'_j(dy)\\
&\geq C_G^{-1}5^{2-\alpha-d}\left(1\wedge\frac{\delta_D(x)^{\alpha-1}}{\rho_j^{\alpha-1}}\right)\rho_j^{\alpha-d}\inf_{y\in B(\widetilde{x}_j,8^{-1}r_j^*)}\left(1\wedge\frac{\delta_D(y)^{\alpha-1}}{\rho_j^{\alpha-1}}\right)\mu'_j(B(\widetilde{x}_j,8^{-1}r_j^*))\,.
\end{aligned}\tag{5.3}$$

Similarly,

$$\begin{aligned}
G^D\mu_j(x) &\le C_G \int_{B(x_j,r_j)} \left(1\wedge \frac{\delta_D(x)^{\alpha-1}}{|x-y|^{\alpha-1}}\right)\left(1\wedge \frac{\delta_D(y)^{\alpha-1}}{|x-y|^{\alpha-1}}\right)|x-y|^{\alpha-d}\mu_j(dy)\\
&\le C_G\left(1\wedge \frac{\delta_D(x)^{\alpha-1}}{\rho_j^{\alpha-1}}\right)\rho_j^{\alpha-d}\int_{B(x_j,r_j)}\left(1\wedge \frac{\delta_D(y)^{\alpha-1}}{\rho_j^{\alpha-1}}\right)\mu_j(dy)\\
&\le C_G\left(1\wedge \frac{\delta_D(x)^{\alpha-1}}{\rho_j^{\alpha-1}}\right)\rho_j^{\alpha-d}\sup_{y\in B(x_j,r_j)}\left(1\wedge \frac{\delta_D(y)^{\alpha-1}}{\rho_j^{\alpha-1}}\right)\mu_j(B(x_j,r_j))\,. \qquad (5.4)
\end{aligned}$$

Let $y_1\in B(x_j,r_j)$ and $y_2\in B(\widetilde{x}_j, 8^{-1}r_j^*)$. Then

$$|y_1-y_2|\le r_j+\frac{3r_j^*}{4}+\frac{r_j^*}{8}\le \frac{r_j^*}{16}+\frac{7r_j^*}{8}=\frac{15r_j^*}{16}\le 15(r_j^*-r_j)\le 15\delta_D(y_2)$$

implying

$$\delta_D(y_1)\le \delta_D(y_2)+|y_1-y_2|\le 16\delta_D(y_2)\,.$$

Hence, by (5.3), (5.4) and the last display,

$$\begin{aligned}
G^D\mu_j(x) &\le C_G^2 5^{d+\alpha-2}16^{\alpha-1}\frac{\mu_j(B(x_j,r_j))}{\mu_j'(B(\widetilde{x}_j,8^{-1}r_j^*))}G^D\mu_j'(x)\\
&= C_G^2 5^{d+\alpha-2}16^{\alpha-1}\frac{|B(x_j,r_j^*)|}{|B(\widetilde{x}_j,8^{-1}r_j^*)|}G^D\mu_j'(x)\\
&= C_G^2 5^{d+\alpha-2}16^{\alpha-1}8^d G^D\mu_j'(x)\,.
\end{aligned}$$

□

6 Proof of Theorem 1.1

The constant function 1 is harmonic with respect to Y, hence there exists a measure μ on ∂D such that

$$1=\int_{\partial D} M^D(x,z)\,\mu(dz)\,.$$

Since $M^D(x_0,z)=1$ for all $z\in\partial D$, we have that $1=\int_{\partial D}\mu(dz)=\|\mu\|$. Let $E\subset D$: Then

$$R_1^E(x_0)=R^E_{\int_{\partial D}M^D(\cdot,z)\,\mu(dz)}(x_0)=\int_{\partial D}R^E_{M^D(\cdot,z)}(x_0)\,\mu(dz)\,.$$

Since $R^E_{M^D(\cdot,z)}(x_0)\le M^D(x_0,z)=1$, we see that $R_1^E(x_0)=1$ if and only if $R^E_{M^D(\cdot,z)}(x_0)=1$ for μ-a.e. $z\in\partial D$. Thus, $R_1^E(x_0)=1$ if and only if E is not minimally thin at z for μ-a.e. $z\in\partial D$.

