On the representation theory of affine vertex algebras and W-algebras

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Plan of the talk

- Vertex algebras: Main definitions and problems
- C_2–cofinite vertex algebras
- Example: triplet vertex algebras and generalizations
- Affine vertex algebras
- Wakimoto and Whittaker modules for affine vertex algebras.
- Affine $\mathcal{W}$–algebras and conformal embeddings

Main litaruture

- D. Adamović, A. Milas, *On the triplet vertex algebra $W(p)$*, Advances in Mathematics 217 (2008), 6; 2664–2699
- D. Adamović, R. Lu, K. Zhao, *Whittaker modules for the affine Lie algebra $A_1^{(1)}$*, Advances in Mathematics 289 (2016) 438–479
- D. Adamović, *A realization of certain modules for the $N = 4$ superconformal algebra and the affine Lie algebra $A_2^{(1)}$*, Transformation Groups (2016)
- D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, *Conformal embeddings of affine vertex algebras in minimal W-algebras I, II*, arXiv:1602.04687, arXiv:1604.00893.

Definition of vertex algebra

[Borcherds 1986], [Frenkel-Lepowsky-Meurman 1988]

Vertex algebra is a triple $(V, Y, \mathbf{1})$ where

V complex vector space

$\mathbf{1}$ *vacuum* vector,

Y is a linear map

$$
\begin{aligned}
Y(\cdot, z): \qquad & V \to (\mathrm{End}\ V)[[z, z^{-1}]]; \\
& a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in (\mathrm{End}\ V)[[z, z^{-1}]]
\end{aligned}
$$

which satisfies the following conditions on $a, b \in V$:

Definition of a vertex algebra

$a_n b = 0$ for n sufficiently large.

$[D, Y(a,z)] = Y(D(a), z) = \frac{d}{dz} Y(a,z)$,

where $D \in \mathrm{End}\ V$ is defined by $D(a) = a_{-2}\mathbf{1}$.

$Y(\mathbf{1}, z) = I_V$.

$Y(a,z)\mathbf{1} \in V[[z]]$ and $\lim_{z \to 0} Y(a,z)\mathbf{1} = a$.

There exist $N \geq 0$ (which depends on a and b) such that

$$(z_1 - z_2)^N [Y(a, z_1), Y(b, z_2)] = 0 \quad \text{(locality)}.$$

Representations of vertex algebras

Representation (module) for vertex algebra V is a pair (M, Y_M) where

M is a complex vector space , and $Y_M(\cdot, z)$ is a linear map

$$Y_M : V \to \mathrm{End}(M)[[z, z^{-1}]], \ a \mapsto Y_M(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},$$

which satisfies the following conditions for $a, b \in V$ and $v \in M$:

Representations of vertex algebras

$Y_M(\mathbf{1}, z) = I_M$.

$a_n v = 0$ for n sufficiently large.

The following Jacobi identity holds:

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1) Y_M(b, z_2) \\ & -z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(b, z_2) Y_M(a, z_1) \\ = \quad & z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(a, z_0)b, z_2), \end{aligned}$$

Vertex superalgebras

Similarly we define notion of a vertex superalgebra $(V, Y, \mathbf{1})$ on a $\mathbb{Z}_2$-graded vector space $V = V^{\bar{0}} \oplus V^{\bar{1}}$.
Let $Y(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$. Define the λ–bracket

$$a_\lambda b = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_j b.$$

Rational vertex algebras

- A vertex algebra V is called **rational** if it has finitely many irreducible modules and if the category of V–modules is semisimple.
- Rational vertex algebras correspond to **rational conformal field theory**

  Examples:

  Vertex algebras associated to integrable representations of affine Kac–Moody Lie algebras

  Minimal models for Virasoro algebras, superconformal algebras, W-algebras

C_2–cofinite vertex algebras

Let V be a vertex algebra.

Define

$$C_2(V) = \mathrm{span}_{\mathbb{C}}\{a_{-2}b \mid a, b \in V\}.$$

Let $\mathcal{P}(V) = V/C_2(V)$.

For $a \in V$ we set $\overline{a} = a + C_2(V)$.

$\mathcal{P}(V)$ is an algebra with multiplication:

$$\overline{a} \cdot \overline{b} = \overline{a_{-1}b}, \qquad (a, b \in V).$$

V is called C_2–cofinite if $\dim \mathcal{P}(V) < \infty$.

