One-Dimensional Dyadic Wavelets

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Abstract

The theory of wavelets has been thoroughly studied by many authors; standard references include books by I. Daubechies, by Y. Meyer, by R. Coifman and Y. Meyer, by C.K. Chui, and by M.V. Wickerhauser. In addition, the development of wavelets influenced the study of various other reproducing function systems. Interestingly enough, some open questions remained unsolved or only partially solved for more than twenty years even in the most basic case of dyadic orthonormal wavelets in a single dimension. These include issues related to the MRA structure (for example, a complete understanding of filters), the structure of the space of negative dilates (in particular, with respect to what is known as the Baggett problem), and the variety of resolution structures that may occur. In this article we offer a comprehensive, yet technically fairly elementary approach to these questions. On this path, we present a multitude of new results, resolve some of the old questions, and provide new advances for some problems that remain open for the future.

In this study, we have been guided mostly by the philosophy presented some twenty years ago in a book by E. Hernandez and G. Weiss (one of us), in which the orthonormal wavelets are characterized by four basic equations, so that the interplay between wavelets and Fourier analysis provides a deeper insight into both fields of research. This book has influenced hundreds of researchers, and their effort has produced a variety of new techniques, many of them reaching far beyond the study of one-dimensional orthonormal wavelets. Here we are trying to close the circle in some sense by applying these new techniques to the original subject of one-dimensional wavelets. We are primarily interested in the quality of new results and their clear presentations; for this reason, we keep our study on the level of a single dimension, although we are aware that many of our results can be extended beyond that case.

Given ψ, a square integrable function on the real line, we want to address the following question: What sort of structures can one obtain from the affine wavelet family $\{2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ associated with ψ? It may be too difficult to directly attack this problem via the function ψ. We argue in this article that the appropriate object to study is the principal shift invariant space generated by ψ (these spaces were introduced by H.Helson decades ago and applied very successfully in the approximation theory by C. de Boor, R.A. DeVore, and A. Ron, with more recent applications to wavelets introduced by A. Ron and Z. Shen). With this goal in mind, in Chapter 1, we present a very detailed study of principal

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shift invariant spaces and their generating families. These include the relationships between principal shift invariant spaces, various basis-like and frame-like properties of their generating families, their classifications based on additional translation invariances, convergence properties of various reproducing families with emphasis on the case of unconditional convergence, and the special properties of maximal principal shift invariant spaces.

Given a principal shift invariant space V and the dyadic dilation D, our approach is that the entire theory can be developed by considering two basic relationships between V and $D(V)$. Chapter 2 is devoted to the first of these two cases, the one in which the space V is contained within $D(V)$. In this chapter, we completely resolve this case via an emphasis on generalized filter studies. We show that the entire generalized MRA theory is a natural consequence of this approach, with a detailed classification of all the special cases of what we term as Pre-GMRA structures. Special attention is devoted to the analysis of the general form of a filter associated with the space V. The theory splits into two subcases, based on the filter properties with respect to dyadic orbits; we distinguish the "full-orbit" case and the "non full-orbit" case. In both cases we introduce new Tauberian conditions which provide complete characterizations of "usable" filters. This approach further splits into the analysis of low frequencies versus high frequencies. There is a fundamental new result here which shows that, based on the "ergodic properties" of $\widehat{\psi}$, the two frequency regimes exhibit radically different behavior; low frequencies allow completely localized adjustments while high frequencies can only be treated in a global sense. Various known results, like the Smith-Barnwell condition, the Cohen condition and its generalizations, the Lawton condition and its generalizations, are extracted naturally from our general approach. A multitude of new technical results are presented, with many examples and counter-examples exhibited to illustrate various subtle points of the theory.

The third and final chapter is devoted to the second case, i.e., when the space V is not contained in $D(V)$. This naturally leads to the space of negative dilates, and the theory again splits into two subcases, based on whether the original function ψ is contained within the space of its negative dilates or not. This is very much in the spirit of the Baggett problem, and we find it somewhat striking that the entire theory can be built based on such a simple property. We end the article with a partial resolution of the Baggett problem, but the problem remains open when taken in its full scope.

CHAPTER 1

Principal Shift-Invariant Spaces: Preliminaries and Auxiliary Results

1. Basic Notions and Set Inclusion

Let $L^2(\mathbb{R})$ denote the usual space of square-integrable, complex-valued functions on $\mathbb{R}$. We denote the usual inner product and norm on $L^2(\mathbb{R})$ by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. For a general Hilbert space X, we denote the inner product and norm by $\langle \cdot, \cdot \rangle_X$ and $\| \cdot \|_X$, respectively. For a measurable subset A of $\mathbb{R}$, we use $|A|$ to denote the Lebesgue measure of A.

For a function $\psi \in L^2(\mathbb{R})$, we study reproducing function systems built from ψ using translations and dilations. We begin with the study of translations. For $a \in \mathbb{R}$, we denote by $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the unitary operator defined pointwise by $(T_a f)(x) := f(x-a)$. For the majority of existing applications, one is only interested in a discrete family of translations, and so we focus on the integer translations first; write

$$\mathcal{B}_\psi := \{T_k \psi : k \in \mathbb{Z}\}. \tag{1.1}$$

The family $\mathcal{B}_\psi$ generates a closed subspace of $L^2(\mathbb{R})$ defined by

$$\langle \psi \rangle := \overline{\text{span} \mathcal{B}_\psi}, \tag{1.2}$$

where the closure is with respect to the usual norm topology on $L^2(\mathbb{R})$. The space $\langle \psi \rangle$ is obviously invariant under integer translations in the sense that $T_k \langle \psi \rangle \subseteq \langle \psi \rangle$ for each integer k. We shall say that, for a general closed subspace V of $L^2(\mathbb{R})$ is a *shift-invariant space* (or SIS for short) if $T_k V \subseteq V$ for every integer k. Clearly, spaces of the form $\langle f \rangle$ represent the "smallest" non-trivial shift-invariant spaces, and so we call $\langle \psi \rangle$ the *principal shift-invariant space generated by* ψ. These spaces have been studied by many authors over the last several decades: for early developments of these spaces, see [**Hel64**]; for applications in approximation theory see [**dBDR94**]; for more recent connections with wavelet theory, see [**RS95**], [**Bow00**], [**WW01**], [**HŠWW10b**], [**HŠWW10a**].

A principal shift-invariant space $\langle \psi \rangle$ is a closed subspace of $L^2(\mathbb{R})$; hence, $\langle \psi \rangle$ is a Hilbert space and $\mathcal{B}_\psi$ is its generating set. In the trivial case when $\psi \equiv 0$, we have $\mathcal{B}_\psi = \{0\} = \langle \psi \rangle$. If $\psi \not\equiv 0$, then $\mathcal{B}_\psi$ is linearly independent (see [**ŠS07**] for a simple argument). It follows then that $\langle \psi \rangle$ is an infinite-dimensional subspace of $L^2(\mathbb{R})$ when $\psi \not\equiv 0$. However, the standard notion of dimension is not the most suitable one for the study of shift-invariant spaces. As we discuss below, there is a natural notion of "SIS dimension" such that the principal shift-invariant spaces are "one-dimensional". We need the Fourier transform to describe it. We will use the standard version of the Fourier transform, i.e., for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

The Plancherel formula guarantees that $\widehat{\cdot}$ is an isometry on $L^2(\mathbb{R})$ and $(\widehat{f})^\vee = f$, where $\cdot^\vee$ denotes the inverse Fourier transform.

Recall that $\widehat{T_a f}(\xi) = e^{-2\pi i a \xi} \widehat{f}(\xi)$. For $a = k \in \mathbb{Z}$, we denote the function $\xi \mapsto e^{2\pi i k \xi}$ by e_k. Observe that $e_k : \mathbb{R} \to \mathbb{C}$ is a 1-periodic function and, as such, can be also considered as a function on the one-dimensional torus, $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ (we shall employ such "dual" treatment whenever we are dealing with 1-periodic functions on $\mathbb{R}$).

Given an SIS $V \subseteq L^2(\mathbb{R})$ and a countable generating set Φ for V, there is a measurable, 1-periodic function $\dim_{V,\Phi} : \mathbb{R} \to \mathbb{N} \cup \{\infty\}$ whose value at ξ is defined to be the dimension of the subspace of $\ell^2(\mathbb{Z})$ spanned by $\{(\widehat{\varphi}(\xi+k))_{k\in\mathbb{Z}} : \varphi \in \Phi\}$. It is not hard to show that, actually, $\dim_{V,\Phi}$ is independent of the choice of Φ; that is, if Φ and Φ' are both countable generating sets for V, then $\dim_{V,\Phi} = \dim_{V,\Phi'}$ almost everywhere. Thus we define[1] $\dim_V$, the *dimension function of V*, to be $\dim_{V,\Phi}$ for some choice of countable generating set Φ:

$$\dim_V := \dim_{V,\Phi} \text{ which is independent of the choice of } \Phi \tag{1.3}$$

It is of interest to consider the following "1-periodic" sets (most of our formulae are in the a.e. sense, and we shall omit the "a.e." to simplify the notation):

$$\begin{aligned} Z_V &:= \{\xi \in \mathbb{R} : \dim_V(\xi) = 0\} \\ U_V &:= Z_V^c = \{\xi \in \mathbb{R} : \dim_V(\xi) \neq 0\} \\ I_V &:= \{\xi \in \mathbb{R} : \dim_V(\xi) = \infty\}. \end{aligned}$$

Obviously, for $V = \{0\}$, we have $Z_V = \mathbb{R}$, while for $V = L^2(\mathbb{R})$, we have $I_V = \mathbb{R}$. It is well known (see [**ŠSW08**] for example) that $I_{\langle\psi\rangle} = \emptyset$ and

$$\dim_{\langle\psi\rangle} = \chi_{U_{\langle\psi\rangle}}, \tag{1.4}$$

i.e. $\dim_{\langle\psi\rangle}$ takes only the values 0 and 1. This shows that principal shift-invariant spaces are "one-dimensional" in the sense of $\dim_V$ — and, in particular, $\langle\psi\rangle$ is "much smaller" than $L^2(\mathbb{R})$.

EXAMPLE 1.5. For a measurable set $E \subseteq \mathbb{R}$, we denote by $L^2(E)^\vee$ the space defined by

$$L^2(E)^\vee := \{f \in L^2(\mathbb{R}) : \text{the support of } \widehat{f} \subseteq E\}.$$

Since $\widehat{T_k f} = e_{-k}\widehat{f}$ for each $k \in \mathbb{Z}$, it is easy to see that $L^2(E)^\vee$ is an SIS. For $E = [a,b]$ with $a, b \in \mathbb{R}$ such that $a < b$ and $b - a \in \mathbb{N}$,

$$\dim_{L^2([a,b])^\vee}(\xi) \equiv b - a. \tag{1.6}$$

In particular, if $a < b \in \mathbb{R}$ with $a < b$ and $b - a \leq 1$, then $L^2([a,b])^\vee = L^2((a,b])^\vee = L^2([a,b))^\vee = L^2((a,b))^\vee$ is a principal shift-invariant space. If $b - a = 1$, then $\dim_{L^2([a,b])^\vee} \equiv 1$. Observe also that we have the following orthogonal sum

$$L^2(\mathbb{R}) := \bigoplus_{n\in\mathbb{Z}} L^2([n, n+1))^\vee.$$

$\diamondsuit$

[1] For an alternative definition of the dimension function, see the end of Remark 2.9

It follows from Equation (1.4) that the analysis of the function $\dim_{\langle\psi\rangle}$ is essentially the analysis of the set $Z_{\langle\psi\rangle}$. Although this is helpful, it is often "too crude" a tool for the analysis of the finer properties of $\langle\psi\rangle$. With this purpose in mind, we go back to some ideas from [**dBDR94**] (see also [**HŠWW10a**] for general properties). We consider the *bracket product* which introduces an "inner-product like" structure into an SIS. For $\varphi, \psi \in L^2(\mathbb{R})$, we make the following definition:

$$[\varphi, \psi](\xi) := \sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi + k)\overline{\widehat{\psi}(\xi + k)} \text{ for } \xi \in \mathbb{R}. \tag{1.7}$$

It is well known that $[\varphi, \psi]$ is a 1-periodic function, belongs to $L^1(\mathbb{T})$ and exhibits many properties analogous to inner products, although the bracket product is now function-valued in the sense that $[\cdot, \cdot] : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^1(\mathbb{T})$.

We now take a moment to suggest a replacement "scalar multiplication" for this bracket product. For $\varphi \in L^2(\mathbb{R})$ and $m \in L^\infty(\mathbb{T})$, we define $m \bullet \varphi$ via

$$m \bullet \varphi = (m\widehat{\varphi})^\vee.$$

The above formula actually makes sense for m in a larger class of 1-periodic functions; it is not hard to see that if m is 1-periodic, the function $m \bullet \varphi$ will be well defined if and only if $m \in L^2(\mathbb{T}, [\varphi, \varphi])$, which is the collection of functions on $\mathbb{T}$ which are square-integrable with respect to the measure $[\varphi, \varphi](\xi)d\xi$ (see [**LWW15**] for more details). In this context, a shift-invariant space can be viewed as a module over $L^\infty(\mathbb{T})$ (or some potentially larger collection of functions), with the scalar multiplication given by this $\bullet$ operation. Many computations point to the utility of this point of view; for example, consider that for $m \in L^\infty(\mathbb{T})$

$$[m \bullet \varphi, \psi] = m[\varphi, \psi] \tag{1.8}$$

and

$$[\varphi, m \bullet \psi] = \overline{m}[\varphi, \psi]. \tag{1.9}$$

We emphasize that these two identities make sense for m in spaces larger than $L^\infty(\mathbb{T})$ — the first identity holds precisely when $m \in L^2(\mathbb{T}, [\varphi, \varphi])$, and the second identity holds when $m \in L^2(\mathbb{T}, [\psi, \psi])$.

Since $[\varphi, \psi] \in L^1(\mathbb{R})$, it has well-defined Fourier coefficients. In fact, for each $k \in \mathbb{Z}$,

$$[\varphi, \psi]^\wedge(k) = \langle \varphi, T_{-k}\psi\rangle_{L^2(\mathbb{R})}. \tag{1.10}$$

We will often use $[\psi, \psi]$ and so we introduce the following special notation for it: $p_\psi := [\psi, \psi]$ whenever $\psi \in L^2(\mathbb{R})$ and we call it *the periodization function for* ψ (see [**ŠSW08**] for several basic properties).

Remark 1.11. There is a natural question to ask here regarding the relationship of $\langle\psi\rangle$ and p_ψ. On the one hand, we will see many examples of φ and ψ such that $\langle\varphi\rangle = \langle\psi\rangle$ and $p_\varphi \neq p_\psi$ as well as examples of φ and ψ so that $p_\varphi = p_\psi$ but $\langle\varphi\rangle \neq \langle\psi\rangle$.

On the other hand, however, it is well known (e.g. [**WW01**], for example) that there exists an isometric isomorphism $\mathcal{I} = \mathcal{I}_\psi : L^2(\mathbb{T}, p_\psi) \to \langle\psi\rangle$ given by

$$\mathcal{I}(m) = m \bullet \psi; \tag{1.12}$$

$L^2(\mathbb{T}, p_\psi)$ is the class of square integrable functions with respect to the measure $p_\psi(\xi)d\xi$. Furthermore, for every $k \in \mathbb{Z}$,

$$\mathcal{I}_\psi(e_{-k}) = T_k\psi. \tag{1.13}$$

Hence the analysis of $\mathcal{B}_\psi$ within $\langle\psi\rangle$ is equivalent to the analysis of the exponentials $\{e_k : k \in \mathbb{Z}\}$ within weighted space $L^2(\mathbb{T}, p_\psi)$. For more details, generalizations, and some extensions, one may consult [**HŠWW10b**], [**HŠWW10a**], [**MŠWW13**]. ■

Observe that, despite the previous remark, we do have some precise connections between $\dim_{\langle\psi\rangle}$ and p_ψ in the sense that

$$Z_{\langle\psi\rangle} = \{\xi \in \mathbb{R} : p_\psi(\xi) = 0\}. \tag{1.14}$$

Beyond that, $\dim_{\langle\psi\rangle}$ and p_ψ may differ substantially; for example, the first function is integer-valued while the second need not be.

We now turn our attention to set inclusion properties of principal SIS. Observe that

$$(\{\langle\psi\rangle : \psi \in L^2(\mathbb{R})\}, \subseteq)$$

forms a partially ordered set. There is a single minimal element, $\langle 0\rangle = \{0\}$, but there are many maximal elements. Recall $\langle\psi\rangle$ is *a maximal principal SIS* if the following is valid:

$$\text{if } \varphi \in L^2(\mathbb{R}) \text{ and } \langle\psi\rangle \subseteq \langle\varphi\rangle, \text{ then } \langle\psi\rangle = \langle\varphi\rangle. \tag{1.15}$$

It follows directly from [**HŠWW10b**] that it is easy to characterize the maximal principal SIS:

PROPOSITION 1.16. *The following are equivalent:*

(1) $\langle\psi\rangle$ *is a maximal principal SIS;*
(2) $\dim_{\langle\psi\rangle} \equiv 1$*;*
(3) $p_\psi(\xi) > 0$ *for almost every* $\xi \in \mathbb{R}$*.*

The following example shows that the conclusion of Zorn's Lemma is trivially fulfilled in this partially ordered set.

EXAMPLE 1.17. Given $\psi \in L^2(\mathbb{R})$, define $\varphi \in L^2(\mathbb{R})$ via

$$\widehat{\varphi}(\xi) := \begin{cases} 1 & \text{if } \xi \in [0,1) \cap Z_{\langle\psi\rangle} \\ \widehat{\psi}(\xi) & \textit{otherwise.} \end{cases}$$

It follows that $\langle\psi\rangle \subseteq \langle\varphi\rangle$, that $\langle\varphi\rangle$ is maximal and that $\langle\psi\rangle = \langle\varphi\rangle$ if and only if $\langle\psi\rangle$ is a maximal principal SIS. Observe that, in any case, if we take $m(\xi) := \chi_{U_{\langle\psi\rangle}}(\xi)$, then we have $m \bullet \varphi = \psi$. ♢

Regarding set inclusions, we need to first characterize when it is that $\langle\varphi\rangle \subseteq \langle\psi\rangle$ for $\varphi, \psi \in L^2(\mathbb{R})$. In essence, the answer is given already in [**dBDR94**], and we provide a brief sketch with some additional details.

Obviously, given $\varphi, \psi \in L^2(\mathbb{R})$, one has that $\langle\varphi\rangle \subseteq \langle\psi\rangle$ if and only if $\varphi \in \langle\psi\rangle$, i.e. if and only if there exists an $m \in L^2(\mathbb{T}, p_\psi)$ such that $\varphi = m \bullet \psi$. Furthermore, it is not difficult to find m in terms of φ and ψ. Using (1.8) we have

$$[\varphi, \psi] = [m \bullet \psi, \psi] = mp_\psi.$$

This directly proves the following lemma.

LEMMA 1.18. *If $\varphi, \psi \in L^2(\mathbb{R})$ and $\langle\varphi\rangle \subseteq \langle\psi\rangle$, then $\varphi = m \bullet \psi$, where m is given by*

$$m := \frac{[\varphi, \psi]}{p_\psi} \chi_{U_{\langle\psi\rangle}}.$$

We would like to emphasize a small (but somewhat subtle) issue regarding how m depends on our knowledge of ψ. Suppose that $\psi \in L^2(\mathbb{R})$ is given. If we want to construct functions φ within $\langle\psi\rangle$, then, as described above, we have to make sure that we consider 1-periodic functions m which belong to $L^2(\mathbb{T}, p_\psi)$ and then apply the formula $\varphi := m \bullet \psi$. If, however, the function $\varphi \in L^2(\mathbb{R})$ is also given, then we observe that we can always calculate the right hand side in Lemma 1.18. Hence we can produce a 1-periodic, measurable function m such that $m = \frac{[\varphi,\psi]}{p_\psi} \chi_{U_{\langle\psi\rangle}}$. First of all, it is not a priori clear whether this m actually belongs to $L^2(\mathbb{T}, p_\psi)$ or not. In order to understand this issue we apply the Cauchy-Schwarz inequality for the bracket product:

$$\left| \frac{[\varphi, \psi]}{p_\psi} \chi_{U_{\langle\psi\rangle}} \right|^2 p_\psi = |[\varphi, \psi]|^2 \frac{1}{p_\psi} \chi_{U_{\langle\psi\rangle}} \leq p_\varphi.$$

Since $p_\varphi \in L^1(\mathbb{T})$, we deduce that, for every $\varphi, \psi \in L^2(\mathbb{R})$,

$$m := \frac{[\varphi, \psi]}{p_\psi} \chi_{U_{\langle\psi\rangle}} \in L^2(\mathbb{T}, p_\psi). \tag{1.19}$$

In particular, $m \bullet \psi \in \langle\psi\rangle$. Observe, though, that $m \bullet \psi$ may not equal φ. Thus in order to understand whether φ belongs to $\langle\psi\rangle$ we first ask whether there is a 1-periodic, measurable function $\widetilde{m}$ such that $\widehat{\varphi} = \widetilde{m}\widehat{\psi}$, and, if the answer to this question is positive, then we would like to know if this is enough to deduce that φ belongs to $\langle\psi\rangle$. Actually, it is since

$$\int_{\mathbb{T}} |\widetilde{m}|^2 p_\psi = \int_{\mathbb{R}} |\widetilde{m}|^2 |\widehat{\psi}|^2 = \int |\widehat{\varphi}|^2 = \|\varphi\|_{L^2(\mathbb{R})} < \infty.$$

In particular, it has to be that $\widetilde{m}\chi_{U_{\langle\psi\rangle}} = m\chi_{U_{\langle\psi\rangle}}$, while the values of $\widetilde{m}$ on $Z_{\langle\psi\rangle}$ are irrelevant.

To this short analysis, we add a discussion of what one can conclude in the case of equality in the Cauchy–Schwarz inequality.

PROPOSITION 1.20. *If $\varphi, \psi \in L^2(\mathbb{R})$, then the following are equivalent:*

(a) $\langle\varphi\rangle \subseteq \langle\psi\rangle$;
(b) $\varphi \in \langle\psi\rangle$;
(c) There exists a 1-periodic, measurable function $m : \mathbb{R} \to \mathbb{C}$ such that $\widehat{\varphi} = m\widehat{\psi}$;
(d) $U_{\langle\varphi\rangle} \subseteq U_{\langle\psi\rangle}$ *and* $|[\varphi, \psi]|^2 = p_\varphi p_\psi$.

The case of equality, when $\langle\varphi\rangle = \langle\psi\rangle$, is then an easy consequence of this proposition:

COROLLARY 1.21. *If $\varphi, \psi \in L^2(\mathbb{R})$, then the following are equivalent:*

(a) $\langle\varphi\rangle = \langle\psi\rangle$;
(b) $U_{\langle\varphi\rangle} = U_{\langle\psi\rangle}$ *and there exists a 1-periodic, measurable function $m : \mathbb{R} \to \mathbb{C}$ such that $\widehat{\varphi} = m\widehat{\psi}$;*
(c) $U_{\langle\varphi\rangle} = U_{\langle\psi\rangle}$ *and* $|[\varphi, \psi]|^2 = p_\varphi p_\psi$.

Let us briefly comment on the function m which appears in 1.21b. Using Lemma 1.18 we obtain that, for $\xi \in U_{\langle\psi\rangle} = U_{\langle\varphi\rangle}$, we have $m(\xi) \neq 0$ and

$$m(\xi) = \frac{[\varphi,\psi](\xi)}{p_\psi(\xi)} \text{ and } \frac{1}{m(\xi)} = \frac{[\psi,\varphi](\xi)}{p_\varphi(\xi)}.$$

Since $\varphi = m \bullet \psi$ and $\psi = \frac{1}{m} \bullet \psi$, for $\xi \in U_{\langle\psi\rangle} = U_{\langle\varphi\rangle}$, we obtain

$$\frac{\widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}}{[\varphi,\psi](\xi)} = \frac{|\widehat{\psi}(\xi)|^2}{p_\psi(\xi)} \text{ and } \frac{\widehat{\psi}(\xi)\overline{\widehat{\varphi}(\xi)}}{[\psi,\varphi](\xi)} = \frac{|\widehat{\varphi}(\xi)|^2}{p_\varphi(\xi)}.$$

Observe that on the right sides of these two equalities we have real values and that, using $[\psi,\varphi] = \overline{[\varphi,\psi]}$, we obtain

$$\frac{|\widehat{\psi}(\xi)|^2}{p_\psi(\xi)} = \frac{|\widehat{\varphi}(\xi)|^2}{p_\varphi(\xi)}.$$

In other words, we have proved the following result.

LEMMA 1.22. *If $\varphi,\psi \in L^2(\mathbb{R})$ satisfy $\langle\varphi\rangle = \langle\psi\rangle$, then for every $\xi \in U_{\langle\varphi\rangle} = U_{\langle\psi\rangle}$,*

$$\frac{|\widehat{\psi}(\xi)|^2}{p_\psi(\xi)} = \frac{\widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}}{[\varphi,\psi](\xi)} = \frac{\widehat{\psi}(\xi)\overline{\widehat{\varphi}(\xi)}}{[\psi,\varphi](\xi)} = \frac{|\widehat{\varphi}(\xi)|^2}{p_\varphi(\xi)}.$$

In particular, $\operatorname{ssupp}\widehat{\varphi} = \operatorname{ssupp}\widehat{\psi}$, where by ssupp we denote the *set support*, defined by

$$\operatorname{ssupp} f := \{\xi \in \mathbb{R} : f(\xi) \neq 0\}.$$

It follows from Lemma 1.22 that both $\operatorname{ssupp}\widehat{\psi}$ and the function $\xi \mapsto \frac{|\widehat{\psi}(\xi)|^2}{p_\psi(\xi)}\chi_{U_{\langle\psi\rangle}}(\xi)$ really only depend on the principal shift-invariant space $\langle\psi\rangle$ rather than the function ψ. This relationship has been employed by several authors, including, for example, [**dBDR94**], [**BMM99**], [**Rze00**], and [**BR05**]. Using the notation from [**Rze00**] and [**BR05**], for $\psi \in L^2(\mathbb{R})$, we denote *the spectral function of* $\langle\psi\rangle$ by $\sigma_{\langle\psi\rangle}$ and define it by

$$\sigma_{\langle\psi\rangle}(\xi) := \frac{|\widehat{\psi}(\xi)|^2}{p_\psi(\xi)}\chi_{U_{\langle\psi\rangle}}(\xi) \text{ for } \xi \in \mathbb{R}. \tag{1.23}$$

Observe that $\sigma_{\langle\psi\rangle} : \mathbb{R} \to [0,\infty)$, that $\sigma_{\langle\psi\rangle} \in L^1(\mathbb{R})$, and

$$\sum_{k\in\mathbb{Z}} \sigma_{\langle\psi\rangle}(\xi + k) = \chi_{U_{\langle\psi\rangle}}(\xi) = \dim_{\langle\psi\rangle}(\xi). \tag{1.24}$$

One can also check some of the basic properties of the spectral function (see [**BR05**] for details). If $\varphi,\psi \in L^2(\mathbb{R})$, then

$$\langle\varphi\rangle \subseteq \langle\psi\rangle \text{ implies that } \sigma_{\langle\varphi\rangle} \leq \sigma_{\langle\psi\rangle}; \tag{1.25}$$

furthermore, if, for some $\xi \in \mathbb{R}$, one has $\sigma_{\langle\varphi\rangle}(\xi) < \sigma_{\langle\psi\rangle}(\xi)$, then $\sigma_{\langle\varphi\rangle}(\xi) = 0$. Using the spectral function we can also test the inclusion property for principal shift-invariant spaces. More precisely, if $\varphi,\psi \in L^2(\mathbb{R})$, then $\langle\varphi\rangle \subseteq \langle\psi\rangle$ if and only if

$$\{\xi \in \mathbb{R} : \widehat{\psi}(\xi) = 0\} \cup \{\xi \in \mathbb{R} : [\varphi,\psi](\xi) = 0\} \subseteq \{\xi \in \mathbb{R} : \widehat{\varphi}(\xi) = 0\}, \text{ and}$$

$$\text{for every } \xi \in \mathbb{R} \text{ such that } [\varphi,\psi](\xi) \neq 0, \text{ one has } \frac{\widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}}{[\varphi,\psi](\xi)} = \sigma_{\langle\psi\rangle}(\xi). \tag{1.26}$$

Similarly, we can test the equality case as well. If $\varphi, \psi \in L^2(\mathbb{R})$, then $\langle\varphi\rangle = \langle\psi\rangle$ if and only if

$$U_{\langle\varphi\rangle} = U_{\langle\psi\rangle} = \{\xi \in \mathbb{R} : [\varphi, \psi](\xi) \neq 0\}, \text{ and for every } \xi \in U_{\langle\psi\rangle}$$

$$\text{one has } \sigma_{\langle\varphi\rangle}(\xi) = \frac{\widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}}{[\varphi, \psi](\xi)} = \sigma_{\langle\psi\rangle}(\xi). \tag{1.27}$$

If we already have an inclusion, then the equality case is very easy to check. More precisely, if $\varphi, \psi \in L^2(\mathbb{R})$ are such that $\langle\varphi\rangle \subseteq \langle\psi\rangle$, then

$$\langle\varphi\rangle = \langle\psi\rangle \text{ if and only if } \sigma_{\langle\varphi\rangle} = \sigma_{\langle\psi\rangle} \text{ if and only if } U_{\langle\varphi\rangle} = U_{\langle\psi\rangle}. \tag{1.28}$$

Observe also that if $\varphi, \psi \in L^2(\mathbb{R})$ and $\langle\varphi\rangle = \langle\psi\rangle$, then

$$\text{ssupp}\widehat{\varphi} = \text{ssupp}\widehat{\psi} = \text{ssupp}\sigma_{\langle\psi\rangle}; \tag{1.29}$$

we denote any of the above equal sets by $\text{ssupp}\langle\psi\rangle$. Obviously,

$$\xi \in U_{\langle\psi\rangle} \text{ if and only if there exists a } k \in \mathbb{Z} \text{ such that } \xi + k \in \text{ssupp}\langle\psi\rangle. \tag{1.30}$$

Although these properties are often very useful, one must be careful in applying them properly. Consider the following example.

Example 1.31. Take $\psi \in L^2(\mathbb{R})$ and define $\varphi \in L^2(\mathbb{R})$ so that $\widehat{\varphi} = |\widehat{\psi}|$. It is easy to see that, for many choices of ψ, we will have $\langle\varphi\rangle \neq \langle\psi\rangle$. On the other hand it is obvious that for any choice of ψ, this choice of φ satisfies $p_\varphi = p_\psi$, $U_{\langle\varphi\rangle} = U_{\langle\psi\rangle}$, and $\sigma_{\langle\varphi\rangle} = \sigma_{\langle\psi\rangle}$. Observe also that

$$\varphi \in \langle\psi\rangle \text{ if and only if } \langle\varphi\rangle = \langle\psi\rangle \text{ if and only if } \psi \in \langle\varphi\rangle.$$

$\diamondsuit$

2. Relationship Between Two Principal Shift-Invariant Spaces

In this section we analyze the relationship between a pair of principal shift-invariant spaces, $\langle\varphi\rangle$ and $\langle\psi\rangle$, generated by $\varphi, \psi \in L^2(\mathbb{R})$. We begin by studying the intersection of two such spaces. Recall that if V and W are shift-invariant spaces, then $V \cap W$ is a shift-invariant space as well. In particular, $\langle\varphi\rangle \cap \langle\psi\rangle$ is a shift-invariant space, and

$$\dim_{\langle\varphi\rangle\cap\langle\psi\rangle} \leq \min\{\dim_{\langle\varphi\rangle}, \dim_{\langle\psi\rangle}\} \leq 1.$$

It follows that $\langle\varphi\rangle \cap \langle\psi\rangle$ is a principal shift-invariant space. It makes sense to consider $U_{\langle\varphi\rangle\cap\langle\psi\rangle}$. Using this approach we can describe $\langle\varphi\rangle \cap \langle\psi\rangle$ completely, as the following theorem shows.

Theorem 2.1. *If $\varphi, \psi \in L^2(\mathbb{R})$, then*
(a) $U_{\langle\varphi\rangle\cap\langle\psi\rangle} = \{\xi \in \mathbb{R} : 0 \neq |[\varphi, \psi](\xi)|^2 = p_\varphi(\xi) \cdot p_\psi(\xi)\}$.
(b) $\langle\varphi\rangle \cap \langle\psi\rangle = \langle\chi_{U_{\langle\varphi\rangle\cap\langle\psi\rangle}} \bullet \varphi\rangle = \langle\chi_{U_{\langle\varphi\rangle\cap\langle\psi\rangle}} \bullet \psi\rangle$.

Proof. Let us denote by $H = H(\varphi, \psi)$ the set

$$H(\varphi, \psi) = \{\xi \in \mathbb{R} : 0 \neq |[\varphi, \psi](\xi)|^2 = p_\varphi(\xi) \cdot p_\psi(\xi)\}.$$

Obviously, this is a measurable and 1-periodic set. Keeping in mind the fact that $\langle\varphi\rangle \cap \langle\psi\rangle$ is a principal shift-invariant space, observe that, in order to prove the entire theorem, it is enough to establish that

$$\langle\varphi\rangle \cap \langle\psi\rangle = \langle\chi_H \bullet \varphi\rangle$$

since one can freely interchange the roles of φ and ψ and $U_{\langle\varphi\rangle\cap\langle\psi\rangle}$ is precisely the set where the periodization function of $\langle\varphi\rangle\cap\langle\psi\rangle$ is positive (here by periodization function of $\langle\varphi\rangle\cap\langle\psi\rangle$, we mean p_α, where $\langle\varphi\rangle\cap\langle\psi\rangle = \langle\alpha\rangle$, keeping in mind the comment immediately preceding the statement of the theorem).

Using Proposition 1.20d, we obtain $\langle\chi_H \bullet \varphi\rangle \subseteq \langle\varphi\rangle\cap\langle\psi\rangle$. For the reverse inclusion, consider $\eta \in \langle\varphi\rangle\cap\langle\psi\rangle$, and, without loss of generality, we may assume $\eta \neq 0$. By Lemma 1.18, there exist $\nu \in L^2(\mathbb{T}, p_\varphi)$ and $\mu \in L^2(\mathbb{T}, p_\psi)$ such that $\eta = \nu \bullet \varphi = \mu \bullet \psi$. It follows that

$$p_\eta = |\nu|^2 p_\varphi = |\mu|^2 p_\psi.$$

Hence, if $p_\eta(\xi) > 0$, then $p_\varphi(\xi) > 0$ and $p_\psi(\xi) > 0$, and, by Lemma 1.18, we also obtain

$$\begin{aligned} 0 \neq \mu(\xi) &= \frac{[\eta,\psi](\xi)}{p_\psi(\xi)} = \frac{\nu(\xi)[\varphi,\psi](\xi)}{p_\psi(\xi)} = \frac{\frac{[\eta,\varphi](\xi)}{p_\varphi(\xi)}[\varphi,\psi](\xi)}{p_\psi(\xi)} \\ &= \mu(\xi)\frac{[\psi,\varphi](\xi)[\varphi,\psi](\xi)}{p_\varphi(\xi)p_\psi(\xi)} = \mu(\xi)\frac{|[\varphi,\psi](\xi)|^2}{p_\varphi(\xi)p_\psi(\xi)}. \end{aligned}$$

Thus if $p_\eta(\xi) > 0$, then $\xi \in H$. It follows that if $p_\eta(\xi) > 0$, then

$$\begin{aligned} |[\eta, \chi_H \bullet \varphi](\xi)|^2 &= |\mu(\xi)|^2 |[\psi, \chi_H \bullet \varphi](\xi)|^2 \\ &= |\mu(\xi)|^2 \chi_H^2 |[\psi,\varphi](\xi)|^2 \\ &= |\mu(\xi)|^2 p_\psi(\xi) p_\varphi(\xi) \text{ (by definition of } H) \\ &= p_\eta(\xi) p_\varphi(\xi) \\ &= p_\eta(\xi) p_{\chi_H \bullet \varphi}(\xi) \text{ (since } \xi \in H). \end{aligned}$$

By Proposition 1.20d, we conclude that $\eta \in \langle\chi_H \bullet \varphi\rangle$.

□

COROLLARY 2.2. *If $\varphi, \psi \in L^2(\mathbb{R})$, then one has $\langle\varphi\rangle\cap\langle\psi\rangle = \{0\}$ if and only if $|U_{\langle\varphi\rangle\cap\langle\psi\rangle}| = 0$.*

REMARK 2.3. (i) Obviously, one always has $U_{\langle\varphi\rangle\cap\langle\psi\rangle} \subseteq U_{\langle\varphi\rangle} \cap U_{\langle\psi\rangle}$. The reverse inclusion is not necessarily true (see the example in (iii), below). We will discuss some special cases when the reverse inclusion is true later in this section.

(ii) The spectral function does not seem particularly helpful in characterizing the intersection, $\langle\varphi\rangle\cap\langle\psi\rangle$. It is easy to see that $\xi \in U_{\langle\varphi\rangle\cap\langle\psi\rangle}$ implies $\sigma_{\langle\varphi\rangle}(\xi) = \sigma_{\langle\psi\rangle}(\xi)$. However, the reverse implication is not necessarily true; see (iii), below.

(iii) Consider $\varphi, \psi \in L^2(\mathbb{R})$ such that

$$\widehat{\varphi} = \frac{1}{\sqrt{2}}\left(\chi_{[0,1/2)} + \chi_{[1,3/2)}\right)$$

and

$$\widehat{\psi} = \frac{1}{\sqrt{2}}\left(\chi_{[0,1/2)} - \chi_{[1,3/2)}\right).$$

It follows that $p_\varphi = p_\psi = \chi_{[0,1/2)}$, that $[\varphi,\psi] \equiv 0$, that $\sigma_{\langle\varphi\rangle} = \sigma_{\langle\psi\rangle} = \frac{1}{2}\left(\chi_{[0,1/2)} + \chi_{[1,3/2)}\right)$, and that $U_{\langle\varphi\rangle} = U_{\langle\psi\rangle} = \bigcup_{k\in\mathbb{Z}}([0,1/2)+k)$. Nonetheless, one also has $U_{\langle\varphi\rangle\cap\langle\psi\rangle} = \emptyset$.

(iv) Considering the other extreme, observe that we obtain easily the following: $U_{\langle\varphi\rangle\cap\langle\psi\rangle} = \mathbb{R}$ if and only if both $\langle\varphi\rangle$ and $\langle\psi\rangle$ are maximal and $\langle\varphi\rangle = \langle\psi\rangle$.

(v) Furthermore, it is also easy to see that $\langle\varphi\rangle \neq \langle\psi\rangle$ implies $|Z_{\langle\varphi\rangle\cap\langle\psi\rangle}| > 0$. In particular, unless both $\langle\varphi\rangle$ and $\langle\psi\rangle$ are maximal and equal, it is impossible for $\langle\varphi\rangle \cap \langle\psi\rangle$ to be a maximal principal shift-invariant space.

■

Consider now a very special case, where $\varphi, \psi, \eta \in L^2(\mathbb{R})$, and $\langle\varphi\rangle \subseteq \langle\eta\rangle$ and $\langle\psi\rangle \subseteq \langle\eta\rangle$. Then there exist $\nu, \mu \in L^2(\mathbb{T}, p_\eta)$ such that $\varphi = \nu \bullet \eta$ and $\psi = \mu \bullet \eta$. It follows that

$$|[\varphi,\psi]|^2 = |\nu\overline{\mu}[\eta,\eta]|^2 = |\nu|^2 p_\eta |\mu|^2 p_\eta = p_\varphi p_\psi.$$

Hence, in this special case, the expression above is non-zero if and only if both p_φ and p_ψ are positive. Furthermore, if $p_\varphi(\xi) > 0$ and $p_\psi(\xi) > 0$, then $\xi \in U_{\langle\varphi\rangle\cap\langle\psi\rangle}$. In other words, we have proven the following:

Proposition 2.4. *Suppose that $\varphi, \psi, \eta \in L^2(\mathbb{R})$ satisfy $\langle\varphi\rangle \subseteq \langle\eta\rangle$ and $\langle\psi\rangle \subseteq \langle\eta\rangle$. Then*

$$U_{\langle\varphi\rangle\cap\langle\psi\rangle} = U_{\langle\varphi\rangle} \cap U_{\langle\psi\rangle}.$$

Moving beyond the intersection relationship, one should study another important relationship between principal shift-invariant spaces: orthogonality. The condition for orthogonality in the setting of principal shift-invariant spaces has been known for a long time (see, for example, [**dBDR94**]): $\langle\varphi\rangle \perp \langle\psi\rangle$ if and only if $[\varphi,\psi] \equiv 0$ almost everywhere. Using this, we abuse notation somewhat and denote, for $\varphi, \psi \in L^2(\mathbb{R})$, the set

$$U_{\langle\varphi\rangle\perp\langle\psi\rangle} := \{\xi \in \mathbb{R} : [\varphi,\psi](\xi) = 0\}. \tag{2.5}$$

Obviously, $U_{\langle\varphi\rangle\perp\langle\psi\rangle}$ is measurable, 1-periodic, and satisfies

$$U_{\langle\varphi\rangle\perp\langle\psi\rangle} \cap U_{\langle\varphi\rangle\cap\langle\psi\rangle} = \emptyset; \tag{2.6}$$

moreover, $\langle\varphi\rangle$ and $\langle\psi\rangle$ are orthogonal to one another if and only if $U_{\langle\varphi\rangle\perp\langle\psi\rangle} = \mathbb{R}$. Using this criterion, it is straightforward to prove that

$$\langle\chi_{U_{\langle\varphi\rangle\perp\langle\psi\rangle}} \bullet \varphi\rangle \perp \langle\psi\rangle \text{ and } \langle\chi_{U_{\langle\varphi\rangle\perp\langle\psi\rangle}} \bullet \psi\rangle \perp \langle\varphi\rangle, \tag{2.7}$$

and

$$\langle\chi_{U_{\langle\varphi\rangle\perp\langle\psi\rangle}} \bullet \varphi\rangle \perp \langle\varphi\rangle \cap \langle\psi\rangle \perp \langle\chi_{U_{\langle\varphi\rangle\perp\langle\psi\rangle}} \bullet \psi\rangle. \tag{2.8}$$

Remark 2.9. Orthogonality is often used in the theory of shift-invariant spaces. It is crucial that the bracket product, $[\cdot,\cdot]$, has many properties analogous to the inner product — this enables one to apply the Gram–Schmidt process. We illustrate this briefly here. Given shift-invariant spaces $V, W \subseteq L^2(\mathbb{R})$, it is obvious that an ordinary sum $V + W$ (we will use $\oplus$ to denote the orthogonal sum) of these spaces is again a shift-invariant space. In particular, for $\varphi, \psi \in L^2(\mathbb{R})$ we have

$$\langle\varphi\rangle + \langle\psi\rangle = \langle\varphi,\psi\rangle,$$

where, for any $\mathcal{E} \subseteq L^2(\mathbb{R})$, we denote by $\langle\mathcal{E}\rangle$ the intersection of all shift-invariant spaces which contain $\mathcal{E}$. We define $\eta_1, \eta_2 \in L^2(\mathbb{R})$ by

$$\eta_1 := \varphi + \chi_{Z_{\langle\varphi\rangle}\cap U_{\langle\psi\rangle}} \bullet \psi$$

and

$$\eta_2 := \chi_{(U_{\langle\varphi\rangle}\cap U_{\langle\psi\rangle})\setminus U_{\langle\varphi\rangle\cap\langle\psi\rangle}} \bullet \left(\psi - \frac{[\psi,\varphi]}{p_\varphi} \bullet \varphi\right),$$

and obtain the following properties:
(1) $\eta_1 \perp \eta_2$;
(2) $\langle\varphi\rangle + \langle\psi\rangle = \langle\varphi, \psi\rangle = \langle\eta_1, \eta_2\rangle = \langle\eta_1\rangle \oplus \langle\eta_2\rangle$.
(3) $U_{\langle\eta_2\rangle} \subseteq U_{\langle\eta_1\rangle} = U_{\langle\varphi\rangle} \cup U_{\langle\psi\rangle}$.
The dimension function is particularly suitable for orthogonal sums of shift-invariant spaces. If $V, W \subseteq L^2(\mathbb{R})$ are orthogonal shift-invariant spaces, then (see, for example [**Bow00**])

$$\dim_{V\oplus W} \equiv \dim_V + \dim_W . \tag{2.10}$$

In particular,

$$\dim_{\langle\varphi,\psi\rangle} = \dim_{\langle\eta_1\rangle\oplus\langle\eta_2\rangle} = \dim_{\langle\eta_1\rangle} + \dim_{\langle\eta_2\rangle},$$

and it follows easily that

$$\dim_{\langle\varphi,\psi\rangle}(\xi) = \begin{cases} 0 & \text{if } \xi \in Z_{\langle\varphi\rangle} \cap Z_{\langle\psi\rangle}; \\ 1 & \text{if } \xi \in \left(Z_{\langle\varphi\rangle} \cap U_{\langle\psi\rangle}\right) \cup \left(U_{\langle\varphi\rangle} \cap Z_{\langle\psi\rangle}\right) \cup \left(U_{\langle\varphi\rangle\cap\langle\psi\rangle}\right); \\ 2 & \text{if } \xi \in \left(U_{\langle\varphi\rangle} \cap U_{\langle\psi\rangle}\right) \setminus U_{\langle\varphi\rangle\cap\langle\psi\rangle} \end{cases} \tag{2.11}$$

Orthogonal sums may also be used to extend the spectral function from principal shift-invariant spaces to general shift-invariant spaces. Recall (see, for example [**Bow00**]) that, for every shift-invariant space $V \subseteq L^2(\mathbb{R})$, there exists a countable family $\mathcal{F} \subset L^2(\mathbb{R})$ such that

$$V = \bigoplus_{f\in\mathcal{F}} \langle f\rangle. \tag{2.12}$$

We may then define the spectral function, σ_V, by

$$\sigma_V = \sum_{f\in\mathcal{F}} \sigma_{\langle f\rangle}. \tag{2.13}$$

For example, by selecting θ_1 and θ_2 via

$$\theta_i := \left(\frac{1}{\sqrt{p_{\eta_i}}}\chi_{U_{\langle\eta_i\rangle}}\right) \bullet \eta_i \text{ for } i = 1, 2,$$

we obtain

$$\sigma_{\langle\varphi,\psi\rangle} = \sigma_{\langle\eta_1\rangle\oplus\langle\eta_2\rangle} = \sigma_{\langle\theta_1\rangle\oplus\langle\theta_2\rangle} = |\widehat{\theta_1}|^2 + |\widehat{\theta_2}|^2. \tag{2.14}$$

This same structure provides us an alternative definition for the dimension function. Suppose that, as above, $V = \bigoplus_{f\in\mathcal{F}}\langle f\rangle$. We assume that $|\mathcal{F}| = \infty$; the case when $\mathcal{F}$ has finite cardinality is handled the same way with minor changes in notation. Let $(f_i)_{i=1}^{\infty}$ denote an enumeration of $\mathcal{F}$. We define the family $\mathcal{G} = \{g_i : i = 1, 2, ...\}$ via the formula

$$g_i := \left(\frac{1}{\sqrt{p_{f_i}}}\chi_{\langle f_i\rangle}\right) \bullet f_i.$$

Then

$$\dim_V = \sum_{i=1}^{\infty} p_{g_i}.$$

In the specific case when $V = \langle\varphi, \psi\rangle$, one obtains

$$\dim_{\langle\varphi,\psi\rangle} = p_{\theta_1} + p_{\theta_2}. \tag{2.15}$$

We can think of $\{\theta_1, \theta_2\}$ as being something like a "shift-invariant space analog of an orthonormal basis" for $\langle\varphi, \psi\rangle$. ■

With the above discussion in mind, it is straightforward to prove the following list of results:

Corollary 2.16. *If $\varphi, \psi \in L^2(\mathbb{R})$ are such that $\langle\varphi\rangle$ and $\langle\psi\rangle$ are both maximal principal shift-invariant spaces, then*

$$\dim_{\langle\varphi,\psi\rangle}(\xi) = \begin{cases} 1 & \text{if } \xi \in U_{\langle\varphi\rangle\cap\langle\psi\rangle}; \\ 2 & \text{otherwise} \end{cases}.$$

Corollary 2.17. *If $\varphi, \psi \in L^2(\mathbb{R})$, then $\langle\varphi,\psi\rangle$ is a principal shift-invariant space if and only if*

$$U_{\langle\varphi\rangle\cap\langle\psi\rangle} = U_{\langle\varphi\rangle} \cap U_{\langle\psi\rangle}$$

Compare the above result to Proposition 2.4

Corollary 2.18. *If $\varphi, \psi \in L^2(\mathbb{R})$ are such that $\langle\varphi,\psi\rangle$ is a principal shift-invariant space, then $\langle\varphi,\psi\rangle = \langle\eta\rangle$, where*

$$\eta = \varphi + \chi_{Z_{\langle\varphi\rangle}\cap U_{\langle\psi\rangle}} \bullet \psi.$$

Furthermore, in this case the following are equivalent:
(a) $\langle\eta\rangle = \langle\varphi\rangle \oplus \langle\psi\rangle$;
(b) $\langle\varphi\rangle \perp \langle\psi\rangle$;
(c) $\langle\varphi\rangle \cap \langle\psi\rangle = \{0\}$;
(d) $U_{\langle\varphi\rangle} \cap U_{\langle\psi\rangle} = \emptyset$.

