Bayesian Truth Serum and Information Theory: Game of Duels

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Abstract— **This paper aims to develop insights into Bayesian Truth Serum algorithm from the perspective of Shannon theory. We postulate a natural sequence of seven axioms that produce Bayesian Truth Serum scoring rule in such a way that it reflects quality of information. This makes it possible to regard Bayesian Truth Serum as a measure of combined information-prediction quality in situations where respondents are asked to choose an alternative from a finite set and provide predictions of their peers' propensities to choose. This is possible for finite and infinite sets of respondents.**

Index Terms— **Bayesian Truth Serum, information entropy, Bayesian game, Shannon theory**

I. INTRODUCTION

Bayesian Truth Serum algorithm was developed in [1]. The Bayesian Truth Serum requires a single multiple-choice question. The respondents are asked, in addition to providing their personal answer, to predict the percentage distribution of answers in the entire sample. The algorithm has applications in studies of public or expert opinions: voting intentions, product ratings, expert forecasting, and various crowdsourcing applications. Published studies of Bayesian Truth Serum include application to assess the degree of learning in design courses [2], validity of deterrence in criminology [3], incentivizing digital pirates' confession [4], and new product adoption in pharmaceutical setting [5].

The algorithm has been studied from the economics point of view, its scoring rule aspects have been explored, but as far as we know it has not been approached from the Shannon theory point of view. The purpose of this paper is to emphasize that there is a

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connection between the Bayesian Truth Serum algorithm and the Shannon Theory and that it lies at the very essence of the algorithm itself.

There are two fundamental features of the Bayesian Truth Serum algorithm. One is that it is (truth) incentive-compatible (i.e. assuming that all other respondents tell the truth, it is in expectation most profitable for the selected respondent to tell the truth as well). Although highly desirable and important from the general point of view, this aspect is not of interest here.

The second fundamental feature of the Bayesian Truth Serum algorithm is that its scoring rule ranks players according to their posterior probability on the actual percentage distribution of answers in the sample [1]. If that distribution is regarded as the 'true state of nature,' this ranking may be interpreted as a ranking according to domain expertise [6, 7]. The Bayesian Truth Serum algorithm is not the only incentive-compatible algorithm with this ranking property, as shown in [6].

In this paper we focus on rankings, but approach the problem without using Bayesian game theory. Instead of having one game where many players provide their one-time answers and predictions, we view this situation in terms of pairwise duels between players. These duels result in transfer of points between the players according to some function P. We use simple simultaneous duels to show that under natural set of axioms compatible with Shannon theory we can derive the same scoring rule as [1].

In the rest of the paper we present the Bayesian Truth Serum algorithm formally in Section II. Our axiomatic system and main results are presented in Section III. We conclude the paper with the discussion in Section IV.

II. BAYESIAN TRUTH SERUM ALGORITHM

By R we denote the set of players (respondents). We assume that R is not empty, not a singleton, and at most countable (i.e. the cardinal number of the set R satisfies $2 \leq card(R) \leq \aleph_0$). Suppose that the players are presented with a multiple choice question, offering a choice of $m \in \mathbb{N} \setminus \{1\}$ answers (we use the standard mathematical notation where $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ is the set of real numbers, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$). Each player picks a simple answer (the one s/he thinks is the correct one) and gives a prediction in terms of probabilities on the distribution of m answers within R.[5] More precisely, we present the answer of a player $r \in R$ as a pair of ordered m-tuples

$$((x_1^r, \dots, x_m^r); (y_1^r, \dots, y_m^r)) \tag{1}$$

where $x_1^r, \dots, x_m^r \in \{0,1\}$, and $y_1^r, \dots, y_m^r \in [0,1]$ such that $\sum_{k=1}^m x_k^r = 1$ and $\sum_{k=1}^m y_k^r = 1$.

[5] The latter question is usually asked in the following way: "please estimate the percentage of your peers who will choose answer *k*", the question is repeated for each *k=1,..,m*.

Exactly one of x_k^r is equal to one (the non-zero term which corresponds to the selected answer), while $(y_1^r, \dots, y_m^r)$ is a probability distribution on $\{1,2,\dots,m\}$. As a consequence, a complete data containing the answers of all players can be presented as a (finite or infinite) matrix $(X;Y)$; it is of the order $card(R)\times 2m$ and its r[th] row, $r \in R$, is given by (1).