If vertex algebra V is C_2–cofinite, then V has finitely many irreducible modules.

Virasoro vertex algebras

- Let $Vir = \oplus_{n \in \mathbb{Z}} \mathbb{C} L(n) \oplus \mathbb{C} C$ be a Virasoro algebra, i.e.,
- $[L(m), L(n)] = (m-n) L(m+n) + \frac{m^3 - m}{12} \delta_{m+n,0} C$
- C is central element.

- Let $L(c,h)$ be irreducible highest weight module of Vir with central charge c and highest weight h.
- $L(c,0)$ is a simple vertex algebra

- $L(c,0)$ is rational $\iff$ $L(c,0)$ is C_2–cofinite $\iff$ $c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$, $(p,q) = 1$.

Non-rational, C_2–cofinite vertex algebras and LCFT

- There exists non-rational, C_2–cofinite vertex algebras.
- These vertex algebras are related to logarithmic conformal field theory LCFT in theoretical physics
- Module categories for these vertex algebras are related to certain quantum groups (more A. Milas talk tomorrow)

Triplet vertex algebra $\mathcal{W}(p)$

- Introduced by H. Kausch, 1991.
- Investigated by "physicists" B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, I.Yu. Tipunin, Gaberidel, Fuchs (and many others)
- $\mathcal{W}(p)$ is uniquely realized on the following module for the Virasoro algebra

$$\mathcal{W}(p) = \bigoplus_{m=0}^{\infty} (2m+1) L(c_{1,p}, m^2 p + mp - m),$$

where $c_{1,p} = 1 - \frac{6(p-1)^2}{p}$, $p \geq 2$.

Triplet vertex algebra $\mathcal{W}(p)$

Theorem (D.A, A.Milas, Advances in Math. 2008; 2011)

.

(i) *$\mathcal{W}(p)$ has exactly $2p$ irreducible modules, explicitly constructed using lattice vertex algebras.*

(ii) *$\mathcal{W}(p)$ is C_2–cofinite.*

(iii) *$\mathcal{W}(p)$ admits $2p-2$ indecomposable, logarithmic modules, realized as projective covers of $2p-2$–irreducible $\mathcal{W}(p)$–modules.*

(iv) *The vector space of irreducible characters and generalized characters (pseudotraces) span*

- *$3p-1$- dimensional representation of $SL(2,\mathbb{Z})$,*
- *a fundamental system of a modular differential equation of order $3p-1$.*

More examples of C_2–cofinite logarithmic vertex algebras

- $N = 1$ triplet vertex superalgebras [D.A, A.M, CMP 2009]
- logaritmic extensions of Virasoro minimal modules [D.A, A.M IMRN 2010] , [A. Tsuchiya, S. Wood IMRN 2014]
- Orbifolds [Miyamoto 2015] , [Adamovic-Milas-Lin 2013-2015]
- Some conjectures are presented, but so far there are no new examples.
- Classification of irreducible modules and construction of projective covers is related to new constant term identities which we discover

Logarithmic constant term identity: Example

Theorem (D.A, A. M, 2010)

Let $p \geq 3$ be an odd integer. Then

$$-\mathrm{Res}_{x_1,x_2,x_3}\left(\frac{1}{x_1x_2x_3}\right)^{3p}\ln\left(1-\frac{x_2}{x_1}\right)(x_1-x_2)^p(x_1-x_3)^p(x_2-x_3)^p$$
$$(1+x_1)^t(1+x_2)^t(1+x_3)^t$$
$$= A_p\binom{t}{2p-1}\binom{t+p}{2p-1}\binom{t+p/2}{2p-1},$$

where

$$A_p = \frac{(-1)^{(p+1)/2}}{6}\frac{(3p)!(\frac{p-1}{2})!^3}{\binom{3p-1}{2p-1}\binom{\frac{5}{2}p-1}{2p-1}(p!)^3(\frac{3p-1}{2})!}$$

Whittaker modules : general definition

Assume that $\mathfrak{g}$ is a Lie algebra (possible infinite dimensional) with triangular decomposition:

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+.$$

Let $\lambda : \mathfrak{n} \to \mathbb{C}$ be a Lie algebra homomorphism. The universal Whittaker modules with Whittaker function λ is defined as

$$\widetilde{W}(\lambda) := U(\mathfrak{g})/J(\lambda)$$

where $J(\lambda) := U(\mathfrak{g}).\{x - \lambda(x)1 \, | x \in \mathfrak{n}_+\}$.