As the last step of our analysis, consider the 1-periodic, measurable set given by

$$\begin{aligned} \mathbb{R} \setminus \left(U_{\langle\varphi\rangle\cap\langle\psi\rangle} \cup U_{\langle\varphi\rangle\perp\langle\psi\rangle}\right) &= \left(U_{\langle\varphi\rangle} \cap U_{\langle\psi\rangle}\right) \setminus \left(U_{\langle\varphi\rangle\cap\langle\psi\rangle} \cup U_{\langle\varphi\rangle\perp\langle\psi\rangle}\right) \\ &= \{\xi \in \mathbb{R} : 0 < |[\varphi,\psi](\xi)|^2 < p_\varphi(\xi) \cdot p_\psi(\xi)\}. \end{aligned} \tag{2.19}$$

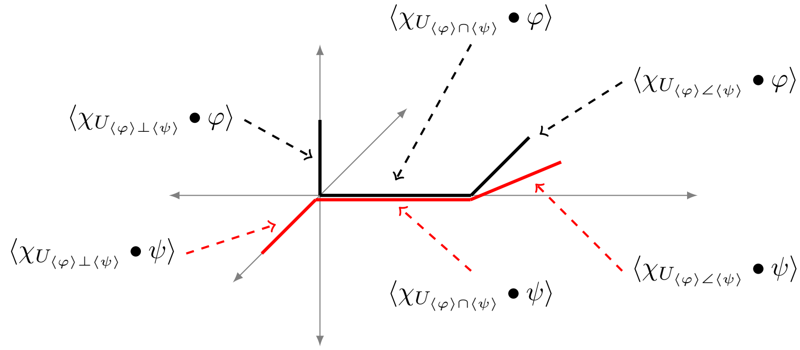
We shall say that the set in Equation (2.19) contains those points ξ at which the spaces $\langle\varphi\rangle$ and $\langle\psi\rangle$ are "at an angle" and denote this set with the following suggestive notation:

$$U_{\langle\varphi\rangle\angle\langle\psi\rangle} := \mathbb{R} \setminus \left(U_{\langle\varphi\rangle\cap\langle\psi\rangle} \cup U_{\langle\varphi\rangle\perp\langle\psi\rangle}\right). \tag{2.20}$$

Clearly, we have

$$\langle\varphi\rangle = \langle\chi_{U_{\langle\varphi\rangle\cap\langle\psi\rangle}} \bullet \varphi\rangle \oplus \langle\chi_{U_{\langle\varphi\rangle\perp\langle\psi\rangle}} \bullet \varphi\rangle \oplus \langle\chi_{U_{\langle\varphi\rangle\angle\langle\psi\rangle}} \bullet \varphi\rangle, \tag{2.21}$$

and a similar formula holds for ψ. For generic functions $\varphi, \psi \in L^2(\mathbb{R})$, there will be sets of positive measure at which the spaces $\langle\varphi\rangle$ and $\langle\psi\rangle$ are "parallel" ($\xi \in U_{\langle\varphi\rangle\cap\langle\psi\rangle}$), "perpendicular" ($\xi \in U_{\langle\varphi\rangle\perp\langle\psi\rangle}$), and "at an angle" ($\xi \in U_{\langle\varphi\rangle\angle\langle\psi\rangle}$). It is sometimes helpful to think of principal shift-invariant spaces as analogs of vectors in a vector space, but one must be careful: unlike vectors in a vector space, principal shift-invariant spaces can, in some sense, be parallel, perpendicular, and at an angle simultaneously. This fact underlies many of the difficulties related to the analysis of shift-invariant spaces. We provide the following diagram as a visualization tool to aid in understanding these relationships — technically, it would be more accurate to draw this scheme in 4-dimensional space, but we must settle for a 3-dimensional approximation.



3. Three Types of Principal Shift-Invariant Spaces

It is sometimes useful to consider translations other than translations by integers. One natural question to ask is, given a $\psi \in L^2(\mathbb{R})$ and an $\alpha \in \mathbb{R}$, when will $T_\alpha \psi \in \langle\psi\rangle$? In other words, when will $\langle\psi\rangle$ be invariant under T_α? Without loss of generality, since $\langle\psi\rangle$ is invariant under integer translations already and $T_k T_\alpha = T_\alpha T_k$, we may restrict our attention to those $\alpha \in (0,1)$. The aforementioned question was first treated in [**ACH$^+$10**]. Extensions and generalizations to groups and lattices can be found in [**ACP11**], [**ACP10**], and [**ŠW11**] (see also [**Ive15**] and [**BHP15**] and the references therein). We shall follow the approach given in [**ŠW11**], including the notation. For example, if $\mathcal{L} \subseteq \mathbb{R}$ is a countable lattice, then we make the following definition:

$$\langle\psi\rangle_{\mathcal{L}} := \overline{\operatorname{span}\{T_\ell \psi : \ell \in \mathcal{L}\}}.$$

This space obviously satisfies $T_\ell \langle\psi\rangle_{\mathcal{L}} = \langle\psi\rangle_{\mathcal{L}}$, and so we refer to $\langle\psi\rangle_{\mathcal{L}}$ as the principal $\mathcal{L}$-invariant subspace generated by ψ. In particular, $\langle\psi\rangle = \langle\psi\rangle_{\mathbb{Z}}$, and principal shift-invariant spaces are principal $\mathbb{Z}$-invariant spaces in this context.

Suppose that $\psi \in L^2(\mathbb{R})$ and that $\alpha = m/n$, with m, n positive integers such that $m < n$. Since, in this case, $T_\alpha = \left(T_{1/n}\right)^m$, it is really enough to consider the case where $\alpha = 1/n$ with $n > 1$. First, observe that the question under consideration depends on the principal shift-invariant space rather than the generator; in particular, if $\langle\psi\rangle = \langle\varphi\rangle$, then $T_\alpha\varphi \in \langle\varphi\rangle$ if and only if $T_\alpha\psi \in \langle\psi\rangle$. Second, notice that our present analysis requires understanding the relationship between the lattice $\mathbb{Z}$ and the "super-lattice" $\frac{1}{n}\mathbb{Z}$. Obviously, it is always true that

$$\langle\psi\rangle = \langle\psi\rangle_{\mathbb{Z}} \subseteq \langle\psi\rangle_{\frac{1}{n}\mathbb{Z}}. \tag{3.1}$$

Furthermore, $T_{1/n}\psi \in \langle\psi\rangle$ if and only if $\langle\psi\rangle = \langle\psi\rangle_{\frac{1}{n}\mathbb{Z}}$.

Following [**ŠW11**], for a function $\psi \in L^2(\mathbb{R})$, we define the following set:

$$\mathcal{T}_{\langle\psi\rangle} := \left\{ n \in \{2,3,4,...\} : T_{1/n}\psi \in \langle\psi\rangle \right\}. \tag{3.2}$$

Based on the properties of this set, we distinguish *three types* of principal shift-invariant spaces:

- Type 1: $\mathcal{T}_{\langle\psi\rangle} = \emptyset$;
- Type 2: $\mathcal{T}_{\langle\psi\rangle} \neq \emptyset$, with $\mathcal{T}_{\langle\psi\rangle}$ a finite set;
- Type 3: $\mathcal{T}_{\langle\psi\rangle}$ is an infinite set.

There is a complete characterization of these three types given in [**ŠW11**] as well as techniques for constructing examples of each type. Let us emphasize only a few details. If ssupp $\langle\psi\rangle$ contains the interval $[0,2)$ (recall that ssupp $\langle\psi\rangle$ is defined to be ssupp $\widehat{\psi}$), then $\langle\psi\rangle$ is a Type 1 space. In particular,

$$\text{ssupp } \langle\psi\rangle = \mathbb{R} \text{ implies } \langle\psi\rangle \text{ is of Type 1,} \tag{3.3}$$

so that, for example, $\langle\chi_{[0,1]}\rangle$ is of Type 1 since $\widehat{\chi_{[0,1]}} = \text{sinc}(x)$. It turns out that $\langle\psi\rangle$ is of Type 2 if and only if there exists an integer $n > 1$ such that

$$\mathcal{T}_{\langle\psi\rangle} = \{m \in \{2,3,4,...\} : m|n\}, \tag{3.4}$$

where $m|n$ is the usual notation for "n is divisible by m". Let us also point out that $\langle\psi\rangle$ is of Type 3 if and only if

$$T_\alpha\psi \in \langle\psi\rangle \text{ for every } \alpha \in \mathbb{R}. \tag{3.5}$$

it is particularly simple to construct ψ so that $\langle\psi\rangle$ is of Type 3. For example, take an arbitrary $f \in L^2(\mathbb{R})$ and suppose that $\{A_k : k \in \mathbb{Z}\}$ forms a partition of $[0,1)$. Define the set H by

$$H := \bigcup_{k\in\mathbb{Z}} (A_k + (k-1)).$$

Then if we let $\psi := \chi_H \bullet f$, one has that $\langle\psi\rangle$ is of Type 3.

Remark 3.6. The characterization of these three types of behavior depends on the properties of ssupp $\langle\psi\rangle$. However, it is not possible to distinguish the type of a space $\langle\psi\rangle$ based solely on the properties of p_ψ or $U_{\langle\psi\rangle}$. For example, it is not hard to construct square-integrable functions ψ_1, ψ_2, and ψ_3 so that $U_{\langle\psi_i\rangle} = \mathbb{R}$ and $p_{\psi_i} \equiv 1$ for $i = 1,2,3$ but so that

- ssupp $\langle\psi_1\rangle = [0,2)$ so that $\langle\psi_1\rangle$ is of Type 1.
- ssupp $\langle\psi_2\rangle = [0,1/2) \cup [3/2,5/2)$ so that $\langle\psi_2\rangle$ is of Type 2.
- ssupp $\langle\psi_3\rangle = [0,1)$ so that $\langle\psi_3\rangle$ is of Type 3.

The fact that the type of a shift-invariant space $\langle\psi\rangle$ cannot be distinguished solely on the basis of p_ψ may seem counterintuitive, particularly when we think of $\langle\psi\rangle$ in terms of its image under the isometry with $L^2(\mathbb{T}, p_\psi)$. Observe, however, that, for a non-integer α, the function $T_\alpha\psi$ might not lie within $\langle\psi\rangle$. The best one can do, at least a priori, is to attempt to understand $\langle\psi\rangle$ in the wider context of $L^2(\mathbb{R})$, in which case p_ψ is very likely to be insufficient. ■

Let us at this point rephrase the necessary and sufficient condition under which one has $\langle\psi\rangle_{\mathbb{Z}} = \langle\psi\rangle_{\frac{1}{2}\mathbb{Z}}$. As before, we follow the terminology and notation from [**ŠW11**].

Lemma 3.7. *If $\psi \in L^2(\mathbb{R})$, then $\langle\psi\rangle_{\mathbb{Z}} = \langle\psi\rangle_{\frac{1}{2}\mathbb{Z}}$ if and only if for almost every $\xi \in \mathbb{R}$ and for every $m \in 2\mathbb{Z}+1$ one has $\widehat{\psi}(\xi) \cdot \widehat{\psi}(\xi+m) = 0$.*

Proof. Observe that our condition in this lemma requires that the set $A := \{\xi \in \mathbb{R} : \text{ there exists } m \in 2\mathbb{Z}+1 \text{ so that } \xi, \xi+m \in \text{ssupp } \langle\psi\rangle\}$ has Lebesgue measure zero. We recall some terminology from [**ŠW11**]. For each integer j, we let

$$E_1^j := \text{ssupp } \langle\psi\rangle \cap ([0,1)+2j)$$

and

$$E_2^j := \text{ssupp } \langle\psi\rangle \cap ([0,1)+2j+1).$$

We define F_1 and F_2 via

$$F_1 := \bigcup_{j\in\mathbb{Z}} (E_1^j - 2j)$$

and

$$F_2 := \bigcup_{j\in\mathbb{Z}} (E_2^j - (2j+1))$$

In the terminology of [**ŠW11**], ssupp $\langle\psi\rangle$ is called a $\frac{1}{2}$-translation system if $F_1 \cap F_2$ has Lebesgue measure zero, and, moreover, it was proven that $\langle\psi\rangle_{\mathbb{Z}} = \langle\psi\rangle_{\frac{1}{2}\mathbb{Z}}$ is equivalent to ssupp $\langle\psi\rangle$ being a $\frac{1}{2}$-translation system.

So, suppose first that ssupp $\langle\psi\rangle$ is a $\frac{1}{2}$-translation system and consider $\xi \in$ ssupp $\langle\psi\rangle$. Then, this ξ belongs either to some set $E_1^{j_0}$ or to some set $E_2^{j_0}$ but not both. Without loss of generality, suppose $\xi \in E_1^{j_0}$. In particular, $\xi - 2j_0 \in F_1$. Consider $m \in 2\mathbb{Z}+1$, i.e. $m = 2k_0+1$ for some $k_0 \in \mathbb{Z}$. If $\xi + m \in$ ssupp $\langle\psi\rangle$, then $\xi + m \in E_2^{j_0+k_0}$, i.e. $\xi + m - (2(j_0+k_0)+1) \in F_2$. Since we are assuming ssupp $\langle\psi\rangle$ is a $\frac{1}{2}$-translation system, the set $F_1 \cap F_2$ has Lebesgue measure zero, and $2\mathbb{Z}+1$ is a countable set, whence A must have Lebesgue measure zero.

Now, suppose that $|F_1 \cap F_2|$ has nonzero measure. Take $u \in F_1 \cap F_2$. Then there exist integers j_0 and k_0 so that $u + 2j_0 \in$ ssupp $\langle\psi\rangle$ and $u + 2k_0 + 1 \in$ ssupp $\langle\psi\rangle$. Let $\xi = u + 2j_0$ and observe that $u + 2k_0 + 1 - \xi = 2(k_0 - j_0) + 1 \in 2\mathbb{Z}+1$. It follows that $\xi \in A$. As $F_1 \cap F_2$ was assumed to have positive measure, it follows that A has positive measure as well.

□

Certainly, if $\langle\psi\rangle_{\mathbb{Z}} = \langle\psi\rangle_{\frac{1}{2}\mathbb{Z}}$, then $\mathcal{T}_{\langle\psi\rangle}$ is nonempty; thus $\langle\psi\rangle$ must be either of Type 2 or Type 3. Using the ideas from [**ŠW11**], it is straightforward to see that for every $\varphi, \psi \in L^2(\mathbb{R})$, one has

$$\langle\varphi\rangle \subseteq \langle\psi\rangle \text{ implies } \mathcal{T}_{\langle\varphi\rangle} \supseteq \mathcal{T}_{\langle\psi\rangle}; \tag{3.8}$$

in particular, taking a shift-invariant subspace of a principal shift-invariant space can only increase the type or keep the type the same. Various possibilities occur, as the following example indicates.

EXAMPLE 3.9. In the following, we always have non-trivial $\varphi_i, \psi_i \in L^2(\mathbb{R})$ with $\langle\varphi_i\rangle \subseteq \langle\psi_i\rangle$. Let ε be any element of the interval $(0, 1/2]$. We only give the relevant support sets, but the construction of the generating functions follows directly from them.

1.
   - ssupp $\langle\psi_1\rangle = [0,\varepsilon] \cup [1+\varepsilon, 1+2\varepsilon] \cup [2, 2+\varepsilon] \cup [3+\varepsilon, 3+2\varepsilon] \cup [4+\varepsilon, 4+2\varepsilon]$, with $\mathcal{T}_{\langle\psi_1\rangle} = \emptyset$.
   - ssupp $\langle\varphi_1\rangle = [0,\varepsilon] \cup [2, 2+\varepsilon]$, with $\mathcal{T}_{\langle\varphi_1\rangle} = \{2\}$.
2.
   - ssupp $\langle\psi_2\rangle = [0,\varepsilon] \cup [1, 1+\varepsilon] \cup [2+\varepsilon, 2+2\varepsilon]$, with $\mathcal{T}_{\langle\psi_2\rangle} = \emptyset$.
   - ssupp $\langle\varphi_2\rangle = [2+\varepsilon, 2+2\varepsilon]$, with $\mathcal{T}_{\langle\varphi_2\rangle} = \mathbb{R}$.
3.
   - ssupp $\langle\psi_3\rangle = [0,\varepsilon] \cup [1+\varepsilon, 1+2\varepsilon] \cup [3+\varepsilon, 3+2\varepsilon] \cup [6, 6+\varepsilon]$, with $\mathcal{T}_{\langle\psi_3\rangle} = \{2\}$.
   - ssupp $\langle\varphi_3\rangle = [0,\varepsilon] \cup [6, 6+\varepsilon]$, with $\mathcal{T}_{\langle\varphi_3\rangle} = \{2, 3, 6\}$.
4.
   - ssupp $\langle\psi_4\rangle = [0,\varepsilon] \cup [1+\varepsilon, 1+2\varepsilon] \cup [2, 2+\varepsilon]$, with $\mathcal{T}_{\langle\psi_4\rangle} = \{2\}$.
   - ssupp $\langle\varphi_4\rangle = [1+\varepsilon, 1+2\varepsilon]$, with $\mathcal{T}_{\langle\varphi_4\rangle} = \mathbb{R}$.

◇

Observe also that (3.8) implies that for every $\varphi, \psi \in L^2(\mathbb{R})$, one has

$$\mathcal{T}_{\langle\varphi\rangle\cap\langle\psi\rangle} \supseteq \mathcal{T}_{\langle\varphi\rangle} \cup \mathcal{T}_{\langle\psi\rangle}. \tag{3.10}$$

Let us also recall a useful tool for analyzing larger lattices; see [**ŠW11**] for a detailed account and further generalizations. For a function $\psi \in L^2(\mathbb{R})$, consider an integer $n > 1$ and the $n\mathbb{Z}$-periodization of $|\widehat{\psi}|^2$, i.e.

$$P_{\psi,\frac{1}{n}\mathbb{Z}}(\xi) := \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi + nk)|^2 \text{ for } \xi \in \mathbb{R}. \tag{3.11}$$

Obviously,

$$p_\psi(\xi) = \sum_{j=0}^{n-1} P_{\psi,\frac{1}{n}\mathbb{Z}}(\xi + j) \text{ for } \xi \in \mathbb{R}. \tag{3.12}$$

Furthermore (see [**ŠW11**]), $T_{\frac{1}{n}}\psi \in \langle\psi\rangle$ if and only if for almost every $\xi \in \mathbb{R}$ and for every $j \in \{1, ..., n-1\}$ one has that

$$P_{\psi,\frac{1}{n}\mathbb{Z}}(\xi) \cdot P_{\psi,\frac{1}{n}\mathbb{Z}}(\xi + j) = 0. \tag{3.13}$$

For the special case $n = 2$, compare (3.13) to Lemma 3.7.

Finally, (see [**ŠW11**]), observe that for $\langle\psi\rangle$ of Type 3, we have that, for almost every $\xi \in \mathbb{R}$, there exists exactly one $k = k(\xi) \in \mathbb{Z}$ such that

$$p_\psi(\xi) = |\widehat{\psi}(\xi + k)|^2; \tag{3.14}$$

in other words, for $j \in \mathbb{Z} \setminus \{k\}$, we have $\widehat{\psi}(\xi + j) = 0$.

4. Coefficients

In the analysis-synthesis approach to the study of functions, it is important to understand the representation of a function $\varphi \in \langle\psi\rangle$ in terms of the elements of $\mathcal{B}_\psi$. With this point of view, it is then natural and essential to study the corresponding coefficients. Here we follow results from [**HŠWW10b**] with some simple modifications (see also the historical remarks in Section 4 of [**HŠWW10b**]).

Let $\psi \in L^2(\mathbb{R})$ and consider a function $\varphi \in \langle\psi\rangle$. There is then a sequence $(\varphi_n)_{n\in\mathbb{N}} \subseteq \operatorname{span} \mathcal{B}_\psi$ such that φ_n converges to φ in the $L^2(\mathbb{R})$ topology. Certainly, then, for each $n \in \mathbb{N}$, there is a finite set $C_{\varphi,n} \subseteq \mathbb{Z}$ such that φ_n is a linear combination of the elements of $\{T_k\psi : k \in C_{\varphi,n}\}$. In this analysis we shall always impose a particular structure on these sets $C_{\varphi,n}$: we will impose the standard symmetric ordering from Fourier series. More precisely, we consider the following ordering of $\mathbb{Z}$ given by

$$0, 1, -1, 2, -2, 3, -3, ... \tag{4.1}$$

It is not difficult to see that we can always modify the sequence (φ_n) so that $C_{\varphi,n} = \{-n, -n+1, ..., -1, 0, 1, ..., n-1, n\}$, allowing some coefficients to be zero if necessary. Hence for $\varphi \in \langle\psi\rangle$, there are coefficients $c_k^n = c_k^n(\varphi)$ for $n \in \mathbb{N}$ and $-n \le k \le n$ such that

$$\varphi_n = \sum_{|k|\le n} c_k^n T_k\psi. \tag{4.2}$$

Among such partial sums, we shall pay special attention to the case when the coefficients (c_k^n) do not depend on n; we shall see that this is enough to cover all the interesting cases. More precisely, we consider the special case where there is a sequence $c_k = c_k(\varphi)$ such that, for every $n \in \mathbb{N}$,

$$\varphi_n = \sum_{|k|\le n} c_k T_k\psi. \tag{4.3}$$

REMARK 4.4. There is a subtle issue concerning the convergence of (4.3) which should be resolved before proceeding further. Let $(f_\ell)_{\ell\in\mathbb{N}}$ be an enumeration of $\mathcal{B}_\psi$ according to the ordering given in (4.1). If $(c_k)_{k\in\mathbb{Z}}$ denotes a sequence of coefficients for $\mathcal{B}_\psi$, let $(\widetilde{c}_\ell)_{\ell\in\mathbb{N}}$ denote the enumeration of this sequence according to (4.1). Our question now is whether the convergence of the sums in (4.3) is equivalent to the convergence of the sums $\sum_{\ell=1}^N \widetilde{c}_\ell f_\ell$. Obviously, the sequence of sums in (4.3) forms a subsequence of the sums $\sum_{\ell=1}^N \widetilde{c}_\ell f_\ell$, and thus convergence of the latter implies convergence of the former. However, the fact that convergence of the former implies convergence of the latter requires a short argument.

The result is obvious when ψ is the zero function, so assume $\psi \not\equiv 0$. Suppose that the sums $(\varphi_n)_{n\in\mathbb{N}}$ in (4.3) converge to a function φ. It is worth stressing here that we are not imposing any summability criterion on the coefficients $(c_k)_{k\in\mathbb{Z}}$; we only require the convergence in $L^2(\mathbb{R})$ of the partial sums from (4.3) for some fixed ψ. In order to show that $\left(\sum_{\ell=1}^L \widetilde{c}_\ell f_\ell\right)_{L\in\mathbb{N}}$ also converge to φ, it is sufficient to prove that $\lim_{n\to\pm\infty} |c_n| = 0$ (this follows from the fact that $\|T_k\psi\| = \|\psi\|$ since translations are unitary). Without loss of generality, we establish only that $\lim_{n\to\infty} |c_n| = 0$. Suppose to the contrary that $|c_n|$ does not converge to 0. Then there exists some $\varepsilon > 0$ and some subsequence $(|c_{n_k}|)$ such that $|c_{n_k}|^2 \geq \varepsilon$. Using (4.3), one obtains

$$\begin{aligned}
0 \leftarrow &\int_{\mathbb{T}} |c_{n_k} e^{-2\pi i n_k \xi} + c_{-n_k} e^{2\pi i n_k \xi}|^2 p_\psi(\xi) d\xi \\
&= \int_{\mathbb{T}} |c_{n_k}|^2 |e^{-2\pi i n_k \xi}|^2 \left|1 + \frac{c_{-n_k}}{c_{n_k}} e^{4\pi i n_k \xi}\right|^2 p_\psi(\xi) d\xi \\
&\geq \varepsilon \int_{\mathbb{T}} \left|1 + \frac{c_{-n_k}}{c_{n_k}} e^{4\pi i n_k \xi}\right|^2 p_\psi(\xi) d\xi.
\end{aligned}$$

Hence there is a subsequence (p_n) and coefficients (a_{p_n}) with the property that

$$\int_{\mathbb{T}} \left|1 - a_{p_n} e^{2\pi i p_n \xi}\right|^2 p_\psi(\xi) d\xi \to 0.$$

Using the mapping $\mathcal{I}_\psi$, this means that $a_{p_n} T_{p_n}\psi \to \psi$ in $L^2(\mathbb{R})$. Taking norms, this guarantees that $|a_{p_n}| \to 1$, and, in particular, that (a_{p_n}) is bounded. Since $p_\psi \in L^1(\mathbb{T})$, one can use Fourier series to deduce that

$$\int_{\mathbb{T}} \left|1 - a_{p_n} e^{2\pi i p_n \xi}\right|^2 p_\psi(\xi) d\xi = (1 + |a_{p_n}|^2)\widehat{p_\psi}(0) - a_{p_n}\widehat{p_\psi}(-p_n) - \overline{a_{p_n}}\widehat{p_\psi}(p_n).$$

Since $|p_n| \to \infty$, the Riemann–Lebesgue Lemma guarantees that $\widehat{p_\psi}(\pm p_n) \to 0$, and, since (a_{p_n}) is bounded, this means that $a_{p_n}\widehat{p_\psi}(-p_n)$ and $\overline{a_{p_n}}\widehat{p_\psi}(p_n)$ both tend to zero. Using this and the fact that $|a_{p_n}|^2 \to 1$, it follows that

$$0 = 2\widehat{p_\psi}(0) = 2\int p_\psi(\xi) d\xi = 2\|\psi\|^2_{L^2(\mathbb{R})},$$

which implies that $\psi \equiv 0$, contradicting our initial assumption.

Based on the above observation, whenever we have a sum ordered by (4.1), we shall denote it by

$$\lim_{n\to\infty} \sum_{|k|\leq n} c_k T_k \psi. \tag{4.5}$$

■

Observe that the sum in (4.5) depends on a particular ordering of the vectors $T_k\psi$. Of particular importance in the present work are those sums for which the ordering is irrelevant. More precisely, if $\varphi = \lim_{n\to\infty} \sum_{|k|\le n} c_k T_k \psi$ exists and the convergence is *unconditional* (see [**HW96**] and [**Sin70**] for more details on unconditional convergence), we shall write

$$\varphi = \sum_{k\in\mathbb{Z}} c_k T_k \psi. \tag{4.6}$$

The following result shows that such unconditional convergence is only possible for a special class of coefficients.

Lemma 4.7. *If $\sum_{k\in\mathbb{Z}} c_k T_k \psi$ converges unconditionally and $\psi \not\equiv 0$, then $(c_k) \in \ell^2(\mathbb{Z})$.*

Proof. Since $\langle \psi \rangle$ is a Hilbert space, we may apply the Orlicz theorem on the unconditional sum $\sum_{k\in\mathbb{Z}} c_k T_k \psi$ to obtain

$$\infty > \sum_{k\in\mathbb{Z}} \|c_k T_k \psi\|^2_{L^2(\mathbb{R})} = \|\psi\|^2_{L^2(\mathbb{R})} \sum_{k\in\mathbb{Z}} |c_k|^2.$$

Since $\|\psi\|_{L^2(\mathbb{R})} > 0$, the result follows.

□

This leads to the following natural question. Given $0 \not\equiv \psi \in L^2(\mathbb{R})$ and $(c_k) \in \ell^2(\mathbb{Z})$, what can be said about the convergence of the sums $\sum_{|k|\le n} c_k T_k \psi$? We observe first that if the limit exists, then it must lie within a particular subset of $\langle \psi \rangle$; the precise statement is as follows[2]:

Lemma 4.8. *If $(c_k) \in \ell^2(\mathbb{Z})$ and the limit $\lim_{n\to\infty} \sum_{|k|\le n} c_k T_k \psi = \varphi$ exists in the L^2 sense, then there is a function $f \in L^2(\mathbb{T})$ such that $f\chi_{U_{\langle\psi\rangle}} = \mathcal{I}_\psi^{-1}(\varphi)\chi_{U_{\langle\psi\rangle}}$ almost everywhere.*[3]

Now that we have a basic understanding of the limit when it exists, one is obviously left with the issue of determining when convergence occurs. We begin by exploring the situation when ψ is such that convergence occurs whenever the coefficient sequence is in $\ell^2(\mathbb{Z})$. One is lead to the following notion (used already in [**HŠWW10b**]) which we borrow from the theory of bases in Banach spaces; see [**Sin70**] or [**TL77**] for numerous results.

Definition 4.9. Suppose that $\psi \in L^2(\mathbb{R})$. We shall say that $\mathcal{B}_\psi$ has the (H)-property[4] or has property (H) if $(c_k) \in \ell^2(\mathbb{Z})$ guarantees that $\sum_{|k|\le n} c_k T_k \psi$ converges.

Observe that Lemma 4.8 directly proves the following:

Lemma 4.10. *If $\mathcal{B}_\psi$ satisfies the (H)-property, then $L^2(\mathbb{T}) \subseteq L^2(\mathbb{T}, p_\psi)$; that is, if $f \in L^2(\mathbb{T})$, then $f \in L^2(\mathbb{T}, p_\psi)$.*

---

[2]This lemma can be proven by a simple modification of a subsequence argument presented in Theorem 3.10 of [**HŠWW10b**].

[3]We know apriori that f must be in the weighted space $L^2(\mathbb{T}, p_\psi)$; the lemma guarantees that f lies in $L^2(\mathbb{T})$.

[4]The letter H here stands for Hilbertian.

The statement of this lemma makes sense whether we consider elements f as functions or equivalence classes. More precisely, if g and h are in the same equivalence class in $L^2(\mathbb{T})$, then they belong to the same class in $L^2(\mathbb{T}, p_\psi)$. The converse of this statement is, of course, not necessarily true.

Using Lemma 3.6 from [**HŠWW10b**], it is easy to obtain the following result:

LEMMA 4.11. *Suppose that v, w are strictly positive, integrable, and 1-periodic functions (i.e. v and w are in equivalence classes of elements of $L^1(\mathbb{T})$). If $L^2(\mathbb{T}, w) \subseteq L^2(\mathbb{T}, v)$ as sets (not necessarily with respect to any topologies), then there exists a constant $0 < C < \infty$ so that $v \leq Cw$ almost everywhere (with respect to Lebesgue measure).*

PROOF. Observe that $\frac{1}{\sqrt{w}} \in L^2(\mathbb{T}, w)$. Under our assumptions, we find that $\frac{1}{\sqrt{w}} \in L^2(\mathbb{T}, v)$, which implies that $f := \frac{v}{w} \in L^1(\mathbb{T})$.

It is now enough to prove that f satisfies the requirement of Lemma 3.6 from [**HŠWW10b**]. Take $0 \leq g \in L^1(\mathbb{T})$. It follows that $\sqrt{\frac{g}{w}}$ belongs to $L^2(\mathbb{T}, w)$. By our assumption, this guarantees that $\sqrt{\frac{g}{w}} \in L^2(\mathbb{T}, v)$. Hence

$$\infty > \int_{\mathbb{T}} \left(\sqrt{\frac{g}{w}} \right)^2 v = \int_{\mathbb{T}} g \frac{v}{w} = \int_{\mathbb{T}} gf.$$

In other words, $fg \in L^1(\mathbb{T})$.

$\square$

It will be useful to consider the following inclusion operator (see also [**HŠWW10b**]). Denote by $\mathcal{M}_\psi$ the quotient space of classes of measurable functions on $\mathbb{T}$ modulo $p_\psi(\xi)d\xi$-almost everywhere equivalence. Consider the inclusion operator

$$\begin{aligned} &J_\psi : L^2(\mathbb{T}) \to \mathcal{M}_\psi \\ &J_\psi(f) := f. \end{aligned} \tag{4.12}$$

Observe that the definition makes sense, since, if $f = g$ $d\xi$-almost everywhere, then $f = g$ $p_\psi(\xi)d\xi$-almost everywhere. Observe also that when the range of J_ψ is contained in $L^2(\mathbb{T}, p_\psi)$, we may also consider $J_\psi : L^2(\mathbb{T}) \to L^2(\mathbb{T}, p_\psi)$. Combining the techniques given above with [**HŠWW10b**], we obtain a somewhat stronger version of Theorem 3.10 from [**HŠWW10b**].

THEOREM 4.13. *Let ψ be a nonzero element of $L^2(\mathbb{R})$. Then the following are equivalent.*

(a) There is a constant B with $0 < B < \infty$ so that $p_\psi(\xi) \leq B$ for almost every ξ;
(b) $\mathcal{B}_\psi$ is a Besselian family;
(c) $\mathcal{B}_\psi$ satisfies the (H)-property;
(d) $(c_k) \in \ell^2(\mathbb{Z})$ if and only if $\sum_{k \in \mathbb{Z}} c_k T_k \psi$ converges unconditionally;
(e) The range of J_ψ is contained in $L^2(\mathbb{T}, p_\psi)$;
(f) $J_\psi : L^2(\mathbb{R}) \to L^2(\mathbb{T}, p_\psi)$ is a bounded, linear operator.

For a discussion of all the above terminology, consult [**HŠWW10b**].

REMARK 4.14. Let us emphasize several details.

(i) Using Lemma 4.8, we obtain that, for every $\psi \in L^2(\mathbb{R})$,

$$\begin{aligned}&\{(c_k) \in \ell^2(\mathbb{Z}) : \lim_{n\to\infty} \sum_{|k|\le n} c_k T_k \psi \text{ exists}\}\\ =&\{(c_k) \in \ell^2(\mathbb{Z}) : \sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k\xi} = f \text{ in } L^2(\mathbb{T}) \text{ and } f \in L^2(\mathbb{T}) \cap L^2(\mathbb{T}, p_\psi)\}.\end{aligned}$$

Furthermore, from Lemma 4.7, we know that for every nonzero ψ in $L^2(\mathbb{R})$ one has that

$$\begin{aligned}&\{(c_k) \in \mathbb{C}^{\mathbb{Z}} : \sum_{k\in\mathbb{Z}} c_k T_k \psi \text{ converges unconditionally}\}\\ \subseteq&\{(c_k) \in \ell^2(\mathbb{Z}) : \lim_{n\to\infty} \sum_{|k|\le n} c_k T_k \psi \text{ exists}\}.\end{aligned}$$

An open question at this point is whether we have an equality in the last set inclusion. Obviously, if $\mathcal{B}_\psi$ satisfies the (H)-property, then the answer is positive.

(ii) Given a nonzero function $\psi \in L^2(\mathbb{R})$ such that p_ψ is *not* bounded above, we conclude that the (H)-property is not satisfied. Hence, in this case, there are sequences $(c_k) \in \ell^2(\mathbb{Z})$ such that the limit $\lim_{n\to\infty} \sum_{|k|\le n} c_k T_k \psi$ does not exist.

(iii) Observe that $L^2(\mathbb{T}) \cap L^2(\mathbb{T}, p_\psi)$ can be considered as a vector space (via $d\xi$-almost everywhere equivalence classes) which contains all of $L^\infty(\mathbb{T})$. It can also be considered as a dense subspace in both $L^2(\mathbb{T})$ and $L^2(\mathbb{T}, p_\psi)$, with respect to the norm topologies. Hence we can think of

$$J_\psi\big|_{L^2(\mathbb{T})\cap L^2(\mathbb{T},p_\psi)} : L^2(\mathbb{T}) \cap L^2(\mathbb{T}, \psi) \to L^2(\mathbb{T}, \psi)$$

as a densely defined linear operator from $L^2(\mathbb{T})$ into $L^2(\mathbb{T}, p_\psi)$. Obviously, it can be extended to a bounded operator from $L^2(\mathbb{T})$ to $L^2(\mathbb{T}, p_\psi)$ if and only if

$$J_\psi\big|_{L^2(\mathbb{T})\cap L^2(\mathbb{T},p_\psi)} \text{ is bounded}$$

or

$$J_\psi\big|_{L^\infty(\mathbb{T})} \text{ is bounded.}$$

■

The above remark suggests that it is of interest to consider the vector space of coefficients, which we denote by

$$\mathrm{Cof}_\psi := \{(c_k) : \lim_{n\to\infty} \sum_{|k|\le n} c_k T_k \psi \text{ exists}\}. \tag{4.15}$$

Using the standard notation $(c_k) \in c_0(\mathbb{Z})$ if and only if $\lim_{|n|\to\infty} c_n = 0$, it is easy to see that

$$\ell^1(\mathbb{Z}) \subseteq \mathrm{Cof}_\psi \subseteq c_0(\mathbb{Z}). \tag{4.16}$$

Observe also that Cof_ψ really depends on p_ψ only, rather than ψ; in other words, if ψ_1 and ψ_2 are functions in $L^2(\mathbb{R})$ such that $p_{\psi_1} = p_{\psi_2}$, then $\mathrm{Cof}_{\psi_1} = \mathrm{Cof}_{\psi_2}$.

Observe that unconditional convergence occurs for the coefficients within $\ell^2(\mathbb{Z}) \cap \mathrm{Cof}_\psi$ (note that $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}) \cap \mathrm{Cof}_\psi$). Consider $(c_k) \in \ell^2(\mathbb{Z}) \cap \mathrm{Cof}_\psi$. Since $(c_k) \in \ell^2(\mathbb{Z})$, there exists an $f \in L^2(\mathbb{T})$ such that $f = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k x}$ within

$L^2(\mathbb{T})$. Since $(c_k) \in \mathrm{Cof}_\psi$, we obtain that $f \in L^2(\mathbb{T}) \cap L^2(\mathbb{T}, p_\psi)$, i.e. that $J_\psi(f) \in L^2(\mathbb{T}, p_\psi)$. Hence we have a mapping

$$\ell^2(\mathbb{Z}) \cap \mathrm{Cof}_\psi \ni (c_k) \to J_\psi(f) \in L^2(\mathbb{T}, p_\psi).$$

It is natural to wonder under which conditions such a mapping is a bijection. Observe that this is equivalent to determining under which conditions the following property holds:

$$(c_k) \in \ell^2(\mathbb{Z}) \text{ and } \lim_{n\to\infty} \sum_{|k|\le n} c_k T_k \psi = 0 \text{ implies } (c_k) \equiv 0. \tag{4.17}$$

This property is well known (see [**Sin70**] and [**TL77**]); it states that $\mathcal{B}_\psi$ (with the ordering of $\mathbb{Z}$ given in (4.1)) is $\ell^2(\mathbb{Z})$*-linearly independent*. The question of the characterization of $\ell^2(\mathbb{Z})$-linear independence of $\mathcal{B}_\psi$ proved to be a rather demanding one. It turns out that

$$\mathcal{B}_\psi \text{ is } \ell^2(\mathbb{Z})\text{-linearly independent if and only if } p_\psi > 0 \text{ almost everywhere.} \tag{4.18}$$

This result was conjectured more than ten years ago at the Wavelet Seminar at Washington University in St. Louis. If $\mathcal{B}_\psi$ is a Besselian family, then the proof is quite easy (see [**ŠS07**]). However, the general case was significantly more difficult and was proved by Saliani in [**Sal13**]; she applied the celebrated Carleson theorem on the almost everywhere convergence of Fourier series and some Menšov-type techniques. For related results, see also [**Pal10**] and [**ŠS12**].

Remark 4.19. (i) Observe that (4.17) is directly related to the question of *redundancy* of the family $\mathcal{B}_\psi$. Recall that for a nonzero ψ, the set $\mathcal{B}_\psi$ is always linearly independent, so finite sums of elements of $\mathcal{B}_\psi$ have no hope of achieving redundancy. Under property (4.17), one can achieve redundancy of countable linear combinations of elements of $\mathcal{B}_\psi$ only via conditional convergence with coefficients outside $\ell^2(\mathbb{Z})$.

(ii) The condition "$p_\psi > 0$" can be rephrased in terms of the measure of $Z_{\langle\psi\rangle}$, namely $|Z_{\langle\psi\rangle}| = 0$. Hence property (4.17) is not just a property of ψ or $\mathcal{B}_\psi$ but rather of the space $\langle\psi\rangle$. This fact may seem somewhat surprising.

(iii) Obviously, the ℓ^2-linear independence of $\mathcal{B}_\psi$ is equivalent to $\langle\psi\rangle$ being a maximal principal shift-invariant space. As we have already indicated, such spaces will play a special role in our theory.

■

5. Maximal Principal Shift-invariant Spaces

As we have seen in previous sections, of particular importance is the case of a principal shift-invariant space $\langle\psi\rangle$ which is also maximal, i.e. for which $U_{\langle\psi\rangle} = \mathbb{R}$. Again, we mostly follow the discussion from [**HŠWW10b**] while providing some additional details. Throughout this section, we assume that $\psi \in L^2(\mathbb{R})$ so that $p_\psi > 0$. Thinking of p_ψ as a weight (see [**Ste93**] or [**Gra04**] for the basic theory of weights), this assumption allows us to consider $1/p_\psi$ as a (finite-valued) weight as well. Furthermore, in some cases we may consider a power of $1/p_\psi$, which, when viewed as a Fourier multiplier, has $\langle\psi\rangle$ as an invariant subspace. With this in mind, observe that, for $\alpha \ge 0$, we have

$$\frac{1}{p_\psi^\alpha} \in L^2(\mathbb{T}, p_\psi) \text{ if and only if } p_\psi^{1-2\alpha} \in L^1(\mathbb{T}). \tag{5.1}$$

In particular, one always has, for those $\alpha \in [0, 1/2]$,

$$\frac{1}{p_\psi^\alpha} \bullet \psi \in \langle \psi \rangle. \tag{5.2}$$

For those $\alpha > 1/2$, one has $1 - 2\alpha < 0$, so $p_\psi^{1-2\alpha}$ may or may not be in $L^1(\mathbb{T})$.

Generally speaking, as properties of $1/p_\psi$ "improve", $\mathcal{B}_\psi$ has stronger "basis-like" properties — in particular, it becomes less possible for $\mathcal{B}_\psi$ to have redundancy. As we have seen, if $1/p_\psi < \infty$ almost everywhere, which is equivalent to the maximality of $\langle \psi \rangle$, then $\mathcal{B}_\psi$ is ℓ^2-linearly independent. Yet further, if one imposes the stronger condition

$$\frac{1}{p_\psi} \in L^1(\mathbb{T}), \tag{5.3}$$

then $\mathcal{B}_\psi$ is *minimal* (see [**HŠWW10b**] and the references therein for definitions and historical remarks); that is, for every $k \in \mathbb{Z}$,

$$T_k\psi \notin \overline{\operatorname{span}(\mathcal{B}_\psi \setminus \{T_k\psi\})}. \tag{5.4}$$

Observe that (5.4) excludes any practical possibility of redundancy.

Remark 5.5. It is at this point still an open question as to what happens "between" ℓ^2-linear independence and minimality. One possible direction is to explore other notions of linear independence. Observe that for $2 < p < \infty$, we have

$$\ell^2(\mathbb{Z}) \subsetneqq \ell^p(\mathbb{Z}) \subsetneqq c_0(\mathbb{Z}).$$

Let ℓ denote either $\ell^p(\mathbb{Z})$ or $c_0(\mathbb{Z})$. Then we say that $\mathcal{B}_\psi$ is ℓ-linearly independent if the following condition holds:

$$(c_k) \in \ell \text{ and } \lim_{n\to\infty} \sum_{|k|\le n} c_k T_k \psi = 0 \text{ implies that } (c_k) \equiv 0. \tag{5.6}$$

If $\mathcal{B}_\psi$ is minimal, then $\mathcal{B}_\psi$ is ℓ^p-linearly independent and c_0-linearly independent. However, the complete characterization of these notions for $\mathcal{B}_\psi$ is still an open problem (see [**Sla14**] and [**ŠS12**]). ■

It is well known that minimality is equivalent to the existence of a biorthogonal pair (see [**HŠWW10b**] and the references therein); this means that, if (5.3) holds, there is a *dual function* $\widetilde{\psi} \in \langle \psi \rangle$ so that $(\mathcal{B}_\psi, \mathcal{B}_{\widetilde{\psi}})$ forms a biorthogonal pair. In particular, one has

$$\widetilde{\psi} = \frac{1}{p_\psi} \bullet \psi. \tag{5.7}$$

It is also well known that the following proposition holds (see, e.g. [**HŠWW10b**]).

Proposition 5.8. *If $\psi \in L^2(\mathbb{R})$ satisfies (5.3), then the dual function $\widetilde{\psi}$ given in (5.7) satisfies the following properties:*
(a) $\widetilde{\psi} = \mathcal{I}_\psi(\frac{1}{p_\psi})$;
(b) $\langle \widetilde{\psi} \rangle = \langle \psi \rangle$;
(c) $p_{\widetilde{\psi}} = \frac{1}{p_\psi}$;
(d) $\widetilde{\psi}$ satisfies (5.3) and $\widetilde{\widetilde{\psi}} = \psi$;
(e) $\widetilde{\psi} = \psi$ if and only if $p_\psi \equiv 1$ if and only if $\mathcal{B}_\psi$ is an orthonormal basis for $\langle \psi \rangle$.

REMARK 5.9. Minimal systems do not allow for any redundancy. On the other hand, they allow for much better control of the coefficients. Suppose that $\varphi = \lim_{n\to\infty} \sum_{|k|\le n} c_k T_k \psi$, for some sequence of scalars (c_k). Obviously, $\varphi \in \langle \psi \rangle$, and we also have $c_k = \langle \varphi, T_k \widetilde{\psi} \rangle_{L^2(\mathbb{R})}$. ■

Hence, for minimal $\mathcal{B}_\psi$ it is of interest to study the convergence of the sequence of partial sums,

$$\sum_{|k|\le n} \langle \varphi, T_k \widetilde{\psi} \rangle T_k \psi, \tag{5.10}$$

for a given $\varphi \in \langle \psi \rangle$. It is easy to see that if the sequence converges, then its limit must be φ. Since $\varphi = \mathcal{I}_\psi(g)$ for some unique element $g \in L^2(\mathbb{T}, p_\psi)$, it makes sense to consider the problem at the level of weighted spaces. Observe also that, for every $k \in \mathbb{Z}$,

$$T_k \widetilde{\psi} = \mathcal{I}_\psi \left(\frac{e^{-2\pi i k \xi}}{p_\psi} \right) \tag{5.11}$$

For the following lemma, it is useful to notice that, under our assumptions, the three measures, $d\xi$, $p_\psi(\xi) d\xi$, and $\frac{1}{p_\psi} d\xi$, all generate the same family of null sets.