Based on $(X;Y)$ we want to assign a numerical score for each player, say

$$u^r = u^r(X;Y) \tag{2}$$

for player $r \in R$. Eventually we expect our scores to be real-valued, but here at the outset we shall not restrict ourselves and in principle we allow even for infinite values, i.e.

$$u^r\,(X,Y) \in \overline{\mathbb{R}} \tag{3}$$

A. Classic definition of the score in Bayesian Truth Serum

Before stating our axioms we present how the score is defined by the Bayesian Truth Serum.

We shall use the notation $\sum_{s\in R}$ in both finite and infinite case. If R is finite, then $\sum_{s\in R}$ has its usual meaning of the sum over all elements of R. If R is infinite, then we consider $R = \bigcup_{n\in\mathbb{N}} R_n$, where $card(R_n) = n$, and the meaning of $\sum_{s\in R}$ is in the sense of $\lim_{n\to\infty} \sum_{s\in R_n}$; the notation comes together with an assumption that the limit exists within $\overline{\mathbb{R}}$. Similarly, the meaning of $(av)\sum_{s\in R}$ is $\frac{1}{card(R)}\sum_{s\in R}$ in the case of a finite *R,* while in the case of an infinite *R* it is $\lim_{n\to\infty} \frac{1}{n}\sum_{s\in R_n}$. With this notation in mind, we consider $\overline{x} := (\overline{x_1}, \dots, \overline{x_m})$ where, for *k=1,…,m*

$$\overline{x_k} := (av)\sum_{s\in R} x_k^r$$

i.e. arithmetic means of *X*-columns, and $\hat{y} := (\widehat{y_1}, \dots, \widehat{y_m})$ where, for *k=1,…,m*

$$\ln(\widehat{y_k}) := (av)\sum_{s\in R} \ln\,(y_k^s)$$

i.e. geometric means of *Y*-columns.

Using the notation above, the algorithm in [1] is given as

$$u^r(\mathrm{X},\mathrm{Y}) := \sum_{k=1}^m x_k^r \,\ln\frac{\overline{x_k}}{\widehat{y_k}} + \sum_{k=1}^m \overline{x_k}\,\ln\frac{y_k^r}{\overline{x_k}} \tag{4}$$

where $r \in R$. The first part of the sum is the information score, while the second one is the prediction score [1].

III. AXIOMATIC SYSTEM

Our goal in this paper is develop an axiomatic system for (4) that is compatible with Shannon theory. In our approach players choose an expert among themselves via simultaneous conceptual duels. Each duel has a "challenger", say player $r \in R$, and an "offender", say player $s \in R$.[6] We denote such duel as $r \to s$. Each respondent plays a duel with every other respondent, including oneself.

Each duel $r \to s$ ends with a transfer of points from player r to player s. We denote the number of transferred points by

$$T^{r \to s} = T^{r \to s}\,(X;Y) \in \mathbb{R} \quad . \tag{5}$$

We can think of positive $T^{r \to s}$ as the winning case for the offender, while negative $T^{r \to s}$ means that the challenger prevails. All the possible duels are to be performed (including the duel with oneself) in order to determine scores u^r for all respondents $r \in R$. In particular, if R is finite, there will be $[card(R)]^2$ duels.

Let us introduce the basic rule for a duel. For every $r \in R$ the score u^r equals the number of received points minus the number of given points, i.e.

$$u^r = u^r(\mathrm{X}, \mathrm{Y}) = \sum_{s \in R} T^{s \to r}\,(X;Y) - \sum_{s \in R} T^{r \to s}\,(X;Y) \tag{6}$$

There are two immediate important consequences of (6). First, assuming that all the sums are finite-valued (which is the only interesting case), the duel is a zero-sum game,

$$\sum_{r \in R} u^r = \sum_{r \in R} \sum_{s \in R} T^{s \to r} - \sum_{r \in R} \sum_{s \in R} T^{r \to s} = 0 \tag{7}$$

The second consequence of (6) is that the description of u^r reduces to the description of $T^{r \to s}$. Hence, we present a set of axioms about $T^{r \to s}$ that generate the Bayesian Truth Serum algorithm (4). For each axiom we give an intuitive justification (which may include some ideas from statistics) and a formal statement (which is always going to be deterministic).