We note further that by [14, Lemma 3.1] μ is in fact the harmonic measure for Y in D: $\mu(dz)=\mathbb{P}_{x_0}(Y_{\zeta-}\in dz)$. It is proved in [14, Theorem 3.14] that the harmonic measure μ is mutually absolutely continuous with respect to the surface measure σ on ∂D. We conclude that $R_1^E(x_0)=1$ if and only if E is not minimally thin at z for σ-a.e. $z\in\partial D$.

Let $\{\overline{B}(x_k,r_k)\}_{k\geq 1}$ be a family of disjoint closed balls in D, and let $A:=\cup_{k\geq 1}\overline{B}(x_k,r_k)$. Then the family of balls is unavoidable if $R_1^A(x_0)=1$, or, equivalently, if A is not minimally thin at z for σ-a.e. $z\in\partial D$. As before, let $\{Q_j\}_{j\geq 1}$ be a Whitney decomposition of D. We will need the following simple geometric lemma whose proof is omitted.

Lemma 6.1 *Assume that* $\sup_{k\geq 1} r_k/\delta_D(x_k)<1$. *There exists a constant* $C_1\geq 1$ *such that for any* Q_j *and any* $\overline{B}(x_k,r_k)$ *which intersects* Q_j

$$C_1^{-1}\leq \frac{\mathrm{dist}(Q_j,\partial D)}{\delta_D(x_k)}\leq C_1 \quad \textit{and} \quad C_1^{-1}\leq \frac{\mathrm{dist}(z,Q_j)}{|x_k-z|}\leq C_1 \quad \textit{for all } z\in\partial D\,. \tag{6.1}$$

We first note that the number of cubes Q_j which intersect a given ball $\overline{B}(x_k,r_k)$ is bounded above by a constant c_2 (independent of k). Next note that if $B(x,r)$ is an open ball and Q a closed cube, then $\mathrm{Cap}^D(B(x,r)\cap Q)=\mathrm{Cap}(\overline{B}(x,r)\cap Q)$. Indeed, every point in $\partial B(x,r)\cap Q$ is regular for $B(x,r)\cap Q$, hence $\mathbb{P}_y(T_{B(x,r)\cap Q}<\infty)=\mathbb{P}_y(T_{\overline{B}(x,r)\cap Q}<\infty)$ for all $y\in\mathbb{R}^d$. This shows that $B(x,r)\cap Q$ and $\overline{B}(x,r)\cap Q$ have the same capacitary measures, hence equal capacities.

Recall that $R=R(D)>0$ is the constant from the interior ball condition.

Lemma 6.2 *Let* $\{\overline{B}(x_k,r_k)\}_{k\geq 1}$ *be a family of closed balls in* D *such that* $\sup r_k/\delta_D(x_k)<1/2$, $\delta_D(x_k)<R/2$ *and* $\eta^*(r_k;x_k)<R/2$ *for all* $k\geq 1$. *Assume that*

(i) $r_k\leq (16^d C\sigma_d)^{-1/\alpha}$ *for all* $k\geq 1$;

(ii) $\dfrac{|x_j-x_k|}{r_k^{1-\alpha/d}}\geq 2C^{1/d}\sigma_d^{-1/d}$, $j\neq k$.

Then there exists $c_3=c_3(d)>0$ *such that for every* $j\geq 1$,

$$\mathrm{Cap}^D(A\cap Q_j)\geq c_3\sum_{k\geq 1}\mathrm{Cap}^D(\overline{B}(x_k,r_k)\cap Q_j)\,.$$

Proof. The first condition $r_k\leq (16^d C\sigma_d)^{-1/\alpha}$ is equivalent to $16r_k\leq C^{-1/d}\sigma_d^{-1/d}r_k^{1-\alpha/d}=\eta_l(r_k;x_k)\leq \eta(r_k;x_k)$ which implies that $\eta^*(r_k;x_k)=\eta(r_k;x_k)$. The second condition implies that

$$|x_j-x_k|\geq 2C^{1/d}\sigma_d^{-1/d}r_k^{1-\alpha/d}=2\eta_u(r_k;x_k)\geq 2\eta(r_k;x_k)\,,$$

that is, the balls $B(x_j,\eta^*(r_j;x_j))$ are disjoint. The claim now follows from Proposition 5.2 and the fact that $\mathrm{Cap}^D(B(x_k,r_k)\cap Q_j)=\mathrm{Cap}^D(\overline{B}(x_k,r_k)\cap Q_j)$. □