LEMMA 5.12. *Let $\psi \in L^2(\mathbb{R})$ satisfy (5.3). If f belongs to one of the three spaces, $L^2(\mathbb{T})$, $L^2(\mathbb{T}, p_\psi)$, and $L^2(\mathbb{T}, \frac{1}{p_\psi})$, then f belongs to $L^1(\mathbb{T})$, as well. In particular, f has well-defined Fourier coefficients via the usual integral formula.*

PROOF. Without loss of generality, consider the case $f \in L^2(\mathbb{T}, p_\psi)$. It follows that $|f|\sqrt{p_\psi} \in L^2(\mathbb{T})$. By (5.3), we have that $\frac{1}{\sqrt{p_\psi}} \in L^2(\mathbb{T})$. Hence

$$|f| = |f|\sqrt{p_\psi} \frac{1}{\sqrt{p_\psi}} \in L^1(\mathbb{T}).$$

□

Suppose now that $\psi \in L^2(\mathbb{R})$ satisfies (5.3) and consider $g \in L^2(\mathbb{T}, p_\psi)$. By Lemma 5.12, the k^{th} Fourier coefficient of g exists for every $k \in \mathbb{Z}$ and is given by

$$\begin{aligned} \widehat{g}(k) &= \int_{\mathbb{T}} g(\xi) e^{-2\pi i k \xi} \frac{1}{p_\psi(\xi)} p_\psi(\xi) d\xi \\ &= \left\langle g, \frac{e^{2\pi i k \xi}}{p_\psi} \right\rangle_{L^2(\mathbb{T}, p_\psi)} = \langle \mathcal{I}_\psi(g), T_{-k} \widetilde{\psi} \rangle_{L^2(\mathbb{R})}. \end{aligned} \tag{5.13}$$

Thus

$$\mathcal{I}_\psi \left(\widehat{g}(-k) e^{-2\pi i k \xi} \right) = \widehat{g}(-k) T_k \psi = \left\langle \mathcal{I}_\psi g, T_k \widetilde{\psi} \right\rangle T_k \psi.$$

From this it immediately follows that the question of convergence of the sequence (5.10) is actually a question about the convergence of

$$\lim_{n\to\infty} \sum_{|k|\le n} \widehat{g}(-k) e^{-2\pi i k \xi} = \lim_{n\to\infty} \sum_{|k|\le n} \widehat{g}(k) e^{2\pi i k \xi} \tag{5.14}$$

within $L^2(\mathbb{T}, p_\psi)$, where $g = \mathcal{I}_\psi^{-1}(\varphi)$. In particular, under the assumption (5.3) we have

$$\mathrm{Cof}_\psi = \{ (\langle \varphi, T_k \widetilde{\psi} \rangle)_{k\in\mathbb{Z}} : \varphi \in \langle \psi \rangle, g = \mathcal{I}_\psi^{-1}(\varphi), \text{ and the limit in (5.14) exists} \}. \tag{5.15}$$

Furthermore, the unconditional convergence of (5.10) occurs if and only if

$$(\widehat{g}(k))_{k\in\mathbb{Z}} \in \mathrm{Cof}_\psi \cap \ell^2(\mathbb{Z}). \tag{5.16}$$

Such considerations show that it is natural to consider the following notion, which is well known (see [**You01**]). The (RF) in the definition stands for "Riesz–Fischer".

Definition 5.17. Given $\psi \in L^2(\mathbb{R})$, we shall say that $\mathcal{B}_\psi$ satisfies the (RF)-property if for every $(c_k) \in \ell^2(\mathbb{Z})$, there exists a $\varphi \in \langle\psi\rangle$, such that for every $k \in \mathbb{Z}$, $c_k = \langle \varphi, T_k\psi\rangle_{L^2(\mathbb{R})}$.

It is well known (see [**You01**]) that for biorthogonal systems, the original sequence is a Besselian family if and only if the dual sequence satisfies the (RF)-property. Hence, using this result (or proving it directly since this is not so difficult within this framework) we obtain that, for every $\psi \in L^2(\mathbb{R})$ which satisfies (5.3),

$$\mathcal{B}_\psi \text{ is a Besselian family if and only if } \mathcal{B}_{\widetilde{\psi}} \text{ satisfies the (RF)-property.} \tag{5.18}$$

Furthermore, if any of the properties in (5.18) holds, then

$$\ell^2(\mathbb{Z}) \subseteq \mathrm{Cof}_\psi. \tag{5.19}$$

Remark 5.20. (i) Observe that, even for a minimal and Besselian family $\mathcal{B}_\psi$, we have, in principle, $\mathrm{Cof}_\psi \setminus \ell^2(\mathbb{Z}) \neq \emptyset$, and this set contains precisely those coefficients for which we have only conditional convergence.

(ii) Observe also that under the condition that $\mathcal{B}_\psi$ is minimal and Besselian, we still have functions (unless $\mathcal{B}_\psi$ is a basis) $\varphi \in \langle\psi\rangle$ such that the limit

$$\lim_{n\to\infty} \sum_{|k|\le n} \langle \varphi, T_k\widetilde{\psi}\rangle T_k\psi$$

does not exist.

■

Let us now reverse our point of view and consider the (RF)-property for $\mathcal{B}_\psi$. Using (5.18) and the fact that, for minimal systems, we have $p_{\widetilde{\psi}} = \frac{1}{p_\psi}$, we obtain the following result. If $\psi \in L^2(\mathbb{R})$ satisfies (5.3), then $\mathcal{B}_\psi$ satisfies the (RF)-property if and only if

$$\text{there exists an } A \text{ with } 0 < A < \infty \text{ such that } p_\psi \ge A \text{ almost everywhere.} \tag{5.21}$$

Remark 5.22. Observe that the last statement makes sense even without assuming (5.3). It is natural to wonder whether this statement is true without assumption (5.3). As we shall see, we have a positive answer here. Observe however, that (5.21) implies (5.3), so we do end up eventually within the framework of minimal $\mathcal{B}_\psi$. ■

Theorem 5.23. *If $\psi \in L^2(\mathbb{R})$ is such that $\mathcal{B}_\psi$ satisfies the (RF)-property, then p_ψ satisfies (5.21).*

Proof. We divide the proof into two steps. First, we reformulate the (RF)-property. Observe that having $(c_k) \in \ell^2(\mathbb{Z})$ is equivalent to having an $f \in L^2(\mathbb{T})$ with $\widehat{f}(k) = c_{-k}$. Observe also that $\varphi \in \langle\psi\rangle$ is equivalent to $g = \mathcal{I}_\psi^{-1}(\varphi) \in L^2(\mathbb{T}, p_\psi)$. Using the isometry $\mathcal{I}_\psi$ we also have

$$c_k = \langle \varphi, T_k\psi\rangle_{L^2(\mathbb{R})} = \langle g, e^{-2\pi i k\xi}\rangle_{L^2(\mathbb{T},p_\psi)}.$$

Hence the (RF)-property can be reformulated as follows:

$$\begin{aligned} &\text{for every } f \in L^2(\mathbb{T}), \text{ there exists a } g \in L^2(\mathbb{T}, p_\psi) \text{ so that,} \\ &\text{for every } k \in \mathbb{Z}, \text{ one has } \widehat{f}(k) = \langle g, e^{2\pi i k\xi}\rangle_{L^2(\mathbb{T},p_\psi)}. \end{aligned} \tag{5.24}$$

Since

$$\langle g, e^{2\pi i k\xi}\rangle_{L^2(\mathbb{T},p_\psi)} = \int_{\mathbb{T}} g(\xi)e^{-2\pi i k\xi}p_\psi(\xi)d\xi,$$

and $g \in L^2(\mathbb{T}, p_\psi)$ implies that $g \in L^1(\mathbb{T}, p_\psi)$, since $p_\psi(\xi)d\xi$ is a finite measure; thus we obtain that $gp_\psi \in L^1(\mathbb{T})$. Moreover, we have that $\widehat{f}(k) = \widehat{gp_\psi}(k)$ for all $k \in \mathbb{Z}$, and so f and gp_ψ are $L^1(\mathbb{T})$ functions with equal Fourier coefficients, which guarantees that $f = gp_\psi$ almost everywhere. This then proves that the (RF)-property implies that

$$\begin{aligned} &\text{For every } f \in L^2(\mathbb{T}), \text{ there exists } g \in L^2(\mathbb{T}, p_\psi) \\ &\text{such that } f = gp_\psi\ d\xi\text{-almost everywhere.} \end{aligned} \tag{5.25}$$

There are several consequences of (5.25). We claim that (5.25) implies that $p_\psi > 0$ $d\xi$-almost everywhere. To see this, suppose, to the contrary, that $|Z_{\langle\psi\rangle}| > 0$ and let $f := \chi_{Z_{\langle\psi\rangle}}$. It follows that f is a nonzero function in $L^2(\mathbb{T})$. By (5.25), we would get a $g \in L^2(\mathbb{T}, p_\psi)$ such that $f = gp_\psi$ $d\xi$-almost everywhere. It follows that $g\chi_{U_{\langle\psi\rangle}} \equiv 0$ $d\xi$-almost everywhere, and, as a consequence, we obtain that $f = gp_\psi = 0$ $d\xi$-almost everywhere, which is a contradiction.

Since we have proved that $p_\psi > 0$ $d\xi$-almost everywhere, it makes sense to consider the function $\frac{1}{p_\psi}$ on $\mathbb{T}$. Take $f \equiv 1 \in L^2(\mathbb{T})$ and apply (5.25) to obtain a $g \in L^2(\mathbb{T}, p_\psi)$ such that $1 = gp_\psi$ $d\xi$-almost everywhere. It follows that $\frac{1}{p_\psi} \in L^2(\mathbb{T}, p_\psi)$ and, by (5.1), that $\frac{1}{p_\psi} \in L^1(\mathbb{T})$.

Thus we have proved that $\mathcal{B}_\psi$ is a minimal system. We can now apply (5.18) to conclude the proof; however, in the name of completeness, we give a short proof using the techniques from these notes. Start with $f \in L^2(\mathbb{T})$ and use (5.25) to conclude that $\frac{f}{p_\psi} \in L^2(\mathbb{T}, p_\psi)$, which implies that $f \in L^2(\mathbb{T}, \frac{1}{p_\psi})$. Thus we have proved that $w :\equiv 1$ and $v := \frac{1}{p_\psi}$ are both strictly positive, integrable, 1-periodic weights such that $L^2(\mathbb{T}, w) \subseteq L^2(\mathbb{T}, v)$. By Lemma 4.11, we conclude that there exists a constant C with $0 < C < \infty$ so that $v \leq Cw$. Hence (5.21) follows with $A := \frac{1}{C}$.

$\square$

This result essentially completes the characterization of the (RF)-property. It is easy to see that we have the following list of results.

Corollary 5.26. *If $\psi \in L^2(\mathbb{R})$, then the following are equivalent:*

(a) $\mathcal{B}_\psi$ satsfies the (RF)-property;
(b) There exists a constant $A \in (0, \infty)$ so that $p_\psi \geq A$;
(c) $L^2(\mathbb{T}, p_\psi) \subseteq L^2(\mathbb{T})$ (in the set sense);
(d) $\frac{1}{p_\psi} \in L^1(\mathbb{T})$ and $L^2(\mathbb{T}) \subseteq L^2(\mathbb{T}, \frac{1}{p_\psi})$ (in the set sense);
(e) $\mathcal{B}_\psi$ is a minimal system and $\mathcal{B}_{\widetilde{\psi}}$ satisfies the (H)-property.

Remark 5.27. Observe that a certain “asymmetry” exists between the (RF)-property and the (H)-property. The (H)-property may hold even in a non-maximal principal shift-invariant space, $\langle\psi\rangle$ — in particular, also for non-minimal $\mathcal{B}_\psi$. By comparison, the (RF)-property is possible only within the minimal systems $\mathcal{B}_\psi$. ■

COROLLARY 5.28. *If $\psi \in L^2(\mathbb{R})$ is such that $\mathcal{B}_\psi$ is minimal, then the following are equivalent:*

(a) $\mathcal{B}_\psi$ satisfies the (RF)-property;
(b) $\mathcal{B}_{\widetilde{\psi}}$ satisfies the (H)-property;
(c) $L^2(\mathbb{T}, p_\psi) \subseteq L^2(\mathbb{T})$ (in the set sense);
(d) $L^2(\mathbb{T}) \subseteq L^2(\mathbb{T}, p_{\widetilde{\psi}})$ (in the set sense);
(e) $\ell^2(\mathbb{Z}) \subseteq Cof_{\widetilde{\psi}}$.

Let us at this point introduce yet another natural counterpart to the (H)-property; it is also a well-known notion, see [**Sin70**] and [**HŠWW10b**] for more details.

DEFINITION 5.29. Let $\psi \in L^2(\mathbb{R})$. We shall say that $\mathcal{B}_\psi$ satisfies the (B)-property if, for some sequence (c_k) of coefficients, whenever the limit

$$\lim_{n\to\infty} \sum_{|k|\le n} c_k T_k \psi$$

exists, one must have that $(c_k) \in \ell^2(\mathbb{Z})$.

REMARK 5.30. Observe that the (B)-property can be expressed in the form

$$\mathrm{Cof}_\psi \subseteq \ell^2(\mathbb{Z}); \tag{5.31}$$

one may expect this to appear as one of the characterizing statements in Corollary 5.28. However, at this point we do not know if such a result is valid or not. We know that any of the properties listed in Corollary 5.28 implies the (B)-property, within the context of minimal systems (see [**HŠWW10b**] for the proof). We also know (it is easy to see this directly) that (5.21), by itself, also implies the (B)-property. The problem lies in the reverse implication. We know that it holds in the case when $\mathcal{B}_\psi$ is a Schauder basis (see [**Sin70**] for the proof). In the more general situation, we have a couple of open questions. The first one is as follows. If $\psi \in L^2(\mathbb{R})$ and $\mathcal{B}_\psi$ satisfies the (B)-property, does it then follow that $p_\psi > 0$ $d\xi$-almost everywhere? The second question is as follows. If $\psi \in L^2(\mathbb{R})$, is such that $p_\psi > 0$ $d\xi$-almost everywhere and $\mathcal{B}_\psi$ satisfies the (B)-property, does it then follow that $\mathcal{B}_\psi$ is a minimal system? In one special case, we can answer this question; see the following proposition. ■

PROPOSITION 5.32. *Let $\psi \in L^2(\mathbb{R})$ be such that $p_\psi > 0$ $d\xi$-almost everywhere and such that $\mathcal{B}_\psi$ satisfies the (B)-property. If, for every $\varphi \in \langle\psi\rangle$ there exists a sequence $(c_k) \in c_0(\mathbb{Z})$ such that $\varphi = \lim_{n\to\infty}\sum_{|k|\le n} c_k T_k\psi$, then $\mathcal{B}_\psi$ is a Schauder basis for $\langle\psi\rangle$, with respect to the ordering of $\mathbb{Z}$ according to (4.1), and $\mathcal{B}_{\widetilde{\psi}}$ satisfies the (H)-property.*

PROOF. Consider $\varphi \in \langle\psi\rangle$. By our assumption, there exists $(c_k) \in c_0(\mathbb{Z})$ such that $\varphi = \lim_{n\to\infty}\sum_{|k|\le n} c_k T_k\psi$. By the (B)-property we obtain that $(c_k) \in \ell^2(\mathbb{Z})$. Suppose now that another sequence (d_k) satisfies $\varphi = \lim_{n\to\infty}\sum_{|k|\le n} d_k T_k\psi$. By the (B)-property, we obtain $(d_k) \in \ell^2(\mathbb{Z})$. Hence we have that $(c_k - d_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and $0 \equiv \lim_{n\to\infty}\sum_{|k|\le n}(c_k - d_k)T_k\psi$. Since $p_\psi > 0$ $d\xi$-almost everywhere implies that $\mathcal{B}_\psi$ is ℓ^2-linearly independent (again according to the ordering of $\mathbb{Z}$ in (4.1)), we conclude that $c_k = d_k$ for every $k \in \mathbb{Z}$.

It follows that $\mathcal{B}_\psi$, ordered by (4.1) is a basis (and, therefore, a Schauder basis) for $\langle\psi\rangle$. As is well known (see [**NŠ07**], for example) the dual basis is precisely $\mathcal{B}_{\widetilde{\psi}}$ (in particular, (5.3) holds) and by general results on bases (see [**Sin70**]), $\mathcal{B}_{\widetilde{\psi}}$ satisfies the (H)-property (which implies that $p_\psi \geq A > 0$ $d\xi$-almost everywhere). $\square$

Remark 5.33. It is possible to construct $\psi \in L^2(\mathbb{R})$ such that p_ψ satisfies (5.21) but is not an A_2-weight in the sense of Muckenhoupt; in other words, $\mathcal{B}_\psi$ is not a basis for $\langle\psi\rangle$ (see [**HŠWW10b**] and [**NŠ07**] for more details). It then follows that $\mathcal{B}_\psi$ is a minimal system which satisfies the (B)-property but does not satisfy the assumption in Proposition 5.32 of every $\varphi \in \langle\psi\rangle$ being represented as a limit from $\mathcal{B}_\psi$.

This also leads to a third question about the (B)-property. If $\psi \in L^2(\mathbb{R})$ is such that $\frac{1}{p_\psi} \in L^1(\mathbb{T})$ and $\mathcal{B}_\psi$ satisfies the (B)-property, does it then follow that p_ψ satisfies (5.21)? $\blacksquare$

In the basis case, all of these properties fit nicely together. Recall first that $\mathcal{B}_\psi$ is a Schauder basis (in the ordering given in (4.1)) if and only if p_ψ is an A_2-weight in the sense of Muckenhoupt (see [**NŠ07**] and [**HŠWW10b**] for details and further references). This is equivalent to $\frac{1}{p_\psi}$ being an A_2-weight and, indeed, $\mathcal{B}_{\widetilde{\psi}}$ acts as a dual basis for $\mathcal{B}_\psi$. Using standard results (see [**Sin70**]) or our discussion above, one obtains, without much effort, the following result.

Corollary 5.34. *If $\psi \in L^2(\mathbb{R})$ is such that $\mathcal{B}_\psi$, ordered according to (4.1), is a Schauder basis for $\langle\psi\rangle$, then the following are equivalent.*

(a) $\mathcal{B}_\psi$ satisfies the (B)-property;
(b) $\mathcal{B}_{\widetilde{\psi}}$ satisfies the (H)-property;
(c) $Cof_\psi \subseteq \ell^2(\mathbb{Z})$;
(d) $\ell^2(\mathbb{Z}) \subseteq Cof_{\widetilde{\psi}}$.

In the terminology of Banach space basis theory, these conditions characterize when $\mathcal{B}_\psi$ is a *Besselian Schauder basis* (not a Besselian family, though; see [**HŠWW10b**]), and, at the same time, when $\mathcal{B}_{\widetilde{\psi}}$ is a *Hilbertian Schauder basis*.

We assume that our reader is familiar with the notion of a *Riesz basis* (see [**HW96**] for details and further references). It is well known (see, for example, [**HŠWW10b**]) that $\mathcal{B}_\psi$ is a Riesz basis for $\langle\psi\rangle$ if and only if there are constants A and B with $0 < A \leq B < \infty$ so that $A \leq p_\psi \leq B$ almost everywhere. Observe that in this case all three measures on the torus, $d\xi$, $p_\psi(\xi)d\xi$, and $p_{\widetilde{\psi}}(\xi)d\xi$, have exactly the same family of negligible sets. The following result then follows easily from our discussion.

Corollary 5.35. *If $\psi \in L^2(\mathbb{R})$ is such that $\mathcal{B}_\psi$, ordered according to (4.1), is a Schauder basis for $\langle\psi\rangle$, then the following are equivalent:*

(a) $\mathcal{B}_\psi$ is a Riesz basis for $\langle\psi\rangle$;
(b) $\mathcal{B}_{\widetilde{\psi}}$ is a Riesz basis for $\langle\psi\rangle$;
(c) $\mathcal{B}_\psi$ satisfies the (H)-property and the (B)-property;
(d) $L^2(\mathbb{T}) = L^2(\mathbb{T}, p_\psi)$ on the set level;
(e) $L^2(\mathbb{T}) = L^2(\mathbb{T}, p_{\widetilde{\psi}})$ on the set level;

(f) $Cof_{\psi} = \ell^2(\mathbb{Z})$;
(g) $Cof_{\widetilde{\psi}} = \ell^2(\mathbb{Z})$.

REMARK 5.36. As it is well known, when $p_\psi \equiv 1$ (or, equivalently, when $\psi = \widetilde{\psi}$) we obtain that $\mathcal{B}_\psi$ is an orthonormal basis for $\langle\psi\rangle$. Observe that, solely on the level of the set Cof_ψ, we cannot distinguish this case from the Riesz basis case.

It is, of course, also well known that, for any maximal principal shift-invariant space $\langle\psi\rangle$, it is easy to construct a generating function φ such that $\mathcal{B}_\varphi$ is an orthonormal basis for $\langle\psi\rangle$; simply take

$$\varphi := \frac{1}{\sqrt{p_\psi}} \bullet \psi.$$

■

6. Direct Analysis Sum

As we have seen in the previous section, it is natural to consider the limits of the sums with coefficients $\langle\varphi, T_k\widetilde{\psi}\rangle_{L^2(\mathbb{R})}$, i.e.

$$\lim_{n\to\infty} \sum_{|k|\le n} \langle\varphi, T_k\widetilde{\psi}\rangle_{L^2(\mathbb{R})} T_k\psi, \tag{6.1}$$

assuming the dual function $\widetilde{\psi}$ exists, of course.

Observe, however, that from the point of view of applications, this may not be so useful. First of all, the dual function $\widetilde{\psi}$ does not exist for many interesting choices ψ. Secondly, even when it does exist, it may not be easy to determine whether this is true or not (assuming that ψ is not given explicitly to us). Thirdly, the limit in (6.1) may not exist, and, even if it does exist, we may only have conditional convergence to this limit.

One way to simplify this approach may be by using a kind of "direct method" in selecting "analysis coefficients", i.e. to consider

$$\lim_{n\to\infty} \sum_{|k|\le n} \langle\varphi, T_k\psi\rangle_{L^2(\mathbb{R})} T_k\psi. \tag{6.2}$$

Note that in the theory of Parseval frame wavelets, exactly such a choice is being utilized.

Observe that the nth partial sum in (6.2) exists for every $\psi \in L^2(\mathbb{R})$ (since one does not need the dual function to exist as one does in (6.1)). Obviously, there are some basic questions about (6.2). When does it converge? What type of convergence can one expect? What is the limit? We treat the question of what the limit is first.

LEMMA 6.3. *Let $\psi \in L^2(\mathbb{R})$, $\varphi \in \langle\psi\rangle$, and $g \in L^2(\mathbb{T}, p_\psi)$ such that $\mathcal{I}_\psi(g) = \varphi$. If the limit $\lim_{n\to\infty}\sum_{|k|\le n}\langle\varphi, T_k\psi\rangle_{L^2(\mathbb{R})}T_k\psi$ exists, then it must equal $p_\psi \bullet \varphi$. (In particular, this means that $p_\psi \bullet \varphi \in \langle\psi\rangle$, i.e. $gp_\psi \in L^2(\mathbb{T}, p_\psi)$.)*

PROOF. If $\psi = 0$, then the statement is not interesting and trivially true. So, assume that $\psi \neq 0$ so that $U_{\langle\psi\rangle}$ is a set of positive Lebesgue measure. By our assumptions, the limit

$$\lim_{n\to\infty} \sum_{|k|\le n} \langle g, e^{2\pi i k\xi}\rangle_{L^2(\mathbb{T},p_\psi)} e^{2\pi i k\xi}$$

exists in the sense of the norm of $L^2(\mathbb{T}, p_\psi)$; let us denote this limit by h.

Since $g \in L^2(\mathbb{T}, p_\psi)$ and $p_\psi(\xi)d\xi$ is a finite measure on $\mathbb{T}$, we obstain that $g \in L^1(\mathbb{T}, p_\psi)$, i.e. $gp_\psi \in L^1(\mathbb{T})$. Hence $\langle g, e^{2\pi i k\xi}\rangle_{L(\mathbb{T},p_\psi)} = \widehat{gp_\psi}(k)$ for every $k \in \mathbb{Z}$. It follows that

$$h_n := \sum_{|k|\leq n} \langle g, e^{2\pi i k\xi}\rangle_{L^2(\mathbb{T},p_\psi)} e^{2\pi i k\xi} = \sum_{|k|\leq n} \widehat{gp_\psi}(k) = S_n(gp_\psi),$$

where $S_n(gp_\psi)$ denotes the n^{th} symmetric partial sum of the Fourier series of the $L^1(\mathbb{T})$ function gp_ψ.

We know that h_n converges to h in $L^2(\mathbb{T}, p_\psi)$, and so $h_n\sqrt{p_\psi}$ converges to $h\sqrt{p_\psi}$ in $L^2(\mathbb{T})$. We also know that $h_n = S_n(gp_\psi)$, but without assuming stronger integrability conditions on gp_ψ ($L^p(\mathbb{T})$ for some $p \geq 2$), we cannot guarantee that $S_n(gp_\psi)$ converges to g in $L^2(\mathbb{T})$. It may also be tempting to invoke the Carleson–Hunt theorem to get almost everywhere pointwise convergence of the Fourier series of gp_ψ to gp_ψ, but Carleson–Hunt also requires stronger integrability conditions for gp_ψ ($L^p(\mathbb{T})$ for some $p > 1$). However, there is a relatively simple technical workaround for this issue which we describe now.

We recall that the Césaro average of a sequence $(f_k)_{k=0}^{\infty}$ is simply the sequence of the partial averages,

$$\frac{f_0 + f_1 + ... + f_{n-1}}{n}.$$

The Césaro averages of the Fourier series of gp_ψ are given by

$$\frac{S_0(gp_\psi) + S_1(gp_\psi) + ... + S_{n-1}(gp_\psi)}{n} = (gp_\psi) * F_n,$$

where F_n is the Féjer kernel. Classical Fourier analysis guarantees that not only does $(gp_\psi) * F_n$ converge to gp_ψ in $L^1(\mathbb{T})$, but $(gp_\psi) * F_n$ converges to gp_ψ almost everywhere[5] (see, e.g., Theorem 3.3.3 from [**Gra04**]).

Now, since

$$\frac{h_0\sqrt{p_\psi} + ... + h_{n-1}\sqrt{p_\psi}}{n} = \frac{S_0(gp_\psi) + ... + S_{n-1}(gp_\psi)}{n}\sqrt{p_\psi} = ((gp_\psi) * F_n) \cdot \sqrt{p_\psi},$$

it follows that the Césaro averages of $h_n\sqrt{p_\psi}$ converge to $gp_\psi\sqrt{p_\psi}$ almost everywhere. Since $h_n\sqrt{p_\psi}$ converges in $L^2(\mathbb{T})$ to $h\sqrt{p_\psi}$, the Césaro averages of $h_n\sqrt{p_\psi}$ must also converge to $h\sqrt{p_\psi}$ in $L^2(\mathbb{T})$. Hence some subsequence of the Césaro averages of $h_n\sqrt{p_\psi}$ must converge to $h\sqrt{p_\psi}$ almost everywhere. But this subsequence must also converge to $gp_\psi\sqrt{p_\psi}$ almost everywhere. Hence $gp_\psi(\xi) = h(\xi)$ for almost every $\xi \in U_{\langle\psi\rangle}$. In particular, we conclude that $gp_\psi \in L^2(\mathbb{T}, p_\psi)$ and that

$$\lim_{n\to\infty} \sum_{|k|\leq n} \langle \varphi, T_k\psi\rangle T_k\psi = \mathcal{I}_\psi(gp_\psi) = p_\psi \bullet \varphi.$$

□

[5]The proof of this fact is much, much easier than the Carleson–Hunt theorem — because the Féjer kernel, unlike the Dirichlet kernel, is an approximate identity, the proof that the Césaro averages of Fourier series converge almost everywhere is at the level of proving the Lebesgue Differentiation Theorem from the boundedness of the Hardy–Littlewood maximal function from $L^1(\mathbb{T})$ to weak-$L^1(\mathbb{T})$.

REMARK 6.4. It is perhaps worth mentioning that, under the assumption that $\mathcal{B}_\psi$ is a minimal system (i.e. (5.3) holds), the proof of Lemma 6.3 is much simpler. Here is the brief argument, using the same notation as in the previous Lemma. From our assumption of convergence, it follows that, for every $k \in \mathbb{Z}$,

$$\langle \mathcal{I}_\psi(h), T_k\widetilde{\psi}\rangle = \langle \varphi, T_k\psi\rangle.$$

By Lemma 5.12, since $h \in L^1(\mathbb{T})$, we have, for every $k \in \mathbb{Z}$, the following two equalities:

$$\langle \varphi, T_k\psi\rangle = \widehat{gp_\psi}(-k) \text{ and}$$
$$\langle \mathcal{I}(h), T_k\widetilde{\psi}\rangle = \widehat{h}(-k).$$

Since $gp_\psi \in L^1(\mathbb{T})$ as well, we must conclude that $gp_\psi = h$ almost everywhere. ∎

THEOREM 6.5. *If $\psi \in L^2(\mathbb{R})$, then the following are equivalent:*

(a) There is a constant B with $0 < B < \infty$ so that $p_\psi \leq B$ a.e.;
(b) For every $\varphi \in \langle\psi\rangle$, one has $p_\psi \bullet \varphi \in \langle\psi\rangle$;
(c) For every $\varphi \in \langle\psi\rangle$, one has $(\langle\varphi, T_k\psi\rangle)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$;
(d) For every $\varphi \in \langle\psi\rangle$, the following limit exists in $L^2(\mathbb{R})$:

$$\lim_{n\to\infty} \sum_{|k|\leq n} \langle \varphi, T_k\psi\rangle T_k\psi.$$

Furthermore, if any of these equivalent statements hold, then the sequence in (d) converges unconditionally.

PROOF. First, the theorem is trivial if $\psi \equiv 0$, so we may assume that $\psi \not\equiv 0$. We prove that (a) implies (b), (c), (d), and that the convergence in (d) is unconditional. Let us assume that (a) holds. Consider $\varphi \in \langle\psi\rangle$ and take $g \in L^2(\mathbb{T}, p_\psi)$ such that $\mathcal{I}_\psi(g) = \varphi$. Then, by the following trivial calculation, $g\sqrt{p_\psi} \in L^2(\mathbb{T}, p_\psi)$:

$$\int_{\mathbb{T}} |g\sqrt{p_\psi}|^2 p_\psi = \int_{\mathbb{T}} |g|^2 p_\psi \cdot p_\psi \leq B \int_{\mathbb{T}} |g|^2 p_\psi < \infty.$$

Observe now that $g\sqrt{p_\psi} \in L^2(\mathbb{T}, p_\psi)$ is equivalent to $gp_\psi \in L^2(\mathbb{T})$. Since $(c_k)_{k\in\mathbb{Z}} := \langle\varphi, T_k\psi\rangle = \widehat{gp_\psi}(-k)$ and $gp_\psi \in L^2(\mathbb{T})$, we conclude that $(c_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, which gives (c). Applying Theorem 4.13d, we obtain that

$$\sum_{k\in\mathbb{Z}} c_k T_k\psi = \sum_{k\in\mathbb{Z}} \langle\varphi, T_k\psi\rangle T_k\psi$$

converges unconditionally. We may now apply Lemma 6.3 to conclude that the limit is $p_\psi \bullet \varphi$, and thus (b).

Second, we establish that (d) and (c) independently imply (b). Certainly, (d) implies (b) by Lemma 6.3. We now prove that (c) implies (a) and therefore (b). Let us assume (c). Consider $\varphi \in \langle\psi\rangle$ and take $g \in L^2(\mathbb{T}, p_\psi)$ such that $\mathcal{I}_\psi(g) = \varphi$. As in the proof of Lemma 6.3, we conclude that $gp_\psi \in L^1(\mathbb{T})$ and that, for every $k \in \mathbb{Z}$, one has $\langle\varphi, T_k\psi\rangle = \widehat{gp_\psi}(-k)$. By (c) it follows that $((\widehat{gp_\psi}(k))_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ which implies that $gp_\psi \in L^2(\mathbb{T})$, i.e. $g \in L^2(\mathbb{T}, p_\psi^2)$. Hence we have proved that, in the set sense,

$$L^2(\mathbb{T}, p_\psi) \subseteq L^2(\mathbb{T}, p_\psi^2).$$

Now, observe that Lemma 4.11 holds for weights w, v with exactly the same zero sets (the proof goes verbatim with $1/w$ considered outside the zero set). Therefore, this minor modification of Lemma 4.11 shows that there exists a constant $0 < C < \infty$ so that $p_\psi^2 \le Cp_\psi$. This obviously implies (a) and thus (b).

Finally, we prove that (b) implies (a). Let us assume that (b) is valid. Consider $g \in L^2(\mathbb{T}, p_\psi)$. Since $\mathcal{I}_\psi(g) \in \langle\psi\rangle$, it follows by (b) that $p_\psi \bullet \mathcal{I}_\psi(g) \in \langle\psi\rangle$, which implies that $gp_\psi \in L^2(\mathbb{T}, p_\psi)$, i.e. that $g \in L^2(\mathbb{T}, p_\psi^3)$. Hence, we have shown that, at the set level,

$$L^2(\mathbb{T}, p_\psi) \subseteq L^2(\mathbb{T}, p_\psi^3).$$

Using the same modification of the Lemma 4.11 we used above, we conclude that there exists a C so that $p_\psi^3 \le Cp_\psi$. This then implies that $p_\psi \le \sqrt{C}$, which guarantees (a).

□

It is easy to compute the $\ell^2(\mathbb{Z})$-norm of the sequence $(\langle\varphi, T_k\psi\rangle)_{k\in\mathbb{Z}}$ in the context of the previous theorem. With this purpose in mind, consider $\psi \in L^2(\mathbb{T})$ such that $p_\psi \le B$ almost everywhere. Take any $\varphi \in \langle\psi\rangle$; hence, there is a $g \in L^2(\mathbb{T}, p_\psi)$ such that $\mathcal{I}_\psi(g) = \varphi$ (or, equivalently, $\varphi = g \bullet \psi$). Since the inner product is continuous, using Theorem 6.5 we obtain

$$\langle p_\psi \bullet \varphi, \varphi\rangle_{L^2(\mathbb{R})} = \sum_{k\in\mathbb{Z}} \langle\varphi, T_k\psi\rangle\langle T_k\psi, \varphi\rangle = \sum_{k\in\mathbb{Z}} |\langle\varphi, T_k\psi\rangle_{L^2(\mathbb{R})}|^2. \tag{6.6}$$

Observe that the left side of (6.6) can be expressed in several ways:

$$\begin{aligned}\langle p_\psi \bullet \varphi, \varphi\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2 p_\psi(\xi)d\xi = \int_{\mathbb{T}} p_\varphi(\xi)p_\psi(\xi)d\xi \\ &= \int_{\mathbb{T}} |[\varphi, \psi](\xi)|^2 d\xi = \int_{\mathbb{T}} |g(\xi)p_\psi(\xi)|^2 d\xi.\end{aligned} \tag{6.7}$$

The function $\xi \mapsto p_\varphi(\xi)\cdot p_\psi(\xi)$ is in $L^1(\mathbb{T})$ since p_ψ is bounded above and $p_\varphi \in L^1(\mathbb{T})$.

Remark 6.8. (i) Comparing the sums in (6.1) and (6.2) we can see that the "natural" choice of coefficients, $\langle\varphi, T_k\widetilde{\psi}\rangle$, is actually at a disadvantage. For convergence of (6.2), the condition $p_\psi \le B$ is sufficient, while for (6.1), one requires minimality of $\mathcal{B}_\psi$ which excludes any possibility of redundancy for the system $\mathcal{B}_\psi$. Under the requirement that $p_\psi \le B$, the sums in (6.2) converge unconditionally for every $\varphi \in \langle\psi\rangle$; assuming minimality for $\mathcal{B}_\psi$, the sums in (6.1) may or may not converge for a particular $\varphi \in \langle\psi\rangle$ and, even if they do converge, the convergence may only be conditional. For (6.2) (with $p_\psi \le B$), the coefficients are always in $\ell^2(\mathbb{Z})$, while, for (6.1), the coefficients may not be in $\ell^2(\mathbb{Z})$. The price we pay in (6.2) is that the limit may not be φ. However, we do have an explicit form for the limit.

(ii) Observe that (c) in Theorem 6.5 is the "dual property" to the (RF)-property.

(iii) If $\psi \in L^2(\mathbb{R})$ is a Parseval frame wavelet (see [**ŠSW08**] for definitions and details), then $p_\psi \le 1$.

■

In order to completely understand Theorem 6.5, we need to explore what happens with various conditions in individual cases (notice that the statements in the theorem are given in the form "for every $\varphi \in \langle\psi\rangle$"). So, consider $\psi \in L^2(\mathbb{R})$, $\varphi \in \langle\psi\rangle$, and $g \in L^2(\mathbb{T}, p_\psi)$ such that $\mathcal{I}_\psi(g) = \varphi$. As we have seen in Lemma 6.3, the convergence of the sum in (6.1) immediately implies that $p_\psi \bullet \varphi \in \langle\psi\rangle$, i.e., that

$$gp_\psi \in L^2(\mathbb{T}, p_\psi). \tag{6.9}$$

Let us explore the implications of the condition (6.9). Observe first that, taken by itself, this condition will not imply that $gp_\psi \in L^2(\mathbb{T})$ — take, for example, ψ such that $p_\psi > 0$ and $p_\psi^{-1} \notin L^1(\mathbb{T})$, then choose $g := p_\psi^{-3/2}$ to obtain $|g|^2 p_\psi^3 \in L^1(\mathbb{T})$ and $|g|^2 p_\psi^2 = p_\psi^{-1} \notin L^1(\mathbb{T})$. However, if we know, as assumed above, the condition $g \in L^2(\mathbb{T}, p_\psi)$ also holds, then (6.9) implies that $|g|\sqrt{p_\psi} \in L^2(\mathbb{T})$ and $|g|p_\psi^{3/2} \in L^2(\mathbb{T})$, which implies that $|g|^2 p_\psi^2 \in L^1(\mathbb{T})$, i.e. that

$$gp_\psi \in L^2(\mathbb{T}). \tag{6.10}$$

Obviously, since $\langle \varphi, T_k\psi\rangle = \widehat{gp_\psi}(-k)$, one has that (6.10) is equivalent (under the assumption $g \in L^2(\mathbb{T}, p_\psi)$) to the condition

$$(\langle \varphi, T_k\psi\rangle)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}). \tag{6.11}$$

However, although (6.10) (or, equivalently, (6.11)) will ensure that

$$\sum_{k\in\mathbb{Z}} \langle \varphi, T_k\psi\rangle e^{-2\pi i k\xi} = gp_\psi \text{ in } L^2(\mathbb{T}),$$

it will not necessarily ensure the convergence in (6.1). Using Lemma 6.3, this follows from the fact that (6.10) does not imply (6.9) — take $\psi \in L^2(\mathbb{R})$ such that $p_\psi > 0$ and $p_\psi \notin L^2(\mathbb{T})$, then consider $g := p_\psi^{-1/2}$ to conclude that $|g|^2 p_\psi^2 = p_\psi \in L^1(\mathbb{T})$, but $|g|^2 p_\psi^3 = p_\psi^2 \notin L^1(\mathbb{T})$.

Let us briefly explore the mapping

$$g \mapsto gp_\psi \tag{6.12}$$

within the space $L^2(\mathbb{T}, p_\psi)$. First of all, it is not difficult to check that the domain of the mapping is a subspace of $L^2(\mathbb{T}, p_\psi)$ given by

$$\left\{ \frac{h}{p_\psi}\chi_{U_{\langle\psi\rangle}} : h \in L^2(\mathbb{T}, p_\psi) \cap L^2(\mathbb{T}, p_\psi^{-1}\chi_{U_{\langle\psi\rangle}}) \right\}. \tag{6.13}$$

Observe that the measures $p_\psi d\xi$ and $p_\psi^{-1}(\xi)\chi_{U_{\langle\psi\rangle}}(\xi)d\xi$ generate the same family of null sets (however, the second measure might not be a finite measure). The following example shows that the subspace given by (6.13) is never a trivial one (unless ψ itself is trivial).

Example 6.14. Consider $\psi \not\equiv 0$ with $\psi \in L^2(\mathbb{R})$. Define the function h on $\mathbb{T}$ by

$$h(\xi) := \begin{cases} \frac{1}{\sqrt{p_\psi(\xi)}} & \text{if } \xi \in \{p_\psi > 1\} \\ \sqrt{p_\psi(\xi)} & \text{if } \xi \in \{p_\psi \le 1\} \end{cases}.$$

Obviously $h \le 1$ and is non-trivial since $h(\xi) > 0$ for $\xi \in U_{\langle\psi\rangle}$, which is presumed to be a set of positive Lebesgue measure. The integrals of $h^2 p_\psi$ over $\{p_\psi \le 1\}$ and of $h^2 p_\psi^{-1}\chi_{U_{\langle\psi\rangle}}$ over $\{p_\psi > 1\}$ are obviously finite. Moreover, one has

$$\int_{\{p_\psi>1\}} h^2 p_\psi = \int_{\{p_\psi>1\}} \frac{1}{p_\psi} p_\psi \le \int_{\mathbb{T}} 1 = 1$$

and

$$\int_{\{p_\psi\le 1\}} h^2 \frac{1}{p_\psi}\chi_{U_{\langle\psi\rangle}} = \int_{U_{\langle\psi\rangle}\cap\{p_\psi\le 1\}} p_\psi \frac{1}{p_\psi} \le \int_{\mathbb{T}} 1 = 1.$$

It follows that $hp_\psi^{-1}\chi_{U_{\langle\psi\rangle}}$ is a non-trivial element of the subspace given by (6.13).

$\diamondsuit$

Consider any $h \in L^2(\mathbb{T}, p_\psi) \cap L^2(\mathbb{T}, p_\psi^{-1}\chi_{U_{\langle\psi\rangle}})$. We obtain

$$\begin{aligned}\int_{\mathbb{T}} |h|^2 \chi_{U_{\langle\psi\rangle}} &= \int_{\{p_\psi \geq 1\}} |h|^2 + \int_{U_{\langle\psi\rangle}\cap\{p_\psi<1\}} |h|^2 \\ &\leq \int_{\{p_\psi \geq 1\}} |h|^2 p_\psi + \int_{U_{\langle\psi\rangle}\cap\{p_\psi<1\}} |h|^2 \frac{1}{p_\psi} \\ &\leq \int_{\mathbb{T}} |h|^2 p_\psi + \int_{\mathbb{T}} |h|^2 \frac{1}{p_\psi}\chi_{U_{\langle\psi\rangle}} \\ &< \infty.\end{aligned}$$

Hence we always have (in the sense of set containment)

$$L^2(\mathbb{T}, p_\psi) \cap L^2(\mathbb{T}, p_\psi^{-1}\chi_{U_{\langle\psi\rangle}}) \subseteq L^2(\mathbb{T}, \chi_{U_{\langle\psi\rangle}}). \tag{6.15}$$

Observe that, for a maximal shift-invariant space $\langle\psi\rangle$ we have the set on the right side of (6.15) is $L^2(\mathbb{T})$. Using these observations together with Theorem 6.5, parts (a) and (b), it is easy to see that the following result holds.

Corollary 6.16. *The domain given in (6.13) equals the entire space $L^2(\mathbb{T}, p_\psi)$ if and only if there is a constant $B \in (0, \infty)$ so that $p_\psi \leq B$ almost everywhere. Furthermore, if this is the case, then*

$$L^2(\mathbb{T}, p_\psi^{-1}\chi_{U_{\langle\psi\rangle}}) \subseteq L^2(\mathbb{T}, \chi_{U_{\langle\psi\rangle}}) \subseteq L^2(\mathbb{T}, p_\psi).$$

It is also easy to see that the image of the mapping given in (6.12) is equal to the following subspace of $L^2(\mathbb{T}, p_\psi)$:

$$L^2(\mathbb{T}, p_\psi) \cap L^2(\mathbb{T}, p_\psi^{-1}\chi_{U_{\langle\psi\rangle}}). \tag{6.17}$$

Obviously then, the image of the mapping in (6.12) is equal to the entire space $L^2(\mathbb{T}, p_\psi)$ (or, in other words, the mapping in (6.12) is surjective) if and only if

$$L^2(\mathbb{T}, p_\psi) \subseteq L^2(\mathbb{T}, p_\psi^{-1}\chi_{U_{\langle\psi\rangle}}). \tag{6.18}$$

Using now the modification of Lemma 4.11 for weights with the same zero sets (as was already used in the proof of Theorem 6.5) we obtain that (6.18) is equivalent to

(6.19)
there exists a constant $A \in (0, \infty)$ so that $p_\psi \geq A\chi_{U_{\langle\psi\rangle}}$ almost everywhere.

Observe also that if (6.19) holds, then

$$L^2(\mathbb{T}, p_\psi) \subseteq L^2(\mathbb{T}, \chi_{U_{\langle\psi\rangle}}) \subseteq L^2(\mathbb{T}, p_\psi^{-1}\chi_{U_{\langle\psi\rangle}}). \tag{6.20}$$

Since the mapping given in (6.12) is always injective (within $L^2(\mathbb{T}, p_\psi)$), we have proved the following result.

Corollary 6.21. *The mapping given in (6.12) is a bijection from $L^2(\mathbb{T}, p_\psi)$ onto $L^2(\mathbb{T}, p_\psi)$ if and only if there exist constants $0 < A \leq B < \infty$ so that*

$$A\chi_{U_{\langle\psi\rangle}} \leq p_\psi \leq B\chi_{U_{\langle\psi\rangle}} \text{ almost everywhere.} \tag{6.22}$$

We assume that our reader is familiar with the notion of a *frame* (see, for example, [**HW96**] and [**HŠWW10b**] for definitions and further references). It is well-known (see, for example, [**HŠWW10b**]) that (6.22) is equivalent to $\mathcal{B}_\psi$ being a frame for $\langle\psi\rangle$. If, moreover, $A = B = 1$, then $\mathcal{B}_\psi$ is called a *Parseval frame* (it is also fairly common, especially in older literature, for this to be called a *normalized* *tight frame*; the term *tight frame* relaxes the condition in a Parseval frame to only requiring that $A = B$). Obviously, if $\langle\psi\rangle$ is a maximal shift-invariant space, then (6.22) is equivalent to $\mathcal{B}_\psi$ being a Riesz basis for $\langle\psi\rangle$ (the case $A = B = 1$ then corresponds to $\mathcal{B}_\psi$ being an orthonormal basis for $\langle\psi\rangle$). In conclusion, using Theorem 6.5, we can rephrase Corollary 6.21 as follows.