Our first axiom is very much in the spirit of medieval duels. We can interpret it as "the offender chooses the playground for the duel".

Axiom 1. *The challenger r will transfer points to the offender s, based on the x answer of the offender s. More precisely, for every $r, s \in R$ and for every $k \in \{1, \dots, m\}$ there is $P_k^{rs}(X;Y) \in \mathbb{R}$ such that*

[6] We use traditional duel terminology, where one player (offender) offends the other (challenger), who in turn challenges the first player to a duel

$$T^{r\to s}\,(X;Y) = \sum_{k=1}^{m} x_k^s\ P_k^{rs}(X;Y) \tag{8}$$

Observe that, among others, Axiom 1 reduces our analysis from $T^{r\to s}$ to P_k^{rs}. Observe also that, for every $s \in R$, there is exactly one $k \in \{1,\dots,m\}$ such that $x_k^s = 1$. Hence, we can think of that k as being the function of s, i.e. $k = k(s)$. It follows then that (8) becomes

$$T^{r\to s}\,(X;Y) = P_{k(s)}^{rs}(X;Y) \tag{9}$$

In order to understand the second axiom, we introduce the following partition of *R*

$$R_k := \{s\ \in R \mid x_k^s = 1\},\quad k = 1,\dots,m \tag{10}$$

Obviously, the partition $R = R_1 \cup \dots . \cup R_m$ is a function of *X*. Fix *k* for a moment and consider R_k, which is a subset of players who choose the same answer k. In general, the number of points P_k^{rs} may vary as *s* changes within R_k. The purpose of our second axiom is to prevent this from happening, i.e. that axiom can be thought of as "the egalitarian principle within R_k."

Axiom 2. *Given* $r \in R$ *and* $k \in \{1,\dots,m\}$ *we have*

$$(s, s' \in R_k \ \Rightarrow\ P_k^{rs}(X;Y) = P_k^{rs'}(X;Y)\,).$$

Axiom 2 says that if the offenders $s, s' \in R$ choose the same answer, then in a duel with every challenger they will receive the same number of points. Observe that Axiom 2 includes even the cases when for some *k* the set R_k may be an empty set; in this case the implication in Axiom 2 is true, since the premise of the implication is never true. Using a slight abuse of notation (think of *k*=*k* (*s*)), Axiom 2 implies that

$$P_k^{rs}(X;Y) = P_k^{r}(X;Y) \tag{11}$$

In order to understand the third axiom, observe that by choosing the answer *k*, the offender *s* decides (given that *r* is known) on a type of function P_k^r that will be used in the duel $r \to s$. However, the P_k^r will in general still depend on $(X;Y)$. Our next axiom can be thought of as strengthening the Axiom 1. The offender *s* chooses the playground *k*, and in doing so it reduces the variable dependence accordingly.

Axiom 3. *For every* $r \in R$ *and for every* $k \in \{1,\dots,m\}$*,*

$$P_k^r(X;Y) = P_k^r((x_k^q)_{q\in R}; (y_k^q)_{q\in R})$$

Next we turn to Axiom 4 which has deterministic form, and which can be justified using some ideas from statistics. One of the main problems in statistical analysis is to make inference about some unknown parameter θ. The inference is based on the information given in a sample $X_1, \dots, X_n$. In most cases the observed sample $x_1, \dots, x_n$ is just a long list of numbers, which can be difficult to interpret directly. Therefore, we employ the principle of data reduction to obtain some statistic $t = T(X_1, \dots, X_n)$ which simplifies our analysis. There are various data reduction principles: among others these are sufficiency principle, the equivariance principle and the likelihood principle. In this paper we are particularly interested in the sufficiency principle, which relies on the notion of sufficient statistic for θ. If t is such a statistic, then whenever we have two sample points $x = (x_1, \dots, x_n)$ and $x' = (x'_1, \dots, x_n')$ with the property $T(x) = T(x')$, then the inference about θ is the same regardless whether x or x' is observed. Typical example is a Bernoulli sample from which we infer about the "probability of success", p. It is well known and easy to understand intuitively that $\bar{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$ is an example of a sufficient statistic regarding p.

We would argue here that the X-part of our data is akin to the Bernoulli sample set-up. We are interested in $\omega = (\omega_1, \dots, \omega_n)$, where ω_k gives the actual fraction of the population that thinks k is the correct answer to the original question. Hence, since we are interested in ω_k, then the average value gives as much information about ω_k as the entire k-th column of the matrix X, i.e. $(x_k^q)_{q \in R}$. Therefore, we term our fourth axiom "the data reduction principle for X".