Let $W(\lambda)$ be its simple quotient.

Problem: Determine the structure of $W(\lambda)$.

For $\mathfrak{g}$ be a semisimple complex Lie algebra, Whittaker modules are studied by Kostant, McDowell, Miličić, Soergel and others.

Affine Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $(\cdot,\cdot)$ be a nondegenerate symmetric bilinear form on $\mathfrak{g}$.

The affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

where K is the canonical central element and the Lie algebra structure is given by

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{n+m,0}K.$$

We will say that M is a $\hat{\mathfrak{g}}$–module of level k if the central element K acts on M as a multiplication with k.

Affine vertex algebras

Set $x(n) = x \otimes t^n$, for $x \in \mathfrak{g}$, $n \in \mathbb{Z}$, and identify $\mathfrak{g}$ as the subalgebra $\mathfrak{g} \otimes t^0$.

Define the field $x(z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}$ which acts on any highest weight $\hat{\mathfrak{g}}$–module of level k.

- Let $V^k(\mathfrak{g})$ be the universal vertex algebra generated by fields $x(z)$, $x \in \mathfrak{g}$.
- The category of $V^k(\mathfrak{g})$–modules contains all $\hat{\mathfrak{g}}$–modules from the category $\mathcal{O}$ and all Whittaker modules of level k.
- $V_k(\mathfrak{g})$ the unique graded simple quotient of $V^k(\mathfrak{g})$.
- Basic Problem: Investigate the tensor category of $V_k(\mathfrak{g})$–modules.

Category of $V_k(\mathfrak{g})$–modules

- Let $h^\vee$ denotes the dual Coxeter number ($h^\vee = n$ for $\mathfrak{g} = sl(n)$)
- If $k < -h^\vee$ or $k \notin \mathbb{Q}$, then the category of $V_k(\mathfrak{g})$–modules is described by D. Kazhdan and G. Lusztig and it is equivalent to module category of certain quantum groups
- $V_k(\mathfrak{g})$ is C_2–cofinite $\iff$ $V_k(\mathfrak{g})$ is rational $\iff$ $k \in \mathbb{Z}_{>0}$.
- If k is admissble ($k + h^\vee = \frac{p'}{p}$, $(p, p') = 1$, $p' \geq h^\vee$ for $\mathfrak{g} = sl(n)$) then
- every $V_k(\mathfrak{g})$–module from the category $\mathcal{O}$ is completely reducible (conjectured in [D.A., A. M., MRL 1995] proved in [T. Arakawa DMJ 2016]) .

Critical and non-critical levels

Level $k = -h^\vee$ is called **critical level**.

Let x_i, y_i, $i = 1, \dots, \dim \mathfrak{g}$ be dual bases of $\mathfrak{g}$ with respect to form $(\cdot,\cdot)$, and let

$$sug = \sum_{i=1}^{\dim \mathfrak{g}} x_i(-1) y_i(-1)\mathbf{1} \in V^k(\mathfrak{g}).$$

If $k \neq -h^\vee$, then $\omega_{sug} = \frac{1}{2(k+h^\vee)} sug$ is Sugawara Virasoro vector

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} \quad \text{Virasoro field}$$

Let $k = -h^\vee$ and $Y(sug, z) = \sum_{n \in \mathbb{Z}} S(n) z^{-n-1}$

$$S(n) \quad \text{are in the center of } V^{-h^\vee}(\mathfrak{g}).$$

- Center of $V^{-h^\vee}(\mathfrak{g})$ is a commuative vertex algebra called Feigin-Frenkel center.