COROLLARY 6.23. *Let $\psi \in L^2(\mathbb{R})$. Then we have*

(i) for every $\varphi \in \langle\psi\rangle$

$$p_\psi \bullet \varphi = \sum_{k\in\mathbb{Z}} \langle \varphi, T_k\psi\rangle T_k\psi;$$

and

(ii) for every $\eta \in \langle\psi\rangle$ there exists a unique $\varphi \in \langle\psi\rangle$ so that

$$\eta = \sum_{k\in\mathbb{Z}} \langle \varphi, T_k\psi\rangle T_k\psi$$

if and only if $\mathcal{B}_\psi = \{T_k\psi : k \in \mathbb{Z}\}$ is a frame for $\langle\psi\rangle$. Furthermore if this is the case, then (on the set level)

$$L^2(\mathbb{T}, p_\psi^{-1}\chi_{U_{\langle\psi\rangle}}) = L^2(\mathbb{T}, \chi_{U_{\langle\psi\rangle}}) = L^2(\mathbb{T}, p_\psi).$$

7. Redundancy Remarks

For non-trivial $\psi \in L^2(\mathbb{R})$, the set $\mathcal{B}_\psi$ is linearly independent, so the only way to achieve redundancy is via infinite sums, i.e.

$$\lim_{n\to\infty} \sum_{|k|\le n} c_k T_k\psi \equiv 0 \tag{7.1}$$

so that $(c_k)_{k\in\mathbb{Z}}$ is a non-trivial sequence of scalars. Particularly useful are non-trivial unconditional sums of the form

$$\sum_{k\in\mathbb{Z}} c_k T_k\psi \equiv 0. \tag{7.2}$$

Observe that such sums can be used "to recover" any coefficients since we have, for every $\ell \in \mathbb{Z}$,

$$\text{if } \lim_{n\to\infty} \sum_{|k|\le n} d_k T_k\psi = \varphi \in \langle\psi\rangle, \text{ then } \lim_{n\to\infty} \sum_{|k|\le n} (d_k + c_{k+\ell}) T_k\psi = \varphi. \tag{7.3}$$

However, the unconditional convergence in (7.2) implies that $(c_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$; as a consequence non-trivial sums of the form (7.2) are possible only if the Lebesgue measure of the zero set of p_ψ is positive, i.e.

$$|Z_{\langle\psi\rangle}| > 0, \tag{7.4}$$

i.e. if $\langle\psi\rangle$ is *not* a maximal principal shift-invariant space.

REMARK 7.5. As we have seen in Section 5, there is no redundancy of any kind if $\mathcal{B}_\psi$ is maximal, i.e. if $p_\psi^{-1} \in L^1(\mathbb{T})$. In particular, this also shows that the (B)-property of $\mathcal{B}_\psi$ is incompatible with the idea of redundancy. The same holds for the (RF)-property of $\mathcal{B}_\psi$.

Let us also mention that for a maximal principal shift-invariant space $\langle\psi\rangle$ such that $\mathcal{B}_\psi$ is not minimal, we have non-trivial sums of the form (7.1), but the coefficients $(c_k)_{k\in\mathbb{Z}} \notin \ell^2(\mathbb{Z})$. Unfortunately, we cannot say much more about these types of spaces $\langle\psi\rangle$ (see also Remark 5.5).

■

REMARK 7.6. In order to understand redundancy issues for maximal principal shift-invariant spaces $\langle\psi\rangle$, it may be interesting to consider the following open question. Given $\psi \in L^2(\mathbb{R})$ such that $\langle\psi\rangle$ is a maximal principal shift-invariant space, is there a non-trivial sequence

$$(c_k)_{k\in\mathbb{Z}} \in \bigcap_{p>2} \ell^p(\mathbb{Z})$$

such that

$$\lim_{n\to\infty} \sum_{|k|\le n} c_k T_k \psi \equiv 0?$$

■

We turn our attention to the non-maximal case, i.e. to a principal shift-invariant space $\langle\psi\rangle$ such that p_ψ satisfies (7.4). If there exists a sum of the form (7.2), i.e. (c_k) is non-trivial and $\sum_{k\in\mathbb{Z}} c_k T_k\psi \equiv 0$, then we must have $(c_k) \in \ell^2(\mathbb{Z})$. It follows that there exists $f \in L^2(\mathbb{Z})$ with $(c_k)_{k\in\mathbb{Z}}$ being its Fourier coefficients. By the Carleson theorem, for Lebesgue-almost everywhere $\xi \in \mathbb{T}$,

$$\sum_{|k|\le n} c_{-k} e^{2\pi i k\xi} \to f(\xi). \tag{7.7}$$

Since, by (7.2), we have that

$$\left\| \sum_{|k|\le n} c_{-k} e^{2\pi i k\xi} \right\| \to 0,$$

we conclude that f has to satisfy (in the almost everywhere sense)

$$\text{ssupp } (f) \subseteq Z_{\langle\psi\rangle} \tag{7.8}$$

This means that our candidates for sums of the form (7.2) are to be selected via functions $f \in L^2(\mathbb{T})$ such that (7.8) holds. Unfortunately, in general such a function does not have to generate convergent sums in (7.2). This obviously changes if p_ψ is bounded above. In other words, we have the following result.

PROPOSITION 7.9. *Let $\psi \in L^2(\mathbb{R})$ such that $\psi \not\equiv 0$, $|Z_{\langle\psi\rangle}| > 0$ and there exists $B \in (0,\infty)$ so that $p_\psi \le B$. Then $\sum_{k\in\mathbb{Z}} c_k T_k\psi \equiv 0$ (in the sense of $L^2(\mathbb{R})$-norm) with non-trivial coefficients $(c_k)_{k\in\mathbb{Z}}$ if and only if $f \in L^2(\mathbb{T})$ such that ssupp $(f) \subseteq Z_{\langle\psi\rangle}$ and, for every $k \in \mathbb{Z}$, $\widehat{f}(k) = c_{-k}$.*

REMARK 7.10. Despite the fact that Proposition 7.9 completely characterizes "unconditional redundancy sums" in the case of a bounded p_ψ, the actual choice of such sums may be quite limited. It will depend on the properties of the set $Z_{\langle\psi\rangle}$. Recall that $\ell^2(\mathbb{Z})$ contains every $\ell^p(\mathbb{Z})$ for $1 \le p < 2$, but, despite ψ satisfying (7.4) we may not be able to select our coefficients within some of these spaces. More precisely (consult Remark 5.5), assuming that p_ψ is bounded above, we have (see [**ŠS12**]) that $\mathcal{B}_\psi$ is $\ell^p(\mathbb{Z})$-linearly independent, $1 \le p \le 2$, if and only if $Z_{\langle\psi\rangle}$ is an $\ell^p(\mathbb{Z})$-*set of uniqueness* (see [**ŠS12**] and [**Kat04**] for definitions and details).

Recall (see [**Kat04**] and [**HK65**]) that for every $p < 2$ there are ℓ^p-sets of uniqueness of positive Lebesgue measure. For $1 \le q < p \le 2$, every ℓ^p-set of uniqueness is also an ℓ^q-set of uniqueness, and there exists an ℓ^q-set of uniqueness which is not an ℓ^p-set of uniqueness. Furthermore, given $\varepsilon > 0$, there exists a measurable set $E \subseteq \mathbb{T}$ such that $1-\varepsilon < |E| \leq 1$ and E is an ℓ^p-set of uniqueness for every $p < 2$; in particular, if $Z_{\langle\psi\rangle}$ is such a set, then candidates for the sum of the form (7.2) could be selected only among

$$(c_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \setminus \left(\bigcup_{p<2} \ell^p(\mathbb{Z})\right).$$

■

Let us also comment on the possibility that every principal shift-invariant space $\langle\psi\rangle$ with the property (7.4) can be continuously embedded into infinitely many maximal principal shift-invariant space. To simplify matters, we can define a particular choice via the following function $\psi_{\max} \in L^2(\mathbb{R})$, defined by

$$\widehat{\psi_{\max}}(\xi) := \begin{cases} 1 & \text{if } \xi \in Z_{\langle\psi\rangle} \cap [0,1] \\ \widehat{\psi}(\xi) & \text{otherwise} \end{cases} . \tag{7.11}$$

Obviously, then, $\langle\psi\rangle \subseteq \langle\psi_{\max}\rangle$ and $\langle\psi_{\max}\rangle$ is a maximal principal shift-invariant space. Furthermore, for $\varphi \in \langle\psi_{\max}\rangle$ we define its projection on $\langle\psi\rangle$ via

$$\chi_{U_{\langle\psi\rangle}} \bullet \varphi; \tag{7.12}$$

see [**LWW15**] for the related idea of the projection. If p_ψ is bounded above, then our "redundancy sums" candidates are also given via

$$\chi_{Z_{\langle\psi\rangle}} \bullet \varphi, \varphi \in \langle\psi_{\max}\rangle. \tag{7.13}$$

Particularly nice is the case when $\mathcal{B}_\psi$ is a frame for $\langle\psi\rangle$. In that case, $L^2(\mathbb{T}, p_{\psi_{\max}})$ can be identified, in the set sense, with

$$L^2(\mathbb{T}, \chi_{U_{\langle\psi\rangle}}) \oplus L^2(\mathbb{T}, \chi_{Z_{\langle\psi\rangle}}). \tag{7.14}$$

Every $\varphi \in \langle\psi\rangle$ can be represented as $\mathcal{I}_\psi(g)$, for some $g \in L^2(\mathbb{T}, \chi_{U_{\langle\psi\rangle}})$, and expanded in terms of $\mathcal{B}_\psi$ as an unconditional sum with $\ell^2(\mathbb{Z})$-coefficients, and every "unconditional redundancy sum" of the form (7.2) is given via some $h \in L^2(\mathbb{T}, \chi_{Z_{\langle\psi\rangle}})$.

CHAPTER 2

MRA Structure

1. Dilations and Shift-invariant Spaces

We shall use D to denote the *dyadic dilation operator*; that is, $D : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the unitary operator given by $D\psi(x) = \sqrt{2}\psi(2x)$ for $\psi \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. The operator $T = T_1$ does not commute with D, but the two operators do share the following relationship:

$$TD = DT^2. \tag{1.1}$$

As in the case of T, we shall denote the k-th integer power of D by D_k. Since the order in which we apply T and D matters, we shall introduce the following notation. For $j, k \in \mathbb{Z}$,

$$\begin{aligned} \psi_{jk} &:= D_j T_k \psi \\ \psi^{jk} &:= T_k D_j \psi. \end{aligned} \tag{1.2}$$

Thus for $x \in \mathbb{R}$, we have $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$ and $\psi^{jk}(x) = 2^{j/2}\psi(2^j(x-k))$. In particular,

$$\psi^{jk} = \psi_{j,2^j k}. \tag{1.3}$$

The family $\{2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ is the main object of our study. This family is, of course, fundamentally important in the study of wavelet theory, and we assume that our readers are familiar with the basic notions from the theory of wavelets, such as bases, orthonormal wavelets, MRA wavelets, frames, Parseval frames, Parseval frame wavelets, and the like. The theory of wavelets has been thoroughly studied by many authors; standard references include the books by I. Daubechies [**Dau92**], Y. Meyer [**Mey90**], R. Coifman and Y. Meyer [**MC97**], C.K. Chui [**Chu92**], and M.V. Wickerhauser [**Wic94**].

In this section we shall review and further explore the action of D on shift-invariant spaces. Recall first (see, for example, Theorem 3.3 in [**Bow00**]) that, for every shift-invariant space $V \subset L^2(\mathbb{R})$, there exists a countable family $\mathcal{F} \subset L^2(\mathbb{R})$ such that V is equal to the following orthogonal sum:

$$\bigoplus_{f \in \mathcal{F}} \langle f \rangle. \tag{1.4}$$

Given a unitary operator $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, we obtain

$$U(V) = \bigoplus_{f \in \mathcal{F}} U(\langle f \rangle). \tag{1.5}$$

Hence, U "preserves the shift-invariant space property" if and only if $U(\langle \psi \rangle)$ is a shift-invariant space for every $\psi \in L^2(\mathbb{R})$. Since

$$U(\langle \psi \rangle) = \overline{\text{span}}\langle UT_k\psi : k \in \mathbb{Z}\},$$

it follows that $U(\langle\psi\rangle)$ is a shift-invariant space if and only if, for every $\ell \in \mathbb{Z}$,

$$TUT_\ell\psi \in \overline{\operatorname{span}}\langle UT_k\psi : k \in \mathbb{Z}\}. \tag{1.6}$$

For example, if $U = D_{-1}$, then $TD_{-1}T_\ell\psi = D_{-1}T_{\ell+\frac{1}{2}}\psi$ — obviously $\ell + \frac{1}{2} \notin \mathbb{Z}$ — and it is not difficult to see that $D_{-1}(\langle\psi\rangle)$ is not necessarily a shift-invariant space. However, in the case of $U = D$, we obtain $TDT_\ell\psi = DT_{\ell+2}\psi$, which shows that $D(\langle\psi\rangle)$ is always a shift-invariant space. Consequently, we obtain the well-known result that, for every shift-invariant space $V \subset L^2(\mathbb{R})$

$$D(V) \text{ is a shift-invariant space.} \tag{1.7}$$

There is the following well-known formula (see, for example, [**BR05**]) for the dimension function:

$$\dim_{D(V)}(2\xi) = \dim_V(\xi) + \dim_V(\xi + 1/2), \tag{1.8}$$

for $\xi \in \mathbb{R}$. Indeed, we shall see that a 2-valued mapping

$$\xi \mapsto \frac{\xi}{2} \text{ and } \frac{\xi}{2} + \frac{1}{2}, \tag{1.9}$$

for $\xi \in \mathbb{T}$ plays an important role in our analysis.

Let us explore $D(\langle\psi\rangle)$ for $\psi \in L^2(\mathbb{R})$ in detail. First of all, $D(\langle\psi\rangle)$ is the shift-invariant space generated by $D\psi$ and $DT\psi$, i.e.

$$D(\langle\psi\rangle) = \langle\psi_{10}, \psi_{11}\rangle. \tag{1.10}$$

Identifying $\mathbb{T}$ with $[0, 1)$, we obtain by (1.8) that for $0 \le \xi < 1$,

$$\dim_{D(\langle\psi\rangle)} = \chi_{2U_{\langle\psi\rangle}\cap[0,1)}(\xi) + \chi_{[2U_{\langle\psi\rangle}\cap[1,2)]-1}(\xi). \tag{1.11}$$

Using the notation $U_L = U_L(\langle\psi\rangle) := 2U_{\langle\psi\rangle} \cap [0, 1)$ and $U_R = U_R(\langle\psi\rangle) := [2U_{\langle\psi\rangle} \cap [1, 2)] - 1$, we obtain, for $0 \le \xi < 1$,

$$\dim_{D(\langle\psi\rangle)}(\xi) = \begin{cases} 0 & \text{if } \xi \notin U_L \cup U_R \\ 1 & \text{if } \xi \in U_L \triangle U_R \\ 2 & \text{if } \xi \in U_L \cap U_R \end{cases} \tag{1.12}$$

Here, $\triangle$ denotes the symmetric difference of two sets. As always, the dim function is 1-periodic and can be extended from (1.12) to $\mathbb{R}$.

Remark 1.13. Observe that one can construct $\dim_{D(\langle\psi\rangle)}$ knowing only $U_{\langle\psi\rangle}$. Indeed, for $0 \le \xi < 1$,

$$\dim_{D(\langle\psi\rangle)}(\xi) = 0 \text{ if and only if } \frac{\xi}{2}, \frac{\xi}{2} + \frac{1}{2} \notin U_{\langle\psi\rangle};$$

$$\dim_{D(\langle\psi\rangle)}(\xi) = 1 \text{ if and only if exactly one of } \frac{\xi}{2}, \frac{\xi}{2} + \frac{1}{2} \text{ is in } U_{\langle\psi\rangle};$$

$$\dim_{D(\langle\psi\rangle)}(\xi) = 2 \text{ if and only if } \frac{\xi}{2}, \frac{\xi}{2} + \frac{1}{2} \in U_{\langle\psi\rangle}.$$

Some specific cases now follow easily. ■

Corollary 1.14. *Let* $\psi \in L^2(\mathbb{R})$.

(a) $\dim_{D(\langle\psi\rangle)}$ *takes only the values* 0 *and* 2 *if and only if* $U_{\langle\psi\rangle}$ *is* $1/2$*-periodic.*
(b) $\dim_{D(\langle\psi\rangle)} \equiv 2$ *if and only if* $U_{\langle\psi\rangle} = \mathbb{R}$ *(which is equivalent to* $\langle\psi\rangle$ *being a maximal shift-invariant space). In this case,* $\dim_{D_j(\langle\psi\rangle)} \equiv 2^j$*, for every* $j \in \mathbb{N} \cup \{0\}$.

The following examples demonstrate that various other configurations for $\dim_{D(\langle\psi\rangle)}$ are possible.

EXAMPLE 1.15. Consider $\psi \in L^2(\mathbb{R})$ such that $U_{\langle\psi\rangle} \cap [0,1) = [1/4, 3/4)$ — for example, one could take $\widehat{\psi} = \chi_{[1/4,3/4)}$. Then one obtains $U_L = [1/2, 1)$ and $U_R = [0, 1/2)$, i.e.

$$\dim_{D(\langle\psi\rangle)} \equiv 1.$$

It follows that $D(\langle\psi\rangle)$ is a maximal principal shift-invariant space. By Corollary 1.14b we obtain, for every $j \in \mathbb{N}$,

$$\dim_{D_j(\langle\psi\rangle)} \equiv 2^{j-1}$$

$\diamondsuit$

EXAMPLE 1.16. Consider $\psi \in L^2(\mathbb{R})$ such that $U_{\langle\psi\rangle} \cap [0,1) = [a,b) \cup [c,d)$, where $0 \le a \le b \le 1/2 \le c \le d \le 1$. We denote the (essential) range of $\dim_{D(\langle\psi\rangle)}$ by $\mathcal{R} \subset \{0,1,2\}$. The following choices of a, b, c, and d provide us with all possible options for $\mathcal{R}$.

- $a = b$ and $c = d$ yields $\mathcal{R} = \{0\}$ — obviously this means $\psi \equiv 0$.
- $a = 1/4$, $b = c$ and $d = 3/4$ yields $\mathcal{R} = \{1\}$. This is Example 1.15.
- $a = 0$, $b = c = 1/2$, and $d = 1$ yields $\mathcal{R} = \{2\}$.
- $a = 0$, $b = 1/3$, $c = 2/3$, and $d = 1$ yields $\mathcal{R} = \{1,2\}$.
- $a = 1/8$, $b = 1/4$, $c = 3/4$, and $d = 7/8$ yields $\mathcal{R} = \{0,1\}$.
- $a = 1/4$, $b = 1/3$, $c = 3/4$, and $d = 5/6$ yields $\mathcal{R} = \{0,2\}$.
- $a = 1/4$, $b = 1/3$, $c = 2/3$, and $d = 5/6$ yields $\mathcal{R} = \{0,1,2\}$.

$\diamondsuit$

REMARK 1.17. Recall the notation $U_V = \{\xi \in \mathbb{R} : \dim_V(\xi) > 0\}$ for any shift-invariant space V. In comparing $U_{\langle\psi\rangle}$ and $U_{D(\langle\psi\rangle)}$, the intuition is that "the action of D increases the measure of the set U_V". We give a precise statement here. Consider the partition of the set $U_{\langle\psi\rangle} \cap [0,1)$ given by $\{U_{\langle\psi\rangle} \cap [0,1/2), U_{\langle\psi\rangle} \cap [1/2,1)\}$. Since $U_{D(\langle\psi\rangle)} \cap [0,1) = U_L \cup U_R$, we obtain

$$\begin{aligned} |U_L \cup U_R| &\ge \max\{|U_L|, |U_R|\} \\ &= 2\max\{|U_{\langle\psi\rangle} \cap [0,1/2)|, |U_{\langle\psi\rangle} \cap [1/2,1)|\} \\ &\ge |U_{\langle\psi\rangle} \cap [0,1/2)| + |U_{\langle\psi\rangle} \cap [1/2,1)| \\ &= |U_{\langle\psi\rangle} \cap [0,1)|. \end{aligned}$$

This shows that, for every $\psi \in L^2(\mathbb{R})$,

$$|U_{D(\langle\psi\rangle)} \cap [0,1)| \ge |U_{\langle\psi\rangle} \cap [0,1)|. \tag{1.18}$$

Let us also observe the extreme case of this inequality. If $U_{\langle\psi\rangle}$ is 1/2-periodic, then (1.18) becomes an equality. If $\dim_{D(\langle\psi\rangle)} \le 1$, then

$$|U_{D(\langle\psi\rangle)} \cap [0,1)| = 2|U_{\langle\psi\rangle} \cap [0,1)|. \tag{1.19}$$

$\blacksquare$

Recall that (1.10) shows that $D(\langle\psi\rangle)$ contains and is generated (as a shift-invariant space) by two principal shift-invariant spaces, $\langle\psi_{10}\rangle$ and $\langle\psi_{11}\rangle$. Our next task is to describe the relationship between $\langle\psi_{10}\rangle$ and $\langle\psi_{11}\rangle$. We start with the following list of formulas which are well-known or elementary to derive. For all $\psi \in L^2(\mathbb{R})$ and almost every $\xi \in \mathbb{R}$, we have

$$\widehat{D\psi}(\xi) = \widehat{\psi_{10}}(\xi) = \frac{1}{\sqrt{2}}\widehat{\psi}(\xi/2) = D_{-1}\widehat{\psi}(\xi) \text{ and} \tag{1.20}$$
$$\widehat{DT\psi}(\xi) = \widehat{\psi_{11}}(\xi) = e^{-\pi i\xi}\frac{1}{\sqrt{2}}\widehat{\psi}(\xi/2) = e^{-\pi i\xi}\widehat{\psi_{10}}.$$

$$p_{\psi_{10}}(\xi) = p_{\psi_{11}}(\xi) = \frac{1}{2}\left(p_\psi(\xi/2) + p_\psi(\xi/2+1/2)\right), \tag{1.21}$$

and similar formulae hold for $\sigma_{\psi_{10}}$ and $\sigma_{\psi_{11}}$.

$$[\psi_{10}, \psi_{11}](\xi) = \frac{e^{\pi i\xi}}{2}\left(p_\psi(\xi/2) - p_\psi(\xi/2+1/2)\right). \tag{1.22}$$

$$|[\psi_{10}, \psi_{11}](\xi)|^2 = \frac{1}{4}\left(p_\psi(\xi/2) - p_\psi(\xi/2+1/2)\right)^2 \tag{1.23}$$
$$p_{\psi_{10}} \cdot p_{\psi_{11}} = \frac{1}{4}\left(p_\psi(\xi/2) + p_\psi(\xi/2+1/2)\right)^2.$$

Using these formulae and the notation and results of Section I.2, we obtain the following characterization of the relationship between $\langle\psi_{10}\rangle$ and $\langle\psi_{11}\rangle$:

Corollary 1.24. *If $\psi \in L^2(\mathbb{R})$, then*

(a)

$$\begin{aligned} Z_{D(\langle\psi\rangle)} &\subseteq U_{\langle\psi_{10}\rangle\perp\langle\psi_{11}\rangle} \\ &= \{\xi : p_\psi(\xi/2) = p_\psi(\xi/2+1/2)\} \\ &= 2\{\xi : p_\psi(\xi) \textit{ is } 1/2\textit{-periodic}\} \\ &\subseteq \{\xi : \dim_{D(\langle\psi\rangle)}(\xi) = 0 \textit{ or } 2\}; \end{aligned}$$

(b)

$$U_{\langle\psi_{10}\rangle\cap\langle\psi_{11}\rangle} = \{\xi : \mathit{dim}_{D(\langle\psi\rangle)}(\xi) = 1\};$$

(c)

$$\begin{aligned} U_{\langle\psi_{10}\rangle\angle\langle\psi_{11}\rangle} &= \{\xi : 0 \neq p_\psi(\xi/2) \neq p_\psi(\xi/2+1/2) \neq 0\} \\ &\subseteq \{\xi : \mathit{dim}_{D(\langle\psi\rangle)}(\xi) = 2\}. \end{aligned}$$

Corollary 1.25. *If $\psi \in L^2(\mathbb{R})$ and $\langle\psi\rangle$ is a maximal principal shift-invariant space, then*

$$\begin{aligned} |U_{\langle\psi_{10}\rangle\cap\langle\psi_{11}\rangle}| &= 0 \\ |Z_{D(\langle\psi\rangle)}| &= 0 \\ U_{\langle\psi_{10}\rangle\perp\langle\psi_{11}\rangle} &= \{\xi : p_\psi(\xi/2) = p_\psi(\xi/2+1/2)\} \\ U_{\langle\psi_{10}\rangle\angle\langle\psi_{11}\rangle} &= \{\xi : p_\psi(\xi/2) \neq p_\psi(\xi/2+1/2)\}. \end{aligned}$$

Corollary 1.26. *If $\psi \in L^2(\mathbb{R})$ and $\mathcal{B}_\psi$ forms a Parseval frame for $\langle\psi\rangle$, then*

$$\begin{aligned} |U_{\langle\psi_{10}\rangle\angle\langle\psi_{11}\rangle}| &= 0 \\ U_{\langle\psi_{10}\rangle\cap\langle\psi_{11}\rangle} &= U_L \triangle U_R \\ U_{\langle\psi_{10}\rangle\perp\langle\psi_{11}\rangle} &= \{\xi : \mathit{dim}_{D(\langle\psi\rangle)}(\xi) = 0 \textit{ or } 2\}, \end{aligned}$$

where $\triangle$ denotes the symmetric difference for sets.

COROLLARY 1.27. *If $\psi \in L^2(\mathbb{R})$, then $D(\langle\psi\rangle) = \langle\psi_{10}\rangle \oplus \langle\psi_{11}\rangle$ if and only if p_ψ is $1/2$-periodic.*

The last result deserves additional attention in the particular case where $\mathcal{B}_\psi$ forms a Parseval frame for $\langle\psi\rangle$ (or, equivalently, when $p_\psi = \chi_{U_{\langle\psi\rangle}}$). Using (1.21) it is easy to prove the following result.

COROLLARY 1.28. *If $\psi \in L^2(\mathbb{R})$ is such that $\mathcal{B}_\psi$ forms a Parseval frame for $\langle\psi\rangle$, then the following are equivalent:*
(a) $\mathcal{B}_{D\psi}$ forms a Parseval frame for $\langle D\psi\rangle$;
(b) $\mathcal{B}_{DT\psi}$ forms a Parseval frame for $\langle DT\psi\rangle$;
(c) $U_{\langle\psi\rangle}$ is $1/2$-periodic;
(d) $D(\langle\psi\rangle) = \langle D\psi\rangle \oplus \langle DT\psi\rangle$.
If any of the above equivalent conditions is fulfilled, then $\mathcal{B}_{D\psi} \cup \mathcal{B}_{DT\psi}$ forms a Parseval frame for $D(\langle\psi\rangle)$. If $\mathcal{B}_\psi$ forms an orthonormal basis for $\langle\psi\rangle$ (i.e. $U_{\langle\psi\rangle} = \mathbb{R}$ and is $1/2$-periodic), then $\mathcal{B}_{D\psi}$, $\mathcal{B}_{DT\psi}$, and $\mathcal{B}_{D\psi} \cup \mathcal{B}_{DT\psi}$ are, respectively, orthonormal bases for $\langle D\psi\rangle$, $\langle DT\psi\rangle$, and $D(\langle\psi\rangle) = \langle D\psi\rangle \oplus \langle DT\psi\rangle$.

REMARK 1.29. In order to understand previous results more completely, we add several observations.

(i) For every $\psi \in L^2(\mathbb{R})$ and for every $\varphi \in \langle\psi\rangle$, we have
$$\sum_{k\in\mathbb{Z}} |\langle\varphi, T_k\psi\rangle|^2 = \sum_{k\in\mathbb{Z}} |\langle D\varphi, DT_k\psi\rangle|^2 = \sum_{m\in\mathbb{Z}} |\langle D\varphi, T_m D\psi\rangle|^2 + \sum_{n\in\mathbb{Z}} |\langle D\varphi, T_n DT\psi\rangle|^2.$$
In particular, if $\mathcal{B}_\psi$ is a Parseval frame for $\langle\psi\rangle$, then $\mathcal{B}_{D\psi} \cup \mathcal{B}_{DT\psi}$ is a Parseval frame for $D(\langle\psi\rangle)$ (i.e. always, irrespective of the $1/2$-periodicity of $U_{\langle\psi\rangle}$).

(ii) Take $0 < \varepsilon < 1/2$ and $\psi \in L^2(\mathbb{R})$ such that $\widehat{\psi} = \chi_{[0,\varepsilon)}$. It is easy to see that $D(\langle\psi\rangle) = \langle D\psi\rangle = \langle DT\psi\rangle$, and, in particular, $\langle D\psi\rangle$ is not orthogonal to $\langle DT\psi\rangle$. Moreover, $\mathcal{B}_{D\psi}$ is not a Parseval frame for $\langle D\psi\rangle$, though $\mathcal{B}_{D\psi}$ is a tight frame for $\langle D\psi\rangle$ (with constant $1/2$).

(iii) It is easy to see that if $\mathcal{B}_\psi$ is a frame for $\langle\psi\rangle$ (with frame bounds $0 < A \leq B$), then $\mathcal{B}_{D\psi} \cup \mathcal{B}_{DT\psi}$ is a frame for $D(\langle\psi\rangle)$, with the same frame bounds A and B, while $\mathcal{B}_{D\psi}$ (or, respectively, $\mathcal{B}_{DT\psi}$) is a frame for $\langle D\psi\rangle$ (respectively, $\langle DT\psi\rangle$) with frame bounds between $A/2 < B$.

■

REMARK 1.30. Regarding the case of a maximal principal shift-invariant space $\langle\psi\rangle$, we observe that D is a bounded, invertible operator $D : \langle\psi\rangle \to D(\langle\psi\rangle)$ and that most properties of $\mathcal{B}_\psi$ are preserved.

(i) If $\mathcal{B}_\psi$ is a Riesz basis for $\langle\psi\rangle$, then $\mathcal{B}_{D\psi} \cup \mathcal{B}_{DT\psi} = D(\mathcal{B}_\psi)$ is a Riesz basis for $D(\langle\psi\rangle)$. Furthermore, it follows directly from (1.21) that, in this case, $\mathcal{B}_{D\psi}$ (respectively, $\mathcal{B}_{DT\psi}$) is a Riesz basis for $\langle D\psi\rangle$ (respectively, $\langle DT\psi\rangle$).

(ii) If $\mathcal{B}_\psi$ (ordering $\mathbb{Z}$ as before by $\mathbb{Z} = \{0, 1, -1, 2, -2, ...\}$) is a Schauder basis for $\langle\psi\rangle$, then $\mathcal{B}_{D\psi} \cup \mathcal{B}_{DT\psi}$ is a Schauder basis for $D(\langle\psi\rangle)$ (with an ordering of the form $\{D\psi, DT\psi, DT_{-1}\psi = T_{-1}DT\psi, TD\psi, T_{-1}D\psi, ...\}$). Using the fact that the sum of two A_2-weights is again an A_2-weight, it is easy to see directly from (1.21) that, in this case, $\mathcal{B}_{D\psi}$ (with an ordering of the form $\{D\psi, TD\psi, T_{-1}D\psi, T_2D\psi, T_{-2}D\psi, ...\}$) is a Schauder basis for $\langle D\psi\rangle$. The analogous statement holds for $\mathcal{B}_{DT\psi}$.

(iii) If $\mathcal{B}_\psi$ is a minimal system (within $\langle\psi\rangle$ and with the dual function $\widetilde{\psi} = p_\psi^{-1} \bullet \psi$), then $\mathcal{B}_{D\psi}$, $\mathcal{B}_{DT\psi}$, and $\mathcal{B}_{D\psi} \cup \mathcal{B}_{DT\psi}$ are minimal systems within, respectively, $\langle D\psi\rangle$, $\langle DT\psi\rangle$, and $D(\langle\psi\rangle)$. Observe that $D\widetilde{\psi}$ may serve as the dual function in all three cases, but it is not necessarily contained within $\langle D\psi\rangle$ (similarly for $\langle DT\psi\rangle$), so for $\mathcal{B}_{D\psi}$, we have $\widetilde{\psi_{10}} = p_{\psi_{10}}^{-1} \bullet \psi_{10}$ (analogously for $\widetilde{\psi_{11}}$).

(iv) As in (ii), the property of ℓ^2-linear independence is preserved, but with the proper ordering as described in (ii).

■

Given a function $\varphi \in \langle\psi\rangle$ such that

$$p_\psi \bullet \varphi = \sum_{k\in\mathbb{Z}} \langle \varphi, T_k\psi\rangle T_k\psi,$$

we obtain directly that (observe the Fourier transform of $D(p_\psi \bullet \varphi)$ is $p_\psi(\xi/2)\widehat{D\varphi}(\xi)$)

$$\begin{aligned} D(p_\psi \bullet \varphi) &= \sum_{k\in\mathbb{Z}} \langle D\varphi, \psi_{1k}\rangle \psi_{1k} \\ &= \sum_{\ell\in\mathbb{Z}} \langle D\varphi, T_\ell D\psi\rangle T_\ell D\psi + \sum_{m\in\mathbb{Z}} \langle D\varphi, T_m DT\psi\rangle T_m DT\psi, \end{aligned} \tag{1.31}$$

by splitting the first sum over k into sums over even and odd values of k, respectively.

Let us explore the condition of 1/2-periodicity of p_ψ. Given $\psi \in L^2(\mathbb{R})$, it is easy to find a generator φ of $\langle\psi\rangle$ such that $\mathcal{B}_\varphi$ is a Parseval frame for $\langle\psi\rangle$; simply use

$$\varphi := \left(\frac{1}{\sqrt{p_\psi}} \chi_{U_{\langle\psi\rangle}}\right) \bullet \psi. \tag{1.32}$$

In general, the positions of $\langle D\psi\rangle$ and $\langle DT\psi\rangle$ within $D(\langle\psi\rangle) = D(\langle\varphi\rangle)$ is different from the positions of $\langle D\varphi\rangle$ and $\langle DT\varphi\rangle$ therein. It is of interest to find out when it is that this interior structure of $D(\langle\psi\rangle)$ is unchanged.

Lemma 1.33. *Let $\psi \in L^2(\mathbb{R})$ and φ be given by (1.32). If p_ψ is 1/2-periodic, then $\langle D\psi\rangle = \langle D\varphi\rangle$. If $\langle\psi\rangle$ is a maximal principal shift-invariant space, then*

$$\langle D\psi\rangle = \langle D\varphi\rangle \text{ if and only if } p_\psi \text{ is } 1/2\text{-periodic.}$$

Proof. It is always true that

$$\begin{aligned} U_{\langle D\psi\rangle} &= Z^c_{\langle D\psi\rangle} \\ &= \{\xi : p_\psi(\xi/2) = 0 = p_\psi(\xi/2 + 1/2)\}^c \\ &= \{\xi : \xi/2 \notin U_{\langle\psi\rangle} \text{ and } \xi/2 + 1/2 \notin U_{\langle\psi\rangle}\}^c \\ &= U_{\langle D\varphi\rangle}. \end{aligned}$$

Observe that, in general, $U_{\langle D\psi\rangle}$ is not necessarily 1/2-periodic. Comparing $\widehat{D\psi}$ and $\widehat{D\varphi}$, we obtain

$$\widehat{D\psi}(\xi) = \sqrt{p_\psi(\xi/2)}\,\widehat{D\varphi}(\xi).$$

If p_ψ is 1/2-periodic, then, by Corollary 1.1.21b, we conclude that $\langle D\psi\rangle = \langle D\varphi\rangle$. If $\langle\psi\rangle$ is maximal, then $U_{\langle D\psi\rangle} = U_{\langle D\varphi\rangle} = \mathbb{R}$ (which is, of course, a 1/2-periodic set). If one also has that $\langle D\psi\rangle = \langle D\varphi\rangle$, then the same corollary guarantees that $\sqrt{p_\psi(\xi/2)}$ must be equal almost everywhere to a 1-periodic function, whence p_ψ must be 1/2-periodic. □

EXAMPLE 1.34. Consider any $\psi \in L^2(\mathbb{R})$ so that ssupp $(\widehat{\psi}) \subseteq [0, 1/2)$. Then p_ψ is not 1/2-periodic; $p_\psi \equiv 0$ on $[1/2, 1)$ and p_ψ is not identically zero on $[0, 1/2)$, unless ψ is the zero function. However, since $U_{\langle D\psi\rangle} = U_{\langle D\varphi\rangle}$, where φ is given by 1.32 from this ψ, and ssupp $(\widehat{D\psi}) =$ ssupp $(\widehat{D\varphi}) \subseteq [0, 1)$, we obtain, using Corollary 1.1.21b, that $\langle D\psi\rangle = \langle D\varphi\rangle$.

$\diamondsuit$

Applying now Corollary 1.27 and Lemma 1.33, it is easy to see that in the case where p_ψ is 1/2-periodic, transferring from our generator ψ to the one given in (1.32) does not disrupt the inner structure of $D(\langle\psi\rangle)$. More precisely, we have the following result.

THEOREM 1.35. *Let $\psi \in L^2(\mathbb{R})$ be such that p_ψ is 1/2-periodic. Let $\varphi := \left(p_\psi^{-1/2} \chi_{U_{\langle\psi\rangle}}\right) \bullet \psi$. Then*

(a) $\langle\psi\rangle = \langle\varphi\rangle$, $\langle D\psi\rangle = \langle D\varphi\rangle$, and $\langle DT\psi\rangle = \langle DT\varphi\rangle$;
(b) $D(\langle\psi\rangle) = \langle D\psi\rangle \oplus \langle DT\psi\rangle = \langle D\varphi\rangle \oplus \langle DT\varphi\rangle$;
(c) $dim_{D(\langle\psi\rangle)} = 2\chi_{2U_{\langle\psi\rangle}}$;
(d) For every $f \in \langle\psi\rangle$,

$$Df = \sum_{k\in\mathbb{Z}} \langle Df, \varphi_{1k}\rangle \varphi_{1k}$$

and

$$f = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{0k}\rangle \varphi_{0k}.$$

As much as the case of 1/2-periodicity of p_ψ is connected to orthogonality of the spaces $\langle\psi_{10}\rangle$ and $\langle\psi_{11}\rangle$, the case of "1/2-antiperiodicity" of p_ψ is connected to the equality of the spaces $\langle\psi_{10}\rangle$ and $\langle\psi_{11}\rangle$. Observe first that, by Corollary 1.25, this case is only possible within non-maximal principal shift-invariant spaces. We have the following result.

THEOREM 1.36. *If $\psi \in L^2(\mathbb{R})$, then the following are equivalent:*

(a) $\langle D\psi\rangle = \langle DT\psi\rangle$;
(b) $dim_{D(\langle\psi\rangle)} \leq 1$;
(c) $D(\langle\psi\rangle)$ is a principal shift-invariant space;
(d) $|[(U_{\langle\psi\rangle} \cap [0, 1/2)) + 1/2] \cap [U_{\langle\psi\rangle} \cap [1/2, 1)]| = 0$;
(e) $|U_{D(\langle\psi\rangle)} \cap [0, 1)| = 2|U_{\langle\psi\rangle} \cap [0, 1)|$;
(f) $\langle D\psi\rangle = D(\langle\psi\rangle)$;
(g) $D(\langle\psi\rangle) \subseteq \langle D\psi\rangle$;
(h) $\langle D\psi\rangle \subseteq \langle DT\psi\rangle$.

PROOF. Observe that (1.10) implies $(f) \Leftrightarrow (g)$ as well as $(a) \Rightarrow (f) \Rightarrow (h)$. Using Proposition 1.2.4 and the fact that $U_{\langle D\psi\rangle} = U_{\langle DT\psi\rangle}$ (see (1.21)), we obtain $(h) \Rightarrow (a)$. Certainly, $(b) \Leftrightarrow (c)$ and $(f) \Rightarrow (c)$. Remark 1.13 (or (1.12)) explains why $(b) \Leftrightarrow (d)$. Computations similar to those in Remark 1.17 show that (1.19) (which is the same equality as (e)) is equivalent to

$$|U_L \cup U_R| = |U_L| + |U_R|, \tag{1.37}$$

which in turn is equivalent to (d).

In order to complete the proof, it suffices to prove that $(d) \Rightarrow (a)$. Assume (d), and recall that $U_{\langle D\psi\rangle} = U_{\langle DT\psi\rangle}$. Hence, using Corollary 1.1.21b, in order to conclude (a), it is enough to find a 1-periodic, measurable function m so that $\psi_{11} = m \bullet \psi_{10}$. However, by (1.20), we know that $\widehat{\psi_{11}}(\xi) = e^{-\pi i\xi}\widehat{\psi_{10}}(\xi)$, and $e^{-\pi i\xi}$ is 2-periodic, not 1-periodic. Here is where (d) becomes crucial. Choose $m : \mathbb{R} \to \mathbb{C}$ to be 1-periodic such that, for $0 \le \xi < 1$,

$$m(\xi) := \begin{cases} e^{-i\pi\xi} & \text{if } \xi \in 2U_{\langle\psi\rangle} \\ e^{-i\pi(\xi+1)} & \text{if } \xi \in 2U_{\langle\psi\rangle} + 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.38}$$

This function is only well-defined since (d) holds (if (d) did not hold, the first two lines of the definition of $m(\xi)$ would conflict on a set of positive measure). Clearly, m is measurable. Observe also that this choice of m satisfies $\psi_{11} = m \bullet \psi_{10}$ — this is because when $\xi/2 \notin U_{\langle\psi\rangle}$, we have $\widehat{\psi_{11}}(\xi) = \widehat{\psi_{10}}(\xi) = 0$. □

Remark 1.39. Observe that property (d) in the previous theorem enables a simple construction of all $\psi \in L^2(\mathbb{R})$ such that $\langle D\psi\rangle = \langle DT\psi\rangle$. Take any $\varphi \in L^2(\mathbb{R})$ and any measurable sets A, B such that $A \subseteq [0, 1/2)$, $B \subseteq [1/2, 1)$, $|(A + 1/2) \cap B| = 0$ and $|A| + |B| \le 1/2$. Take any two families of measurable sets $\{A_n : n \in \mathbb{Z}\}$ and $\{B_n : n \in \mathbb{Z}\}$ so that, for every $n \in \mathbb{Z}$, $A_n \subseteq A$, $B_n \subseteq B$, $A = \bigcup_{n\in\mathbb{Z}} A_n$, and $B = \bigcup_{n\in\mathbb{Z}} B_n$. Define $\psi \in L^2(\mathbb{R})$ so that

$$\widehat{\psi} = \sum_{n\in\mathbb{Z}} \widehat{\phi}(\chi_{n+A_n} + \chi_{n+B_n}).$$

Such ψ has the property that $\langle D\psi\rangle = \langle DT\psi\rangle$, and every ψ with this property can be constructed using this algorithm. ■

Remark 1.40. It is perhaps worth noticing the following difference between the cases described in Theorem 1.35 and Theorem 1.36. In order to characterize the first theorem, one needs to know the values of p_ψ, while, for the second theorem, it is enough to know the set $U_{\langle\psi\rangle}$. This is essentially the consequence of the fact that $U_{\langle\psi_{10}\rangle\cap\langle\psi_{11}\rangle}$ is completely determined by $U_{\langle\psi\rangle}$, but determining $U_{\langle\psi_{10}\rangle\perp\langle\psi_{11}\rangle}$ and $U_{\langle\psi_{10}\rangle\angle\langle\psi_{11}\rangle}$ requires knowledge of p_ψ. ■

Finally, let us explore the effect of D on the lattice structure of $\langle\psi\rangle$ (recall Section 1.3). It is easy to see that, for every $\psi \in L^2(\mathbb{R})$ (see (1.1))

$$D(\langle\psi\rangle) = \langle D\psi, DT\psi\rangle = \langle D\psi, T_{1/2}D\psi\rangle = \langle D\psi\rangle_{(1/2)\mathbb{Z}}.$$

Using a simple induction argument, we obtain that, for every $j \in \mathbb{N} \cup \{0\}$,

$$D_j(\langle\psi\rangle) = \langle D_j\psi, T_{1/2^j}D_j\psi, T_{2/2^j}D_j\psi, ..., , T_{(2^j-1)/2^j}D_j\psi\rangle = \langle D_j\psi\rangle_{(1/2^j)\mathbb{Z}}. \tag{1.41}$$

Remark 1.42.
(i) Formula (1.41) provides us with yet another equivalent condition for Theorem 1.36: $\langle D\psi\rangle = \langle D\psi\rangle_{\frac{1}{2}\mathbb{Z}}$ (or, in other words, $2 \in \mathcal{T}_{\langle D\psi\rangle}$). Obviously, in this case $\langle D\psi\rangle$ is either of Type-2 or Type-3.
(ii) Obviously, $D_j(\langle\psi\rangle) = \langle D_j\psi\rangle$ if and only if $2^j \in \mathcal{T}_{\langle D_j\psi\rangle}$. Recall that in this case $2^\ell \in \mathcal{T}_{\langle D_j\psi\rangle}$ for every $\ell \in \{1, 2, ..., j\}$.
(iii) If there exists $j \in \mathbb{N} \cup \{0\}$ such that $2^\ell \in \mathcal{T}_{\langle D_j\psi\rangle}$ for every $\ell \in \mathbb{N}$, then $\langle D_j\psi\rangle$ is of Type-3.

■

The intuition of Remark 1.17 would suggest that the Type of $\langle D\psi\rangle$ should be at most the Type of $\langle\psi\rangle$. This is "almost true", but there are exceptions. We have the following results.

EXAMPLE 1.43. For $0 < \varepsilon < 1/2$, consider $\psi \in L^2(\mathbb{R})$ defined by

$$\widehat{\psi} = \chi_{[0,\varepsilon)} + \chi_{[1,1+\varepsilon)}.$$

Using Theorem 2.2 from [**ŠW11**] it is easy to see that $\mathcal{T}_{\langle\psi\rangle} = \emptyset$ (following the notation in [**ŠW11**], we have that $F_1 \cap F_2 = [0,\varepsilon)$, i.e. a set of positive measure for all $n \in \mathbb{N} \setminus \{1\}$). Obviously, ssupp $\widehat{D\psi} = [0, 2\varepsilon) \cup [2, 2+2\varepsilon)$, so we obtain that $\mathcal{T}_{\langle D\psi\rangle} = \{2\}$. Hence $\langle D\psi\rangle$ is of Type-2 and thus of strictly greater type than $\langle\psi\rangle$, which is of Type-1. $\diamondsuit$

As we show in the next lemma, the number 2 in the previous example does not appear by mere coincidence.

LEMMA 1.44. *Let $\psi \in L^2(\mathbb{R})$ and $n \in \mathbb{N} \setminus \{1\}$ an odd number. If $n \in \mathcal{T}_{\langle D\psi\rangle}$, then $n \in \mathcal{T}_{\langle\psi\rangle}$.*

PROOF. Since $n \in \mathcal{T}_{\langle D\psi\rangle}$, we obtain

$$D(T_{2/n}\psi) = T_{1/n}(D\psi) \in \langle D\psi\rangle.$$

Hence there exists a sequence $(\varphi_m)_{m\in\mathbb{N}} \subseteq \operatorname{span}(\mathcal{B}_{D\psi})$ such that $D(T_{2/n}\psi) = \lim_{m\to\infty} \varphi_m$. Every φ_m is a finite linear combination of the form

$$\sum_k \lambda_k T_k D\psi = \sum_k \lambda_k D(T_{2k}\psi) = D\left(\sum_k \lambda_k T_{2k}\psi\right).$$

Hence for every $m \in \mathbb{N}$, there exists $\eta_m \in \langle\psi\rangle$ such that $\varphi_m = D(\eta_m)$. It follows that

$$T_{2/n}\psi = D^{-1}D(T_{2/n}\psi) = D^{-1}(\lim_{m\to\infty} \varphi_m) = \lim_{m\to\infty} \eta_m \in \langle\psi\rangle.$$

Since $T_{2/n}\psi \in \langle\psi\rangle$, it follows that $T_{2/n}(\langle\psi\rangle) \subseteq \langle\psi\rangle$. Let us write $n = 2\ell + 1$, with $\ell \in \mathbb{N}$ since $n \in \mathbb{N} \setminus \{1\}$ is odd. Observe that

$$T_{(2\ell+2)/n}\psi = \underbrace{T_{2/n}T_{2/n}...T_{2/n}}_{\ell+1 \text{ times}}\psi \in \langle\psi\rangle.$$

Since $(2\ell+2)/n = 1 + (1/n)$, we obtain

$$T_{1/n}(T\psi) \in \langle\psi\rangle = \langle T\psi\rangle,$$

which guarantees that $T_{1/n}(\langle\psi\rangle) \subseteq \langle\psi\rangle$. In other words, $n \in \mathcal{T}_{\langle\psi\rangle}$. $\square$

The number 2 plays yet another important role in these considerations.