Axiom 4. *For every $r \in R$ and for every $k \in \{1, \dots, m\}$,*

$$P_k^r\left(\left(x_k^q\right)_{q\in R}; \left(y_k^q\right)_{q\in R}\right) = P_k^r(\overline{x_k}; (y_k^q)_{q\in R})$$

Our second data reduction principle deals with Y. Our axioms so far provided the offender s with the advantage to "choose the playground" k. In the next axiom we give an advantage to the challenger r by giving him/her an option to "choose the weapon". We can think of it as allowing the challenger to select some information from the k^{th} column of Y in order to predict ω_k. We assume that the challenger is very self-confident and always keeps with his/her own choice i.e. y_k^r. This gives us the data reduction principle for Y.

Axiom 5. *For every $r \in R$ and for every $k \in \{1, \dots, m\}$,*

$$P_k^r(\overline{x_k}; (y_k^q)_{q\in R}) = P_k^r(\overline{x_k}; y_k^r)$$

Observe that our axioms have reduced a function defined on a matrix $(X;Y)$ to a function defined on a pair of numbers $(\overline{x_k}; y_k^r)$ which are between 0 and 1. However, on this level of generality we still allow the form of the function to change with r or with k (i.e. the function can vary with the choice of different players or answers). A system that would allow for such level of generality would not be very practical, as for every k and every r we would have a different function P_k^r. Hence we opt for a more robust selection and introduce the following "universality axiom".

Axiom 6. *There exists a function* $P: [0,1]\times[0,1] \to \mathbb{R}$ *such that for every* $r \in R$ *and for every* $k \in \{1, \dots, m\}$ *we have* $P_k^r = P$.

In other words, Axiom 6 insures that function P_k^r is the same for every player *r* and for every answer *k*.

We believe that the first six axioms are natural and easy to accept. Their combined effect is that, for every $r, s \in R$

$$T^{r\to s}\ (X; Y) = \sum_{k=1}^{m} x_k^s\ P(\overline{x_k}; y_k^r) \tag{12}$$

Let us now turn our attention to the last and the most demanding axiom. In order to justify it, we borrow ideas from information theory[7]. We employ again a comparison with the Bernoulli probabilistic model; we think of both $\overline{x_k}$ and y_k^r as estimates of the probability of success, say p_k.

Intuitively speaking, let us think of $P(\overline{x_k}; y_k^r)$ as an estimate of some function $h(p_k)$, where h awards points with the intention of measuring the uncertainty of the associated Bernoulli model. If we have two independent Bernoulli random variables *U* (with success probability *p*) and *V* (with success probability *q*), then the probability of joint success is *pq*. Following the same argument as in [8], page 6, it is natural to require $h(pq) = h(p) + h(q)$. By translating this requirement to the language of *P*, we obtain

$$P\big(x(p)\cdot x(q);\ y(p)\cdot y(q)\big) = P\big(x(p);\ y(p)\big) + P\big(x(q);\ y(q)\big)$$

As in [8] we exclude the case of zero and treat it separately (see also [1]). Hence, we introduce the additivity property axiom in the following form.

Axiom 7. *The restriction* $P|_{\langle 0,1]\times\langle 0,1]}$ *of the function P given in (12) is a continuous function such that, for every* $u \in \langle 0, 1]$, $P(u, u) = 0$, *and for every* $u_1, u_2, v_1, v_2 \in \langle 0, 1]$,

$$P(u_1u_2, v_1v_2) = P(u_1, v_1) + P(u_2, v_2).$$

Observe that if the selected "playground information" of the offender results in the $\overline{x_k}$ which is exactly equal to the "challenger information", then the natural outcome is "a draw", i.e. $P(u, u) = 0$. Obviously, as in the Shannon theory, the consequence of the Axiom 7 is that one can rely on one of the well-known functional equations from mathematics. More precisely, the following result is standard and well-known.

Lemma. *If* $h: \langle 0, 1] \to \mathbb{R}$ *is continuous and such that, for every* $u, v \in \langle 0, 1]$, $h(uv) = h(u) + h(v)$, *then* $h(u) = a \cdot ln(u)$, *where* $a = -h(e^{-1})$.