Lemma 6.3 *Let $\{\overline{B}(x_k, r_k)\}_{k\geq 1}$ be a family of closed balls in D such that $\sup r_k/\delta_D(x_k) < 1/2$, $\delta_D(x_k) < R/2$ and $\eta^*(r_k; x_k) < R/2$ for all $k \geq 1$. Assume that*

$$\frac{|x_j - x_k|}{r_k^{1-\alpha/d}\delta_D(x_k)^{\alpha/d}} \geq 32C^{2/d}C_1^{2\alpha/d}, \quad j \neq k, \tag{6.2}$$

where $C_1 \geq 1$ is the constant from Lemma 6.2. Then there exists a constant $c_4 = c_4(d, D)$ such that

$$\mathrm{Cap}^D(A \cap Q_j) \geq c_4 \sum_{k\geq 1} \mathrm{Cap}^D(\overline{B}(x_k, r_k) \cap Q_j). \tag{6.3}$$

Proof. Let Q_j be a Whitney cube such that

$$\mathrm{dist}(Q_j, \partial D) \leq C_1^{-1}(16^d C\sigma_d)^{-1/\alpha}. \tag{6.4}$$

Define the scaling constant $a > 0$ by

$$a := \frac{\left(16^d C\sigma_d\right)^{-1/\alpha}}{C_1 \mathrm{dist}(Q_j, \partial D)}.$$

By (6.4) it holds that $a \geq 1$.

Let $y_k = ax_k$, $\rho_k = ar_k$ and $F := (\cup_{k=1}^{\infty} B(y_k, \rho_k)) \cap (aQ_j)$. Since $\delta_{aD}(y_k) = a\delta_D(x_k)$, the condition $\sup \rho_k/\delta_{aD}(y_k) < 1/2$ is satisfied. Let $R^a := R(aD) = aR$. Then $\delta_{aD}(y_k) = a\delta_D(x_k) < aR/2 = R^a/2$. We now check that $\eta_a^*(\rho_k; y_k) \leq R^a/2$ where $\eta_a(\rho_k; y_k)$ is computed with respect to Cap^{aD}, the capacity with respect to the censored stable process in aD. Indeed, by (2.7)

$$|B(y_k, \eta_a(\rho_k; y_k))| = \mathrm{Cap}^{aD} B(y_k, \rho_k) = a^{d-\alpha}\mathrm{Cap}^D B(x_k, r_k) = a^{d-\alpha}|B(x_k, \eta(r_k; x_k))|.$$

This implies that $\eta_a(\rho_k; y_k) = a^{1-\alpha/d}\eta(r_k; x_k) \leq a\eta(r_k; x_k) \leq aR/2 = R^a/2$ since $a \geq 1$. Clearly, $16\rho_k = 16ar_k \leq aR/2 = R^a/2$.

Suppose that $B(x_k, r_k) \cap Q_j \neq \emptyset$. Then

$$\rho_k = ar_k \leq a\delta_D(x_k) \leq aC_1 \mathrm{dist}(Q_j, \partial D) = \left(16^d C\sigma_d\right)^{-1/\alpha}.$$

Further, for $l \neq k$,

$$\begin{aligned}
\frac{|y_l - y_k|}{\rho_k^{1-\alpha/d}} &= \frac{a|x_l - x_k|}{a^{1-\alpha/d}\, r_k^{1-\alpha/d}} = a^{\alpha/d}\,\frac{|x_l - x_k|}{r_k^{1-\alpha/d}} \\
&\geq 32C^{2/d}C_1^{2\alpha/d}\delta_D(x_k)^{\alpha/d}a^{\alpha/d} \\
&= 2C^{1/d}\sigma_d^{-1/d}\left(C_1\frac{\delta_D(x_k)}{\mathrm{dist}(Q_j, \partial D)}\right)^{\alpha/d} \\
&\geq 2C^{1/d}\sigma_d^{-1/d}.
\end{aligned}$$