LEMMA 1.45. *If $\psi \in L^2(\mathbb{R})$ is such that $2 \in \mathcal{T}_{\langle D\psi\rangle}$, then $\langle\psi\rangle$ is not a maximal shift-invariant space.*

PROOF. If $2 \in \mathcal{T}_{\langle D\psi\rangle}$, then (by Remark 1.42i) we have $\langle D\psi\rangle = \langle DT\psi\rangle$ and, consequently (by Theorem 1.36), $\dim_{D(\langle\psi\rangle)} \leq 1$. By Corollary 1.14b, the only way $\langle\psi\rangle$ could be maximal is if $\dim_{D(\langle\psi\rangle)} \equiv 2$, which is not the case. Hence $\langle\psi\rangle$ must not be maximal. $\square$

THEOREM 1.46. *Let $\psi \in L^2(\mathbb{R})$.*

(a) If $\langle D\psi\rangle$ is of Type-3, then $\langle\psi\rangle$ must also be of Type-3.

(b) If $\langle\psi\rangle$ is a maximal shift-invariant space which is of Type-3, then $\langle D\psi\rangle$ is either of Type-1 or of Type-2.

Proof. (a) If $\langle D\psi\rangle$ is of Type-3, then (by 1.3.5), every $n \in \mathbb{N} \setminus \{1\}$ belongs to $\mathcal{T}_{\langle D\psi\rangle}$. If n is odd, then (by Lemma 1.44) $n \in \mathcal{T}_{\langle\psi\rangle}$. It follows that $\mathcal{T}_{\langle\psi\rangle}$ is infinite, whence $\langle\psi\rangle$ must be of Type-3.

(b) We prove this by contrapositive. Suppose that $\langle D\psi\rangle$ is of Type-3. Then (by 1.3.5) it must be that $2 \in \mathcal{T}_{\langle D\psi\rangle}$. By Lemma 1.45, it follows that $\langle\psi\rangle$ is not a maximal shift-invariant space.

□

Hence if $\langle\psi\rangle$ is of Type-3, then the type of $\langle D\psi\rangle$ could be 1, 2, or 3. If, however, we know that $\langle\psi\rangle$ is maximal, then $\langle D\psi\rangle$ is either 1 or 2, which is strictly less than the type of $\langle\psi\rangle$. The following examples complete the picture for Type-3 principal shift-invariant spaces.

Example 1.47. (i) Consider $0 < \varepsilon \leq 1/2$ and ψ such that $\widehat{\psi} = \chi_{[0,\varepsilon)}$. Then $\langle\psi\rangle$ is of Type-3, and ssupp $(\widehat{D\psi}) = [0, 2\varepsilon)$. Hence $\langle D\psi\rangle$ is of Type-3 as well.

(ii) Consider $1/2 < \varepsilon \leq 1$ and ψ such that $\widehat{\psi} = \chi_{[0,\varepsilon)}$. Then $\langle\psi\rangle$ is of Type-3 — and, in the case that $\varepsilon = 1$, $\langle\psi\rangle$ is maximal. Since ssupp $(\widehat{D\psi}) = [0, 2\varepsilon)$, which strictly contains the interval $[0, 1)$, it is easy to see that $\langle D\psi\rangle$ is of Type-1: the Fourier transforms of functions in $\langle D\psi\rangle$ are simply Fourier series multiplied by $\chi_{[0,2\varepsilon)}$ and thus are 1-periodic functions multiplied by $\chi_{[0,2\varepsilon)}$, but, for $n \geq 2$, the functions $e^{2\pi ix/n}\chi_{[0,2\varepsilon)}$ are not of this form.

(iii) Take ψ so that $\widehat{\psi} = \chi_{[0,1/2)\cup[3/2,2)}$. It is easy to check that $\langle\psi\rangle$ is a maximal principal shift-invariant space of Type-3. Since ssupp $(\widehat{D\psi}) = [0, 1) \cup [3, 4)$, we obtain $\mathcal{T}_{\langle D\psi\rangle} = \{3\}$, so that $\langle D\psi\rangle$ is of Type-2.

♢

Observe that a consequence of Theorem 1.46 is that, if $\langle\psi\rangle$ is of Type-2, then $\langle D\psi\rangle$ is of Type-1 or of Type-2. The following example shows that maximality assumptions does not improve this property (unlike in the case of $\langle\psi\rangle$ being Type-3).

Example 1.48. (i) Take ψ so that $\widehat{\psi} = \chi_{[1/2,1)\cup[2,5/2)\cup[13/2,7)}$. It is not difficult to check that $\langle\psi\rangle$ is maximal and $\mathcal{T}_{\langle\psi\rangle} = \{2, 3, 6\}$ so that $\langle\psi\rangle$ is of Type-2. Since ssupp $(\widehat{D\psi}) = [1, 2) \cup [4, 5) \cup [13, 14)$, it is not difficult to check that $\mathcal{T}_{\langle D\psi\rangle} = \{3\}$. Hence $\langle D\psi\rangle$ is of Type-2 and $\mathcal{T}_{\langle D\psi\rangle} \subsetneq \mathcal{T}_{\langle\psi\rangle}$.

(ii) Take ψ so that $\widehat{\psi} = \chi_{[1/2,1)\cup[2,5/2)\cup[7/2,4)}$. Again, $\langle\psi\rangle$ is maximal and of Type-2, with $\mathcal{T}_{\langle\psi\rangle} = \{3\}$. Since ssupp $(\widehat{D\psi}) = [1, 2) \cup [4, 5) \cup [7, 8)$, we conclude that $\mathcal{T}_{\langle D\psi\rangle} = \{3\}$. Thus $\langle D\psi\rangle$ is of Type-2 with $\mathcal{T}_{\langle\psi\rangle} = \mathcal{T}_{\langle D\psi\rangle}$.

♢

Example 1.43 shows that Type-1 $\langle\psi\rangle$ may result in Type-2 $\langle D\psi\rangle$. However, this is not possible if $\langle\psi\rangle$ is maximal.

Corollary 1.49. *If $\psi \in L^2(\mathbb{R})$ is such that $\langle\psi\rangle$ is maximal and of Type-1, then $\langle D\psi\rangle$ is also of Type-1.*

Proof. According to the properties of $\mathcal{T}_{\langle\psi\rangle}$ given in 1.3.4, it is enough to show that $p \notin \mathcal{T}_{\langle D\psi\rangle}$ for any prime p. If $p = 2$, then $2 \notin \mathcal{T}_{\langle D\psi\rangle}$ by Lemma 1.45. If $p \neq 2$, then p is odd. Since $\mathcal{T}_{\langle\psi\rangle} = \emptyset$, then, by Lemma 1.45, $p \notin \mathcal{T}_{\langle D\psi\rangle}$.

□

2. Dilation Invariances

The basic theme of our approach is to built theories from various relationships between $\langle\psi\rangle$ and $D(\langle\psi\rangle)$. We focus first on the inclusion-type relationship. At the core of wavelet systems are the combined actions of translations and dilations, so we find the following well-known result very useful.

LEMMA 2.1 ([**BRS01**]). *If $E \subseteq \mathbb{R}$ is a measurable, 1-periodic set such that $2E \subseteq E$, then either $E = \mathbb{R}$ or $E = \emptyset$ (modulo sets of measure zero).*

We start our analysis with a result that "D-invariance" is "not possible" for principal shift-invariant spaces.

PROPOSITION 2.2. *If $\psi \in L^2(\mathbb{R})$ is such that $D(\langle\psi\rangle) \subseteq \langle\psi\rangle$, then $\psi \equiv 0$.*

PROOF. Since both $D(\langle\psi\rangle)$ and $\langle\psi\rangle$ are shift-invariant spaces, the inclusion $D(\langle\psi\rangle) \subseteq \langle\psi\rangle$ guarantees that $\dim_{D(\langle\psi\rangle)} \leq \dim_{\langle\psi\rangle}$. Consider the 1-periodic, measurable set $U_{\langle\psi\rangle} \subseteq \mathbb{R}$. Given $\xi \in 2U_{\langle\psi\rangle}$, we obtain $\dim_{\langle\psi\rangle}(\xi/2) = 1$. By (1.8), it follows that $\dim_{D(\langle\psi\rangle)}(\xi) > 0$, which implies $\dim_{\langle\psi\rangle}(\xi) > 0$. Hence $\xi \in U_{\langle\psi\rangle}$. Thus $U_{\langle\psi\rangle}$ satisfies the hypotheses of Lemma 2.1 and is either $\mathbb{R}$ or $\emptyset$. If $U_{\langle\psi\rangle} = \mathbb{R}$, then, by Corollary 1.14b, $\dim_{D(\langle\psi\rangle)} \equiv 2 > \dim_{\langle\psi\rangle}$, which contradicts our assumptions. Therefore, $U_{\langle\psi\rangle} = \emptyset$ a.e., and $\psi \equiv 0$.

□

Interestingly enough, taking D^{-1}, instead of D, leads to a rich theory. Many results are well-known, but we shall revisit them from a somewhat different point of view. We shall say that a shift-invariant space $V \subseteq L^2(\mathbb{R})$ is D^{-1}-invariant if $D^{-1}(V) \subseteq V$. Other authors have used different names for this notion; for example in [**Rze00**] the author uses the term *refinability*.

Here we focus first on the case of a principal shift-invariant space V. For reasons of tradition, we use φ as the notation for our basis function. The following result is well-known (see, for example, [**HW96**], [**Rze00**], and the references therein) and easily deducible from Proposition 1.1.20.

PROPOSITION 2.3. *If $\varphi \in L^2(\mathbb{R})$, then the following are equivalent:*
(a) $\langle\varphi\rangle$ is D^{-1}-invariant;
(b) $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$;
(c) $\varphi \in D(\langle\varphi\rangle)$;
(d) $D^{-1}\varphi \in \langle\varphi\rangle$;
(e) There exists a 1-periodic, measurable function $m : \mathbb{R} \to \mathbb{C}$ such that

$$\widehat{\varphi}(2\xi) = m(\xi)\widehat{\varphi}(\xi), \tag{2.4}$$

for almost every $\xi \in \mathbb{R}$.

REMARK 2.5. (i) Observe that $D(\langle\varphi\rangle)$ is always a shift-invariant space, while $D^{-1}(\langle\varphi\rangle)$ may or may not be a shift-invariant space.
(ii) Observe that Proposition 2.2 implies that whenever we have a D^{-1}-invariant $\langle\varphi\rangle$ such that $\varphi \not\equiv 0$, we must have that

$$\langle\varphi\rangle \subsetneq D(\langle\varphi\rangle). \tag{2.6}$$

(iii) Equation (2.4) is known as the "two-scale equation" and has been studied by numerous authors on various levels of generality. Observe that the function m which appears in (2.4) is not necessarily uniquely determined by φ (see, for example, [**PŠWX01**] for more details).

(iv) Equation (2.4) introduces a "dyadic orbit" as an important element of study. Hence, for $\xi \in \mathbb{R} \setminus \{0\}$, we denote the "dyadic orbit" by

$$\operatorname{orb}(\xi) := \{2^j \xi : j \in \mathbb{Z}\}. \tag{2.7}$$

It is often useful to single out representatives of "dyadic orbits" in an organized manner. If we let

$$I := [-1, 1) \setminus [-1/2, 1/2), \tag{2.8}$$

we will take our representative of $\operatorname{orb}(\xi)$ to be the unique element of $\operatorname{orb}(\xi) \cap I$. ■

It follows that the question of characterizing all D^{-1}-invariant principal shift-invariant spaces $\langle \varphi \rangle$ leads to the following problem on functions. Find all pairs (φ, m), where $\varphi \in L^2(\mathbb{R})$ and $m : \mathbb{R} \to \mathbb{C}$ is measurable and 1-periodic, such that (2.4) holds; we shall term this the (φ, m)-Problem. Various versions of this problem have been treated by numerous authors (for a start, one could check [**Dau92**] or [**HW96**] and the references therein), mostly from the point of view of scaling functions and filters of orthonormal and Parseval frame wavelets. Here we take a slightly more general position. As the following example shows, there are many (φ, m)-Problems with nontrivial solutions.

EXAMPLE 2.9. Consider any function $\varphi \in L^2(\mathbb{R})$ with the property that $\widehat{\varphi}(\xi) \neq 0$ for $\xi \in [0, 1)$ and $\widehat{\varphi}(\xi) = 0$ for every $\xi \notin [0, 1)$. Define m on $[0, 1)$ by

$$m(\xi) := \begin{cases} \frac{\widehat{\varphi}(2\xi)}{\widehat{\varphi}(\xi)} & \text{if } \xi \in [0, 1/2) \\ 0 & \text{if } \xi \in [1/2, 1) \end{cases}$$

and extend it 1-periodically to $\mathbb{R}$. Obviously the pair (φ, m) satisfies (2.4). ♢

Following ideas from [**Con98**], [**PŠW99**], and [**PŠWX01**], it is often useful to reduce the problem from the complex field, $\mathbb{C}$, to the positive reals, $[0, \infty)$ (in particular, we get the benefit of a natural ordering of function values). We define $\Phi(\xi) := |\widehat{\varphi}(\xi)|^2$ and $M(\xi) := |m(\xi)|^2$. This leads to the following functional equation problem, which we refer to as the (Φ, M)-Problem: find all pairs (Φ, M), where $\Phi : \mathbb{R} \to [0, \infty)$ is in $L^1(\mathbb{R})$ and $M : \mathbb{R} \to [0, \infty)$ is measurable and 1-periodic, such that, for a.e. $\xi \in \mathbb{R}$,

$$\Phi(2\xi) = M(\xi)\Phi(\xi). \tag{2.10}$$

REMARK 2.11. (i) It is obvious that having a solution to the (φ, m)-Problem gives us solutions to the (Φ, M)-Problem. However, the reverse holds as well — consult [**Con98**], [**PŠW99**], and [**PŠWX01**] for detailed accounts in the wavelet case. Let us recall the basics. We shall say a function $\mu : \mathbb{R} \to \mathbb{C}$ is *unimodular* if it is measurable and, for every $\xi \in \mathbb{R}$, $|\mu(\xi)| = 1$. Given any 1-periodic and unimodular function μ, there are infinitely many unimodular functions $\nu : \mathbb{R} \to \mathbb{C}$ such that, for a.e. $\xi \in \mathbb{R}$,

$$\nu(2\xi)\overline{\nu(\xi)} = \mu(\xi), \tag{2.12}$$

where it is worth mentioning that the unimodularity of ν guarantees that $\overline{\nu(\xi)} = 1/\nu(\xi)$. The choice of ν depends entirely on the choice of the values of ν on I (defined in (2.8)); from I, one extends ν to $\mathbb{R} \setminus \{0\}$ via (2.12). Observe that, by taking all solutions (Φ, M) of the (Φ, M)-Problem and all pairs (ν, μ)

which satisfy (2.12), the family of pairs (φ, m) with $\widehat{\varphi} = \nu\sqrt{\Phi}$ and $m = \mu\sqrt{M}$ will provide us with all solutions of the (φ, m)-Problem.

(ii) One should not deduce too much from part (i) of this remark, though. Suppose we are given $\varphi \in L^2(\mathbb{R})$ and we want to check whether there is an m such that (φ, m) satisfies (2.4). Say that we take $\Phi = |\widehat{\phi}|^2$ and find that there is an M such that (Φ, M) satisfies (2.10). Does it mean that we have a positive answer for φ as well, i.e. that there is an m so that (ϕ, m) satisfies (2.4)? Actually, no. We will have a positive answer if and only if there exists a "phase" ν of $\widehat{\varphi}$ such that $\nu(2\xi)\overline{\nu(\xi)}$ is 1-periodic.

(iii) Observe that $\Phi \equiv 0$ satisfies (2.10) for every M.

(iv) If (Φ, M) is a solution to the (Φ, M)-Problem, then

$$\int_{\mathbb{R}} \Phi(\xi) d\xi = 2\int_{\mathbb{R}} M(\xi)\Phi(\xi) d\xi < \infty. \tag{2.13}$$

■

Both of these problems can be considered from the "position of any member of the pair." For example (and more precisely), suppose that $\Phi \in L^1(\mathbb{R})$ is given. Is there any M such that (Φ, M) satisfies (2.10)? The "direct approach", i.e. checking whether $\Phi(2\xi)/\Phi(\xi)$ is 1-periodic, seems to be the simplest one. We must be careful about the zeroes of Φ. By a slight abuse of notation, we denote the periodization of Φ, i.e. $\sum_{k\in\mathbb{Z}} \Phi(\xi + k)$, by $p_\Phi(\xi)$. We also introduce the following notation:

$$u_\Phi(\xi) := \{k \in \mathbb{Z} : \Phi(\xi + k) \neq 0\}$$

and a function $\tau(\xi) := \xi - \lfloor \xi \rfloor$, for $\xi \in \mathbb{R}$, where $\lfloor\cdot\rfloor$ denotes the "largest integer function". We believe that our readers can check the proof of the following result easily for themselves.

THEOREM 2.14. *Let $\Phi : \mathbb{R} \to [0, \infty)$ be in $L^1(\mathbb{R})$. Then*

(a) There exists at least one M such that (Φ, M) is a solution to the (Φ, M)-Problem if and only if, for a.e. $\xi \in \mathbb{R}$,
   (i) $\Phi(\xi) = 0 \Rightarrow \Phi(2\xi) = 0$;
   (ii) $k, \ell \in u_\Phi(\xi) \Rightarrow \dfrac{\Phi(2\xi + 2k)}{\Phi(\xi + k)} = \dfrac{\Phi(2\xi + 2\ell)}{\Phi(\xi + \ell)}$.

(b) The following are equivalent:
   (i) There exists at most one M such that (Φ, M) is a solution of the (Φ, M)-Problem;
   (ii) For a.e. $\xi \in \mathbb{R}$, $u_\Phi(\xi) \neq \emptyset$;
   (iii) $\tau(ssupp\ \Phi) = [0, 1)$ a.e.;
   (iv) $p_\Phi > 0$ a.e.

Obviously, if (i) and (ii) in Theorem 2.14a are satisfied, we can define M to be measurable and

$$M(\xi) := \begin{cases} \frac{\Phi(2\xi+2k)}{\Phi(\xi+k)} & \text{for } k \in u_\Phi(\xi) \text{ if } u_\Phi(\xi) \neq \emptyset \\ \text{arbitrary value in } [0, \infty) & \text{if } u_\Phi(\xi) = \emptyset \end{cases}, \tag{2.15}$$

where, in the case of $u_\Phi(\xi) = \emptyset$ we need only preserve 1-periodicity in our arbitrary choice (and, of course, not actively try to create a non-measurable function).

It is now trivial to "transfer" Theorem 2.14 from the (Φ, M)-Problem to the (ϕ, m)-problem. We emphasize one consequence.

COROLLARY 2.16. *Let $\varphi \in L^2(\mathbb{R})$ be such that $\langle\varphi\rangle$ is D^{-1}-invariant. Then there is exactly one m such that (φ, m) satisfies (2.4) if and only if $\langle\varphi\rangle$ is maximal.*

REMARK 2.17. Suppose that (Φ, M) is a pair satisfying (2.10). It is useful to "follow" Φ along the orbits $\mathrm{orb}(\xi)$.

(i) For a.e. $\xi \in \mathbb{R} \setminus \{0\}$, there are three options for $\mathrm{orb}(\xi)$:
   (A) For every $j \in \mathbb{Z}$, $\Phi(2^j\xi) = 0$;
   (B) For every $j \in \mathbb{Z}$, $\Phi(2^j\xi) \neq 0$;
   (C) There exists a $j_0 \in \mathbb{Z}$ such that $\Phi(2^j\xi) = 0$ for $j > j_0$ and $\Phi(2^j\xi) \neq 0$ for $j \leq j_0$.

   In case (A), we will say that $\mathrm{orb}(\xi)$ is a zero-orbit, while in cases (B) and (C), we will say that $\mathrm{orb}(\xi)$ is a nonzero orbit. Observe that in case (C) we must have $M(2^j\xi) \neq 0$ for $j < j_0$ and $M(2^{j_0}\xi) = 0$.

(ii) Since $\Phi \in L^1(\mathbb{R})$, it is well-known (see, for example, [**PŠWX01**]) that, for a.e. $\xi \in \mathbb{R}$,
$$\lim_{n\to\infty} \Phi(2^n\xi) = 0.$$

(iii) For a measurable function $f : \mathbb{R} \to [0, \infty)$, we have
$$\int_{\mathbb{R}} f(\xi)d\xi = \sum_{j\in\mathbb{Z}} \int_{2^j I} f(\xi)d\xi = \int_I \sum_{j\in\mathbb{Z}} 2^j f(2^j u)du.$$
   Hence $f \in L^1(\mathbb{R})$ if and only if $u \mapsto \sum_{j\in\mathbb{Z}} 2^j f(2^j u)$ is in $L^1(I)$. It will be useful for us to consider a weighted space, $L^1(I, w)$, the space of integrable functions over I with respect to the measure $w(\xi)d\xi$, where $w(\xi) = \sum_{j\in\mathbb{Z}} 2^j\Phi(2^j\xi)$.

■

Approaching the (Φ, M)-Problem from "the position of M" is actually more demanding. We shall provide additional analysis later in a section on filters. For now, suppose we are given $M : \mathbb{R} \to [0, \infty)$ which is measurable and 1-periodic; we introduce

$$\mathrm{Sol}_M := \left\{\Phi : \mathbb{R} \to [0, \infty) : \Phi \in L^1(\mathbb{R}) \text{ and } (\Phi, M) \text{ satisfies (2.10)}\right\}. \tag{2.18}$$

By Remark 2.11iii, for every M, $0 \in \mathrm{Sol}_M \neq \emptyset$.

EXAMPLE 2.19. Consider $M \equiv a \geq 0$. If $a = 0$, then it is obvious from (2.10) that $\mathrm{Sol}_M = \{0\}$. We claim that the same holds for $a > 0$. Indeed, for $\Phi \in \mathrm{Sol}_M$,
$$\sum_{j\in\mathbb{Z}} 2^j\Phi(2^j\xi) = \Phi(\xi)\left(\sum_{\ell\in\mathbb{N}} \frac{1}{(2a)^\ell} + \sum_{j\in\mathbb{N}\cup\{0\}} (2a)^j\right) < \infty,$$
for a.e. ξ. However, one of the two sums in the parentheses above will be infinite, depending on whether $2a \geq 1$ or $2a \leq 1$, so one can only produce a finite expression if Φ is zero almost everywhere. Hence for $M \equiv a \geq 0$, we have $\mathrm{Sol}_M = \{0\}$. Observe that by (2.13) we also have

$$\begin{aligned} M > 1/2 \text{ a.e.} &\Rightarrow \mathrm{Sol}_M = \{0\} \\ M < 1/2 \text{ a.e.} &\Rightarrow \mathrm{Sol}_M = \{0\}. \end{aligned} \tag{2.20}$$

Observe that all these restrictions in this example stem from the requirement that Φ belongs to $L^1(\mathbb{R})$. Without this, it would be easy to construct infinitely many Φ which satisfy (2.10) for any M: simply define Φ arbitrarily on I and use (2.10) to extend it to $\mathbb{R} \setminus \{0\}$ (completely analogously to how we used (2.12)). ♢

EXAMPLE 2.21. Consider a 1-periodic, measurable function $M : \mathbb{R} \to [0, \infty)$ such that $M|_{[0,1)} = a\chi_{[0,1/2)}$, where $a \geq 0$. By (2.20), $\mathrm{Sol}_M = \{0\}$ for $a < 1/2$. For $a = 1/2$, we obtain from (2.13) that $\Phi \in \mathrm{Sol}_M$ implies that

$$\int_0^1 \Phi(\xi)d\xi = \int_0^{1/2} \Phi(\xi)d\xi.$$

Hence $\Phi|_{[1/2,1)} \equiv 0$. By (2.10) and $M > 0$ on $[0, 1/2)$, we obtain $\Phi = 0$, which means $\mathrm{Sol}_M = \{0\}$.

For $a > 1/2$, we have $2a > 1$. If $\Phi \in \mathrm{Sol}_M$, then $\Phi|_{(-\infty,0)} \equiv 0$, since $M|_{[-1/2,0)} = 0$, and $\Phi|_{[1,\infty)} \equiv 0$, since $M|_{[1/2,1)} \equiv 0$. Hence $\mathrm{ssupp}\, \Phi \subseteq [0, 1)$. Obviously, we must have $\int_{1/2}^1 \Phi(\xi)d\xi < \infty$. Take any $g : [1/2, 1) \to [0, \infty)$ such that $g \in L^1([1/2, 1))$. Define $\Phi_g^a : [0, 1) \to [0, \infty)$ by

$$\Phi_g^a\big|_{[2^{-n-1},2^{-n})}(\xi) = \frac{g(2^n\xi)}{a^n}, \text{ for } n \in \mathbb{N} \cup \{0\}.$$

It is easy to check that Φ_g^a (extended to be zero outside $[0, 1)$) satisfies (2.10). Furthermore,

$$\begin{aligned}
\int_0^1 \Phi_g^a(\xi)d\xi &= \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \frac{g(2^n\xi)}{a^n} d\xi \\
&= \int_{1/2}^1 \left(\sum_{n=0}^{\infty} \frac{1}{(2a)^n} \right) g(u)du \\
&= \frac{1}{1 - \frac{1}{2a}} \int_{1/2}^1 g(u)du \\
&< \infty.
\end{aligned}$$

Hence, for every $a > 1/2$, we have

$$\mathrm{Sol}_M = \left\{ \Phi_g^a : g \in L^1([1/2, 1)), g \geq 0 \right\} \tag{2.22}$$

Observe that for $g = \chi_{[1/2,1)}$ and $a = 1$, we obtain $\Phi_g^a = \chi_{[0,1)}$. $\diamondsuit$

Given $\Phi \in \mathrm{Sol}_M$ and a measurable set $Z \subseteq I$, we can redefine Φ so that it is equal to zero on $\mathrm{orb}(\xi)$, when $\xi \in Z$ and is left unchanged on $\mathrm{orb}(\xi)$ for $\xi \in I \setminus Z$. This new function will again be in Sol_M. Hence, two solutions from Sol_M may have different orbit behavior in the sense that for one solution a particular orbit is a nonzero-orbit while for the other it is a zero-orbit (see Remark 2.17i). However, as the following result shows, we will have the same behavior on nonzero-orbits.

LEMMA 2.23. *Suppose $\Phi_1, \Phi_2 \in Sol_M$. For almost every $\xi \in \mathbb{R}$ the following holds. If orb(ξ) is a nonzero-orbit for both Φ_1 and Φ_2, then (using the notation in Remark 2.17i) either (B) holds for both Φ_1 and Φ_2 or (C) holds for both Φ_1 and Φ_2 (with the same value for j_0).*

PROOF. Observe that in order to prove this lemma, it is enough to show (without loss of generality on the choice of Φ_1 and Φ_2) that there is no $j \in \mathbb{Z}$ such that: $\Phi_1(2^j\xi) \neq 0 \neq \Phi_1(2^{j+1}\xi)$ and $\Phi_2(2^j\xi) \neq 0 = \Phi_2(2^{j+1}\xi)$. Indeed, observe that assumptions on Φ_1 lead to $M(2^j\xi) \neq 0$ while those on Φ_2 lead to $M(2^j\xi) = 0$, a contradiction. $\square$

The following family of "orbit constant" functions will play a useful role in describing the set of solutions Sol_M.

$$\mathrm{OrC}_+ := \{\mathcal{L} : \mathbb{R} \setminus \{0\} \to [0, \infty) : \mathcal{L} \text{ measurable and } \mathcal{L}(2\xi) = \mathcal{L}(\xi) \text{ a.e.}\}\,. \tag{2.24}$$

Observe that $\mathcal{L} \in \mathrm{OrC}_+$ is completely determined by $\mathcal{L}|_I$.

Let us now describe a basic structural feature of Sol_M. It is easy to see that it is a positive cone, i.e. that

$$\Phi_1, \Phi_2 \in \mathrm{Sol}_M \text{ and } \beta_1, \beta_2 \geq 0 \Rightarrow \beta_1\Phi_1 + \beta_2\Phi_2 \in \mathrm{Sol}_M. \tag{2.25}$$

In particular, this shows that either $\mathrm{Sol}_M = \{0\}$ or Sol_M is infinite.

We introduce the following binary relation on Sol_M: for $\Phi_1, \Phi_2 \in \mathrm{Sol}_M$, we shall say that

$$\Phi_1 \prec_M \Phi_2 \tag{2.26}$$

if, for a.e. $\xi \in \mathbb{R}$, the following holds: $\mathrm{orb}(\xi)$ is a zero-orbit for Φ_2 implies that $\mathrm{orb}(\xi)$ is a zero-orbit for Φ_1. It is obvious that $\prec_M$ is reflexive and transitive, but it is not antisymmetric (in particular, $\prec_M$ does not generate a partial ordering). It is easy to see that if $\Phi_1, \Phi_2 \in \mathrm{Sol}_M$, then

$$\begin{aligned} &\max(\Phi_1, \Phi_2) \in \mathrm{Sol}_M \\ &\Phi_1 \prec_M \max(\Phi_1, \Phi_2) \\ &\Phi_2 \prec_M \max(\Phi_1, \Phi_2). \end{aligned} \tag{2.27}$$

Observe that $0 \prec_M \Phi$ for every $\Phi \in \mathrm{Sol}_M$. If $\Phi \in \mathrm{Sol}_M$ satisfies $\Phi \prec_M 0$, then $\Phi \equiv 0$. If there exists $\Phi_0 \in \mathrm{Sol}_M$ such that, for a.e. $\xi \in \mathbb{R}$, $\mathrm{orb}(\xi)$ is a nonzero orbit for Φ_0, then $\Phi \prec_M \Phi_0$ for every $\Phi \in \mathrm{Sol}_M$ (however, as the following result shows, Φ_0 is not unique in this regard).

THEOREM 2.28. *Let $M : \mathbb{R} \to [0, \infty)$ be measurable and 1-periodic. Let $\Phi_0 \in Sol_M$. Then the following hold:*

(a) $\Phi \in Sol_M$ and $\Phi \prec_M \Phi_0$ if and only if there exists

$$\mathcal{L} \in OrC_+ \cap L^1(I, \sum_{j\in\mathbb{Z}} 2^j\Phi_0(2^j u))$$

such that $\Phi = \mathcal{L}\Phi_0$.

(b) $\Phi \in Sol_M$ and $\Phi \prec_M \Phi_0 \prec_M \Phi$ if and only if there exists $\mathcal{L} \in OrC_+ \cap L^1(I, \sum_{j\in\mathbb{Z}} 2^j\Phi_0(2^j u))$ such that $\Phi = \mathcal{L}\Phi_0$ and, for almost every $u \in I$, $\mathcal{L}(u) > 0$.

PROOF. Observe first that, for $\mathcal{L} \in \mathrm{OrC}_+$, we have that $\mathcal{L}\Phi_0 \in L^1(\mathbb{R})$ if and only if $\mathcal{L} \in L^1(I, \sum_{j\in\mathbb{Z}} 2^j\Phi_0(2^j u))$; see Remark 2.17iii.

Observe also that, for $\mathcal{L} \in \mathrm{OrC}_+$ and $\xi \in \mathbb{R} \setminus \{0\}$, $\mathcal{L}$ is equal to a constant $a \geq 0$ on $\mathrm{orb}(\xi)$. Hence if $a = 0$, then $\mathcal{L}(2\xi)\Phi_0(2\xi) = M(\xi)\mathcal{L}(\xi)\Phi_0(\xi)$ for any M. If $a > 0$, then (again, for every M),

$$\begin{aligned} \mathcal{L}(2\xi)\Phi_0(2\xi) =& a\Phi_0(2\xi) = aM(\xi)\Phi_0(\xi) = M(\xi)\mathcal{L}(\xi)\Phi_0(\xi) \\ &\Leftrightarrow \Phi_0(2\xi) = M(\xi)\Phi_0(\xi). \end{aligned} \tag{2.29}$$

These two observations prove that, if $\mathcal{L} \in \mathrm{OrC}_+ \cap L^1(I, \sum_{j\in\mathbb{Z}} 2^j\Phi_0(2^j u))$, then $\Phi = \mathcal{L}\Phi_0 \in \mathrm{Sol}_M$. Observe also that if $a = 0$ in the last two observations, then $\mathrm{orb}(\xi)$ is a zero-orbit for Φ. If $a > 0$, then $\mathrm{orb}(\xi)$ is a nonzero-orbit for Φ. This completes the proof of sufficiency for both (a) and (b).

In order to prove necessity for (a), assume that $\Phi \in \mathrm{Sol}_M$ and $\Phi \prec_M \Phi_0$. If Φ equals zero on the entire orbit $\mathrm{orb}(\xi)$, we define $\mathcal{L}$ to be zero on this entire orbit. If $\mathrm{orb}(\xi)$ is a nonzero-orbit for Φ, then, by $\Phi \prec_M \Phi_0$, it is a nonzero-orbit for Φ_0 (for a.e. ξ). By Lemma 2.23, $\mathrm{orb}(\xi)$ is either in case (B) for both Φ and Φ_0 or case (C) for both Φ and Φ_0. Without loss of generality (the other proof is very similar), consider the latter case, (C). Take $u = 2^j\xi$ with $j < j_0$ (observe $M(u)$ must be nonzero) and we obtain

$$\frac{\Phi(2u)}{\Phi_0(2u)} = \frac{M(u)\Phi(u)}{M(u)\Phi_0(u)} = \frac{\Phi(u)}{\Phi_0(u)}.$$

We define $\mathcal{L}$ on $\mathrm{orb}(\xi)$ to be $\Phi(2^{j_0}\xi)/\Phi_0(2^{j_0}\xi)$, and, since $\Phi(2^j\xi) = 0 = \Phi_0(2^j\xi)$ for $j > j_0$, we obtain that $\Phi(2^j\xi) = \mathcal{L}(2^j\xi)\Phi_0(2^j\xi)$, for every $j \in \mathbb{Z}$. Hence $\mathcal{L} \in \mathrm{OrC}_+$ and $\Phi = \mathcal{L}\Phi_0$. Since $\Phi \in \mathrm{Sol}_M$, we have $\Phi = \mathcal{L}\Phi_0 \in L^1(\mathbb{R})$, which completes the proof of (a).

In order to complete the proof of (b), it is enough to show that $\Phi \prec_M \Phi_0 \prec_M \Phi$ enables us to choose $\mathcal{L}$ so that $\mathcal{L} > 0$ a.e. Indeed, $\Phi \prec_M \Phi_0 \prec_M \Phi$ means that, for a.e. $\xi \in \mathbb{R}\backslash\{0\}$, $\mathrm{orb}(\xi)$ is either a zero-orbit for both Φ and Φ_0 or a nonzero-orbit for both Φ and Φ_0. In the former case, we have a lot of freedom, but we may certainly set $\mathcal{L}$ to be identically 1 on these orbits. In the latter case, we set $\mathcal{L} = \Phi/\Phi_0$ as we did in the preceding paragraph. $\square$

Corollary 2.30. *Let $M : \mathbb{R} \to [0,\infty)$ be measurable and 1-periodic. If there exists $\Phi_0 \in Sol_M$ such that $\Phi \prec_M \Phi_0$ for every $\Phi \in Sol_M$, then*

$$Sol_M = \{\mathcal{L}\Phi_0 : \mathcal{L} \in OrC_+ \cap L^1(I, \textstyle\sum_{j\in\mathbb{Z}} 2^j\Phi_0(2^j u))\}.$$

The following result is a direct consequence of Lemma 2.1 — it has been used in our circle for many years and, perhaps, could be characterized as “folklore”.

Lemma 2.31. *Given a measurable set $A \subseteq I$ and $j_0 \in \mathbb{Z}$, consider the sets*

$$E_A := \bigcup_{j,k\in\mathbb{Z}} (2^j A + k)$$

and

$$E_{(A,j_0)} := \bigcup_{k\in\mathbb{Z}, j\geq j_0} (2^j A + k).$$

If $|A| = 0$ then $|E_A| = |E_{(A,j_0)}| = 0$. If $|A| > 0$, then $E_A = E_{(A,j_0)} = \mathbb{R}$, up to sets of measure zero.

Theorem 2.32. *Let $M : \mathbb{R} \to [0,\infty)$ be measurable and 1-periodic such that Sol_M contains non-trivial solutions. Then either, for every non-trivial $\Phi \in Sol_M$, almost every nonzero-orbit is case (B) (from Remark 2.17(i)), or, for every non-trivial solution $\Phi \in Sol_M$, almost every nonzero-orbit is case (C) (from Remark 2.17(i)). Case (B) occurs if and only if $M > 0$ almost everywhere.*

Proof. For a nontrivial $\Phi \in \mathrm{Sol}_M$, we define

$$A_\Phi^{(2)} := \{\xi \in I : orb(\xi) \text{ is case } (B)\}$$
$$A_\Phi^{(3)} := \{\xi \in I : orb(\xi) \text{ is case } (C)\}.$$

Since Φ is non-trivial, their union, $A_\Phi^{(2)} \cup A_\Phi^{(3)}$, must have positive measure. If $|A_\Phi^{(3)}| > 0$, then $|\{\xi \in \mathbb{R} : M(\xi) = 0\}| > 0$. If $|A_\Phi^{(2)}| > 0$, then (by Lemma 2.31)

$E_{A_\Phi^{(2)}} = \mathbb{R}$, up to sets of measure zero. Observe that $E_{A_\Phi^{(2)}} \subseteq \{\xi \in \mathbb{R} : M(\xi) > 0\}$, up to sets of measure zero. Hence, in this case, $M > 0$ almost everywhere. This implies the following two statements: if $\Phi \in \mathrm{Sol}_M$ is nontrivial, then $|A_\Phi^{(2)}| > 0$ if and only if $|A_\Phi^{(3)}| = 0$; and if $\Phi_1, \Phi_2 \in \mathrm{Sol}_M$ are both nontrivial, then $|A_{\Phi_1}^{(2)}| > 0$ if and only if $|A_{\Phi_2}^{(2)}| > 0$. This completes the proof of the theorem. $\square$

REMARK 2.33. (i) Obviously, we are interested in the class of functions M for which Sol_M contains nontrivial solutions. Theorem 2.32 provides a natural partition of that class into two subclasses. One subclass contains M such that $M > 0$ almost everywhere and is characterized by the property that nonzero-orbits of its solutions are of case (B), i.e. are "full orbits". Hence, we shall say that M is *FO ("of full orbit type")* if $\{0\} \subsetneq \mathrm{Sol}_M$ and, for every nontrivial $\Phi \in \mathrm{Sol}_M$, $|A_\Phi^{(2)}| > 0$. The other subclass contains M such that $|\{M = 0\}| > 0$. Hence, we shall say that M is *non-FO ("not of full orbit type")* if $\{0\} \subsetneq \mathrm{Sol}_M$ and, for every nontrivial $\Phi \in \mathrm{Sol}_M$, $|A_\Phi^{(3)}| > 0$.

(ii) Suppose that $0 \neq \varphi \in L^2(\mathbb{R})$ is such that $\langle \varphi \rangle$ is D^{-1}-invariant. If m satisfies (2.4) for φ, then $M := |m|^2$ has the property that $\{0\} \subsetneq \mathrm{Sol}_M$. Hence M is either FO or non-FO. Therefore, we extend our terminology to m as well; m could be either FO or non-FO. Observe that if $\varphi_1 \in L^2(\mathbb{R})$ such that $\langle \varphi_1 \rangle = \langle \varphi \rangle$, then ssupp $\widehat{\varphi_1}$ = ssupp $\widehat{\varphi}$, which implies that φ_1 and φ have nonzero-orbits of the same type. As a consequence, we may extend our terminology to principal shift-invariant spaces as well. More precisely, given a non-trivial principal shift-invariant space V which is also D^{-1} invariant, we have two possibilities: either all of its generating pairs (φ, m) belong to the FO class (in which case we say that V is FO) or all of its generating pairs (φ, m) belong to the non-FO class (in which case we say that V is non-FO).

(iii) In order to completely resolve the (Φ, m)-Problem, it remains to show when Sol_M is nontrivial and to find, in such a case, at least one $\Phi_0 \in \mathrm{Sol}_M$ such that $\Phi \prec_M \Phi_0$ for every $\Phi_0 \in \mathrm{Sol}_M$. Observe (see Remark 2.11) that this would also complete the (φ, m)-Problem. We shall treat the FO and non-FO cases separately.

■

Suppose first that $M : \mathbb{R} \to [0, \infty)$ is measurable, 1-periodic, and $M > 0$ almost everywhere. We define a function $T = T_M : I \to [0, \infty]$ (here T stands for Tauberian) by

$$T(\xi) := \sum_{j=1}^{\infty} \left[\left(\prod_{k=1}^{j} \frac{1}{2M(2^{-k}\xi)} \right) + \left(\prod_{k=0}^{j-1} 2M(2^k\xi) \right) \right] \text{ for } \xi \in I, \tag{2.34}$$

Observe that T is measurable and may have infinite values. Using the convention that $1/0 = +\infty$ and $1/+\infty = 0$, we define a measurable, non-negative function $\Phi_0 = \Phi_{0,M} : \mathbb{R} \to [0, \infty)$ by

$$\Phi_0(u) := \begin{cases} 1 \text{ if } u = 0; & \\ \frac{1}{1+T(u)} & \text{for } u \in I; \\ \left(\prod_{k=0}^{j-1} M(2^k\xi)\right) \frac{1}{1+T(\xi)} & \text{for } u = 2^j\xi, \xi \in I, j \in \mathbb{N}; \\ \left(\prod_{k=1}^{j} \frac{1}{M(2^{-k}\xi)}\right) \frac{1}{1+T(\xi)} & \text{for } u = 2^{-j}\xi, \xi \in I, j \in \mathbb{N}. \end{cases} \tag{2.35}$$

REMARK 2.36. (i) Observe that in a measure-theoretic setup, the value at any single point is irrelevant (since "the point" is of measure zero). Hence, we can choose $\Phi_0(0)$ to be any positive number. If, on the other hand, we can impose some "continuity at 0" condition on Φ_0, then this value becomes important. As we shall see later, the choice $\Phi_0(0) = 1$ is quite natural.

(ii) Observe that $1 + T \geq 1$, so $0 \leq \frac{1}{1+T} \leq 1$ and $\frac{1}{1+T} \in L^1(I)$.

(iii) Observe that $\Phi_0 = 0$ almost everywhere if and only if $|\{\xi \in I : T(\xi) < \infty\}| = 0$.

(iv) It is easy to check directly that Φ_0 and M satisfy (2.10).

(v) Observe that $A^{(2)}_{\Phi_0} = \{\xi \in I : T(\xi) < \infty\}$. ■

For a complete understanding of the following result, recall Theorem 2.28 and Corollary 2.30. The following theorem provides a complete solution of the (Φ, M)-Problem in the FO case.

THEOREM 2.37. *If $M : \mathbb{R} \to [0, \infty)$ is measurable and 1-periodic and $M > 0$ almost everywhere, then*

(a) $\Phi_{0,M} \in Sol_M$;

(b) For every $\Phi \in Sol_M$, *one has* $\Phi \prec_M \Phi_{0,M}$;

(c)

$$\sum_{j\in\mathbb{Z}} 2^j \Phi_{0,M}(2^j u) = \begin{cases} 0 & \text{if } T(u) = \infty \\ 1 & \text{if } T(u) < \infty; \end{cases}$$

(d) $\{0\} = Sol_M \Leftrightarrow |\{\xi \in I : T(\xi) < \infty\}| = 0$.

PROOF. The formula in (c) follow directly from (2.35), since, for $u \in I$,

$$\sum_{j\in\mathbb{Z}} 2^j \Phi_{0,M}(2^j u) = \frac{1}{1+T(u)} \left[\sum_{j\in\mathbb{N}} 2^j \left(\prod_{k=0}^{j-1} M(2^k u) \right) + 1 + \sum_{j\in\mathbb{N}} 2^{-j} \left(\prod_{k=1}^{j} \frac{1}{M(2^{-k}u)} \right) \right]$$
$$= \frac{1}{1+T(u)} (1 + T(u)),$$

which is equal to 1 if $T(u) < \infty$ and 0 if $T(u) = \infty$.

Since (c) holds, (a) follows by Remark 2.17(iii) and Remark 2.36(iv).

In order to prove (b), suppose to the contrary that there is some $\Phi \in \mathrm{Sol}_M$ and set $A \subseteq \{\xi \in I : T(\xi) = \infty\}$ such that $|A| > 0$ and $\Phi|_A > 0$. We would then have

$$\int_{\mathbb{R}} \Phi(\xi) d\xi \geq \int_{\bigcup_{j\in\mathbb{Z}} 2^j A} \Phi(\xi) d\xi = \sum_{j\in\mathbb{Z}} \int_A 2^j \Phi(2^j u) du$$
$$= \int_A (1 + T(u)) \Phi(u) du = \infty,$$

which contradicts the requirement that elements of Sol_M are integrable.

The proof of (d) now follows directly from Remark 2.36(iii). □

Combining Corollary 2.30 with Theorem 2.37, we obtain that in the FO case

$$\mathrm{Sol}_M = \{\mathcal{L}\Phi_{0,M} : \mathcal{L} \in Orc_+ \cap L^1(I, \chi_{\{T_M<\infty\}})\}, \tag{2.38}$$

and for $\Phi \in \mathrm{Sol}_M$ with $\Phi = \mathcal{L}\Phi_{0,M}$,

$$\int_{\mathbb{R}} \Phi(\xi) d\xi = \int_{\{T_M<\infty\}} \mathcal{L}(u) du. \tag{2.39}$$

We turn our attention to the non-FO case now. Suppose that $M : \mathbb{R} \to [0, \infty)$ is measureable, 1-periodic, and

$$|Z_M| > 0, \text{ where } Z_M := \{\xi \in \mathbb{R} : M(\xi) = 0\}. \tag{2.40}$$

We define, for such an M, the set

$$H = H_M := \{\xi \in \mathbb{R} : M(\xi) = 0, M(2^{-j}\xi) \neq 0, \text{ for every } j \in \mathbb{N}\}, \tag{2.41}$$

and we call H *the horizon of* M.

Remark 2.42. Observe that some orbits may not have "a horizon point". One example is an orbit such that $M(2^k\xi) \neq 0$ for every $k \in \mathbb{Z}$. Consider a set $A \subseteq I$, defined by $A := \{\xi \in I : M(2^k\xi) \neq 0, \text{ for every } k \in \mathbb{Z}\}$. Since M is 1-periodic, it follows that $E_A \subseteq \{\xi : M(\xi) \neq 0\}$. By Lemma 2.31, if $|A| > 0$, then $E_A = \mathbb{R}$ almost everywhere, which is clearly a contradiction to (2.40). Hence, under (2.40), the set A is negligible.