[7] In particular, one may consult a chapter on a measure of information in [8] with the emphasis on section 1.2.

Recall that the additivity property is very strong. The conclusion of the Lemma follows even with much milder requirements than continuity on function *h*; for example it is sufficient to require monotonicity or measurability. Although this would allow us to reduce the requirement on continuity given in Axiom 7, in order to avoid unnecessary mathematical intricacies we presented the Axiom 7 in the above form. Namely, to construct non-measurable additive functions one needs to go into details about “the axiom of choice” and we think that from the point of view of various applications it is perhaps not necessary to go into such “axiom minimization” issues further.

Using Lemma it is not difficult to see that Axiom 7 reduces function *P* to a particular form.

Corollary. *If a function* $P: \langle 0,1] \times \langle 0,1] \to \mathbb{R}$ *satisfies Axiom 7, then there exists* $a \in \mathbb{R}$ *such that, for every* $u, v \in \langle 0,1]$,

$$P(u,v) = a \cdot ln\,(u/v)$$

Proof. Take $u_1 = u$, $u_2 = 1$, $v_1 = v$, $v_2 = 1$ in Axiom 7. We obtain $P(u,v) = P(u,1) + P(1,v)$. We start with the function $u \to P(u,1)$. If we apply Axiom 7 with $v_1 = v_2 = 1$,
we obtain

$$P(u_1 u_2, 1) = P(u_1, 1) + P(u_2, 1).$$

Hence, $u \to P(u,1)$ satisfies the requirement of the Lemma. We conclude that there exists $a \in \mathbb{R}$ such that $P(u,1) = a \cdot ln\,(u)$.

Consider now the function $v \to P(1,v)$. If we apply Axiom 7 with $u_1 = u_2 = 1$, we obtain

$$P(1, v_1 v_2) = P(1, v_1) + P(1, v_2).$$

Again, using Lemma, we conclude that there exists $b \in \mathbb{R}$ such that $P(1,v) = b \cdot ln\,(v)$
.

Finally, using $P(u,u) = 0$ and $P(u,u) = P(u,1) + P(1,u) = a \cdot ln\,(u) + b \cdot ln\,(u)$, we obtain $b = -a$. Hence, for every $u, v \in \langle 0,1]$, it follows $P(u,v) = a \cdot ln\left(\frac{u}{v}\right)$.

Q.E.D.

Remark. We need to decide on a particular choice of the normalizing constant $a \in \mathbb{R}$

from the previous Corollary. Suppose for the moment that the challenger r has selected $y_k^r = 1$, for some k. This implies $y_l^r = 0$ for all $l \neq k$, i.e. the challenger has put his entire trust on k. If, in this case, "the playground chosen by the offender" is indeed k, then it is the challenger who should earn points in this duel. More precisely, if $0 < u < 1$, then $P(u, 1) < 0$, and it follows that

$$a > 0 \tag{13}$$

What is then the natural choice for the constant a? This is now just the matter of normalization. Suppose for the moment that all offenders have chosen playground k. In that case the challenger would receive in total[8] $-a \cdot card(R) \cdot P(\overline{x_k}; 1)$ points in the finite case, and $\lim_{n\to\infty} -a(R_n) \cdot card(R_n) \cdot P(\overline{x_k}; 1)$ points in the infinite case. It is natural to normalize so that the total is $-P(\overline{x_k}; 1)$ points. Hence we define the constant a to be

$$a = \frac{1}{card(R)} \quad \text{in finite case, or}$$

$$a(R_n) = \frac{1}{card(R_n)} \quad \text{in the infinite case.} \tag{14}$$

Theorem. *If the scoring system satisfies Axioms 1-7 and condition (14), then the resulting system is the Bayesian Truth Serum algorithm, i.e.* u^r *satisfies (4).*

Proof. Without loss of generality we present the proof for the finite case. In the infinite case we can use exactly the same proof under the limit sign $\lim_{n\to\infty} \frac{1}{n}\sum_{s\in R_n}$, so we do not consider the infinite case separately. We proceed with the proof in the finite case.