Affine Lie algebra $A_1^{(1)}$

Let now $\mathfrak{g} = sl_2(\mathbb{C})$
with generators e, f, h
and relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$.
The corresponding affine Lie algebra $\hat{\mathfrak{g}}$ is of type $A_1^{(1)}$.
$\widetilde{\mathfrak{n}}_+$ is generated by

$$e_0 = e \otimes t^0; e_1 = f \otimes t^1.$$

Lie algebra homomorphism $\lambda : \widetilde{\mathfrak{n}}_+ \to \mathbb{C}$ is uniquely determined by

$$(\lambda_0, \lambda_1) = (\lambda(e_0), \lambda(e_1)).$$

Let $\widetilde{W}(k, \lambda_1, \lambda_2)$ and $W(k, \lambda_1, \lambda_2)$ denote the universal and simple Whittaker modules of level k and type (λ_1, λ_2).

Whittaker modules for $\widehat{sl_2}$

Theorem (D.A; R. L, K. Z, AIM, 2016)

(1) *Assume that* $k \neq -2$ *and* $\lambda_1 \cdot \lambda_2 \neq 0$. *Then* $\widetilde{W}(k, \lambda_1, \lambda_2)$ *is an irreducible* $\widetilde{sl_2}$*–module.*

(2) *Assume that* $k = -2$ *and* $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 \neq 0$. *Let* $c(z) = \sum_{n \leq 0} c_n z^{-n-2} \in \mathbb{C}((z))$.

$$W(-2, \lambda_1, \lambda_2, c(z)) = \widetilde{W}(-2, \lambda_1, \lambda_2)/\langle (S(n) - c_n) \, | n \leq 0 \rangle$$

is an irreducible $\widehat{sl_2}$*–module.*

Weyl vertex algebra

The Weyl vertex algebra W is generated by the fields

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \quad a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n) z^{-n},$$

whose components satisfy the commutation relations for infinite-dimensional Weyl algebra

$$[a(n), a(m)] = [a^*(n), a^*(m)] = 0, \quad [a(n), a^*(m)] = \delta_{n+m,0}.$$

Wakimoto modules

Assume that $\chi(z) \in \mathbb{C}((z))$.

On the vertex algebra W exists the structure of the $A_1^{(1)}$–module at the critical level defined by

$$\begin{aligned} e(z) &= a(z), \\ h(z) &= -2 : a^*(z)a(z) : -\chi(z) \\ f(z) &= - : a^*(z)^2 a(z) : -2\partial_z a^*(z) - a^*(z)\chi(z). \end{aligned}$$

This module is called the Wakimoto module and it is denoted by $W_{-\chi(z)}$.

Theorem (D.A., CMP 2007, Contemp. Math. 2014)

The Wakimoto module $W_{-\chi}$ is irreducible if and only if $\chi(z)$ satisfies one of the following conditions:

(i) *There is $p \in \mathbf{Z}_{>0}$, $p \geq 1$ such that*

$$\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z)) \quad \text{and} \quad \chi_p \neq 0.$$

(ii) $\chi(z) = \sum_{n=0}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z))$ *and* $\chi_0 \in \{1\} \cup (\mathbb{C} \setminus \mathbb{Z})$.

(iii) *There is $\ell \in \mathbb{Z}_{\geq 0}$ such that*

$$\chi(z) = \frac{\ell+1}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z))$$

and $S_\ell(-\chi) \neq 0$, where $S_\ell(-\chi) = S_\ell(-\chi_{-1}, -\chi_{-2}, \dots)$ is a Schur polynomial.

Whittaker modules for Weyl vertex algebra

Every restricted module for the Weyl algebra is a module for Weyl vertex algebra W.

For $(\lambda, \mu) \in \mathbb{C}^2$ let $M_1(\lambda, \mu)$ be the module for the Weyl algebra generated by the Whittaker vector v_1 such that

$$a(0)v_1 = \lambda v_1, \; a^*(1)v_1 = \mu v_1, \; a(n+1)v_1 = a^*(n+2)v_1 = 0 \; (n \geq 0).$$

$M_1(\lambda, \mu)$ is a W-module.