The other possibility to disrupt "a horizon point" is if there exists a subsequence $\{j_k\} \subseteq \mathbb{N}$ such that $M(2^{-j_k}\xi) = 0$. Observe that any solution of (2.10) would have to be identically zero on such an orbit. Let us denote the generating set for such orbits by

$$A_M^{(1)} := \{\xi \in I : \text{ there exists } \{j_k\} \subseteq \mathbb{N} \text{ with } M(2^{-j_k}\xi) = 0, \text{ for all } k \in \mathbb{N}\}. \tag{2.43}$$

Obviously, for $\xi \in A_M^{(1)}$ the orbit $\operatorname{orb}(\xi)$ does not have a "horizon point". Observe that the families $\{2^jA_M^{(1)} : j \in \mathbb{Z}\}$ and $\{2^jH_M : j \in \mathbb{Z}\}$ consist of disjoint sets. Furthermore, for a non-FO M, one has

$$\begin{gathered} \left| \left(\bigcup_{j\in\mathbb{Z}} 2^jA_M^{(1)} \right) \cap \left(\bigcup_{j\in\mathbb{Z}} 2^kH_M \right) \right| = 0 \\ \text{and } \mathbb{R} = \left(\bigcup_{j\in\mathbb{Z}} 2^jA_M^{(1)} \right) \cup \left(\bigcup_{j\in\mathbb{Z}} 2^kH_M \right) \end{gathered} \tag{2.44}$$

Observe that:

$$M|_{[-1/2,1/2)} = \chi_{[-1/4,1/4)} \Rightarrow \bigcup_{j\in\mathbb{Z}} 2^jH_M = \mathbb{R} \text{ and } |A_M^{(1)}| = 0;$$
$$M|_{[-1/2,1/2)} = \chi_{[-1/2,-1/4)\cup[1/4,1/2)} \Rightarrow |H_M| = 0 \text{ and } A_M^{(1)} = I;$$
$$M|_{[-1/2,1/2)} = \chi_{[0,1/4)} \Rightarrow \bigcup_{j\in\mathbb{Z}} 2^jH_M = \mathbb{R}_+ \text{ and } \bigcup_{j\in\mathbb{Z}} 2^jA_M^{(1)} = \mathbb{R}_-.$$

This analysis of the (Φ, M)-Problem in the non-FO case depends essentially only on H_M since any solution of (2.10) must satisfy

$$\Phi|_{\bigcup_{j\in\mathbb{Z}} 2^jA_M^{(1)}} \equiv 0. \tag{2.45}$$

Hence, in the non-FO case, without loss of generality, we analyze those M which satisfy (2.40) and

$$|H_M| > 0. \tag{2.46}$$

■

Observe that we always have

$$H_M \subseteq Z_M; \tag{2.47}$$

so the requirement given in (2.46) implies (2.40). The analysis of the non-FO case now proceeds following an approach similar to that of the FO case; the main difference here being that in the non-FO case we can focus on low-frequencies only. Given $M : \mathbb{R} \to [0, \infty)$ which is measurable, 1-periodic, and whose associated H_M has positive measure, we define $T = T_M : H_M \to [0, \infty]$ by

$$T_M(\xi) := \sum_{j=1}^{\infty} \frac{1}{2^j} \left(\prod_{k=1}^{j} \frac{1}{M(2^{-k}\xi)} \right) \text{ for } \xi \in H_M. \tag{2.48}$$

Observe that we could extend the definition of T_M to $A_M^{(1)}$, but on $A_M^{(1)}$ we would have $R_M \equiv \infty$, and this would be equivalent to (2.45). As an analog to (2.35), we define $\Phi_0 := \Phi_{0,M} : \mathbb{R} \to [0, \infty)$ in this case by

$$\Phi_0(u) := \begin{cases} 0 & \text{for } u \in \bigcup_{j \in \mathbb{Z}} 2^j A_M^{(1)} \\ 1 & \text{for } u = 0 \\ \frac{1}{1+T_M(u)} & \text{for } u = 2^j \xi, \xi \in H_M, j \in \mathbb{N} \\ \left(\prod_{k=1}^{j} \frac{1}{M(2^{-k}\xi)} \right) \frac{1}{1+T_M(u)} & \text{for } u = 2^{-j}\xi, \xi \in H_M, j \in \mathbb{N} \end{cases} \tag{2.49}$$

Observe that Remark 2.36i–iv apply in this situation as well. Instead of Remark 2.36v we have the following analogous statement:

$$A_{\Phi_0}^{(3)} = \{\xi \in I : \emptyset \neq \operatorname{orb}(\xi) \cap H_M = \{u\}, T_M(u) < \infty\}. \tag{2.50}$$

Hence in the non-FO case, the set I is partitioned into three almost everywhere-disjoint sets, $A_M^{(1)}$, $A_{\Phi_0}^{(3)}$, and

$$\{\xi \in I : \emptyset \neq \operatorname{orb}(\xi) \cap H_M = \{u\}, T_M(u) = \infty\}.$$

Furthermore,

$$\Phi_0(u) > 0 \Leftrightarrow u = 2^{-j}\xi, \xi \in H_M, j \in \mathbb{N}, \operatorname{orb}(u) \cap A_{\Phi_0}^{(3)} \neq \emptyset. \tag{2.51}$$

Using an argument analogous to the one in Theorem 2.37, we obtain the following result.

THEOREM 2.52. *If $M : \mathbb{R} \to [0, \infty)$ is measurable, 1-periodic, and has $|H_M| > 0$, then*

(a) $\Phi_{0,M} \in Sol_M$*;*
(b) For every $\Phi \in Sol_M$*, one has* $\Phi \prec_M \Phi_{0,M}$*;*
(c) For every $u \in H_M$*;*

$$\sum_{j \geq 0} 2^{-j} \Phi_{0,M}(2^{-j}u) = \begin{cases} 0 & \text{if } T_M(u) = \infty \\ 1 & \text{if } T_M(u) < \infty \end{cases};$$

(d) $\{0\} = Sol_M \Leftrightarrow |\{\xi \in H_M : T_M(\xi) < \infty\}| = 0.$

3. Filter Analysis, FO Case.

We begin with the (Φ, M)-Problem in the FO case, i.e. we consider a function $M : \mathbb{R} \to [0, \infty)$ which is measurable and 1-periodic, with $M > 0$ almost everywhere. The Tauberian function, $T = T_M$, given in (2.34), is essential for the analysis of Sol_M. We consider its "low frequency" part and its "high frequency" separately. More precisely, we define $T_- = T_{M,-} : I \to [0, \infty]$ by

$$T_-(\xi) = \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} \frac{1}{2M(2^{-k}\xi)} \right), \text{ for } \xi \in I, \tag{3.1}$$

and $T_+ = T_{M,+} : I \to [0, \infty]$ by

$$T_+(\xi) = \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} 2M(2^{k}\xi) \right), \text{ for } \xi \in I. \tag{3.2}$$

Obviously, $T = T_+ + T_-$.

Consider $T_- = T_{M,-}$ first. It is useful to take into account a function $r_- = r_{M,-} : I \to [0, \infty]$, where, for $\xi \in I$, $r_-(\xi)$ is defined as the radius of convergence of the power series

$$z \mapsto \sum_{j=1}^{\infty} z^j \left(\prod_{k=1}^{j} \frac{1}{M(2^{-k}\xi)} \right). \tag{3.3}$$

Observe that $T_-(\xi)$ equals the value of the power series at $z = 1/2$. Hence

$$r_-(\xi) > 1/2 \Rightarrow T_-(\xi) < \infty \Rightarrow r_-(\xi) \geq 1/2. \tag{3.4}$$

Since

$$\frac{1}{r_-(\xi)} = \limsup_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} \frac{1}{M(2^{-k}\xi)}} = \frac{1}{\liminf\limits_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} M(2^{-k}\xi)}},$$

we obtain the following result.

Lemma 3.5. *If $M : \mathbb{R} \to [0, \infty)$ is measurable and 1-periodic and $M > 0$ almost everywhere, and $T_- = T_{M,-}$, then the following string of implications hold for $\xi \in I$:*

$$\begin{aligned} \liminf_{n\to\infty} M(2^{-n}\xi) > \frac{1}{2} &\Rightarrow \liminf_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} M(2^{-k}\xi)} > \frac{1}{2} \\ &\Rightarrow T_-(\xi) < \infty \\ &\Rightarrow \liminf_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} M(2^{-k}\xi)} \geq \frac{1}{2}. \end{aligned}$$

Remark 3.6. (i) Observe that the set $\{\xi \in I : T_-(\xi) < \infty\}$ is completely determined by the values of M around zero. More precisely, if M_1, M_2 are two functions with the properties listed in Lemma 3.5 and there exists an $\varepsilon > 0$ such that $M_1|_{(-\varepsilon,\varepsilon)} = M_2|_{(-\varepsilon,\varepsilon)}$, then, for all $\xi \in I$, $T_{M_1,-}(\xi) < \infty$ if and only if $T_{M_2,-}(\xi) < \infty$.

(ii) We need to include the analysis of T_+ in order to provide more elaborate examples for the case when the limit in Lemma 3.5 is exactly $1/2$. For now, observe (recall Example 2.19) that, if $M \equiv 1/2$, then $T_- \equiv \infty$ and

$$\lim_{N\to\infty} M(2^{-n}\xi) = \liminf_{n\to\infty} M(2^{-n}\xi) = \liminf_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} M(2^{-n}\xi)} = \frac{1}{2}.$$

(iii) Observe that the finite products from Lemma 3.5 appear also in the following formula; for every $n \in \mathbb{N}$, for every $\Phi \in \mathrm{Sol}_M$, and for almost every $\xi \in I$,

$$\Phi(\xi) = \left(\prod_{k=1}^{n} M(2^{-k}\xi)\right) \Phi(2^{-n}\xi). \tag{3.7}$$

■

Using notation from Theorem 2.32, for an M which is FO and a nontrivial solution $\Phi \in \mathrm{Sol}_M$, the set $A^{(2)}_\Phi$ denotes the set of $\xi \in I$ such that $orb(\xi)$ is a full orbit. In the case of $\Phi = \Phi_{0,M}$, we denote $A^{(2)}_{\Phi_{0,M}}$ by $A^{(2)}_M$. Observe that for every nontrivial solution $\Phi \in \mathrm{Sol}_M$ we have

$$A^{(2)}_\Phi \subseteq A^{(2)}_M \text{ and } 0 < |A^{(2)}_\Phi| \le |A^{(2)}_M| \le 1. \tag{3.8}$$

Furthermore,

$$\Phi_{0,M} > 0 \text{ almost everywhere} \Leftrightarrow |A^{(2)}_M| = 1. \tag{3.9}$$

LEMMA 3.10. *If M is FO and $\Phi \in Sol_M$ is nontrivial, then for almost every $\xi \in A^{(2)}_\Phi$ the following hold:*

(a) If $\limsup_{n\to\infty} \Phi(2^{-n}\xi) < \infty$, then $\liminf_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} M(2^{-k}\xi)} \ge 1$.
(b) If $\liminf_{n\to\infty} \Phi(2^{-n}\xi) > 0$, then $\limsup_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} M(2^{-k}\xi)} \le 1$.
(c) If $\liminf_{n\to\infty} M(2^{-n}\xi) > 1$, then $\lim_{n\to\infty} \Phi(2^{-n}\xi) = 0$.
(d) If $\limsup_{n\to\infty} M(2^{-n}\xi) < 1$, then $\lim_{n\to\infty} \Phi(2^{-n}\xi) = \infty$.
(e) $\lim_{n\to\infty} 2^{-n}\Phi(2^{-n}\xi) = 0$.

PROOF. Observe that for almost every $\xi \in A^{(2)}_\Phi$ we have $\Phi(\xi) > 0$, and, for every $n \in \mathbb{N}$, $\Phi(2^{-n}\xi) > 0$ and $\prod_{k=1}^{n} M(2^{-k}\xi) > 0$. Using standard limit properties (a), (b), (c), and (d) follow directly from (3.7). Since, for $\xi \in A^{(2)}_\Phi$, we have $T_-(\xi) < \infty$, we must have

$$\lim_{n\to\infty} \frac{1}{2^n \prod_{k=1}^{n} M(2^{-k}\xi)} = 0;$$

this, together with (3.7), implies (e). □

REMARK 3.11. (i) Consider an M which is FO and a nontrivial solution $\Phi \in \mathrm{Sol}_M$. For almost every $\xi \in A^{(2)}_\Phi$, there exists $\mathcal{L}(\xi) > 0$ such that $\Phi(2^j\xi) = \mathcal{L}(\xi)\Phi_{M,0}(2^j\xi)$, for every $j \in \mathbb{Z}$. Hence the behavior of any Φ along $orb(\xi)$, with respect to the three "value sets" $\{0\}, (0,\infty)$, and $\{\infty\}$, will be essentially the same as the behavior of $\Phi_{0,M}$.

(ii) If M is FO and $\xi \in A^{(2)}_M$, then the infinite product

$$\prod_{n=1}^{\infty} M(2^{-n}\xi) \tag{3.12}$$

exists (as an element of $[0,\infty]$) if and only if $\lim_{n\to\infty} \Phi_{0,M}(2^{-n}\xi)$ exists (as an element of $[0,\infty]$). Furthermore, observe that for almost every $\xi \in A_M^{(2)}$ we have $\frac{1}{1+T(\xi)} \in (0,1)$. Hence, assuming that the infinite products exist, (3.7) and (2.35) imply

$$\prod_{n=1}^{\infty} M(2^{-n}\xi) = 0 \Leftrightarrow \lim_{n\to\infty} \Phi_{0,M}(2^{-n}\xi) = \infty;$$
$$\prod_{n=1}^{\infty} M(2^{-n}\xi) \in (0,\infty) \Leftrightarrow \lim_{n\to\infty} \Phi_{0,M}(2^{-n}\xi) \in (0,\infty);$$
$$\prod_{n=1}^{\infty} M(2^{-n}\xi) = \infty \Leftrightarrow \lim_{n\to\infty} \Phi_{0,M}(2^{-n}\xi) = 0.$$

(iii) Consider the special case when $M : \mathbb{R} \to (0,1]$ is measurable and 1-periodic. In such a case, the product in (3.12) always exists and, for every $\xi \in I$,

$$\prod_{n=1}^{\infty} M(2^{-n}\xi) \in [0,1]. \tag{3.13}$$

Hence if M is also FO, then for almost every $\xi \in A_M^{(2)}$ the limit $\lim_{n\to\infty} \Phi_{0,M}(2^{-n}\xi)$ exists and is an element of $(0,\infty]$ (recall that, for $\xi \in I \setminus A_M^{(2)}$, $\Phi_{0,M}$ is identically 0 on $orb(\xi)$). An important question for us is whether the infinite product provides us with an element of Sol_M. ■

LEMMA 3.14. *If $M > 0$ is FO such that $M \leq 1$ almost everywhere and, for almost every $\xi \in \mathbb{R}$, $\lim_{n\to\infty} \Phi_{0,M}(2^{-n}\xi) < \infty$, then, for almost every $\xi \in \{T_+ < \infty\}$ the following are equivalent:*

(a) $\prod_{n=1}^{\infty} M(2^{-n}\xi) > 0$*;*

(b) $\lim_{n\to\infty} M(2^{-n}\xi) = 1$*;*

(c) $T_-(\xi) < \infty$.

PROOF. By basic properties of infinite products and by Lemma 3.5 we always have $(a) \Rightarrow (b) \Rightarrow (c)$. In order to prove $(c) \Rightarrow (a)$, without loss of generality, we can assume that $T_M(\xi) < \infty$, that ξ satisfies (3.7) for every $n \in \mathbb{N}$, and that $\lim_{n\to\infty} \Phi_{0,M}(2^{-n}\xi) < \infty$. Hence $\Phi_{0,M}(\xi) > 0$ and (3.7) imply that $\lim_{n\to\infty} \prod_{k=1}^{n} M(2^{-k}\xi) > 0$. □

Before turning our attention to T_+, let us revisit the definition of $\Phi_{0,M}$ given in (2.35). Observe that there is a function "hidden" in (2.35) which serves as a "universal dyadic multiplier for M" of sorts. Let us be more precise. Given $M > 0$ we define $F_M : \mathbb{R} \to [0,\infty)$ by

$$F_M(\xi) := \begin{cases} 1 \text{ if } \xi \in I \cup \{0\}; \\ \displaystyle\prod_{k=0}^{j-1} M(2^k u) & \text{for } \xi = 2^j u, u \in I, j \in \mathbb{N}; \\ \displaystyle\prod_{k=1}^{j} \frac{1}{M(2^{-k}u)} & \text{for } \xi = 2^{-j} u, u \in I, j \in \mathbb{N}. \end{cases} \tag{3.15}$$

Given any function $\Theta : I \to [0, \infty)$ which is measurable, there exists exactly one function $F = F(M, \Theta)$ such that

$$F(2\xi) = M(\xi)F(\xi) \text{ and } F|_I = \Theta;$$

it is given by the formula

$$F(\xi) := F_M(\xi)\Theta(u), \text{ where } \xi = 2^j u, u \in I, j \in \mathbb{Z}. \tag{3.16}$$

Obviously, F is not necessarily a solution in Sol_M. It is easy to see that $F = F(M, \Theta) \in \mathrm{Sol}_M$ if and only if

$$\Theta \in L^1(I, \textstyle\sum_{j\in\mathbb{Z}} 2^j F_M(2^j u)). \tag{3.17}$$

Observe that, if $\Theta \in L^1(I)$ and $\liminf_{n\to\infty} M(2^{-n}\xi) > 1/2$ almost everywhere, then the integral

$$\int_I \Theta(u) \cdot \sum_{j=0}^{\infty} \frac{1}{2^j} F_M(2^{-j}u)du < \infty; \tag{3.18}$$

hence, in this case, we need only to check $F_M(2^j u)$ for $j > 0$. This also leads naturally to the study of T_+.

Regarding $T_+ = T_{M,+}$, there is an important property we want to emphasize at the outset. By (3.2), it is obvious that T_+ is defined via 1-periodic functions (unlike T_-), i.e. we can consider T_+ as a 1-periodic function defined on all of $\mathbb{R}$. Hence the set $\{T_+ < \infty\}$ is a measurable, 1-periodic subset of $\mathbb{R}$. Observe that, for almost every $\xi \in \mathbb{R}$,

$$T_+(\xi) = 2M(\xi)(1 + T_+(2\xi)). \tag{3.19}$$

Since $M > 0$ almost everywhere, it follows that $\{T_+ < \infty\} = 2\{T_+ < \infty\}$, and so the set $\{T_+ < \infty\}$ satisfies the conditions of Lemma 2.1. This proves the following important result.

PROPOSITION 3.20. *If $M : \mathbb{R} \to [0, \infty)$ is measurable, 1-periodic, and $M > 0$ almost everywhere, then either $T_{M,+}(\xi) < \infty$ for almost every $\xi \in \mathbb{R}$ or $T_{M,+}(\xi) = \infty$ for almost every $\xi \in \mathbb{R}$.*

Obviously in the second case, we have $\{0\} = \mathrm{Sol}_M$.

REMARK 3.21. (i) If $\mathrm{Sol}_M \neq \{0\}$, then $T_{M,+} < \infty$ almost everywhere. Hence the size of the "full-orbit set" will be determined by $T_{M,-}$. More precisely, if $\mathrm{Sol}_M \neq \{0\}$, then $A_M^{(2)} = \{T_{M,-} < \infty\}$.

(ii) By changing M near zero, we can always adjust the size of $A_M^{(2)}$. Take any measurable $A \subseteq I$. Take any M which is FO with $A_M^{(2)} = I$ (there are plenty of such examples). Take any $a \in (0, 1/2)$. Define M_1, measurable, 1-periodic by

$$M_1|_{[-1/2,1/2)}(\xi) := \begin{cases} M(\xi) & \text{if } \mathrm{orb}(\xi) \cap A = \emptyset; \\ \min(a, M(\xi)) & \text{if } \mathrm{orb}(\xi) \cap A \neq \emptyset. \end{cases}$$

Obviously, $M_1 \le M$ and, since M is FO, $T_{M_1,+} \le T_{M,+} < \infty$ almost everywhere. If $\mathrm{orb}(\xi) \cap A = \emptyset$, then $T_{M_1,-}(\xi) = T_{M,-}(\xi) < \infty$, i.e. $\mathrm{orb}(\xi)$ is a full orbit. If $\mathrm{orb}(\xi) \cap A \neq \emptyset$, then $\limsup_{n\to\infty} M_1(2^{-n}\xi) \le a < 1/2$, i.e. $T_{M_1,-}(\xi) = \infty$. Hence $A_{M_1}^{(2)} = A$.

(iii) If M satisfies the property that, for almost every $\xi \in I$, $\lim_{n\to\infty} M(2^{-n}\xi) = 1$, then we have two possibilities. Either $T_{M,+} \equiv \infty$ almost everywhere (in which case $\mathrm{Sol}_M = \{0\}$) or $|\{T_{M,+} < \infty\}| > 0$; in such a case, $\mathrm{Sol}_M \neq \{0\}$ and $\Phi_{0,M} > 0$ almost everywhere. ■

Similarly as for T_-, it is useful to consider the function $r_+ = r_{M,+} : I \to [0,\infty]$, where, for $\xi \in I$, $r_+(\xi)$ is the radius of convergence of the power series

$$z \mapsto \sum_{j=1}^{\infty} z^j \left(\prod_{k=0}^{j-1} M(2^k \xi) \right). \tag{3.22}$$

Obviously, $T_+(\xi)$ is the value of the power series at $z = 2$. As in Lemma 3.5, it is easy to see that the following result holds.

LEMMA 3.23. *If $M : \mathbb{R} \to [0,\infty)$ is measurable, 1-periodic, and $M > 0$ almost everywhere, then the following string of implications holds for $\xi \in I$:*

$$\begin{aligned} \limsup_{n\to\infty} M(2^n\xi) < \frac{1}{2} &\Rightarrow \limsup_{n\to\infty} \sqrt[n]{\prod_{k=0}^{n-1} M(2^k\xi)} < \frac{1}{2} \\ &\Rightarrow T_+(\xi) < \infty \\ &\Rightarrow \limsup_{n\to\infty} \sqrt[n]{\prod_{k=0}^{n-1} M(2^k\xi)} \le \frac{1}{2}. \end{aligned}$$

Let us explore the properties of T_+ in more detail. Suppose that for a point $\xi \in I$ we have $\liminf_{n\to\infty} M(2^{-n}\xi) > 1/2$. It follows that there exists an $a > 1/2$ and $n_0 \in \mathbb{N}$ such that $M(2^{-n}\xi) \ge a$ for every integer $n > n_0$. Consider an arbitrary $B > 0$ and $n_1 \in \mathbb{N}$. Recall that a function $u \mapsto T_+(2^{n_1}u)$ is 2^{-n_1}-periodic. Since $2a > 1$, there exists $n_2 \in \mathbb{N}$ such that $(2a)^{n_2} > B$. Take $u = 2^{-n}\xi$ where $n > n_0 + n_1 + n_2$. We obtain

$$\begin{aligned} T_+(2^{n_1}u) &\ge \prod_{k=0}^{n_2-1} 2M(2^{n_1+k}u) \\ &= \prod_{k=0}^{n_2-1} 2M(2^{n_1+k-n}\xi) \\ &\ge \prod_{k=0}^{n_2-1} 2a \\ &= (2a)^{n_2} \\ &> B. \end{aligned}$$

Hence we have proved that

$$\lim_{n\to\infty} T_+(2^{n_1}(2^{-n}\xi)) = \infty,$$

i.e. T_+ is unbounded around zero and $u \mapsto T_+(2^{n_1}u)$ is unbounded around any point $k/2^{n_1}$ for $k \in \{0, 1, ..., 2^{n_1-1}\}$. Since n_1 is arbitrary and every interval contains at least one dyadic point $k/2^n$ for $k, n \in \mathbb{N}$, we have proved the following result.

PROPOSITION 3.24. *If $M : \mathbb{R} \to [0, \infty)$ is measurable, 1-periodic, strictly positive almost everywhere, and*

$$|\{\xi \in I : \liminf_{n\to\infty} M(2^{-n}\xi) > \frac{1}{2}\}| > 0$$

then $T_{M,+}$ is unbounded on any open interval.

REMARK 3.25. The proof of Proposition 3.24 suggests a fractal nature for T_+ (observe that T_- does not have a "fractal nature" at all). We can emphasize this even more. It is easy to see directly from (3.19) that we obtain, for almost every $\xi \in \mathbb{R}$,

$$\frac{1}{2}\left(T_+(\xi) + T_+(\xi + 1/2)\right) = \left[M(\xi) + M(\xi + 1/2)\right]\left(1 + T_+(2\xi)\right). \tag{3.26}$$

Of particular interest are examples of M where there exists $a > 0$ such that, for almost every $\xi \in \mathbb{R}$,

$$M(\xi) + M(\xi + 1/2) = a. \tag{3.27}$$

The case of $a = 1$ is the celebrated Smith–Barnwell condition (see, for example, [**HW96**] for more details and historical references). It is easy to show by induction over n that if (3.27) holds, then for almost every $\xi \in \mathbb{R}$,

$$\frac{1}{2^n}\sum_{k=0}^{2^n-1} T_+\left(\frac{\xi}{2^n} + \frac{k}{2^n}\right) = \begin{cases} \frac{a}{a-1}(a^n - 1) + a^n T_+(\xi) & \text{if } a \neq 1; \\ n + T_+(\xi) & \text{if } a = 1. \end{cases} \tag{3.28}$$

Observe yet another fundamental difference between T_- and T_+. As we have seen (Remark 3.6 and Remark 3.21), T_- depends on (and can be adjusted through) the values of M around zero. On the other hand, values of $T_+(2^n\xi)$, even for large values of n and for a positive measure set of ξ, depend on values of M on the entire domain; see Remark 3.21.

■

Formula (3.26) provides useful information about T_+. However, T_+ cannot be "reconstructed" based on this formula alone; see the following example.

EXAMPLE 3.29. Let us combine (3.26) and (3.27) to consider the functional equation

$$\frac{1}{2}\left(f\left(\frac{\xi}{2}\right) + f\left(\frac{\xi}{2} + \frac{1}{2}\right)\right) = a(1 + f(\xi)),$$

where $a > 0$ is given and $f : \mathbb{R} \to (0, \infty)$ is measurable and 1-periodic. Observe that both sides of the equation are 1-periodic, so it is enough to check the equation for $\xi \in [0, 1)$. There is a simple algorithm to generate solutions of such equations.

Start by defining f on $[1/4, 3/4)$ so that $0 < f(\xi) < 2a$ for every $\xi \in [1/4, 3/4)$. We select $f|_{[1/4,3/4)} \equiv a$. Use the given functional equation to build f iteratively on the sets $E_n = [\frac{2^n-1}{2^n}, \frac{2^{n+1}-1}{2^{n+1}})$ for $n \geq 2$. If we include $E_1 := [1/2, 3/4)$, then for every $n \in \mathbb{N}$, $\xi \in E_n \Leftrightarrow \frac{\xi}{2} + \frac{1}{2} \in E_{n+1}$ (at the same time, $\xi/2$ is always in $[1/4, 1/2)$). Hence we obtain that $f|_{E_n} = \sum_{k=1}^{n} 2^{k-1}a^k$, and that the functional equation holds for every $\xi \in [1/2, 1)$.

We now turn our attention to $[0, 1/2)$. Define the sets $F_n := [\frac{1}{2^{n+1}}, \frac{1}{2^n})$ for $n \in \mathbb{N}$. Define f on F_n inductively via the functional equation. Observe that $\xi \in F_n \Leftrightarrow \xi/2 \in F_{n+1}$ (at the same time $\xi/2 + 1/2$ is always in $[1/2, 3/4)$). We obtain $f|_{F_n} = \sum_{k=1}^{n} 2^{k-1}a^k$ and the equation holds for $\xi \in [0, 1/2)$ as well. Observe that $f|_{[0,1)}$ is symmetric with respect to $\xi = 1/2$ due to the symmetry of the process and the symmetry of the initial solution on $[1/4, 3/4)$. Hence, we have a solution for every $a > 0$, but our solution is not T_+ for any M (recall Proposition 3.24).

$\diamondsuit$

As we have seen T_+ (and therefore, T) is typically an unbounded function. Let us explore conditions under which T is integrable on I. From (3.15) it is easy to see that

$$\int_{\mathbb{R}} F_M(\xi) d\xi = \int_I (1 + T_M(u)) du, \tag{3.30}$$

i.e. $T_M \in L^1(I)$ if and only if $F_M \in \mathrm{Sol}_M$; in other words, we can take $\theta \equiv 1$ on I in (3.16). Observe also that if we split both sides in (3.30) into disjoint sets, we obtain

$$T_{M,-} \in L^1(I) \Leftrightarrow \int_{[-1/2,1/2)} F_M(\xi) < \infty \tag{3.31}$$

and

$$T_{M,+} \in L^1(I) \Leftrightarrow \int_{\mathbb{R}\setminus[-1,1)} F_M(\xi) < \infty \tag{3.32}$$

Remark 3.33. Integrability of T_- and T_+ has an effect on boundedness uniformly over various dyadic orbits. Let us analyze it on a specific set of examples of M. Suppose that there exists $0 < A_1 < B_1 < \infty$ such that $A_1 \leq M \leq B_1$ almost everywhere. We define then, for $n \in \mathbb{N}$

$$A_n := \operatorname*{ess\,inf}_{[-1/2^n, 1/2^n)} M$$

and

$$B_n := \operatorname*{ess\,sup}_{[-1/2^n, 1/2^n)} M.$$

It follows that

$$A_1 \leq A_n \leq A_{n+1} \leq B_{n+1} \leq B_n \leq B_1.$$

Hence there exist $A_\infty := \lim_{n\to\infty} A_n$ and $B_\infty := \lim_{n\to\infty} B_n$, and they satisfy, for every $n \in \mathbb{N}$

$$A_n \leq A_\infty \leq B_\infty \leq B_n.$$

Observe that if M is also continuous at zero, then $M(0) = A_\infty = B_\infty$. $\blacksquare$

Proposition 3.34. *Suppose that $M : \mathbb{R} \to [0, \infty)$ is measurable, 1-periodic and is such that there exist $0 < A_1 < B_1 < \infty$ such that $A_1 \leq M \leq B_1$ almost everywhere. Then*

(a) If $T_{M,-} < \infty$ almost everywhere, then $B_\infty \geq 1/2$.
(b) If $T_{M,+} \in L^1(I)$, then $A_\infty \leq 1$.
(c) If $T_M \in L^1(I)$ and M is continuous at 0 then $1/2 \leq M(0) \leq 1$.
(d) If $A_\infty > 1/2$, then $T_{M,-} \in L^\infty(I)$.

PROOF. (a) Suppose to the contrary that $B_\infty < 1/2$. Then there exists an $n_0 \in \mathbb{N}$ such that $B_{n_0} < 1/2$. Then we have for almost every $\xi \in I$

$$\begin{aligned}
T_{M,-}(\xi) &\geq \sum_{k=n_0+1}^{\infty} 2^{-k} \left(\prod_{\ell=1}^{n_0} \frac{1}{M(2^{-\ell}\xi)} \right) \left(\prod_{\ell=n_0+1}^{k} \frac{1}{M(2^{-\ell}\xi)} \right) \\
&\geq \sum_{k=n_0+1}^{\infty} 2^{-k} B_1^{-n_0} B_{n_0}^{-(k-n_0)} \\
&= \frac{1}{(2B_1)^{n_0}} \sum_{k=n_0+1}^{\infty} \frac{1}{(2B_{n_0})^{k-n_0}} \\
&= \infty.
\end{aligned}$$

(b) Suppose to the contrary that $A_\infty > 1$. Then there exists $n_0 \in \mathbb{N}$ such that $A_{n_0} > 1$. It follows that $M(\xi) > 1$ for almost every $\xi \in (-\delta, \delta)$ where $\delta = 2^{-n_0}$. Observe that for every $k \in \mathbb{N}$ and for every $\xi \in (1 - \delta/2^k, 1)$ we have $2^k\xi \in (2^k - \delta, 2^k)$. Hence

$$\infty > \int_I T_{M,+}(\xi) d\xi \geq \sum_{k=1}^{\infty} 2^k \int_{1-\delta/2^k}^{1} \prod_{\ell=0}^{k-1} M(2^\ell \xi) d\xi \geq \sum_{k=1}^{\infty} 2^k \cdot 1 \cdot \frac{\delta}{2^k} = \infty.$$

(c) This follows directly from (a), (b), and Remark 3.33.

(d) If $A_\infty > 1/2$, then there exists $n_0 \in \mathbb{N}$ such that $A_{n_0} > 1/2$. Then for almost every $\xi \in I$,

$$\begin{aligned}
T_{M,-}(\xi) \leq& \sum_{j=1}^{n_0-1} 2^{-j} \left(\prod_{k=1}^{j} \frac{1}{M(2^{-k}\xi)} \right) + \\
&+ 2^{-(n_0-1)} \left(\prod_{k=1}^{n_0-1} \frac{1}{M(2^{-k}\xi)} \right) \sum_{j=n_0}^{\infty} 2^{-(j-n_0+1)} \frac{1}{(A_{n_0})^{(j-n_0+1)}} \\
\leq& \sum_{j=1}^{n_0-1} 2^{-j} A_1^{-j} + (2A_1)^{-(n_0-1)} \frac{\frac{1}{2A_{n_0}}}{1 - \frac{1}{2A_{n_0}}} \\
<& \infty.
\end{aligned}$$

□

As before, it is easier to control T_- rather than T_+. For the analysis of integrability of T_+, it is useful to consider the following *recurrence formula*. Observe that for positive powers $2^n I$, $n \in \mathbb{N}$, we can define F_M even when M attains zero-values.

PROPOSITION 3.35. *If $M : \mathbb{R} \to [0, \infty)$ is measurable and 1-periodic, then, for every $j \in \mathbb{N}$,*

$$\int_I F_M(2^j\xi) d\xi = \frac{1}{2} \int_I \left(M\left(\frac{\xi}{2}\right) + M\left(\frac{\xi}{2} + \frac{1}{2}\right) \right) F_M(2^{j-1}\xi) d\xi.$$

The proof of this proposition relies on change of variables formulas for functions on the torus (or, equivalently, 1-periodic functions). Similar formulas have been used in the literature on wavelets (see, for example, [**PŠW99**]), but for the reader's convenience and the clarity of our exposition, we provide a quick tour of the results.

Denote $J_1 := [1/2, 3/4)$, $J_2 = [3/4, 1)$, $I_1 = J_1 \cup (-J_1)$ and $I_2 = J_2 \cup (-J_2)$. Define dilation $\rho_1 : I_1 \to I$ and $\rho_2 : I_2 \to I$ by

$$\rho_1(y) := \begin{cases} 2y - 2 & \text{if } y \in J_1 \\ 2y + 2 & \text{if } y \in -J_1 \end{cases}$$

$$\rho_2(y) := \begin{cases} 2y - 1 & \text{if } y \in J_2 \\ 2y + 1 & \text{if } y \in -J_2; \end{cases} \tag{3.36}$$

these are bijections with $\rho_1' = \rho_2' = 2$ (compare with the function ρ in [**PŠW99**]) and it is easy to explicitly write ρ_1^{-1} and ρ_2^{-1}. In the following, consider a function G where G is either $G : \mathbb{R} \to \mathbb{C}$ and locally integrable or $G : \mathbb{R} \to [0, \infty]$ and measurable. For $\ell = 1, 2$ and $J \subset I_\ell$ measurable, the following change of variable formula holds:

$$\int_J G(\xi) d\xi = \frac{1}{2} \int_{\rho_\ell(J)} G(\rho_\ell^{-1}(\xi)) d\xi. \tag{3.37}$$

If, moreover, G is 1-periodic, then for $\ell = 1$ we have

$$\int_J G(\xi) d\xi = \frac{1}{2} \int_{\rho_1(J)} G(\xi/2) d\xi, \tag{3.38}$$

and for $\ell = 2$

$$\int_J G(\xi) d\xi = \frac{1}{2} \int_{\rho_2(J)} G((\xi + 1)/2) d\xi. \tag{3.39}$$

Hence if $J \subseteq I$ is measurable and G is 1-periodic, then

$$\int_J G(\xi) d\xi = \frac{1}{2} \left(\int_{\rho_1(J \cap I_1)} G(\xi/2) d\xi + \int_{\rho_2(J \cap I_2)} G((\xi + 1)/2) d\xi \right). \tag{3.40}$$

In particular, for $J = I$, (3.40) becomes

$$\int_I G(\xi) d\xi = \frac{1}{2} \int_I \left(G(\xi/2) + G((\xi + 1)/2) \right) d\xi. \tag{3.41}$$

If, moreover, G is $1/2$-periodic, then (3.41) becomes

$$\int_I G(\xi) d\xi = \int_I G(\xi/2) d\xi. \tag{3.42}$$

(The formula is correct, despite its "appearance".) The sets $\rho_1(J \cap I_1)$ and $\rho_2(J \cap I_2)$ overlap in principle, so if $G \geq 0$ and G is $1/2$-periodic, then (3.40) becomes

$$\int_J G(\xi) d\xi \geq \frac{1}{2} \int_{\rho_1(J \cap I_1) \cup \rho_2(J \cap I_2)} G(\xi/2) d\xi \tag{3.43}$$

Proof of Proposition 3.35. For every $j \in \mathbb{N}$ consider a function $G_j : \mathbb{R} \to [0, \infty)$ defined by

$$G_j(\xi) := \prod_{k=0}^{j-1} M(2^k \xi),$$

and $G_0 : \mathbb{R} \to [0, \infty)$ so that $G_0 \equiv 1$. Obviously these are non-negative, measurable, 1-periodic functions. For $\xi \in I$, we have $G_j(\xi) = F_M(2^j \xi)$, for every $j \in \mathbb{N}$. Furthermore, for every $j \in \mathbb{N}$ and every $\xi \in I$, we have

$$G_j(\xi/2) = M(\xi/2) G_{j-1}(\xi)$$
$$G_j((\xi + 1)/2) = M((\xi + 1)/2) G_{j-1}(\xi).$$

The proof now follows directly from (3.41). □

It is now easy to obtain, directly from the recurrence formula, the following result.

COROLLARY 3.44. *Let $M : \mathbb{R} \to [0, \infty)$ be measurable, 1-periodic, and $M > 0$ almost everywhere.*

(a) If, for almost every $\xi \in \mathbb{R}$, $M(\xi) + M(\xi + \frac{1}{2}) \geq 1$ (in particular if M satisfies the Smith–Barnwell condition above), then $T_{M,+} \notin L^1(I)$.

(b) If, for almost every $\xi \in \mathbb{R}$, $M(\xi) + M(\xi + \frac{1}{2}) \leq a < 1$, then $T_{M,+} \in L^1(I)$.

At this point we have enough theory to develop several illustrative examples.

EXAMPLE 3.45. Take $a, b, \varepsilon \in (0, 1)$ such that $\varepsilon \leq 1/4$, $b < a < 1$ and $a + b < 1$. Define M to be 1-periodic so that $M|_{[-\varepsilon,\varepsilon)} \equiv a$ and $M|_{[-1/2,1/2)\setminus[-\varepsilon,\varepsilon)} \equiv b$. Since $a + b < 1$, Corollary 3.44b implies that $T_{M,+} \in L^1(I)$. We have several cases.

If $0 < a < 1/2$, then $\mathrm{Sol}_M = \{0\}$ by Lemma 3.5.

If $a = 1/2$, then $\liminf_{n\to\infty} M(2^{-n}\xi) = \liminf_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} M(2^{-j}\xi)} = 1/2$ for every ξ. However, direct calculation from (3.1) shows that $T_{M,-} \equiv \infty$, i.e. $\mathrm{Sol}_M = \{0\}$ (compare with Lemma 3.5).

If $a \in (1/2, 1)$, then $T_M \in L^1(I)$, i.e. $F_M \in \mathrm{Sol}_M$. Observe, however, that the infinite product given in (3.13) is equal to zero for every $\xi \in I$. Hence, although we have non-trivial solutions, the infinite product is not "able to recognize them" (this is one important point where our approach is more general and more comprehensive than the theory previously developed in [**PŠWX01**] and [**PŠWX03**]; we shall revisit this issue in more detail later). Observe also that in this case, $\lim_{n\to\infty} F_M(2^{-n}\xi) = \infty$ for every $\xi \in I$. Nevertheless, F_M is a non-trivial solution and $F_M > 0$ almost everywhere; in particular, $A_M^{(2)} = I$.

Let us now change M on $(-\varepsilon, \varepsilon)$, where we now take $\varepsilon = 1/4$ for simplicity. For $\xi \in I$, we define

$$M\left(\frac{\xi}{2^{2\ell-1}}\right) := \frac{1}{2}$$
$$M\left(\frac{\xi}{2^{2\ell}}\right) := a > \frac{1}{2},$$

for every $\ell \in \mathbb{N}$. Observe that $\liminf_{n\to\infty} M(2^{-n}\xi) = \frac{1}{2}$ for every $\xi \in I$. Still, $\mathrm{Sol}_M \neq \{0\}$ since

$$\liminf_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} M(\xi/2^k)} = \lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} M(\xi/2^k)} = (1/2)^{1/2} a^{1/2} > 1/2;$$

compare to Lemma 3.5.

Take now $\varepsilon = 1/2^5$ (again for simplicity) and define M on $[-\varepsilon, \varepsilon]$ so that for $\xi \in I$ and $n \in \mathbb{N}$ with $n \geq 5$,

$$M(2^{-n}\xi) := \frac{1}{2}\left(\frac{n}{n-1}\right)^2.$$

It follows that for $n \geq 5$ and $\xi \in I$ we have

$$\prod_{k=1}^{n} 2M(2^{-k}\xi) = (2b)^4 \frac{1}{16} n^2 = b^4 n^2.$$

Hence

$$\lim_{n\to\infty} M(2^{-n}\xi) = \lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} M(2^{-k}\xi)} = \lim_{n\to\infty} \sqrt[n]{\frac{b^4n^2}{2^n}} = \frac{1}{2},$$

while $\mathrm{Sol}_M \neq \{0\}$ since, for every $\xi \in I$,

$$T_-(\xi) \sim \sum_{n=5}^{\infty} \frac{1}{n^2} < \infty.$$

$\diamondsuit$

REMARK 3.46. (i) Observe that in the case of $T_M \in L^1(\mathbb{R})$ we have two "natural" candidates for a maximal solution (in terms of $\prec_M$): $\Phi_{0,M}$ and F_M. Since in this case $1 + T_M \in L^1(I)$, we simply take $\mathcal{L} \in OrC_+$ (see (2.38)) to be $\mathcal{L} = 1+T_M$ and obtain $(1+T_M)\Phi_{0,M} = F_M$ (obviously, $F_M \prec_M \Phi_{0,M} \prec_M F_M$).

(ii) If $T_M \in L^1(I)$ and $\lim_{n\to\infty} M(2^{-n}\xi) = a \in (1/2, 1)$ for almost every $\xi \in \mathbb{R}$, we will have the same phenomenon as in Example 3.45; all infinite products in (3.13) are trivial, and any nontrivial $\Phi \in \mathrm{Sol}_M$ will be unbounded around zero. Hence, even in the case of $T_M \in L^1(I)$, bounded, non-trivial solutions are to be expected in the case where $\lim_{n\to\infty} M(2^{-n}\xi) \equiv 1$.

$\blacksquare$

It is of interest to explore whether M can be bigger than 1.

EXAMPLE 3.47. Take $a, b, c \in (0, \infty)$ so that $1/2 < a < 1$, c is arbitrary, and b is small enough that $a + b + 4cb < 1$. Observe that we can take $c > 1$ here. Define M by

$$M(\xi) := \begin{cases} a & \text{if } \xi \in [-1/6, 1/6) \\ c & \text{if } \xi \in [-1/3, 1/3) \setminus [-1/6, 1/6) \\ b & \text{if } \xi \in [-1/2, 1/2) \setminus [-1/3, 1/3) \end{cases}$$

and extend M 1-periodically to $\mathbb{R}$. It follows that

$$M(\xi) + M(\xi + \tfrac{1}{2}) = \begin{cases} a + b & \text{if } \xi \in [-1/6, 1/6) \\ 2c & \text{if } \xi \in [-1/4, 1/4) \setminus [-1/6, 1/6). \end{cases} \tag{3.48}$$

Since we can select c to be arbitrarily large (we just need to adjust b to be small enough), both $M(\xi)$ and $M(\xi) + M(\xi + \frac{1}{2})$ can be as large as desired on a set of positive measure, while we still have that $T_M \in L^1(I)$. For $T_{M,-}$, this follows from Proposition 3.34d, while for $T_{M,+}$ we apply the recurrence formula and (3.48). We obtain for $j \geq 2$

$$\begin{aligned} \int_I F_M(2^j\xi)d\xi &= \frac{a+b}{2} \int_{[-1,-2/3)\cup[2/3,1)} F_M(2^{j-1}\xi)d\xi + \\ &\quad + c \int_{[-2/3,-1/2)\cup[1/2,2/3)} M(\xi) \prod_{k=1}^{j-2} M(2^k\xi)d\xi \\ &\leq \frac{a+b}{2} \int_I F_M(2^{j-1}\xi)d\xi + cb \int_I F_M(2^{j-2}\xi)d\xi. \end{aligned}$$

By (3.32) we need to prove that

$$\sum_{j=1}^{\infty} 2^j \int_I F_M(2^j\xi)d\xi < \infty.$$

Consider the N^{th} partial sum, $L_N = \sum_{j=1}^{N} 2^j \int_I F_M(2^j\xi)d\xi$. Our estimate shows that

$$\begin{aligned}
L_N &\leq 2\int_I F_M(2\xi)d\xi + \sum_{j=2}^{N} \frac{a+b}{2} 2^j \int_I F_M(2^{j-1}\xi)d\xi + \sum_{j=2}^{N} bc2^j \int_I F_M(2^{j-2}\xi)d\xi \\
&= 2\int_I F_M(2\xi)d\xi + (a+b)L_{N-1} + 4bcL_{N-2} + 4bc\int_I F_M(\xi)d\xi \\
&\leq 2\int_I F_M(2\xi)d\xi + 4bc\int_I F_M(\xi)d\xi + (a+b+4bc)L_N.
\end{aligned}$$

It follows that

$$L_N \leq \frac{1}{1-a-b-4bc}\left(2\int_I F_M(2\xi)d\xi + 4bc\int_I F_M(\xi)d\xi\right).$$

The right side of this inequality is, of course, finite and independent of N, which gives the desired result.

$\diamondsuit$

REMARK 3.49. In order to emphasize the differences between $T_{M,-}$ and $T_{M,+}$, let us turn our attention to $r_{M,-}$ and $r_{M,+}$. As we have seen, the value of $r_{M,-}(\xi)$ depends entirely on the orbit $\mathrm{orb}(\xi)$ and can change from orbit to orbit — not so with $r_{M,+}$. Interestingly enough, this is the consequence of ergodicity. The mapping $\xi \mapsto 2\xi \pmod 1$ is a standard example of a measure-preserving transformation (with respect to Lebesgue measure); observe that the mapping $\xi \mapsto \frac{\xi}{2}$ used in $r_{M,-}$ does not have this property. Furthermore, it is well-known (and follows from Lemma 2.1) that $\xi \mapsto 2\xi \pmod 1$ is ergodic (see, for example, [**Wal00**] for an introduction to ergodic theory). Hence the classical Birkhoff's Ergodic Theorem applies and provides us with the following important result.