By Lemma 6.2, there exists $c_3 = c_3(d)$ such that

$$\mathrm{Cap}^{aD}(F \cap aQ_j) \geq c_3 \sum_{k \geq 1} \mathrm{Cap}^{aD}(\overline{B}(y_k, \rho_k) \cap aQ_j)\,,$$

where Cap^{aD} is the capacity with respect to the censored α-stable process in aD. By (2.7), $\mathrm{Cap}^{aD}(F \cap aQ_j) = \mathrm{Cap}^{aD}(a(A \cap Q_j)) = a^{d-\alpha}\mathrm{Cap}^D(A \cap Q_j)$ and $\mathrm{Cap}^{aD}(\overline{B}(y_k, \rho_k) \cap aQ_j) = a^{d-\alpha}\mathrm{Cap}^D(\overline{B}(x_k, r_k) \cap Q_j)$. Therefore

$$\mathrm{Cap}^{D}(A \cap Q_j) \geq c_3 \sum_{k \geq 1} \mathrm{Cap}^{D}(\overline{B}(x_k, r_k) \cap Q_j)\,.$$

For finitely many Whitney cubes that do not satisfy (6.4), one obtains inequalities (6.3) by choosing $c_4 \leq c_3$ small enough. □

Proof of Theorem 1.1. (a) Assume that A is unavoidable. Since $2r_k < \delta_D(x_k)$, we have that $B(x_k, 2r_k) \subset D$. Hence, by (2.6), $\mathrm{Cap}^D \overline{B}(x_k, r_k) \leq C r_k^{d-\alpha}$. If $\overline{B}(x_k, r_k)$ and Q_j intersect, we have that

$$\mathrm{Cap}^D(\overline{B}(x_k, r_k) \cap Q_j) \leq \mathrm{Cap}^D(\overline{B}(x_k, r_k)) \leq C r_k^{d-\alpha}\,.$$

Since the number of cubes Q_j which intersect a given ball $\overline{B}(x_k, r_k)$ is bounded above by a constant c_2, we have for every $z \in \partial D$

$$\begin{aligned}
&\sum_{j \geq 1} \frac{\mathrm{dist}(Q_j, \partial D)^{2(\alpha-1)}}{\mathrm{dist}(z, Q_j)^{d+\alpha-2}}\,\mathrm{Cap}^D(A \cap Q_j)\\
&\quad\leq \sum_{k \geq 1}\sum_{j \geq 1} \frac{\mathrm{dist}(Q_j, \partial D)^{2(\alpha-1)}}{\mathrm{dist}(z, Q_j)^{d+\alpha-2}}\,\mathrm{Cap}^D(\overline{B}(x_k, r_k) \cap Q_j)\\
&\quad\leq C_1^{2\alpha-2+d+\alpha-2} C c_2 \sum_{k \geq 1} \frac{\delta_D(x_k)^{2\alpha-2}}{|x_k - z|^{d+\alpha-2}}\, r_k^{d-\alpha}\,.
\end{aligned}$$

The claim now follows from (4.11) in Proposition 4.4.

(b) Conversely, assume that (1.1) and the separation condition (1.2) hold true. Consider only the balls $B(x_k, r_k)$ such that $\delta_D(x_k) < R/2$. In this way a finite number of balls is omitted. If we show that this smaller collection is unavoidable, the same will be true for the whole collection.

Choose $\delta \in (0, 1]$ small enough so that

$$\frac{|x_j - x_k|}{(\delta r_k)^{1-\alpha/d}\delta_D(x_k)^{\alpha/d}} \geq 32 C^{2/d} C_1^{2\alpha/d}\,, \quad j \neq k\,,$$

and so that $\eta^*(\delta r_k; x_k) < R/2$. Note that the latter is possible because

$$\eta^*(\delta r_k; x_k) \leq \max\left(C^{1/d}\sigma_d^{-1/d}\delta^{1-\alpha/d} r_k^{1-\alpha/d}, 16\delta r_k\right)\,.$$

Let $A_\delta := \cup_{k\geq 1}\overline{B}(x_k,\delta r_k)$. Lemma 6.3 applied to the family of balls $\{B(x_k,\delta r_k)\}$ gives that

$$\mathrm{Cap}^D(A_\delta\cap Q_j)\geq c_4\sum_{k\geq 1}\mathrm{Cap}^D(\overline{B}(x_k,\delta r_k)\cap Q_j)\,.$$