Theorem (D.A; R. Lu, K. Z., Advances in Math. 2016)

For every $\chi(z) \in \mathbb{C}((z))$, $(\lambda, \mu) \in \mathbb{C}^2$, $\lambda \neq 0$ there exists irreducible $\widehat{sl_2}$–module $\overline{M_{Wak}}(\lambda, \mu, -2, \chi(z))$ realized on the W–module $M_1(\lambda, \mu)$ such that

$$
\begin{aligned}
e(z) &= a(z); \\
h(z) &= -2 : a^*(z)a(z) : +\chi(z); \\
f(z) &= - : a^*(z)^2 a(z) : -2\partial_z a^*(z) + a^*(z)\chi(z)
\end{aligned}
$$

Affine W algebra $W^k(\mathfrak{g}, f_\theta)$

- Choose root vectors e_θ and f_θ in simple Lie superalgebra $\mathfrak{g}$ such that $[e_\theta, f_\theta] = x, \ [x, e_\theta] = e_\theta, \ [x, f_\theta] = -f_\theta$.
- ad(x) defines minimal $\frac{1}{2}\mathbb{Z}$–gradation:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1.$$

- Let $\mathfrak{g}^\natural = \{a \in \mathfrak{g}_0 \mid (a|x) = 0\}$.
- $W^k(\mathfrak{g}, f_\theta)$ is strongly generated by vectors
- $G^{\{u\}}$, $u \in \mathfrak{g}_{-1/2}$, of conformal weight $3/2$;
- $J^{\{a\}}$, $u \in \mathfrak{g}^\natural$ of conformal weight 1;
- ω conformal vector of central charge

$$c(\mathfrak{g}, k) = \frac{k \mathrm{sdim} \mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4.$$

- $W_k(\mathfrak{g}, f_\theta)$ simple quotient of $W^k(\mathfrak{g}, f_\theta)$

Affine vertex subalgebra of $W_k(\mathfrak{g}, f_\theta)$

- Assume that $\mathfrak{g}^\natural = \bigoplus_{i \in I} \mathfrak{g}_i^\natural$; and that $\mathfrak{g}_i^\natural$ is either simple or 1–dimensional abelien
- Let $\mathcal{V}^k(\mathfrak{g}^\natural)$ be the vertex subalgebra of $W^k(\mathfrak{g}, f_\theta)$ generated by $\{J^{\{a\}} \mid u \in \mathfrak{g}^\natural\}$. Then:

$$\mathcal{V}^k(\mathfrak{g}^\natural) = \bigotimes_{i \in I} V^{k_i}(\mathfrak{g}_i^\natural).$$

- Let $\mathcal{V}_k(\mathfrak{g}^\natural)$ be the image of $\mathcal{V}^k(\mathfrak{g}^\natural)$ in $W_k(\mathfrak{g}, f_\theta)$.
- Let ω_{sug} be the Sugawara Virasoro vector in $\mathcal{V}_k(\mathfrak{g}^\natural)$.
- Embedding $\mathcal{V}_k(\mathfrak{g}^\natural)$ in $W_k(\mathfrak{g}, f_\theta)$ is called conformal if

$$\omega_{sug} = \omega.$$

A numerical criterion for conformal embedding

Theorem (D.A, Kac, Moseneder-Frajria, Papi, Perše (2016))

Embedding $\mathcal{V}_k(\mathfrak{g}^\natural)$ in $W_k(\mathfrak{g}, f_\theta)$ is conformal if and only if

$$c(sug) = c(\mathfrak{g}, k).$$

Theorem (D.A, Kac, Moseneder-Frajria, Papi, Perše (2016))

Assume that k is conformal level and that $W_k(\mathfrak{g}, f_\theta)$ does not collapse on its affine part. Then

$$k = -\frac{2}{3}h^\vee \quad \text{or} \quad k = -\frac{h^\vee - 1}{2}.$$

Example: $W_k(psl(2|2), f_\theta)$

$W_k(psl(2|2), f_\theta)$ is generated by the Virasoro field L, three primary fields of conformal weight 1, J^0, J^+ and J^- (even part) and four primary fields of conformal weight $\frac{3}{2}$, $G^\pm$ and $\overline{G}^\pm$ (odd part).
The remaining (non-vanishing) λ–brackets are

$$
\begin{array}{rl}
[J^0_\lambda, J^\pm] = \pm 2J^\pm & [J^0_\lambda J^0] = \dfrac{c}{3} \\
[J^+_\lambda J^-] = J^0 + \frac{c}{6}\lambda & [J^0_\lambda G^\pm] = \pm G^\pm \\
[J^0 \overline{G}^\pm] = \pm \overline{G}^\pm & [J^+_\lambda G^-] = G^+ \\
[J^-_\lambda G^+] = G^- & [J^+_\lambda \overline{G}^-] = -\overline{G}^+ \\
[J^-_\lambda \overline{G}^+] = -\overline{G}^- & [G^\pm_\lambda \overline{G}^\pm] = (T + 2\lambda)J^\pm \\
[G^\pm_\lambda \overline{G}^\mp] = & L \pm \dfrac{1}{2} TJ^0 \pm \lambda J^0 + \dfrac{c}{6}\lambda^2
\end{array}
$$

$W_k(psl(2|2), f_\theta)$ is called the $N = 4$ superconformal vertex algebra important in **Mathieu Moonshine**.