Using (12) and the Corollary, we obtain

$$u^r = u^r(\mathrm{X}, \mathrm{Y}) = \sum_{s\in R} T^{s\to r}(X; Y) - \sum_{s\in R} T^{r\to s}(X; Y) =$$

$$= \sum_{s\in R}\sum_{k=1}^{m} x_k^r \frac{1}{card(R)}\left(\ln\frac{\overline{x_k}}{y_k^s}\right) - \sum_{s\in R}\sum_{k=1}^{m} x_k^s \frac{1}{card(R)}\left(\ln\frac{\overline{x_k}}{y_k^r}\right)$$

The first sum becomes

$$\sum_{s\in R}\sum_{k=1}^{m} x_k^r \frac{1}{card(R)}\left(\ln(\overline{x_k}) - \ln(y_k^s)\right) =$$

$$= \sum_{k=1}^{m} x_k^r \left[\frac{1}{card(R)}\sum_{s\in R}\ln(\overline{x_k}) - \frac{1}{card(R)}\sum_{s\in R}\ln(y_k^s)\right]$$

[8] In total here means from all the offenders.

Since the choice of k depends on r (not on s), we obtain

$$\frac{1}{card(R)} \sum_{s \in R} \ln(\overline{x_k}) = \ln(\overline{x_k})$$

On the other hand,

$$\frac{1}{card(R)} \sum_{s \in R} \ln(y_k^s) = (av) \sum_{s \in R} \ln(y_k^s) = \ln(\widehat{y_k})$$

It follows that the first sum equals $\sum_{k=1}^{m} x_k^r \ln\left(\frac{\overline{x_k}}{\widehat{y_k}}\right)$, i.e. equals the information score in (4). For the second sum we obtain

$$-\sum_{s \in R} \sum_{k=1}^{m} x_k^s \frac{1}{card(R)} \left(\ln \frac{\overline{x_k}}{y_k^r}\right) = \sum_{s \in R} \sum_{k=1}^{m} x_k^s \frac{1}{card(R)} \ln \frac{y_k^r}{\overline{x_k}} =$$

$$= \sum_{k=1}^{m} \ln \frac{y_k^r}{\overline{x_k}} \left(\frac{1}{card(R)} \sum_{k=1}^{m} x_k^s\right) = \sum_{k=1}^{m} \overline{x_k} \ln \frac{y_k^r}{\overline{x_k}}$$

This is equal to prediction score in (4).

Q.E.D.

IV. CONCLUSION

Formula (6) describes a large class of ranking systems which are all based on totality of the points players receive after the sequence of duels is performed. Every player performs two duels against every other player: one time as an offender and one time as a challenger. These duels are deterministic and in principle they work in the same way for the finite and the infinite number of players.

As in the Shannon theory, the key function involved is the logarithm. Hence, it seems plausible that the Bayesian Truth Serum may be somewhat connected to the notion of entropy. The connection is rather subtle, and it exists only in the infinite case where we take into account full Bayesian stochastic approach together with the exchangeability assumption on the set of players (respondents). The de Finetti theorem guarantees that there is an underlying governing random variable (we call the outcomes of this variable "states of nature"), which represents various possible belief systems within the population. For each $k \in \{1, \dots, m\}$ and for every state of nature, say i, we consider conditional probabilities $z_k^i = Prob(state\ i|\ response\ k)$, so called posteriors. It is shown in [1] [9] that for the true state of nature, assuming that $x_k^r = 1$, we have that the Bayesian Truth Serum scoring rule is given by

$$u^r = \ln\left(z_k^i\right) + A$$

The term A does not reflect ranking and ensures that the zero-sum property is satisfied. Let us denote $Z_k = \{z_k^i : i\ state\ of\ nature\}$. Hence, the conditional expectation can be expressed as

$$E(u^r|response\ k) = entropy(Z_k) + A$$

Observe however that the u^r ranking is not based on entropy, but rather on z_k^i itself. Thus it more resembles the maximum likelihood estimator based on posteriors.

Observe that an essential feature of the BTS algorithm is that it simultaneously measures the quality of information and prediction. We emphasize that the system of axioms in this paper does not use the zero-sum property as the distinguishing one among various algorithms of the form (6) (compare this to the characterization of BTS in [6]).

So far Bayesian Truth Serum bas been successfully tested for human respondents, hence the importance of incentive-compatibility. By setting aside incentive-compatibility and focusing on the ranking system, we believe that the algorithm can be applied in the context where players are machines instead of humans. One application would be measuring information-prediction capability in such various fields as meteorology, finance, medicine, etc.

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