Combined with (6.1) this yields

$$\begin{aligned}
&\sum_{j\geq 1}\frac{\mathrm{dist}(Q_j,\partial D)^{2(\alpha-1)}}{\mathrm{dist}(z,Q_j)^{d+\alpha-2}}\,\mathrm{Cap}^D(A_\delta\cap Q_j)\\
&\quad\geq\ \sum_{j\geq 1}\sum_{k\geq 1}\frac{\mathrm{dist}(Q_j,\partial D)^{2(\alpha-1)}}{\mathrm{dist}(z,Q_j)^{d+\alpha-2}}\,\mathrm{Cap}^D(\overline{B}(x_k,\delta r_k)\cap Q_j)\\
&\quad\geq\ \sum_{k\geq 1}\sum_{j\geq 1}\frac{c_4}{C_1^{d+3\alpha-4}}\,\frac{\delta_D(x_k)^{2\alpha-2}}{|x_k-z|^{d+\alpha-2}}\,\mathrm{Cap}(\overline{B}(x_k,\delta r_k)\cap Q_j)\,.
\end{aligned}$$

Subadditivity of Cap^D implies that for each k there is j such that

$$\mathrm{Cap}^D(\overline{B}(x_k,\delta r_k)\cap Q_j)\geq c_2^{-1}\mathrm{Cap}^D\overline{B}(x_k,\delta r_k)\geq c_2^{-1}C^{-1}\delta^{d-\alpha}r_k^{d-\alpha}\,,$$

where c_2 is the constant from the proof of part (a) and the last inequality follows from (2.6). Therefore,

$$\begin{aligned}
&\sum_{j\geq 1}\frac{\mathrm{dist}(Q_j,\partial D)^{2(\alpha-1)}}{\mathrm{dist}(z,Q_j)^{d+\alpha-2}}\,\mathrm{Cap}^D(A_\delta\cap Q_j)\\
&\quad\geq\ \frac{c_4\delta^{d-\alpha}}{C_1^{d-\alpha}c_2C}\sum_{k\geq 1}\frac{\delta_D(x_k)^{2\alpha-2}}{|x_k-z|^{d+\alpha-2}}\,r_k^{d-\alpha}=+\infty
\end{aligned}$$

for μ-a.e. $z\in\partial D$. It follows from Proposition 4.4, (4.11), that A_δ is unavoidable, hence the same is true for A. □

7 Proof of Theorem 1.2

Let $B=B(0,1)$, $M^B(x,z)$ the Martin kernel based at $x_0=0$, and μ be the measure on ∂B such that

$$1=\int_{\partial B}M^B(x,z)\,\mu(dz)\,.$$

It holds by (2.4) that

$$C_M^{-1}\frac{(1-|x|)^{\alpha-1}}{|x-z|^{d+\alpha-2}}\leq M^B(x,z)\leq C_M\frac{(1-|x|)^{\alpha-1}}{|x-z|^{d+\alpha-2}}\,,\quad x\in B, z\in\partial B\,.$$

The last two displays imply

$$\frac{C_M^{-1}}{(1-|x|)^{\alpha-1}}\leq\int_{\partial B}\frac{\mu(dz)}{|x-z|^{d+\alpha-2}}\leq\frac{C_M}{(1-|x|)^{\alpha-1}}\,. \tag{7.1}$$

With this inequality at hand, the proof is essentially the same as the proof of [12, Theorem 2]. We provide the proof for readers' convenience. Recall that a, b, c are constants from the statement of the theorem.

Proof of Theorem 1.2. (a) Without loss of generality we may assume that $\phi(|x_k|) \leq 1/2$ for all $k \geq 1$, so that $\sup_{k\geq 1} r_k/\delta_B(x_k) = \sup_{k\geq 1} \phi(|x_k|) \leq 1/2$. It follows from Theorem 1.1 (a) that

$$\sum_{k\geq 1} \frac{\delta_B(x_k)^{2\alpha-2}}{|x_k - z|^{d+\alpha-2}} r_k^{d-\alpha} = \sum_{k\geq 1} \frac{(1-|x_k|)^{d+\alpha-2}}{|x_k - z|^{d+\alpha-2}} \phi(|x_k|)^{d-\alpha} = \infty \quad \mu - \text{a.e. } z \in \partial B$$

Let the center x_k belong to the Whitney cube Q_m. Then

$$1 - |x_k| \leq \operatorname{dist}(Q_m, \partial B) + \operatorname{diam}(Q_m)\,,$$

implying (recall that $\operatorname{diam}(Q_m) \leq \operatorname{dist}(Q_m, \partial B) \leq 4\operatorname{diam}(Q_m)$)

$$1 - |x_k| \leq 5\,\operatorname{diam}(Q_m)\,, \qquad 1 - |x_k| \leq 2(1 - |x|)\,, \quad x \in Q_m\,. \tag{7.2}$$