$W_k(psl(2|2), f_\theta)$

Theorem (D.A., Transformation Groups, 2016)

(i) *The simple affine vertex algebra $V_k(sl_2)$ with $k = -3/2$ is conformally embedded into $W_k(psl(2|2), f_\theta)$.*

(ii) *$W_k(psl(2|2), f_\theta)$ with $k = -3/2$ is completely reducible $\widehat{sl_2}$–module and the following decomposition holds:*

$$W_k(psl(2|2), f_\theta) \cong \bigoplus_{m=0}^{\infty} (m+1) L_{A_1}(-(\frac{3}{2} + m)\Lambda_0 + m\Lambda_1).$$

Hamiltonian reduction functor relates $W_k(psl(2|2), f_\theta)$ with $k = -3/2$ with triplet vertex algebras appearing in LCFT.

Application to the extension theory of affine vertex algebras

- When V is a rational vertex algebra, theory of SCE was developed by H. Li (several papers) Dong-Li-Mason (1997), Huang-Lepowsky-Kirilov,
- When V is not rational, not C_2 cofinite, and M is a SC V–module satisfying certain condition, it is non-trivial to show that $V \oplus M$ is again a vertex algebra.
- " Elegant solution". Realize:

$$W_k(\mathfrak{g}, f_\theta) = V \bigoplus M$$

- In [AKMPP 2016] for every simple Lie superalgebra $\mathfrak{g}$ such that $\mathfrak{g}^\natural$ is a semi–simple Lie algebra, we take suitable conformal k and show that

$$W_k(\mathfrak{g}, f_\theta) = \mathcal{V}_k(\mathfrak{g}^\natural) \bigoplus M$$

where M is a simple $V_k(\mathfrak{g}^\natural)$–module.

Conformal embedding $V_{k+1}(gl(n))$ in $W_k(sl(n+2), f_\theta)$:

Theorem (AKMPP, 2016)

Let $k = -\frac{2}{3}(n+2)$. Then we have conformal embedding $V_{k+1}(gl(n))$ in $W_k = W_k(sl(n+2), f_\theta)$. Assume that $n \geq 3$. Then

$$W_k = \bigoplus_{i \in \mathbb{Z}} W_k^{(i)} \tag{1}$$

and each $W_k^{(i)}$– is irreducible $V_{k+1}(gl(n))$–module.

In all cases, $n \geq 3$ $W_k^{(i)}$ are not admissible $V_{k+1}(gl(n))$–modules. We believe that they are simple currents in a suitably category.

Conformal embeddings with infinite decomposition property

- Conformal embeddings with infinite decomposition property are
- $V_{-3/2}(gl(2))$ in $V_{-3/2}(sl(3))$,
- $V_{-5/3}(gl(2))$ in $W_{-8/3}$.

- Each $V_{-3/2}(sl(3))^{(i)}$ and $W_{-8/3}^{(i)}$ are direct sum of infinitely many irreducible $\widehat{gl}_2$–modules.

- Analysis of these embedding uses explicit realization of $V_{-3/2}(sl(3))$ and $W_{-8/3}(sl(4))$ from D.A, Transform. Groups (2016).

Open problems and new research directions

- Construct higher rank C_2–cofinite logarithmic vertex algebras and superalgebras.
- Prove equivalence of tensor categories between C_2–cofinite vertex algebras and quantum groups at roots of unity
- Explicit realization of projective covers.
- Higher rank generalization of presented results for Wakimoto and Whittaker modules
- Study above problems in the context of quantum vertex algebras
- Study number theoretical version of decompositions appearing in the context of conformal embeddings (connections with Modular mock functions)

Thank you