$\blacksquare$

THEOREM 3.50. *If $M : \mathbb{R} \to [0,\infty)$ is measurable, 1-periodic, and bounded, then for almost every $\xi \in \mathbb{R}$,*

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=0}^{n-1} M(2^k\xi)} = \exp\left(\int_0^1 \ln M(\xi)d\xi\right).$$

PROOF. Suppose first that $\int_0^1 \ln M(\xi)d\xi > -\infty$. Since $\ln M(\xi)$ is bounded above, this implies that $\ln M(\xi) \in L^1(\mathbb{T})$. Observe that

$$\ln \sqrt[n]{\prod_{k=0}^{n-1} M(2^k\xi)} = \frac{1}{n}\sum_{k=0}^{n-1} \ln M(2^k\xi)$$

and apply Birkhoff's Ergodic Theorem to conclude that for almost every $\xi \in [0,1)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \ln M(2^k\xi) = \int_0^1 \ln M(\xi)d\xi.$$

Since $\xi \mapsto \prod_{k=0}^{n-1} M(2^k\xi)$ is 1-periodic, it follows that, for almost every $\xi \in \mathbb{R}$,

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=0}^{n-1} M(2^k\xi)} = \exp\left(\int_0^1 \ln M(\xi)d\xi\right).$$

Now suppose that $\int_0^1 \ln M(\xi)d\xi = -\infty$. For every $\delta > 0$, consider the function $M_\delta(\xi) = M(\xi) + \delta$. Certainly, $\int_0^1 \ln M_\delta(\xi)d\xi > -\infty$. Thus we obtain for every $\delta > 0$ and almost every $\xi \in \mathbb{R}$

$$\limsup_{n\to\infty} \sqrt[n]{\prod_{k=0}^{n-1} M(2^k\xi)} \le \lim_{n\to\infty} \sqrt[n]{\prod_{k=0}^{n-1} M_\delta(2^k\xi)} = \exp\left(\int_0^1 \ln M_\delta(\xi)d\xi\right).$$

A routine monotone convergence argument allows one to conclude

$$\lim_{\delta\to 0^+} \int_0^1 \ln M_\delta(\xi)d\xi = -\infty.$$

Hence

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=0}^{n-1} M(2^k\xi)} = 0.$$

$\square$

Corollary 3.51. *Let $M : \mathbb{R} \to [0,\infty)$ be measurable, 1-periodic, bounded, and $M > 0$ almost everywhere. Then*

(a) For almost every $\xi \in I$,

$$\frac{1}{r_{M,+}(\xi)} = \exp\left(\int_0^1 \ln M(\xi)d\xi\right).$$

(b) The following string of implications holds:

$$\int_0^1 \ln M(\xi)d\xi < \ln\frac{1}{2} \Rightarrow T_{M,+} \text{ is finite almost everywhere}$$
$$\Rightarrow \int_0^1 \ln M(\xi)d\xi \le \ln\frac{1}{2}.$$

Remark 3.52. (i) Recall that by Jensen's inequality,

$$\exp\left(\int_0^1 \ln M(\xi)d\xi\right) \le \int_0^1 M(\xi)d\xi,$$

and equality holds only when M is (almost everywhere) equal to a constant function.

(ii) It follows that if $M : \mathbb{R} \to [0,\infty)$ is measurable, bounded, 1-periodic, $M > 0$ almost everywhere, M is not (almost everywhere) a constant function, and

$$\int_0^1 M(\xi)d\xi \le \frac{1}{2}, \tag{3.53}$$

then $T_{M,+}$ is finite almost everywhere.

(iii) As already mentioned in Remark 3.25, of particular interest are functions $M : \mathbb{R} \to [0,\infty)$ which are measurable, 1-periodic, $M > 0$ almost everywhere, and such that M satisfies the Smith–Barnwell condition

$$M(\xi) + M(\xi + \tfrac{1}{2}) = 1 \text{ almost everywhere.}$$

Observe that, for such functions, we have $0 < M \le 1$ and

$$\int_0^1 M(\xi)d\xi = \frac{1}{2}.$$

Among such functions M, there is only one constant function, $M \equiv \frac{1}{2}$.

■

Corollary 3.54. *Let $M : \mathbb{R} \to [0, \infty)$ be measurable, 1-periodic, and $M > 0$ almost everywhere.*

(a) If $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere and $M \not\equiv \frac{1}{2}$, then $T_{M,+}$ is finite almost everywhere.

(b) If $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere and, for almost every $\xi \in I$, $\liminf_{n\to\infty} M(2^{-n}\xi) > \frac{1}{2}$, then M is FO, $A_M^{(2)} = I$, and

$$\Phi_{0,M}(\xi) > 0 \text{ for almost every } \xi \in \mathbb{R}.$$

Observe that $M \equiv \frac{1}{2}$ is the only $M > 0$ which satisfies the Smith–Barnwell condition and has $T_{M,+} \equiv \infty$.

Remark 3.55. We have now painted a fairly complete picture of T_+ and T_- for the class of functions $M > 0$ which satisfy the Smith–Barnwell condition. Except for $M \equiv \frac{1}{2}$, all such functions will have $T_{M,+}$ which is finite almost everywhere, which is unbounded on any interval, and which is non-integrable. The existence of non-trivial solutions will then depend entirely on $T_{M,-}$.

Under the additional assumption that M is bounded away from zero, we can say even more about the properties of $T_{M,+}$. Since this case is not of immediate interest to us (observe that this conditions prevents M from "reaching" the value 1), we present only the main steps of the techniques required (and are confident that an interested reader would be able to fill in the details easily).

We use the notation developed in (3.36) through (3.43). It is easy to see that for a measurable $K \subseteq I$ we have that

$$|\rho_1(K \cap I_1) \cap \rho_2(K \cap I_2)| = 0 \Rightarrow |\rho_1(K \cap I_1) \cup \rho_2(K \cap I_2)| = 2|K|. \tag{3.56}$$

We denote the set $\rho_1(K \cap I_1) \cup \rho_2(K \cap I_2) = \rho(K)$. We are interested in the dynamics of $\rho^N(K)$, $N \in \mathbb{N}$. In general (i.e. if we only know that $|K| > 0$) we will only be able to conclude that $|I \setminus \rho^N(K)| \to 0$ as $N \to \infty$.

For some special sets we can obtain better results. We will say that $J \subset I$ is a (mod 1)-interval if there exists an interval $L \subset [-2, 2)$ which is "congruent mod 1" to J (i.e. J is an open (mod 1)-interval if it is of the form (a, b), $[-1, a) \cup [b, 1)$, or $(a, -1/2) \cup [1/2, b)$). It is not difficult to show that

$$J \text{ is a (mod 1)-interval} \Rightarrow \rho(J) \text{ is a (mod 1)-interval.} \tag{3.57}$$

Hence, since $|J| > 0$ and $|I| = 1$, it follows that there exists $N_1 \in \mathbb{N}$ such that

$$|\rho_1(\rho^{N_1}(J) \cap I_1) \cap \rho_2(\rho^{N_1}(J) \cap I_2)| > 0;$$

observe that this precludes $\rho^{N_1}(J)$ from being of the form (a, b). Without loss of generality, we consider the case $\rho^{N_1}(J) = [-1, a) \cup [b, 1)$, where $-1 < a \le -1/2$ and $1/2 \le b < 1$. If $b > 3/4$, then $\rho([b, 1)) = \rho_2([b, 1)) = [2b, 2) - 1$; and similarly for a. Therefore, there exists $N_2 \in \mathbb{N}$ such that $\rho^{N_1+N_2}(J) \supseteq [-1, a_1) \cup [b_1, 1)$, where $a_1 \ge -3/4$ and $b_1 \le 3/4$. Observe that $\rho([-1, a_1)) \supseteq \rho_2(-J_2) = [-1, -1/2)$ and $\rho([b_1, 1)) \supseteq \rho_2(J_2) = [1/2, 1)$.

Hence we have proven that, for every $J \subseteq I$, there exists $N = N(J) \in \mathbb{N}$ such that

$$\rho^N(J) = I. \tag{3.58}$$

We also need to extend (3.43). The following statement is easy to prove by induction over N. If $K \subseteq I$ is measurable and $G \geq 0$ is $\frac{1}{2^N}$-periodic for some $N \in \mathbb{N}$, then

$$\int_K G(\xi)d\xi \geq \frac{1}{2^N} \int_{\rho^N(K)} G\left(\frac{\xi}{2^N}\right) d\xi. \tag{3.59}$$

■

Proposition 3.60. *Let $M : \mathbb{R} \to [0, \infty)$ be measurable, 1-periodic, and satisfy $M(\xi) + M(\xi + \frac{1}{2}) \geq 1$ almost everywhere. If there exists $A > 0$ such that $M \geq A$ almost everywhere, then for every interval $J \subseteq I$ we have*

$$\int_J T_{M,+}(\xi)d\xi = \infty.$$

Proof. Take an interval $J \subseteq I$, and $N \in \mathbb{N}$ such that (3.58) holds. For $j > N$, we obtain

$$G_j(\xi) := \left(\prod_{k=0}^{N-1} M(2^k\xi)\right) \cdot \left(\prod_{k=0}^{j-N-1} M(2^{k+N}\xi)\right) \geq A^N G_{j-N}(2^N\xi),$$

and observe that $G_{j-N}(2^N\xi)$ is $\frac{1}{2^N}$-periodic. We apply (3.59) to get

$$\int_J G_j(\xi)d\xi \geq A^N \int_J G_{j-N}(2^N\xi) \geq A^N \frac{1}{2^N} \int_I G_{j-N}(\xi)d\xi.$$

Observe that Proposition 3.35 and our assumptions imply that, for every $\ell \in \mathbb{N}$,

$$2^\ell \int_I G_\ell(\xi) \geq 1.$$

Hence

$$\begin{aligned}
\int_J T_{M,+}(\xi)d\xi &\geq \sum_{j=N+1}^{\infty} 2^j \int_J G_j(\xi)d\xi \\
&\geq A^N \sum_{j=N+1}^{\infty} 2^{j-N} \int_I G_{j-N}(\xi)d\xi \\
&\geq A^N \sum_{j=N+1}^{\infty} 1 \\
&= \infty
\end{aligned}$$

□

The following example shows that having $M(\xi)+M(\xi+\frac{1}{2}) \geq 1$ is not necessarily an obstacle for the existence of nontrivial elements of Sol_M.

Example 3.61. Take $a > 1$ and $b > 0$ such that $ab < \frac{1}{4}$ (i.e. $\sqrt{ab} < \frac{1}{2}$). Observe that by adjusting b small enough, we can select arbitrarily large a. Take any two measurable sets $A, B \subseteq [-1/2, 1/2)$ such that $A \cap B = \emptyset$, $A \cup B = [-1/2, 1/2)$, and $A + \frac{1}{2} = B (\mathrm{mod}\ 1)$. Define M so that M is 1-periodic and

$$M|_{[-1/2,1/2)} = a\chi_A + b\chi_B.$$

It follows that for every $\xi \in \mathbb{R}$

$$M(\xi) + M(\xi + \tfrac{1}{2}) = a + b > 1. \tag{3.62}$$

Observe that

$$\int_0^1 \ln M(\xi) d\xi = \frac{1}{2}(\ln a + \ln b) = \ln \sqrt{ab} < \ln \frac{1}{2}.$$

Hence by Corollary 3.51b, we have that $T_{M,+}$ is finite almost everywhere. Obviously, there are many choices for A and B. One possible choice is to select A so that there exists an $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq A$. With this choice, we have $\lim_{n\to\infty} M(2^{-n}\xi) = a > 1$ for all $\xi \in I$. Observe that under such a choice $M \geq a > 1$ in a neighborhood of 0, we have (3.62), and $\mathrm{Sol}_M \neq \{0\}$.

$\diamondsuit$

We now turn our attention to the class of functions M which are FO and satisfy the Smith–Barnwell condition.

4. Smith–Barnwell Filters, FO Case

In this section, we revisit one of the most studied classes of filters, consisting of filters satisfying the Smith–Barnwell condition. Let $M : \mathbb{R} \to [0, \infty)$ be measurable, 1-periodic, $M > 0$ almost everywhere, $M \not\equiv 1/2$ and $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere. Then

$$\mathrm{Sol}_M \neq \{0\} \Leftrightarrow |\{\xi \in I : T_{M,-}(\xi) < \infty\}| > 0 \tag{4.1}$$

and $A_M^{(2)} = \{T_{M,-} < \infty\}$. In this section, we concentrate on the class

$$\begin{aligned}\mathcal{M}_{SB}^{FO} :=\{&M : \mathbb{R} \to [0, \infty) : M \text{ is measurable, 1-periodic,} \\ &M > 0 \text{ a.e.}, M(\xi) + M(\xi + 1/2) = 1 \text{ a.e., and } \mathrm{Sol}_M \neq \{0\}\}.\end{aligned} \tag{4.2}$$

REMARK 4.3. In the terminology developed in [**PŠW99**], [**PŠWX01**], and [**PŠWX03**], a function $m : \mathbb{R} \to \mathbb{C}$ which is measurable, 1-periodic, and which satisfies $|m(\xi)|^2 + |m(\xi + \frac{1}{2})|^2 = 1$ almost everywhere is called a *generalized filter*. Hence, among other things, in this section we provide a fresh perspective on the theory of generalized filters.

$\blacksquare$

Given $\Phi \in \mathrm{Sol}_M$, where $M \in \mathcal{M}_{SB}^{FO}$, it is easy to see directly (or consult [**PŠW99**]) that for almost every $\xi \in I$ and for every $j \in \mathbb{Z}$,

$$\infty \geq \lim_{n\to\infty} \Phi(2^{-n}\xi) \geq \Phi(2^j\xi) \geq \Phi(2^{j+1}\xi) \geq \lim_{n\to\infty} \Phi(2^n\xi) = 0. \tag{4.4}$$

In particular, we have for almost every $\xi \in I$,

$$\lim_{n\to\infty} \Phi_{0,M}(2^{-n}\xi) > 0 \Leftrightarrow T_{M,-}(\xi) < \infty, \tag{4.5}$$

and

$$0 < \lim_{n\to\infty} \Phi_{0,M}(2^{-n}\xi) < \infty \Leftrightarrow \prod_{n=1}^{\infty} M(2^{-n}\xi) > 0. \tag{4.6}$$

Recall also that (4.6) implies $\lim_{n\to\infty} M(2^{-n}\xi) = 1$.

REMARK 4.7. Recall that properties of $T_{M,-}$ depend only on the values of M in a neighborhood of 0. Observe that we can take any measurable function

$h : [-1/4, 1/4) \to (0, 1]$ such that $|\{\xi : h(\xi) = 1\}| = 0$ and define a 1-periodic, measurable M so that

$$M(\xi) := \begin{cases} h(\xi) & \text{if } \xi \in [-1/4, 1/4) \\ 1 - h(\xi - 1/2) & \text{if } \xi \in [1/4, 1/2) \\ 1 - h(\xi + 1/2) & \text{if } \xi \in [-1/2, -1/4) \end{cases}$$

Then $M > 0$ almost everywhere and M satisfies the Smith–Barnwell condition. It is easy to see that we can adjust h to have various behaviors on different orbits. For example, if $\liminf_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} h(2^{-k}\xi)} < 1/2$, then $\Phi(2^j\xi) = 0$ for every $\Phi \in \mathrm{Sol}_M$ and every $j \in \mathbb{Z}$. If $\liminf_{n\to\infty} h(2^{-n}\xi) \in (1/2, 1)$, then $T_{M,-}(\xi) < \infty$ and $\lim_{n\to\infty} \Phi_{0,M}(2^{-n}\xi) = \infty$.

■

Observe that for $M \leq 1$, we always have $\limsup_{n\to\infty} M(2^{-n}\xi) \leq 1$. Hence, in this case, $\liminf_{n\to\infty} M(2^{-n}\xi) = 1$ if and only if $\lim_{n\to\infty} M(2^{-n}\xi) = 1$. Of particular interest to us are functions M which satisfy

$$\lim_{n\to\infty} M(2^{-n}\xi) = 1 \text{ for almost every } \xi \in I; \tag{4.8}$$

for simplicity we assume that, in such a case, $M(0) = 1$ (although if the only condition on M is that M is measurable, this does not have any appreciable effect) and we call (4.8) *dyadic continuity at 0* — this notation has been used in the Wavelet Seminar at Washington University in St. Louis for at least fifteen years; see also [**Gun00**]. Obviously, if M is 1-periodic, measurable, $M > 0$ almost everywhere, M is dyadically continuous at 0, and $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere, then $M \in \mathcal{M}_{SB}^{FO}$ and $\Phi_{0,M} > 0$ almost everywhere. By adjusting h in Remark 4.7 appropriately, it is not difficult to construct $M \in \mathcal{M}_{SB}^{FO}$ so that M is not dyadically continuous at 0. Similarly, it is easy to construct $M \in \mathcal{M}_{SB}^{FO}$ such that M is dyadically continuous at 0 but not continuous at 0. Obviously, if M is continuous at 0 (and $M(0) = 1$), then M is also dyadically continuous at 0. Moreover, if $M \in \mathcal{M}_{SB}^{FO}$ and there exists $\Phi \in \mathrm{Sol}_M$ such that $0 < \Phi \leq A < \infty$ almost everywhere, then M is dyadically continuous at 0. Actually (recall Lemma 3.14), in this case we also have (4.6); observe that, in general, dyadic continuity at zero by itself does not imply (4.6).

These notions beg several questions. Let us begin with the question as to whether continuity at 0 (which is a stronger requirement than dyadic continuity at 0) implies (4.6). As the following counter-example shows, the answer is negative.

Example 4.9. Define M via h, as outlined in Remark 4.7, so that $h|_{2^{-k}I} \equiv a_k > 0$, $k \in \{2, 3, 4, ...\}$. Obviously, if $h(0) := 1$ and $\lim_{k\to\infty} a_k = 1$, then h is continuous at 0. We also assume that $a_k < 1$ for every k. It follows that $M \in \mathcal{M}_{SB}^{FO}$ and $\Phi_{0,M} > 0$ almost everywhere.

Now select $a_k = e^{-1/k}$. It follows that for every $\xi \in \frac{1}{2}I$,

$$\prod_{n=1}^{\infty} M(2^{-n}\xi) = \prod_{k=2}^{\infty} e^{-1/k} = \exp\left(-\sum_{k=2}^{\infty} \frac{1}{k}\right) = 0.$$

♢

The issue of (4.6) brings back the question raised in Remark 3.11(iii). Under the Smith–Barnwell condition, we can fully answer this question; actually, most of the answer is already provided in [**PŠW99**]. Let us revisit the problem from the point of view developed in this article. If $M : \mathbb{R} \to [0,1]$ is measurable and 1-periodic, then we can define a function $f_M : \mathbb{R} \to [0,1]$ via

$$f_M(\xi) = \prod_{n=1}^{\infty} M(2^{-n}\xi) \text{ for } \xi \in \mathbb{R}. \tag{4.10}$$

Obviously, f_M is measurable and $f_M(\xi) = M(\xi) f_M(\xi)$ for every $\xi \in \mathbb{R}$. Consider a measurable set $K \subseteq \mathbb{R}$ which, up to almost everywhere equality, is 1-translation congruent to $[-1/2, 1/2)$, in the sense that the mapping $\xi \mapsto \xi - \lfloor \xi \rfloor - 1/2$ is a bijection from K to $[-1/2, 1/2)$; see [**PŠW99**] for details. Assume further that $0 \in \operatorname{Int}(K)$. For every $n \in \mathbb{N}$, we define $f_M^{K,n} : \mathbb{R} \to [0,1]$ by

$$f_M^{K,n}(\xi) = \chi_{2^n K}(\xi) \prod_{j=1}^{n} M(2^{-j}\xi) \text{ for } \xi \in \mathbb{R}. \tag{4.11}$$

If, in addition, we assume that $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere, then the well-known "peeling-off argument" (see Lemma 3.15 in [**PŠW99**] and pp. 369–372 in [**HW96**]) shows that:

$$\int_{\mathbb{R}} f_M^{K,n}(\xi) e_\ell(\xi) d\xi = \delta_{0,\ell}. \tag{4.12}$$

$$\lim_{n\to\infty} f_M^{K,n}(\xi) = f_M(\xi) \text{ for almost every } \xi \in \mathbb{R}. \tag{4.13}$$

$$\|f_M\|_{L^1(\mathbb{R})} \le 1. \tag{4.14}$$

It follows that, for M which satisfy the Smith–Barnwell condition, we always have

$$f_M \in \mathrm{Sol}_M. \tag{4.15}$$

However, f_M may be zero on some (or all) orbits, in which case it also does not provide us with much information about the solution of our problem. Take M as in Example 4.9; we have that M is continuous at 0, $M \in \mathcal{M}_{SB}^{FO}$, $\Phi_{0,M} > 0$ almost everywhere, but $f_M \equiv 0$ — observe that in this case $\lim_{n\to\infty} \Phi_{0,M}(2^{-n}\xi) = \infty$ almost everywhere. Combining these results in a single statement, it is now easy to show that the following theorem holds.

THEOREM 4.16. *If $M : \mathbb{R} \to [0, \infty)$ is measurable, 1-periodic, $M > 0$ almost everywhere, and $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere, then the following are equivalent:*

(a) $M \in \mathcal{M}_{SB}^{FO}$ and, for almost every $\xi \in I$, $0 < \lim_{n\to\infty} \Phi_{0,M}(2^{-n}\xi) < \infty$.
(b) $M \in \mathcal{M}_{SB}^{FO}$ and there exists $\Phi \in Sol_M$ such that $0 < \Phi \le 1$ almost everywhere.
(c) $f_M > 0$ almost everywhere.
(d) For almost every $\xi \in I$, $\prod_{k=1}^{\infty} M(2^{-n}\xi) > 0$.
(e) For almost every $\xi \in I$, $\sum_{k=1}^{\infty} \ln M(2^{-n}\xi) > -\infty$.

It is also easy to see directly that, under the assumptions of Theorem 4.16, f_M is a maximal solution with respect to $\prec_M$ if and only if

$$f_M(\xi) > 0 \text{ for almost every } \xi \in A_M^{(2)}. \tag{4.17}$$

REMARK 4.18. Observe that under conditions of Theorem 4.16, i.e. if $f_M > 0$ almost everywhere, then we also have

$$\lim_{n\to\infty} f_M(2^{-n}\xi) = 1 \text{ for almost every } \xi \in I. \tag{4.19}$$

Assume now that $M = |m|^2$ for some generalized filter m. In the terminology of [**PŠW99**], [**PŠWX01**], and [**PŠWX03**], a generalized filter m which satisfies (4.19) is called a *generalized low-pass filter*. It follows that the conditions of Theorem 4.16 are, in this case, equivalent to m being a generalized low-pass filter.

This class of filters is particularly important in the theory of Parseval frame wavelets, as developed in [**PŠWX01**] and [**PŠWX03**]. It is shown there that every MRA Parseval frame wavelet is constructed from a generalized low-pass filter, and, vice-versa, every generalized low-pass filter provides a constructive way to build an MRA Parseval frame wavelet. Hence Theorem 4.16 characterizes all functions which are of the form $|m|^2$ where m is the low-pass filter associated to an MRA Parseval frame wavelet and $m \neq 0$ almost everywhere.

■

As we have seen in Example 4.9, continuity does not necessarily imply that $f_M > 0$ almost everywhere. However, any level of Hölder continuity does imply $f_M > 0$ almost everywhere. This result is essentially contained in Lemma 4.5 in [**PŠW99**]. For the reader's convenience, we state the result here and prove a stronger result later on.

Proposition 4.20 ([**PŠW99**], Lemma 4.5). *Let $M : \mathbb{R} \to [0, \infty)$ be measurable, 1-periodic, $M > 0$ almost everywhere, and $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere. If $M(0) = 1$ and there exist constants $\alpha, A, \varepsilon \in (0, \infty)$ such that, for every $\xi \in (-\varepsilon, \varepsilon)$, one has $|M(0) - M(\xi)| \leq A|\xi|^\alpha$ (i.e. M is Hölder continuous at 0), then $\prod_{k=1}^{\infty} M(2^{-n}\xi) > 0$ for almost every $\xi \in I$.*

One may think that some form of "dyadic Hölder continuity" is necessary for $f_M > 0$ almost everywhere to hold. The following example shows that it is not.

Example 4.21. Use the same construction as in Example 4.9 with $a_k = \exp(-\frac{1}{k^2})$. M is then continuous at 0 and for every $\xi \in \frac{1}{2}I$,

$$\prod_{n=1}^{\infty} M(2^{-n}\xi) = \exp\left(-\sum_{k=2}^{\infty} \frac{1}{k^2}\right) > 0.$$

However, M is not Hölder continuous at 0. Suppose, to the contrary, that it were. Then, for k sufficiently large, we would have

$$0 < 1 - e^{-\frac{1}{k^2}} \leq A\left(\frac{1}{2^k}\right)^\alpha,$$

which leads to

$$1 = \lim_{k\to\infty} \frac{1 - \exp\left(-\frac{1}{k^2}\right)}{\frac{1}{k^2}} \leq A \lim_{k\to\infty} \frac{k^2}{2^{\alpha k}} = 0,$$

which is obviously a contradiction.

Observe that M is an example of a function of the form $M = |m|^2$, where m is a generalized low-pass filter, i.e. m generates an MRA Parseval frame wavelet.

♢

One may ask whether the continuity of the entire M may improve the situation. Consider the following example.

Example 4.22. Take a partition of $[1/2, 1]$ defined by $1/2 = c_0 < c_1 < c_2 < c_3 < c_4 < c_5 = 1$ and consider $C_i := [-c_i, -c_{i-1}] \cup [c_{i-1}, c_i]$ for $i = 1, 2, 3, 4, 5$.

Observe that $[-1/4, 1/4] \setminus \{0\} = \bigcup_{k=2}^{\infty} \bigcup_{i=1}^{5} 2^{-k} C_i$. Consider two sequences (a_k) and (b_k) in $(0,1)$ such that $\lim_{k\to\infty} a_k = 1 = \lim_{k\to\infty} b_k$. Define M via h as outlined in Remark 4.7 in the following way. Let $h(0) := 1$, $h(\pm 1/4) := 1/2$, $h|_{2^{-k}C_2} \equiv a_k$, $h|_{2^{-k}C_4} \equiv b_k$, for $k \in \{2, 3, 4, ...\}$. Define h on $\bigcup_{k=2}^{\infty}(2^{-k}C_1 \cup 2^{-k}c_3 \bigcup 2^{-k}C_5)$ so that the endpoints on every interval are connected by a linear function; for example, $h(\frac{1}{4}c_4) = b_2$ and $h(\frac{1}{4}c_5) = \frac{1}{2}$, so we define h to be linear on $[\frac{1}{4}c_4, \frac{1}{4}c_5] = [\frac{1}{4}c_4, \frac{1}{4}]$ with the given endpoints. It is not difficult to see that $M|_{[-1/2,1/2]}$ is now continuous with $M(0) = 1$ and $M(\pm 1/2) = 0$, so that M can be extended 1-periodically to a function M which is continuous on all of $\mathbb{R}$. Observe also that $M > 0$ almost everywhere, M satisfies the Smith–Barnwell condition for all $\xi \in \mathbb{R}$, Sol_M is non-trivial, and $\Phi_{0,M} > 0$ almost everywhere.

Observe also that if we select increasing sequences (a_k) and (b_k), then, for every $\xi \in [-1/4, 1/4]$,

$$M(\xi) \geq \min\{1/2, a_2, b_2\} > 0. \tag{4.23}$$

As for the condition "$f_M > 0$" is concerned, we can have different behaviors, depending on the choice of (a_k) and (b_k). For example, if we select $a_k := \exp(-\frac{1}{k^2})$ and $b_k = \exp(-\frac{1}{k})$, then f_M will behave differently on different subsets of positive measure. More precisely,

$$f_M|_{C_2} > 0 \text{ almost everywhere and } f_M|_{C_4} \equiv 0.$$

Observe also that, for almost every $\xi \in C_2$, one has that $\lim_{n\to\infty} f_M(2^{-n}\xi) = 1$, while, for every $\xi \in C_4$, it is obviously the case that $\lim_{n\to\infty} f_M(2^{-n}\xi) = 0$. Hence, despite M being continuous, f_M is not continuous at 0.

If, instead, we select $a_k = b_k = \exp(-\frac{1}{k^2})$, then it is not difficult to see that $f_M > 0$ almost everywhere, and $f_M(0) := 1$ gives that f_M is continuous at 0. Observe that, in this case, M is not Hölder continuous at 0 (see Example 4.21). ♢

A modification of the argument of Lemma 4.5 from [**PŠW99**] shows that the following result (which is stronger than Proposition 4.20) is valid. Observe that we do not require $M > 0$ almost everywhere in this statement.

Proposition 4.24. *Let $M : \mathbb{R} \to [0, \infty)$ be measurable, 1-periodic, and $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere. Let us define $f_M(0) = 1$. If $M(0) = 1$ and there exists $\alpha, A, \varepsilon \in (-\varepsilon, \varepsilon)$ such that, for every $\xi \in (-\varepsilon, \varepsilon)$, one has $|M(0) - M(\xi)| \leq A|\xi|^{\alpha}$, then f_M is continuous at 0.*

Proof. (See Lemma 4.5 in [**PŠW99**].) Take any $0 < \delta < 1$. Choose $n_0 \in \mathbb{N}$ such that $2^{-n_0} < \varepsilon$ and

$$\frac{A}{2^{\alpha n_0}(2^{\alpha} - 1)} < \delta.$$

For $n \geq n_0$ and $r \in \mathbb{N}$ with $r > 1$, we obtain, for every $\xi \in I$,

$$
\begin{aligned}
0 &\leq 1 - \prod_{j=n_0+1}^{n_0+r} M(2^{-j}\xi) \\
&= 1 - \prod_{j=n_0+2}^{n_0+r} M(2^{-j}\xi) + (1 - M(2^{-n_0-1}\xi)) \prod_{j=n_0+2}^{n_0+r} M(2^{-j}\xi) \\
&\leq 1 - \prod_{j=n_0+2}^{n_0+r} M(2^{-j}\xi) + A\frac{|\xi|^\alpha}{2^{(n_0+1)\alpha}} \\
&\leq \dots \\
&\leq A\left(\frac{|\xi|}{2^{n_0+1}}\right)^\alpha \sum_{k=0}^{n_0+r-1} \frac{1}{2^{k\alpha}} \\
&\leq \frac{A}{2^{n_0\alpha}} \frac{1}{2^\alpha - 1} \\
&< \delta,
\end{aligned}
$$

and the same inequality holds for every $n \geq n_0$. If we let $r \to \infty$, we obtain, for every $n \geq n_0$ and every $\xi \in I$,

$$0 \leq 1 - f_M(2^{-n}\xi) < \delta.$$

Hence for every $u \in [-2^{-n_0}, 2^{-n_0}]$, we have $0 \leq 1 - f_M(u) < \delta$, i.e. $|f_M(0) - f_M(u)| < \delta$. $\square$

EXAMPLE 4.25. Let us modify Example 4.22 so that we have a partition $1/2 = c_0^k < c_1^k < c_2^k < c_3^k < c_4^k < c_5^k = 1$, for every $k \in \mathbb{N} \setminus \{1\}$. We assume that c_1^k decreases to $1/2$ and that c_2^k increases to 1 as $k \to \infty$. For $k \in \mathbb{N} \setminus \{1\}$ and $i = 1, 2, 3, 4, 5$, we define $C_i^k := [-c_i^k, -c_{i-1}^k] \cup [c_{i-1}^k, c_i^k]$. Again, we have that $[-1/4, 1/4] \setminus \{0\} = \bigcup_{k=2}^\infty \bigcup_{i=1}^5 2^{-k} C_i^k$. Define h in a piece-wise linear fashion as before, but select h so that $h(0) = 1$, $h(\pm 1/4) = 1/2$, $h|_{2^{-k}C_2^k} \equiv \exp(-\frac{1}{k^2})$, and $h|_{2^{-k}C_4^k} \equiv \frac{1}{2}$. Observe that M is then continuous on $[-1/2, 1/2] \setminus \{0\}$, but it is not continuous at 0. Observe that for every ξ in the interior of I, there exists $k_0 \in \mathbb{N}$ such that $\xi \in C_2^k$, for every $k \geq k_0$. Hence $\prod_{k=1}^\infty M(2^{-n}\xi) > 0$ almost everywhere. Observe that for every $\xi \in 2^{-k} C_k^4$ we have $f_M(2\xi) \leq M(\xi) = 1/2$, i.e. f_M is not continuous at 0.

$\diamondsuit$

REMARK 4.26. This completes the analysis of those functions M which are FO-type and are induced by generalized low-pass filters (i.e. $M = |m|^2$ for some generalized low-pass filter m). Our results and examples show that we have the following schematic diagram for functions $M : \mathbb{R} \to [0, \infty)$ which are measurable, 1-periodic, and such that $M(0) = 1$, $M > 0$ almost everywhere, and $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere:

$$\begin{tikzcd}[column sep=small]
& M \text{ Hölder continuous at } 0 \arrow[d, Rightarrow] & & \\
& f_M(0)=1 \text{ and } f_M \text{ continuous at } 0 \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \\
M \text{ continuous at } 0 \arrow[dr, Rightarrow] & & f_M>0 \text{ a.e.} \arrow[r, Leftrightarrow] \arrow[dl, Rightarrow] & f_M \text{ dyadically continuous at } 0 \\
& M \text{ dyadically continuous at } 0 & &
\end{tikzcd}$$

Observe that, in this scheme, whenever we have an implication in one direction, i.e. "$\Rightarrow$", there is a counterexample which shows that the reverse implication does not hold. Recall that continuity of M at 0 does not necessarily imply that $f_M > 0$ almost everywhere (Example 4.22). The condition "$f_M > 0$ almost everywhere" does not necessarily imply the continuity of M at 0 (Example 4.25).

$\diamondsuit$

The condition "$f_M > 0$" is of continuing interest for our analysis. For this reason, we shall go a bit deeper into the schematic diagram given in the previous remark. In some cases it is useful to present the analysis of f_M in terms of

$$\ln \frac{1}{f_M(\xi)} = \sum_{n=1}^{\infty} |\ln M(2^{-n}\xi)|. \tag{4.27}$$

For $\delta > 0$, let $I_\delta := [-\delta, \delta) \setminus [-\delta/2, \delta/2)$. Since $f_M(\xi) \geq f_M(2\xi)$, it is sufficient in many cases to obtain properties of f_M on I_δ. For example, the condition "f_M is continuous at 0" implies (but is not equivalent to; using methods similar to Example 4.25 our readers can construct counterexamples themselves) the condition "f_M is bounded away from zero in a neighborhood of 0", which, in turn, is equivalent to the condition

$$\text{"there exists } \delta > 0 \text{ such that } \ln \frac{1}{f_M} \text{ is essentially bounded on } I_\delta.\text{"} \tag{4.28}$$

Similarly, observe that the condition "$f_M > 0$ almost everywhere" is equivalent to

$$\text{"there exists } \delta > 0 \text{ such that } \ln \frac{1}{f_M(\xi)} < \infty \text{ for almost every } \xi \in I_\delta.\text{"} \tag{4.29}$$

There is another condition which "fits naturally between" conditions (4.28) and (4.29), namely

$$\text{"there exists } \delta > 0 \text{ such that } \ln \frac{1}{f_M} \in L^1(I_\delta).\text{"} \tag{4.30}$$

This leads naturally to the following Dini condition for $\ln M(\xi)$.

Proposition 4.31. *Let $M : \mathbb{R} \to [0, \infty)$ be measurable, 1-periodic, $M > 0$ almost everywhere, and $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere. If there exists a $\delta > 0$ such that*

$$\int_{-\delta/2}^{\delta/2} |lnM(\xi)| \frac{d\xi}{|\xi|} < \infty,$$

then $f_M(\xi) > 0$ for almost every $\xi \in \mathbb{R}$.

PROOF. By our previous discussion, it is enough to show that (4.30) holds. Obviously,

$$\int_{I_\delta} \ln \frac{1}{f_M(\xi)} d\xi = \sum_{n=1}^{\infty} \int_{I_\delta} |\ln M(2^{-n}\xi)| d\xi = \sum_{n=1}^{\infty} 2^n \int_{2^{-n} I_\delta} |\ln M(y)| dy.$$

Observe that $y \in 2^{-n}\overline{I_\delta}$ if and only if

$$\frac{\delta}{2} \frac{1}{|y|} \leq 2^n \leq \delta \frac{1}{|y|}.$$

Hence

$$\int_{I_\delta} \ln \frac{1}{f_M(\xi)} d\xi \asymp \delta \sum_{n=1}^{\infty} \int_{2^{-n} I_\delta} |\ln M(y)| \frac{dy}{|y|} = \delta \int_{-\delta/2}^{\delta/2} |\ln M(y)| \frac{dy}{|y|}.$$

□

REMARK 4.32. As our discussion shows, we have narrowed the analysis of M to understanding those functions "between" the conditions "$f_M > 0$ almost everywhere" and

(4.33) "there exist $\delta, C > 0$ such that $f_M|_{(-\delta,\delta)} \geq C$ almost everywhere"

(observe that (4.33) is equivalent to (4.28)). In this realm of functions, we do *not* necessarily have that M is continuous at 0 (compare also with the schematic diagram in Remark 4.26), but we have a gradual improvement of the behavior of M around 0. Let us be more precise here.

Assuming the Dini condition from Proposition 4.31, we obtain

$$\lim_{\delta \to 0^+} \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} |\ln M(\xi)| d\xi \leq \lim_{\delta \to 0^+} \frac{1}{2} \int_{-\delta/2}^{\delta/2} |\ln M(\xi)| \frac{d\xi}{|\xi|} = 0,$$

i.e., the point $\xi = 0$ is a point of Lebesgue differentiation for the function $\ln M(\xi)$ with value 0. Using Jensen's inequality, we observe that

$$\begin{aligned} 1 &\geq \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} M(\xi) d\xi \\ &= \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \exp(\ln M(\xi)) d\xi \\ &\geq \exp\left(\frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \ln M(\xi) d\xi \right). \end{aligned}$$

Hence $\lim_{\delta \to 0^+} \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} M(\xi) d\xi = 1$, i.e. $\xi = 0$ is a point of Lebesgue differentiation for the function $M(\xi)$ with value 1.

Actually, the Dini condition implies the analogous result for f_M as well. Using the proof of Proposition 4.31, we can see that the Dini condition implies that $\xi = 0$ is a point of Lebesgue differentiation for $\ln f_M(\xi)$ with value 0 (and, therefore, also for $f_M(\xi)$ with value 1):

$$\begin{aligned}\frac{1}{\delta}\int_{-\delta/2}^{\delta/2}|\ln f_M(\xi)|d\xi &= \sum_{n=1}^{\infty}\frac{1}{\delta}\int_{2^{-n}I_\delta}|\ln f_M(\xi)|d\xi \\ &\asymp \sum_{n=1}^{\infty}2^{-n}\int_{-\delta/4}^{\delta/4}|\ln M(y)|\frac{dy}{|y|} \\ &= \int_{-\delta/4}^{\delta/4}|\ln M(y)|\frac{dy}{|y|}.\end{aligned}$$

This leads naturally to the question of the characterization of the condition "$f_M > 0$ almost everywhere" via the behavior of f_M around zero.

■

Let us introduce some additional notation closely related to the notation given in (4.11). We denote the interval $[-1/2, 1/2)$ by K_0. For $n \in \mathbb{N}$, we define $f_M^n : \mathbb{R} \to [0,1]$ by

$$f_M^n(\xi) = \prod_{k=1}^{n} M(2^{-k}\xi) \text{ for } \xi \in \mathbb{R}, \tag{4.34}$$

and $f_M^{n,*} : \mathbb{R} \to [0,\infty)$ by

$$f_M^{n,*}(\xi) = 2^n f_M^n(2^n\xi) = \prod_{k=1}^{n-1} 2M(2^k\xi) \text{ for } \xi \in \mathbb{R}. \tag{4.35}$$

Observe that for every measurable set $E \subseteq \mathbb{R}$, we have

$$\int_E f_M^{n,*}(\xi)d\xi = \int_{2^nE} f_M^n(\xi)d\xi. \tag{4.36}$$

In particular, (4.12) implies

$$\int_{K_0} f_M^{n,*}(\xi)d\xi = \int_{2^nK_0} f_M^n(\xi)d\xi = 1. \tag{4.37}$$

For $a \in (0,1)$, denote $E^a = E_M^a := \{\xi \in K_0 : f_M(\xi) \geq a\}$ and $E_a = E_{a,M} := \{\xi \in K_0 : f_M(\xi) < a\} = K_0 \setminus E^a$.

Proposition 4.38. *Let $M : \mathbb{R} \to [0,\infty)$ be measurable, 1-periodic, $M > 0$ almost everywhere, and $M(\xi)+M(\xi+\frac{1}{2}) = 1$ almost everywhere. Then the following are equivalent:*

(a) $f_M > 0$ *almost everywhere.*
(b) $\lim_{n\to\infty} f_M(2^{-n}\xi) = 1$ *almost everywhere.*
(c) $\lim_{n\to\infty} \int_{K_0} f_M(2^{-n}\xi)d\xi = 1$.
(d) There exists $0 < a < 1$ such that $\lim_{n\to\infty} \chi_{E^a}(2^{-n}\xi) = \lim_{n\to\infty} \chi_{2^nE^a}(\xi) = 1$ *almost everywhere.*
(e) There exists $0 < a < 1$ such that $\lim_{n\to\infty} \int_{K_0} \chi_{E^a}(2^{-n}\xi)d\xi = 1$.

Proof. That $(a) \Leftrightarrow (b)$ is true is well known since [**PŠW99**]. Directly from the monotone convergence theorem, we obtain $(b) \Rightarrow (c)$ and $(d) \Rightarrow (e)$. Observe that the monotone convergence theorem provides the proof of $(c) \Rightarrow (b)$ since

$$1 = \int_{K_0} \lim_{n\to\infty} f_M(2^{-n}\xi)d\xi$$

and the integrand is bounded by 1.

It is straightforward to see that $(b) \Rightarrow (d)$, since $\lim_{n\to\infty} f_M(2^{-n}\xi) = 1$ implies that, for every $0 < a < 1$, there exists $n_0 \in \mathbb{N}$ such that $f_M(2^{-n}\xi) \geq a$ for every $n \geq n_0$.

Finally, $(e) \Rightarrow (b)$, since we know from [**PŠW99**] that $\lim_{n\to\infty} f_M(2^{-n}\xi)$ equals 0 or 1, and the first case occurs only if $f_M|_{\mathrm{orb}(\xi)} \equiv 0$. If this would have happened on a set $D \subseteq K_0$ with $|D| > 0$, then for every $0 < a < 1$, we would have $D \subseteq E_a$. Hence $\chi_{E^a}(2^{-n}\xi) = 0$, for every $n \in \mathbb{N}$ and every $\xi \in D$. The consequence would be that

$$\lim_{n\to\infty} \int_{K_0} \chi_{E^a}(2^{-n}\xi)d\xi \leq |K_0 \setminus D| < 1,$$

which contradicts the assumption in (e). $\square$

Observe that we have provided the characterization of "$f_M > 0$ almost everywhere" already in Theorem 4.16. In the previous proposition we provided a list of technical conditions useful for the analysis of M "in the spirit of" Remark 4.32.

REMARK 4.39. Observe that conditions (d) and (e) (respectively) are equivalent to conditions (d') and (e') where the sentence "there exists $0 < a < 1$" is replaced by "for every $0 < a < 1$".

Furthermore, since $\chi_{E_a} + \chi_{E^a} = \chi_{K_0}$, these conditions are equivalent to (respectively)

$$(f) \text{ For every } 0 < a < 1, \lim_{n\to\infty} \chi_{E_a}(2^{-n}\xi) = 0 \text{ almost everywhere}$$

and

$$(g) \text{ For every } 0 < a < 1, \lim_{n\to\infty} \int_{K_0} \chi_{E_a}(2^{-n}\xi) = 0.$$

Observe that

$$\int_{K_0} f_M(2^{-n}\xi)d\xi = 2^n \int_{-2^{-n-1}}^{2^{-n-1}} f_M(y)dy;$$

therefore (c) is equivalent to

$$(h) \text{ The point } \xi = 0 \text{ is a point of Lebesgue density 1 for } f_M.$$

Similarly, (e') and (g) (respectively) are equivalent to

$$(i) \text{ The point } \xi = 0 \text{ is a point of Lebesgue density 1 for } E^a \text{ for every } 0 < a < 1$$

and

$$(j) \text{ The point } \xi = 0 \text{ is a point of Lebesgue density 0 for } E_a \text{ for every } 0 < a < 1.$$

■

We turn our attention now to one of the most important classes of filters in wavelet theory: the class of (MRA orthonormal wavelet) *low-pass filters* (for historical details and further references, see [**Dau92**] and [**HW96**]). This terminology is also used in [**PŠW99**], [**PŠWX01**], and [**PŠWX03**]. We can rephrase the characterization theorem, Theorem 3.17 from [**PŠW99**], in the following form.

THEOREM 4.40 ([**PŠW99**]). *Let $m : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ be a measurable, 1-periodic function, and let us denote $|m|^2$ by M. Then m is a low-pass filter (for an MRA orthonormal wavelet) if and only if $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere, $f_M(\xi) > 0$ almost everywhere, and*

$$\int_{\mathbb{R}} f_M(\xi) d\xi = 1 \tag{4.41}$$

REMARK 4.42. (i) Observe that the characterization theorem depends completely on the properties of M; the phase of m has no impact on this matter. Hence we denote the class of M that satisfy Theorem 4.40 by $\mathcal{M}_{LPF}^{FO}$. Obviously, $\mathcal{M}_{LPF}^{FO} \subseteq \mathcal{M}_{SB}^{FO}$.

(ii) As it is shown in [**PŠW99**], the Smith–Barnwell condition implies that

$$\int_{\mathbb{R}} f_M(\xi) d\xi \leq 1 \tag{4.43}$$

but does not necessarily imply (4.41). The typical counterexample comes from the filter

$$m_H(\xi) = \frac{1}{2}\left(1 + e^{2\pi i \xi}\right)$$

of the Haar wavelet (see [**HW96**] for more details on the Haar wavelet). We consider the "elongated" Haar wavelet associated with the filter

$$\xi \mapsto m_H(3\xi) = \frac{1}{2}\left(1 + e^{6\pi i \xi}\right) \tag{4.44}$$

As is well known (consult [**PŠWX01**], [**PŠWX03**], and [**ŠSW08**] for details), the corresponding M satisfies the Smith–Barnwell condition as well as $f_M > 0$ almost everywhere but does not satisfy (4.41).

(iii) It is also known (see [**PŠW99**]) that condition (4.41) in Theorem 4.40 can be replaced by the following equivalent condition. For $n, k \in \mathbb{N}$, we denote $B(n,k)$ by

$$B(n,k) := \int_{K_0 \setminus 2^{-k} K_0} f_M^{n,*}(\xi) d\xi.$$

Observe that, for a fixed n, the sequence $k \mapsto B(n,k)$ is non-decreasing. The condition equivalent to (4.41) is

$$\lim_{n_0 \to \infty} \sup_{n > n_0} B(n, n - n_0) = 0. \tag{4.45}$$

■

In some practical situations it may be difficult to check conditions on f_M directly. Therefore, we shall analyze (4.41) in detail in a similar approach as we took to the analysis of "$f_M > 0$ almost everywhere". The first natural question is whether, in the FO case, condition (4.41) implies "$f_M > 0$ almost everywhere". As the following example shows, this is not the case.

EXAMPLE 4.46. Consider the Haar wavelet filter, m_H and denote M_0 by $M_0 := |m_H|^2$. Hence, for $\xi \in K_0$, we have $M_0(\xi) = \cos^2(\pi\xi)$, and, for $\xi \in \mathbb{R}$,

$$f_{M_0}(\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2.$$

Take a small δ, say $0 < \delta < 1/8$, and adjust M_0 near the points 0 and 1/2. More precisely, we define M to be 1-periodic and, on K_0 given by

$$M(\xi) := \begin{cases} M_0(\xi) & \text{if } \xi \in [-1/2, 1/2) \setminus ([-\delta, 0) \cup [1/2 - \delta, 1/2)) \\ M_0(-\delta) & \text{if } \xi \in [-\delta, 0) \\ M_0(1/2 - \delta) & \text{if } \xi \in [1/2 - \delta, 1/2) \end{cases}.$$

Let us denote $M_0(1/2 - \delta)$ by ε. It follows that $M_0(-\delta) = 1 - \varepsilon$. It is easy to see that M satisfies the Smith–Barnwell condition and $M > 0$ almost everywhere. Since $1 - \varepsilon < 1$, it follows that

$$f_M(\xi) = 0 \text{ for every } \xi < 0.$$

Hence M does not satisfy the condition "$f_M > 0$ almost everywhere". It remains to show that f_M satisfies (4.41). We will show first that there exists a constant $0 < c$ such that

$$f_M(\xi) \geq c \text{ for every } \xi \in (0, 1). \tag{4.47}$$

Indeed, since $M = M_0$ on $(0, 1/2 - \delta)$, we obtain that, for every $\xi \in (0, 1 - 2\delta)$,

$$f_M(\xi) = f_{M_0}(\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2 \geq f_{M_0}(1 - 2\delta).$$

If $\xi \in [1 - 2\delta, 1)$, then

$$f_M(\xi) = M(\xi/2) f_{M_0}(\xi/2) = \varepsilon f_{M_0}(\xi/2) \geq \varepsilon f_{M_0}(1/2 - \delta).$$

It follows that $c := \min\{f_{M_0}(1 - 2\delta), \varepsilon f_{M_0}(1/2 - \delta)\}$ satisfies (4.47).