By Lemma 6.1,

$$|z - x_k| \geq \frac{1}{C_1} \operatorname{dist}(z, Q_m) \geq \frac{1}{2C_1} (\operatorname{dist}(z, Q_m) + \operatorname{diam}(Q_m)) \geq \frac{|z - x|}{2C_1}\,, \quad x \in Q_m\,.$$

Together with the fact that ϕ is decreasing, this gives the estimate

$$\frac{(1-|x_k|)^{d+\alpha-2}}{|z - x_k|^{d+\alpha-2}} \phi(|x_k|)^{d-\alpha} \leq (10C_1 \operatorname{diam}(Q_m))^{d+\alpha-2} \frac{\phi((2|x|-1)^+)^{d-\alpha}}{|z-x|^{d+\alpha-2}}\,, \quad x \in Q_m\,.$$

Note that the number of centers x_k that belong to Q_m is bounded from above by $c_1(a, c, d) b M((2|x| - 1)^+)$ for every $x \in Q_m$. By using that $\sup_{x\in Q_m} \operatorname{diam}(Q_m)/(1 - |x|) < 1$, it follows that

$$\begin{aligned}
&\sum_{x_k \in Q_m} \frac{(1-|x_k|)^{d+\alpha-2}}{|x_k - z|^{d+\alpha-2}} \phi(|x_k|)^{d-\alpha} \\
&\leq \quad c_2(a, c, d) b C_1^{d+\alpha-2} (\operatorname{diam}(Q_m))^{\alpha-2} \int_{Q_m} \frac{\phi((2|x|-1)^+)^{d-\alpha} M((2|x|-1)^+)}{|z-x|^{d+\alpha-2}}\, dx \\
&\leq \quad c_3(a, b, c, C_1, d) \int_{Q_m} \frac{\phi((2|x|-1)^+)^{d-\alpha} M((2|x|-1)^+)(1-|x|)^{\alpha-2}}{|z-x|^{d+\alpha-2}}\, dx
\end{aligned}$$

By summing up over all Whitney cubes, we get,

$$\sum_{k\geq 1} \frac{(1-|x_k|)^{d+\alpha-2}}{|x_k - z|^{d+\alpha-2}} \phi(|x_k|)^{d-\alpha} \leq c_3(a, b, c, C_1, d) \int_B \frac{\phi((2|x|-1)^+)^{d-\alpha} M((2|x|-1)^+)(1-|x|)^{\alpha-2}}{|z-x|^{d+\alpha-2}}\, dx$$

Since the left-hand side is infinite for μ-a.e. $z \in \partial B$, the same is true for the right-hand side. By integrating the right-hand side over ∂B with respect to μ, and by using (7.1), we get that

$$\int_B \frac{\phi((2|x|-1)^+)^{d-\alpha} M((2|x|-1)^+)}{1-|x|}\, dx = \infty\,.$$

By switching to polar coordinates (and using that t^{d-1} is bounded near 1) this yields

$$\int_{1/2}^{1} \frac{\phi(2t-1)^{d-\alpha} M(2t-1)}{1-t}\, dt = \infty\,.$$

The change of variables implies (1.4).

(b) Let $(t_i)_{i\geq 1}$ be a sequence defined by $t_i := 1 - \frac{1}{2}\left(\frac{1-a}{1+a}\right)^i$. Then $(t_i)_{i\geq 1}$ is increasing and $t_i < 1$. Let $z \in \partial B$ and define $z_i := t_i z$. Then the balls $B(z_i, a(1-|z_i|))$ are pairwise disjoint. Indeed, this follows from the inequality $t_i + a(1-t_i) < t_{i+1} - a(1-t_{i+1})$. Further, for $x \in B(z_i, a(1-|z_i|))$ it holds that

$$1-|x| \geq 1 - t_i - a(1-t_i) = (1-a)(1-t_i) \quad \text{and} \quad |z-x| \leq 1 - t_i + a(1-t_i) = (1+a)(1-t_i)\,.$$

Therefore,

$$\begin{aligned}
\sum_{k\geq 1} \frac{(1-|x_k|)^{2\alpha-2}}{|z-x_k|^{d+\alpha-2}}\, r_k^{d-\alpha} &= \sum_{k\geq 1} \frac{(1-|x_k|)^{d+\alpha-2}}{|z-x_k|^{d+\alpha-2}}\, \phi(|x_k|)^{d-\alpha} \\
&\geq \sum_{i\geq 1} \sum_{\{k:\, x_k \in B(z_i, a(1-|z_i|))\}} \frac{(1-|x_k|)^{d+\alpha-2}}{|z-x_k|^{d+\alpha-2}}\, \phi(|x_k|)^{d-\alpha} \\
&\geq \left(\frac{1-a}{1+a}\right)^{d+\alpha-2} \sum_{i\geq 1} bM(|z_i|)\phi(1-(1-a)(1-t_i))^{d-\alpha} \\
&\geq c_4(a,b,c,\gamma,d) \sum_{i\geq 1} \phi(1-(1-a)(1-t_i))^{d-\alpha} M(1-(1-a)(1-t_{i+1}))\,,
\end{aligned}$$

where in the last line we have used $|z_i| = t_i = 1 - \frac{1+a}{(1-a)^2}(1-a)(1-t_{i+1})$, (1.3) and the fact that M increases to conclude that $M(|z_i|) \geq c^{-\lceil \log_2 \frac{1+a}{(1-a)^2} \rceil} M(1-(1-a)(1-t_{i+1}))$.

Let $s_i := 1-(1-a)(1-t_i)$. Then $s_{i+1} - s_i = (1-a)(t_{i+1}-t_i) = \frac{2a}{1+a}(1-s_i)$. Rewriting the last expression in terms of s_i we get

$$\begin{aligned}
\sum_{k\geq 1} \frac{(1-|x_k|)^{2\alpha-2}}{|z-x_k|^{d+\alpha-2}}\, r_k^{d-\alpha} &\geq c_4(a,b,c,\gamma,d) \sum_{i\geq 1} \phi(s_i)^{d-\alpha} M(s_{i+1}) \\
&= c_4(a,b,c,\gamma,d) \tfrac{1+a}{2a} \sum_{i=1}^{\infty} \frac{\phi(s_i)^{d-\alpha} M(s_{i+1})}{1-s_i} (s_{i+1}-s_i) \\
&= c_4(a,b,c,\gamma,d) \tfrac{1+a}{2a} \sum_{i=1}^{\infty} \int_{s_i}^{s_{i+1}} \frac{\phi(s_i)^{d-\alpha} M(s_{i+1})}{1-s_i}\, ds \\
&\geq c_4(a,b,c,\gamma,d) \tfrac{1+a}{2a} \sum_{i=1}^{\infty} \int_{s_i}^{s_{i+1}} \frac{\phi(s)^{d-\alpha} M(s)}{1-s}\, ds \\
&= c_4(a,b,c,\gamma,d) \tfrac{1+a}{2a} \int_{s_1}^{1} \frac{\phi(s)^{d-\alpha} M(s)}{1-s}\, ds\,.
\end{aligned}$$

We conclude from the assumption (1.4) that the last display is equal to $+\infty$. Since the separation condition (1.5) is the same as (1.2), the claim follows from Theorem 1.1 (b). □

Corollary 7.1 *Let ϕ be as above and assume that the family of balls $\{\overline{B}(x_k, r_k)\}_{k\geq 1}$ satisfies the separation condition*

$$\inf_{j\neq k} \frac{|x_j - x_k|}{1-|x_k|} > 0\,, \tag{7.3}$$

and that for some $a \in (0,1)$, $N_a(x) \geq 1$ for every $x \in D$. Then the family $\{\overline{B}(x_k, r_k)\}_{k\geq 1}$ is unavoidable if and only if

$$\int_0^1 \frac{\phi(t)^{d-\alpha}}{1-t}\, dt = \infty\,. \tag{7.4}$$

Proof. First note that $N_a(x) \geq 1$ for every $x \in D$ and the separation condition (7.3) imply that there exists $b \geq 1$ such that $1 \leq N_a(x) \leq b$. Hence, by taking $M(t) = 1$ for all $t \in [0,1)$, we see that $M(|x|) \leq N_a(x) \leq bM(|x|)$ for all $x \in D$. Moreover, since $0 < \phi(|x|)^{1-\alpha/d} < 1$, (7.3) implies the weaker separation condition (6.2). The statement now follows immediately from Theorem 1.2 □

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