Since M satisfies the Smith–Barnwell condition, we have (see (4.34)–(4.37)) that f_M^n is 2^n-periodic and

$$1 = \int_{2^n K_0} f_M^n(\xi) d\xi = \int_0^{2^n} f_M^n(\xi) d\xi.$$

Furthermore, (4.47) implies that, for every $\xi \in (0, 2^n)$,

$$f_M(\xi) = f_M^n(\xi) \cdot f_M(\xi/2^n) \geq f_M^n(\xi) \cdot c.$$

Using $g_n(\xi) := f_M^n(\xi) \chi_{(0,2^n)}(\xi)$, we obtain that

$$\lim_{n\to\infty} g_n(\xi) = f_M(\xi) \text{ for every } \xi > 0,$$

$$g_n(\xi) \leq \frac{1}{c} f_M(\xi) \text{ for every } \xi > 0 \text{ (observe that } f_M \in L^1(\mathbb{R}) \text{ by (4.43))},$$

$$\text{and } \int_0^\infty g_n(\xi) d\xi = 1 \text{ for every } n \in \mathbb{N}.$$

By the Lebesgue dominated convergence theorem, we obtain (4.41). $\diamondsuit$

Let us analyze (4.41) from the "point of view of Proposition 4.38"; it gives us an interesting comparison of (4.41) and "$f_M > 0$ almost everywhere". We start with the following simple — but very useful — lemma.

LEMMA 4.48. *If $M : \mathbb{R} \to [0, \infty)$ is measurable, 1-periodic, $M > 0$ almost everywhere, and $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere, then, for every $0 < a < 1$,*

$$\lim_{n\to\infty} f_M(\xi) \chi_{2^n E_a}(\xi) = \lim_{n\to\infty} f_M(\xi) \chi_{E_a}(2^{-n}\xi) = 0 \textit{ almost everywhere.}$$

In particular,

$$\lim_{n\to\infty}\int_{2^nE_a} f_M(\xi)d\xi = \lim_{n\to\infty}\int_{E_a} 2^n f_M(2^n\xi)d\xi = 0.$$

PROOF. If $f_M(\xi) = 0$, then the statement is trivial. If $f_M(\xi) > 0$, then $\lim_{n\to\infty} f_M(2^{-n}\xi) = 1$, which implies (since $a < 1$) that $\lim_{n\to\infty}\chi_{E_a}(2^{-n}\xi) = 0$. Recall that, under our assumptions, $f_M \in L^1(\mathbb{R})$ and f_M dominates all functions $\xi \mapsto f_M(\xi)\chi_{2^nE_a}(\xi)$. The last statement of the lemma now follows from the Lebesgue dominated convergence theorem. □

PROPOSITION 4.49. *Let $M : \mathbb{R} \to [0,\infty)$ be measurable, 1-periodic, $M > 0$ almost everywhere, and $M(\xi)+M(\xi+\frac{1}{2}) = 1$ almost everywhere. Then the following are equivalent.*

(a)

$$\int_{\mathbb{R}} f_M(\xi)d\xi = 1.$$

(b)

$$\lim_{n\to\infty}\int_{E^a} 2^n f_M(2^n\xi)d\xi = 1 \textit{ for some (all) } 0 < a < 1.$$

(c)

$$\lim_{n\to\infty}\int_{E^a} f_M^{n,*}(\xi)d\xi = 1 \textit{ for some (all) } 0 < a < 1.$$

(d)

$$\lim_{n\to\infty}\int_{E_a} f_M^{n,*}(\xi)d\xi = 0 \textit{ for some (all) } 0 < a < 1.$$

Observe that each of $(b), (c)$, and (d) consists of two equivalent statements, one for "some" and one for "all".

PROOF. Observe that (4.37) and $\chi_{E_a} + \chi_{E^a} = \chi_{K_0}$ show that $(c) \Leftrightarrow (d)$. Since $f_M \le f_M^n$, it is obvious that $(b) \Rightarrow (c)$.

Suppose now that (a) holds and consider an arbitrary $0 < a < 1$. From Lemma 4.48 we have

$$\lim_{n\to\infty}\int_{2^nE_a} f_M(\xi)d\xi = 0.$$

By (4.37) we have (using (a)) that

$$\begin{aligned} 1 &\ge \limsup_{n\to\infty}\int_{2^nK_0\setminus 2^nE_a} f_M^n(\xi)d\xi \\ &\ge \liminf_{n\to\infty}\int_{2^nK_0\setminus 2^nE_a} f_M^n(\xi)d\xi \\ &\ge \lim_{n\to\infty}\int_{2^nK_0\setminus 2^nE_a} f_M(\xi)d\xi \\ &= \lim_{n\to\infty}\int_{2^nK_0} f_M(\xi)d\xi \\ &= \int_{\mathbb{R}} f_M(\xi)d\xi \\ &= 1. \end{aligned}$$

Again, using (4.37), this computation shows that

$$\lim_{n\to\infty} \int_{2^n E_a} f_M^n(\xi) d\xi = 0,$$

i.e. we have shown that $(a) \Rightarrow$ "(d) (for all)" statement.

Suppose now that (d) is valid for some $0 < a < 1$. For $\xi \in 2^n K_0 \setminus 2^n E_a$, we have $\xi/2^n \notin E_a$, i.e.

$$f_M(\xi) = f_M^n(\xi) f_M(\xi/2^n) \geq f_M^n(\xi) \cdot a.$$

Considering now a sequence of functions

$$g_n(\xi) := f_M^n(\xi) \cdot \chi_{2^n K_0 \setminus 2^n E_a}(\xi),$$

we obtain that $g_n \leq \frac{1}{a} f_M \in L^1(\mathbb{R})$ and, by Lemma 4.48,

$$\lim_{n\to\infty} g_n = f_M \text{ almost everywhere on } \mathbb{R}.$$

Using (4.37) and (d), we obtain, by the Lebesgue dominated convergence theorem,

$$\int_{\mathbb{R}} f_M(\xi) d\xi = \lim_{n\to\infty} \int_{2^n K_0 \setminus 2^n E_a} f_M^n(\xi) d\xi = 1 - \lim_{n\to\infty} \int_{E_a} f_M^{n,*}(\xi) d\xi = 1.$$

It remains to prove that $(a) \Rightarrow (b)$. This follows directly from Lemma 4.48, since, for all $0 < a < 1$,

$$\int_{\mathbb{R}} f_M(\xi) d\xi = \lim_{n\to\infty} \int_{2^n K_0} f_M(\xi) d\xi = \lim_{n\to\infty} \left(\int_{2^n E^a} f_M(\xi) d\xi + \int_{2^n E_a} f_M(\xi) d\xi \right).$$

□

Remark 4.50. (i) It is of interest to examine condition (4.28) in this context. Observe first that (4.28) is equivalent to the condition

"there exists $0 < a < 1$ such that $\mathrm{dist}(0, E_a) > 0$".

This means that, for almost every $\xi \in \mathbb{R}$, there exists $n_0 = n_0(\xi, a) \in \mathbb{N}$ such that $\xi \notin 2^n E_a$ for every $n \geq n_0$.

(ii) It is obvious that (4.28) implies (4.29), i.e. (4.28) implies "$f_M > 0$ almost everywhere".

(iii) The example given in (4.44) shows that, in general, (4.28) does not imply (4.41). Nevertheless, condition (4.28) is a strong assumption in this direction. As the following results show, assuming (4.28) leads to (4.41) via algebraic conditions connected with simple number theoretic conditions.

■

For $k \in \mathbb{N}$, consider a bijection $\rho^k : 2^{-k} I \to K_0$ given by $\rho^k(\xi) = 2^k \xi - \mathrm{signum}(\xi)$, where $\mathrm{signum}(\xi)$ outputs the sign of ξ (compare with ρ_1 and ρ_2 given in (3.36)).

Proposition 4.51. *Let $M : \mathbb{R} \to [0, \infty)$ be measurable, 1-periodic, $M > 0$ almost everywhere, $M(\xi) + M(\xi + \frac{1}{2}) = 1$ almost everywhere, and such that (4.28) holds. If, for some $0 < a < 1$,*

$$\rho^1(E_a \cap 2^{-1} I) \cap E_a = \emptyset,$$

then

$$\int_{\mathbb{R}} f_M(\xi) d\xi = 1$$

(i.e. M is generated from a low-pass filter).

PROOF. Recall that $f_M(\xi) < a \Rightarrow f_M(2\xi) < a$. Hence our condition implies that, for every $k \in \mathbb{N}$,

$$\rho^k(E_a \cap 2^{-k}I) \cap E_a = \emptyset \tag{4.52}$$

since $\rho^k(E_a \cap 2^{-k}I) \subseteq \rho^1(E_a \cap 2^{-1}I)$.

Observe that (4.28) implies that $E_a \cap 2^{-k}I \neq \emptyset$ for at most finitely many $k \in \mathbb{N}$. By Proposition 4.49 it is enough to check condition (d) in that proposition. Therefore, it is enough to check that, for finitely many $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \int_{E^a \cap 2^{-k}I} f_M^{n,*}(\xi) d\xi = 0.$$

Observe that $(\rho^k)^{-1}(\xi) = \frac{\xi \pm 1}{2^k}$, so, for n sufficiently large,

$$2^{-k} f_M^{n,*}((\rho^k)^{-1}(\xi)) = 2^{n-k} \left(\prod_{\ell=1}^{n-k} M(2^{n-k-\ell}(\xi) \right) M\left(\frac{\xi \pm 1}{2}\right) \cdot A,$$

where $A \le 1$. It follows that

$$2^{-k} f_M^{n,*}((\rho^k)^{-1}(\xi)) \le f_M^{n-k,*}(\xi)(1 - M(\xi/2)).$$

Observe also that for $\xi \notin E_a$ we have

$$f_M^{n-k,*}(\xi) \le \frac{1}{a} 2^{n-k} f_M(2^{n-k}\xi).$$

Using these we obtain

$$\begin{aligned}\int_{E_a \cap 2^{-k}I} f_M^{n,*}(\xi) d\xi &= 2^{-k} \int_{\rho^k(E_a \cap 2^{-k}I)} f_M^{n,*}((\rho^k)^{-1}(\xi)) d\xi \\ &\le \frac{1}{a} \int_{\mathbb{R}} 2^{n-k} f_M(2^{n-k}\xi)(1 - M(\xi/2)) d\xi \\ &= \frac{1}{a} \int_{\mathbb{R}} f_M(u) \left(1 - M\left(\frac{u}{2^{n+1-k}}\right)\right) du.\end{aligned}$$

On the right side we have a sequence of functions dominated by $f_M \in L^1(\mathbb{R})$, and such that the converge pointwise to zero almost everywhere. By the Lebesgue dominated convergence theorem, the right side converges to 0 as $n \to \infty$. Thus the left side of the inequality must also converge to 0 almost everywhere. $\square$

EXAMPLE 4.53. In this example, we would like to emphasize some aspects raised in Remark 4.50(iii) with respect to the "nature" of Proposition 4.51. Consider the family of functions $M : \mathbb{R} \to [0, \infty)$ which are continuous, 1-periodic, $M > 0$ almost everywhere, $M(\xi) + M(\xi + \frac{1}{2}) \equiv 1$, M even, and such that (4.28) holds and $M|_{(0,1/4)}$ has exactly one zero-point, denoted by $z \in [1/8, 1/4)$. It is not difficult to construct a large subfamily of this family. Take any function $h : [0, 1/4] \to [0, 1]$ such that h is continuous, h is Hölder continuous at 0 with $h(0) = 1$, $h(1/4) = 1/2$, $h(z) = 0$, and $h(x) > 0$ for every $x \in [0, 1/4] \backslash \{z\}$. Extend h to $[-1/4, 0]$ so that the function remains even, use the Smith–Barnwell condition to extend it to $[-1/2, 1/2]$ (observe that by Smith–Barnwell, we must have $M(-1/2) = M(1/2) = 0$), and finally extend it to a 1-periodic function. Hölder continuity at 0 will ensure (4.28) (consult Remark 4.26). Consider the condition $\rho^1(E_a \cap 2^{-1}I) \cap E_a = \emptyset$ for M in our family (by Proposition 4.51, such M are developed from orthonormal MRA wavelet low-pass filters). Since a can be arbitrarily small, the M are continuous, and (4.28) holds, $E_a \cap 2^{-1}I$ consists exactly of small open intervals around the points $-2z$ and $2z$, while $E_a \cap 2^{-k}I = \emptyset$ for $k \geq 2$ and a sufficiently small. Hence for our class of M, the condition $\rho^1(E_a \cap 2^{-1}I) \cap E_a = \emptyset$ is equivalent to $2(2z) - 1 \neq -2z$ and $2(-2z) + 1 \neq 2z$. Obviously, then, the condition that $\rho^1(E_a \cap 2^{-1}I) \cap E_a \neq \emptyset$ for every $0 < a < 1$ is equivalent to

$$2(2z) - 1 = -2z,$$

where the last equation has exactly one solution, $z = 1/6$.

Therefore, if $z \neq 1/6$, then M is generated from a low-pass filter. If $z = 1/6$ and M is C^∞ at 0, then M is not generated from a low-pass filter; consult results in [**PŠW99**] to justify this statement (our results below provide justifications, as well). Observe that in all these examples, the condition "$f_M > 0$ almost everywhere" is satisfied. What fails to hold in the case $z = 1/6$ is the condition "$\int_{\mathbb{R}} f_M(\xi)d\xi = 1$". It is perhaps difficult to grasp intuitively that for such a large class of M the condition about the integral of f_M would depend solely on a simple algebraic condition such as the one given above.

Let us conclude this example with a remark that even if we allow $z \in (0, 1/4)$, then the same conclusion holds; we need some additional results (to follow) in order to easily prove this. ♢

Example 4.54. Consider the Haar wavelet filter m_H given in (4.44). For every $\ell \in \mathbb{N} \cup \{0\}$, we define

$$m_\ell^H := |m_H((2\ell + 1)\pi\xi)|^2, \tag{4.55}$$

i.e. we consider the Haar filter and the "elongated" Haar filters. It is not difficult to check that

$$M_\ell^H(\xi) = \cos^2((2\ell + 1)\pi\xi).$$

Hence the functions M_ℓ^H are C^∞ functions with $M_\ell^H(0) = 1$, they all satisfy the Smith–Barnwell condition as well as the condition "$M > 0$ almost everywhere". For all of them, $f_{M_\ell^H}$ is also a C^∞ function and "$f_{M_\ell^H} > 0$ almost everywhere" holds.

Since

$$\int_{\mathbb{R}} f_{M_0^H}(\xi)d\xi = 1,$$

we obtain that, for every $\ell \in \mathbb{N} \cup \{0\}$,

$$\int_{\mathbb{R}} f_{M_\ell^H}(\xi)d\xi = \int_{\mathbb{R}} f_{M_0^H}((2\ell + 1)\xi)d\xi = \frac{1}{2\ell + 1}. \tag{4.56}$$

Therefore, only M_0^h is generated from a low-pass filter, while the M_ℓ^H, $\ell \in \mathbb{N}$, are not (they are generated from the so-called "generalized" low-pass filters which play a role in the theory of Parseval frame wavelets). They fail to be low-pass filters precisely because of the zero points in $[0, 1/4]$ (all these examples have M which is an even function). Observe, though, that only $\ell = 1$ has a single zero point in $[0, 1/4]$, and it is, of course, $z = 1/6$ (compare with the previous example). For $\ell \geq 2$ we have multiple zeros in $[0, 1/4]$; they are all of the form

$$\frac{k}{2(2\ell + 1)} \text{ with } k \text{ even.}$$

Let us recall that, in these examples, our filters are trigonometric polynomials, so their analysis falls under the umbrella of an early version of the celebrated *Cohen condition* — consult [**Coh90**], but also [**Coh92**], [**Mey90**], [**Dau92**], and [**Gun00**]; particularly interesting for us is the discussion in [**Gun00**]. There, the zeros of the form

$$\frac{k}{2^N-1}+\frac{1}{2}(\text{mod } 1) \text{ where } 1 \le k \le 2^N-2$$

are excluded (it may not be immediately visible that $1/2(2\ell+1)$ is of this form, but this boils down to the question of whether $2\ell+1$ divides any number of the form 2^N-1; indeed it does, take $N=\varphi(2\ell+1)$, where φ is the Euler function — recall that Fermat's Little Theorem is the fact that $\varphi(p)=p-1$ when p is prime).

This brings us naturally to the Cohen condition, and we shall improve somewhat on the discussions given in [**PŠW99**] and [**Gun00**]. ♢

The Cohen condition, presented in the form suitable for M, is (see, for example, p. 367 in [**HW96**]):

$$\text{(c) There exists a set } K\subseteq\mathbb{R} \text{ which is a finite union of closed, bounded intervals such that } 0\in\operatorname{int}(K), \sum_{k\in\mathbb{Z}}\chi_K(\xi+k)=1 \text{ for almost every } \xi\in\mathbb{R}, \text{and } M(2^{-j}\xi)>0 \text{ for all } j\in\mathbb{N} \text{ and all } \xi\in K. \tag{4.57}$$

The condition has been mostly applied in a situation where M is a C^1-function (or even a C^∞-function) with $M(0)=1$. It is worth noticing that $M(2^{-j}\xi)>0$ is required for *all* $\xi\in K$ (not just almost every $\xi\in K$). As noted before, we refer to the condition $\sum\chi_K(\xi+k)=1$ as K being "1-congruent to $K_0=[-1/2,1/2)$".

Consider a class of continuous functions M with $M(0)=1$. Hence, there is a $\delta>0$ such that $M(\xi)\ge 1/2$ for every $\xi\in[-\delta,\delta]$. Since K in (4.57) is compact, it is easy to see that for continuous M with $M(0)=1$, $(c)\Leftrightarrow(c+)$, where

$$\text{(c+) There exists a set } K\subseteq\mathbb{R} \text{ which is a finite union of closed, bounded intervals such that } 0\in\operatorname{int}(K),\ K \text{ is 1-translation congruent to } K_0, \text{ and there exists a constant } a \text{ with } 0<a<1 \text{ such that, for all } j\in\mathbb{N} \text{ and for almost every } \xi\in K,\ M(2^j\xi)\ge a. \tag{4.58}$$

There are two sides to the Cohen condition: one is whether it is sufficient for M to be associated with the low-pass filter of an MRA orthonormal wavelet and another whether it is necessary. We shall comment on the sufficiency issue first.

It is well-known (see, for example, Section 7.4 in [**HW96**]) that if $M\in\mathcal{M}_{SB}^{FO}$ is also a C^1-function, then f_M is continuous on $\mathbb{R}$ and the condition (c) (or, equivalently, $(c+)$) is sufficient for M to be associated with the low-pass filter of an MRA orthonormal wavelet. A natural question is whether the theorem still holds if the "C^1 assumption" is replaced by a weaker property such as the "C^0 condition", i.e. the assumption that M is continuous on $\mathbb{R}$. Our Example 4.22 shows that the answer is *negative*. Interestingly enough, it shows even more since the f_M in that example fails to even be dyadically continuous at 0. It is shown in [**PŠW99**] (see also Proposition 4.24) that this situation improves if we assume that M is Hölder continuous at zero. This, however, is a bit too strong an assumption. As explained in [**PŠW99**] and [**Gun00**], all sufficiency theorems really lead to the following condition:

$$\text{(4.59)}\quad \begin{aligned}&\text{There exists a set } K \subseteq \mathbb{R} \text{ which is 1-translation congruent to } K_0 \text{ and has}\\ &0 \in \operatorname{int}(K), \text{ and a constant } 0 < a < 1 \text{ such that, for almost every } \xi \in K,\\ &f_M(\xi) \geq a.\end{aligned}$$

Indeed, it is not difficult to see that (4.59) is a sufficient condition. First of all, since $0 \in \operatorname{int}(K)$, we obtain that "$f_M > 0$ almost everywhere" holds. Using the "peeling-off" argument (consult (4.11)–(4.14)) we obtain that f_M is integrable, while (4.59) implies that, for every $n \in \mathbb{N}$,

$$f_M^{K,n} \leq \frac{1}{a} f_M \text{ almost everywhere on } \mathbb{R}.$$

Applying the Lebesgue dominated convergence theorem gives $\int_{\mathbb{R}} f_M(\xi) d\xi = 1$.

REMARK 4.60. Using an additional assumption, that f_M is a continuous function, it is natural to consider a stronger version of (4.59), i.e.

$$\text{(4.61)}\quad \begin{aligned}&\text{There exists a set } K \subseteq \mathbb{R} \text{ which is 1-translation congruent to } [-1/2, 1/2]\\ &\text{and a constant } 0 < a < 1 \text{ such that, for every } \xi \in K,\ f_M(\xi) \geq a.\end{aligned}$$

This property is analyzed in [**Gun00**] as the ultimate version of the Cohen condition under the assumption that f_M is continuous. Observe that, in particular, the continuity of f_M at zero enables us to select an interval around zero where $f_M >$ is bounded away from zero. Hence one can consider a version of (4.61) where $0 \in \operatorname{int}(K)$ and K is a finite union of intervals (due to the compactness of $[-1/2, 1/2]$). We shall analyze a "relaxed" (measure-theoretic) version of such a condition. ■

We shall say that M satisfies the "continuous f_M condition" or, more briefly, the "(C-f_M) condition" if the following holds:

$$\text{(4.62)}\quad \begin{aligned}&\text{There exists a set } K \subseteq \mathbb{R} \text{ which is a finite union of closed, bounded}\\ &\text{intervals such that } 0 \in \operatorname{int}K,\ K \text{ is 1-translation congruent to } K_0, \text{ and}\\ &\text{there exists a constant } 0 < a < 1 \text{ such that, for almost every } \xi \in K,\\ &f_M(\xi) \geq a.\end{aligned}$$

PROPOSITION 4.63. *Suppose that $M \in \mathcal{M}_{SB}^{FO}$ is continuous and that f_M satisfies (4.28). Then the following are equivalent.*

(a) Condition (c);
(b) Condition (c+);
(c) Condition (C-f_M).

Furthermore, if any of the equivalent conditions hold, then M is associated with a low-pass filter for an MRA orthonormal wavelet.

PROOF. As we have seen, $(a) \Leftrightarrow (b)$ under the condition that M is continuous. Since, for every $j \in \mathbb{N}$, we have $M(2^{-j}\xi) \geq f_M(\xi)$, it follows that (c) always implies (b). It remains to show that $(b) \Rightarrow (c)$. This is the only portion of the proof which requires (4.28). Consider first the set K_1 generated by $(c+)$ and a corresponding constant a_1. By (4.28) there exists a closed interval L with $0 \in \operatorname{int}(L)$ such that, for almost every $\xi \in L$, $f_M(\xi) \geq a_1$. Observe that each interval in K_1 may have a part which is 1-congruent to a part of L, and both parts must be finite unions of intervals. Hence, by keeping L and removing from K_1 those parts which are

1-congruent to L, it is fairly obvious that we can construct a set $K \subseteq \mathbb{R}$ such that $L \subseteq K$, $K \setminus L \subseteq K_1$, K is a finite union of closed, bounded intervals, and K is 1-translation equivalent to K_0, with all these statements being valid in the almost everywhere sense (since we may need to "add" or "subtract" endpoints to close the intervals). Since $0 \in \mathrm{int}L$ and K is bounded, there exists $\ell \in \mathbb{N}$ such that $2^\ell L \subseteq K$. Take now a constant $a := a_1^{\ell+1} < a_1$. For almost every $\xi \in L$, we have $f_M(\xi) \geq a_1 > a$. For almost every $\xi \in K \setminus L \subseteq K_1$, we have (since $2^{-\ell}\xi \in L$)

$$f_M(\xi) = \left(\prod_{j=1}^{\ell} M(2^{-j}\xi) \right) \cdot f_M(2^{-\ell}\xi) \geq a_1^\ell \cdot a_1 = a.$$

This completes the proof of $(b) \Rightarrow (c)$. Since (c) implies (4.59), the proof of the last statement is completed, too. □

REMARK 4.64. (i) If we only have the assumption that M is continuous, then (b) does not imply (c). This follows from Example 4.22.

(ii) If we have the assumption that M is continuous and f_M is dyadically continuous at zero (which is weaker than (4.28)), then (b) still does not imply (c). This follows from Example 4.25.

■

Using Remark 4.26 we obtain directly the following result.

COROLLARY 4.65. *(a) If $M \in \mathcal{M}_{SB}^{FO}$ is continuous and Hölder continuous at zero and satisfies (c), then M is associated with a low-pass filter for an MRA orthonormal wavelet.*

(b) If $M \in \mathcal{M}_{SB}^{FO}$ is continuous, M satisfies (c), and f_M is continuous at zero, then M is associated with a low-pass filter for an MRA orthonormal wavelet.

(c) If $M \in \mathcal{M}_{SB}^{FO}$ is Hölder continuous at zero and satisfies $(c+)$, then M is associated with a low-pass filter for an MRA orthonormal wavelet.

(d) If $M \in \mathcal{M}_{SB}^{FO}$ satisfies $(c+)$ and f_M is continuous at zero, then M is associated with a low-pass filter for an MRA orthonormal wavelet.

REMARK 4.66. Regarding the necessity of the Cohen condition, it fails under fairly general conditions. The discussion in [**Gun00**] covers this issue together with an example provided in [**DGH00**]. This is an example of a low-pass filter such that f_M is continuous on $\mathbb{R}$ but for which (4.61) fails to hold. The example is constructed in such a way that the periodization of f_M has a few exceptional points where it is equal to zero. Using compactness of $[-1/2, 1/2]$, it is not difficult to check that the following necessity result holds. ■

PROPOSITION 4.67. *Suppose that $M \in \mathcal{M}_{SB}^{FO}$ is continuous and such that f_M is continuous on $\mathbb{R}$. If, for every $\xi \in \mathbb{R}$ one has*

$$\sum_{k \in \mathbb{Z}} f_M(\xi + k) > 0,$$

then M satisfies (c) (and $(c+)$ and $(C\text{-}f_M)$)

5. Multiplicative Structure in the Class of FO Filters

Consider the class $\mathcal{M}^{FO}$ of measurable, 1-periodic functions $M : \mathbb{R} \to [0, \infty)$ such that $M > 0$ almost everywhere and $\mathrm{Sol}_M \neq \{0\}$. If $M_1, M_2 \in \mathcal{M}^{FO}$ and

$\Phi_1 \in \mathrm{Sol}_{M_1}$, $\Phi_2 \in \mathrm{Sol}_{M_2}$ are non-trivial solutions, then the pair $(M_1 \cdot M_2, \Phi_1 \cdot \Phi_2)$ will always satisfy equation (2.10). However, it may not be the case that $M_1 \cdot M_2$ belongs to $\mathcal{M}^{FO}$. There are two possible obstructions to $M_1 \cdot M_2$ belonging to $\mathcal{M}^{FO}$: one is that $\Phi_1 \cdot \Phi_2$ will be trivial whenever $|A^{(2)}_{M_1} \cap A^{(2)}_{M_2}| = 0$, the other is that $\Phi_1 \cdot \Phi_2$ may not be integrable. In order to avoid the first of the two problems, we focus on the following class. For a measurable set $A \subset I$ with $|A| > 0$, we define

$$\mathcal{M}^{FO}_A := \{M \in \mathcal{M}^{FO} : A^{(2)}_M = A \text{ almost everywhere}\}. \tag{5.1}$$

Observe that we identify $\mathcal{M}^{FO}_A$ and $\mathcal{M}^{FO}_B$ whenever $A = B$ almost everywhere. Recalling (2.10) and the notation of (2.35), we are confident that our readers can easily prove the following result.

Proposition 5.2. *Let $A \subseteq I$ be measurable with $|A| > 0$, and let $n \in \mathbb{N} \setminus \{1\}$ and $t = (t_1, ..., t_n) \in [0,1]^n$ such that $t_1 + ... + t_n = 1$. If $M_1, ..., M_n \in \mathcal{M}^{FO}_A$, then $M_t = M_1^{t_1} \cdot M_2^{t_2} \cdots M_n^{t_n}$ belongs to $\mathcal{M}^{FO}_A$ and $\Phi_{0,M_t} = \Phi^{t_1}_{0,M_1} \cdots \Phi^{t_n}_{0,M_n}$.*

Corollary 5.3. *Let $A \subseteq I$ be measurable with $|A| > 0$ and let $n \in \mathbb{N} \setminus \{1\}$. If $M_1, ..., M_n \in \mathcal{M}^{FO}_A$, then the geometric mean of the M_i is also, i.e. $\sqrt[n]{M_1 \cdots M_n} \in \mathcal{M}^{FO}_A$.*

Remark 5.4. Consider a measurable set $A \subseteq I$ with $|A| > 0$, $n = 2$, and $t_1 = \lambda$, $t_2 = 1 - \lambda$, where $\lambda \in [0,1]$. It follows from Proposition 5.2 that $M_\lambda = M_1^\lambda M_2^{1-\lambda} \in \mathcal{M}^{FO}_A$ whenever $M_1, M_2 \in \mathcal{M}^{FO}_A$. Furthermore, we also can follow "the path" given by $\lambda \mapsto \Phi_{M_1,M_2}(\lambda) := \Phi_{0,M_\lambda} \in L^1(\mathbb{R})$. Observe that, for every λ, our function $\Phi_{M_1,M_2}(\lambda)$ is dominated by $\Phi_{0,M_1} + \Phi_{0,M_2} \in L^1(\mathbb{R})$. If $\lambda_n \to \lambda$, then $\Phi_{M_1,M_2}(\lambda_n) \to \Phi_{M_1,M_2}(\lambda)$ pointwise almost everywhere — observe that having the same set A is crucial for this convergence, in particular when $\lambda = 0$ and $\lambda = 1$. By the Lebesgue Dominated Convergence Theorem, we obtain

$$\lambda \mapsto \Phi_{M_1,M_2}(\lambda) \text{ is continuous in the } L^1\text{-norm.} \tag{5.5}$$

Hence $\mathcal{M}^{FO}_A$ is logarithmically convex and connected (in the sense of (5.5)). Observe that (5.5) is a "natural path" and compare this with the connectivity theorem in [**Con98**], where it was not possible to achieve such a path. Notice that here (within $\mathcal{M}^{FO}_A$) we do not require the Smith–Barnwell condition, unlike in [**Con98**]. ■

Given $M \in \mathcal{M}^{FO}_A$, it is fairly obvious that the set

$$\{\alpha \in (0, \infty) : \Phi^\alpha_{0,M} \in L^1(\mathbb{R})\} \tag{5.6}$$

is an interval containing 1. Furthermore, α belongs to the set in (5.6) if and only if

$$M^\alpha \in \mathcal{M}^{FO}_A. \tag{5.7}$$

The consequence is that, if $\Phi_{0,M}$ is bounded, then

$$M^\alpha \in \mathcal{M}^{FO}_A \text{ for every } \alpha \geq 1. \tag{5.8}$$

Let us also observe that the entire family $\mathcal{M}^{FO}$ is closed under the operation $\xi \mapsto M(k\xi)$ for $k \in \mathbb{N}$. More precisely, the following result is straightforward.

Lemma 5.9. *If $M \in \mathcal{M}^{FO}$ and $k \in \mathbb{N}$, then $\xi \mapsto M(k\xi)$ is a function belonging to $\mathcal{M}^{FO}$, its maximal solution is the function $\xi \mapsto \Phi_{0,M}(k\xi)$, and*

$$\int_{\mathbb{R}} \Phi_{0,M}(k\xi) d\xi = \frac{1}{k} \int_{\mathbb{R}} \Phi_{0,M}(\xi) d\xi.$$

For the following class, recall the definition of f_M (see (4.10)):

(5.10)
$$\mathcal{M}_*^{FO} := \{M \in \mathcal{M}_I^{FO} : M \leq 1, f_M > 0 \text{ almost everywhere, and } f_M \in L^1(\mathbb{R})\}.$$

Observe that we do not require the Smith–Barnwell condition. Nevertheless, for every $M \in \mathcal{M}_*^{FO}$, the function f_M is a maximal solution to the (Φ, M)-Problem.

REMARK 5.11. It is fairly obvious that on $\mathcal{M}_*^{FO}$ we can perform all the various operations described in this section.

(i) $\mathcal{M}_*^{FO}$ is logarithmically convex, i.e. if $M_1, ..., M_n \in \mathcal{M}_*^{FO}$ and $t = (t_1, ..., t_n) \in [0,1]^n$ with $t_1 + ... + t_n = 1$, then $M_t \in \mathcal{M}_*^{FO}$ and $f_{M_t} = \prod_{\ell=1}^n f_{M_i}^{t_i}$.
(ii) If $M_1, M_2 \in \mathcal{M}_*^{FO}$, then $\lambda \mapsto f_{M_1}^{\lambda} f_{M_2}^{1-\lambda}$ is an L^1-norm continuous path.
(iii) If $M \in \mathcal{M}_*^{FO}$, then $M^\alpha \in \mathcal{M}_*^{FO}$ for every $\alpha \geq 1$.
(iv) If $M \in \mathcal{M}_*^{FO}$, then, for every $k \in \mathbb{N}$, $\xi \mapsto M(k\xi)$ belongs to $\mathcal{M}_*^{FO}$ and its f_M-function is $\xi \mapsto f_M(k\xi)$. Observe also that if $M \in \mathcal{M}_*^{FO} \cap \mathcal{M}_{SB}^{FO}$, then

$$\xi \mapsto M(k\xi) \text{ belongs to } \mathcal{M}_*^{FO} \cap \mathcal{M}_{SB}^{FO} \Leftrightarrow k \text{ is odd.} \tag{5.12}$$

(v) We can apply the transformation analogous to the *semiorthogonalization* from [**ŠSW08**] to "project" filters from $\mathcal{M}_*^{FO}$ into $\mathcal{M}_{SB}^{FO}$. Observe first that whenever a pair (M, Φ) is a solution of the (Φ, M)-Problem, then the periodization of Φ, i.e. $p_\Phi(\xi) := \sum_{k \in \mathbb{Z}} \Phi(\xi + k)$ satisfies the equation

$$p_\Phi(2\xi) = M(\xi)p_\Phi(\xi) + M(\xi + 1/2)p_\Phi(\xi + 1/2) \text{ almost everywhere.} \tag{5.13}$$

Observe that both sides of (5.13) are 1/2-periodic, so it is enough "to check" (5.13) on an interval of size 1/2. Furthermore, since $f_M > 0$ almost everywhere (for $M \in \mathcal{M}_*^{FO}$), it follows that $p_{f_M} > 0$ almost everywhere; therefore, the following definition makes sense. Given $M \in \mathcal{M}_*^{FO}$, we define M_{ORT} on $\mathbb{R}$ by

$$M_{ORT}(\xi) := \frac{p_{f_M}(\xi)}{p_{f_M}(2\xi)} M(\xi) \text{ for } \xi \in \mathbb{R}. \tag{5.14}$$

It is fairly straightforward to check that M_{ORT} is measurable, 1-periodic, $M_{ORT} > 0$ almost everywhere, and that M_{ORT} satisfies the Smith–Barnwell condition. Observe also that

$$f_{M_{ORT}}(\xi) = \frac{1}{p_{f_M}(\xi)} f_M(\xi) \text{ almost everywhere.} \tag{5.15}$$

In particular, $M_{ORT} \in \mathcal{M}_*^{FO} \cap \mathcal{M}_{SB}^{FO}$. However, M_{ORT} has even stronger properties (thus its name "ORT" for orthonormal); for almost every $\xi \in \mathbb{R}$,

$$p_{f_{M_{ORT}}}(\xi) \equiv 1. \tag{5.16}$$

As is well known, (5.16) is equivalent to M_{ORT} being associated with a low-pass filter for an orthonormal MRA wavelet — this is not a property of every element of $\mathcal{M}_*^{FO} \cap \mathcal{M}_{SB}^{FO}$.

■

EXAMPLE 5.17. Consider the Haar wavelet filter $m_H(\xi) = \frac{1}{2}(1 + e^{2\pi i \xi})$ (consult also Remark 4.42(iii)). Let $M = |m_H|^2$, i.e. $M(\xi) = \cos^2(\pi\xi)$. It follows that $f_M(\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2$. In particular, $M \in \mathcal{M}_*^{FO}$ which implies that $M^\alpha \in \mathcal{M}_*^{FO}$ for every $\alpha \geq 1$. Observe that $f_{M^\alpha}(\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^{2\alpha}$. It follows that

$$\frac{p_{f_{M^\alpha}}(\xi)}{p_{f_{M^\alpha}}(2\xi)} = \frac{1}{2^{2\alpha}(\cos^2(\pi\xi))^\alpha} \cdot \frac{\sum_{k\in\mathbb{Z}}[\pi(\xi+k)]^{-2\alpha}}{\sum_{k\in\mathbb{Z}}[\pi(2\xi+k)]^{-2\alpha}}$$

which gives us (with proper adjustment for $\xi \in \mathbb{Z}$)

$$M^\alpha_{ORT}(\xi) = \frac{\sum_{k\in\mathbb{Z}}[\pi(\xi+k)]^{-2\alpha}}{\sum_{k\in\mathbb{Z}}[\pi(\xi+k)]^{-2\alpha} + \sum_{k\in\mathbb{Z}}[\pi(\xi+\frac{1}{2}+k)]^{-2\alpha}}; \tag{5.18}$$

observe that M^α_{ORT} is associated with a low-pass filter for an MRA orthonormal wavelet. Recall also that a function

$$(a, z) \mapsto \sum_{n=0}^{\infty} \frac{1}{(z+n)^a}$$

is known as a *Hurwitz zeta function.*

Scaling functions associated with M^α and M^α_{ORT}, with properly selected phases, will actually generate equal principal shift-invariant spaces. Let us also mention that, for $\alpha = N+1$, $N \in \mathbb{N}$, we have $M^\alpha(\xi) = \left(\cos^2(\pi\xi)\right)^{N+1}$. Recall that *the Daubechies filter* which produces compactly supported wavelets of the order $C^N(\mathbb{R})$ will give the absolute value squared of the form

$$\left(\cos^2(\pi\xi)\right)^{N+1} P_N(\sin^2(\pi\xi)),$$

where P_N is the polynomial of order N which satisfies what is essentially the Smith–Barnwell condition,

$$(\cos^2(\pi\xi))^{N+1}P_N(\sin^2(\pi\xi)) + (\sin^2(\pi\xi))^{N+1}P_N(\cos^2(\pi\xi)) = 1;$$

for details, see, for example, [**Dau92**] and [**HW96**]. The well-known formula for P_N is

$$P_N(x) = \sum_{j=0}^{N} \binom{N+j}{j} x^j.$$

Finally, observe that the "ray" $\alpha \mapsto M^\alpha$ can often be extended to some values $\alpha < 1$, depending on the integrability of f_M. In this example, any $\alpha > 1/2$ will give us $M^\alpha \in \mathcal{M}^{FO}_*$. ♢

If f_M functions remain within a closed ball in $L^1(\mathbb{R})$, then we can treat infinite products as well.

PROPOSITION 5.19. *Suppose that* $(t_n : n \in \mathbb{N}) \subseteq [0,1]$ *with* $\sum_{n=1}^\infty t_n = 1$ *and* $(M_n : n \in \mathbb{N}) \subseteq \mathcal{M}^{FO}_*$ *with*

$$\sup_{n\in\mathbb{N}} \left(\int_{\mathbb{R}} f_{M_n}(\xi)d\xi\right) =: C < \infty.$$

If, for almost every $\xi \in \mathbb{R}$,

$$\prod_{n=1}^{\infty} f_{M_n}^{t_n}(\xi) > 0$$

(observe that this implies $\prod_{n=1}^\infty M_n^{t_n}(\xi) > 0$*), then* $\prod_{n=1}^\infty M_n^{t_n} \in \mathcal{M}^{FO}_*$ *(and the corresponding* f_M *function is* $\prod_{n=1}^\infty f_{M_n}^{t_n}$*).*

Proof. Let us denote $(t_n : n \in \mathbb{N})$ by t and define $M_t := \prod_{n=1}^{\infty} M_n^{t_n}$. It follows from our assumptions that M_t is measurable, 1-periodic, non-negative, and $0 < M_t \leq 1$ almost everywhere. We define $f_t := \prod_{n=1}^{\infty} f_{M_n}^{t_n}$. It follows that f_t is measurable, non-negative, and $0 < f_t \leq 1$ almost everywhere. Using limit properties, it follows that (M_t, f_t) satisfies (2.10) and that $f_{M_t} = f_t$. In order to complete the proof, it remains to show that $f_t \in L^1(\mathbb{R})$. Recall that for numbers $a_n \geq 0$ we have

$$\prod_{n=1}^{\infty} a_n^{t_n} \leq \sum_{n=1}^{\infty} t_n a_n. \tag{5.20}$$

Using (5.20), we obtain

$$\int_{\mathbb{R}} f_t(\xi) d\xi \leq \prod_{n=1}^{\infty} \left(\int_{\mathbb{R}} f_{M_n}(\xi) d\xi \right)^{t_n} \leq C \sum_{n=1}^{\infty} t_n = C < \infty.$$

□

If $M_n(\xi) = M(2^n \xi)$ in Proposition 5.19, then

$$\int_{\mathbb{R}} f_{M_n}(\xi) d\xi = \frac{1}{2^N} \int_{\mathbb{R}} f_M(\xi) d\xi \leq \int_{\mathbb{R}} f_M(\xi) d\xi < \infty,$$

for every $n \in \mathbb{N}$. Hence we obtain the following result.

Corollary 5.21. *Suppose that $(t_n : n \in \mathbb{N}) \subseteq [0,1]$ with $\sum_{n=1}^{\infty} t_n = 1$ and $M \in \mathcal{M}_*^{FO}$. If, for almost every $\xi \in \mathbb{R}$,*

$$\prod_{n=1}^{\infty} f_M^{t_n}(2^n \xi) > 0,$$

then $\xi \mapsto \prod_{n=1}^{\infty} M^{t_n}(2^n \xi)$ belongs to $\mathcal{M}_*^{FO}$ (and the corresponding f_M function is $\xi \mapsto \prod_{n=1}^{\infty} f_M^{t_n}(2^n \xi)$).

Example 5.22. Take $M \in \mathcal{M}_*^{FO} \cap \mathcal{M}_{SB}^{FO}$ and $t_n = 1/2^n$. Assume that $\prod_{n=1}^{\infty} f_M^{1/2^n}(2^n \xi) > 0$ almost everywhere. Define $M^*(\xi) := \prod_{n=1}^{\infty} M^{1/2^n}(2^n \xi)$ and $f^* = f_{M^*}$. Since $\sum_{N=\ell+1}^{\infty} \frac{1}{2^N} = \frac{1}{2^\ell}$, we obtain, for almost every $\xi \in \mathbb{R}$,

$$\begin{aligned}
f^*(\xi) &= \prod_{N=1}^{\infty} \prod_{k=1}^{\infty} M^{1/2^N}(2^{N-k}\xi) \\
&= \prod_{N=1}^{\infty} \prod_{\ell=1-N}^{\infty} M^{1/2^N}(2^{-\ell}\xi) \\
&= \left[\prod_{N=1}^{\infty} \prod_{\ell=1}^{\infty} M^{1/2^N}(2^{-\ell}\xi) \right] \cdot \left[\prod_{N=1}^{\infty} \prod_{\ell=0}^{N-1} M^{1/2^N}(2^{\ell}\xi) \right] \\
&= \left[\prod_{\ell=1}^{\infty} M(2^{-\ell}\xi) \right] \cdot \left[M(\xi) \prod_{\ell=1}^{\infty} \prod_{N=\ell+1}^{\infty} M^{1/2^N}(2^{\ell}\xi) \right] \\
&= f_M(\xi) \cdot M(\xi) \cdot \prod_{\ell=1}^{\infty} [M(2^\ell \xi)]^{\sum_{N=\ell+1}^{\infty} 1/2^N} \\
&= f_M(\xi) \cdot M(\xi) \cdot M^*(\xi).
\end{aligned}$$

Hence, for almost every $\xi \in \mathbb{R}$,

$$\begin{aligned} f^*(\xi) &= M^*(\xi) f_M(2\xi) \\ &= M^*(\xi/2) f^*(\xi/2) \\ &= [M^*(\xi/2)]^2 f_M(\xi). \end{aligned}$$

This leads to the formula

$$M^*(\xi) = \sqrt{M^*(2\xi) M(2\xi)} \text{ almost everywhere.} \tag{5.23}$$

Hence M^* is at the midpoint of the (logarithmically convex) path between $\xi \mapsto M(2\xi)$ and $\xi \mapsto M^*(2\xi)$. In particular, it is not an extreme point of the convex set $\ln\left(\mathcal{M}_*^{FO}\right)$.

Consider the case of $M(\xi) = \cos^2(\pi\xi)$ (i.e. the Haar wavelet filter). In order to show that this M satisfies the condition $\prod_{n=1}^{\infty} f_M^{1/2^n}(2^n\xi) > 0$ almost everywhere, we will show that, for every $p \in \mathbb{Z}$,

$$I_p := \int_p^{p+1} \ln \frac{1}{f^*(\xi)} d\xi < \infty.$$

Observe that

$$I_p = \sum_{n=1}^{\infty} \int_p^{p+1} \frac{1}{2^n} \ln((2^n\pi\xi)^2) d\xi + \sum_{n=1}^{\infty} \int_p^{p+1} \frac{1}{2^n} \ln \frac{1}{\sin^2(2^n\pi\xi)} d\xi.$$

For the first term, observe that $\xi^2 = |\xi|^2 \leq (|p|+1)^2$, which implies that the first term is smaller than

$$C_p \sum_{n=1}^{\infty} \frac{n}{2^n} < \infty,$$

where C_p is a constant depending on p. For the second term, use the 1-periodicity of $\sin^2(\pi\xi)$ and the fact that it is equal to

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{2^n} \int_{2^n p}^{2^n(p+1)} \ln \frac{1}{\sin^2(\pi\xi)} d\xi &= \sum_{n=1}^{\infty} \frac{1}{2^n} \int_{-1/2}^{1/2} \ln \frac{1}{\sin^2(\pi\xi)} d\xi \\ &= \int_{-1/2}^{1/2} \ln \frac{1}{\sin^2(\pi\xi)} d\xi \\ &< \infty. \end{aligned}$$

$\diamondsuit$

Remark 5.24. Examples given in Lemma 5.9 with k even and as in (5.23) provide us with functions M which are 1/2-periodic. Observe first that such functions can satisfy the Smith–Barnwell equation only if $M \equiv 1/2$ almost everywhere.

(i) Observe that if $M \in \mathcal{M}_*^{FO}$ is 1/2-periodic, then p_{f_M} is not 1/2-periodic (in particular, M is never associated with a low-pass filter for an MRA orthonormal wavelet). Indeed, if both M and p_{f_M} are 1/2-periodic, then (5.13) leads to $M_{ORT} \equiv 1/2$ almost everywhere, i.e. $f_{M_{ORT}} \equiv 0$ almost everywhere, which contradicts (5.16).

(ii) If $M \in \mathcal{M}_*^{FO}$ is 1/2-periodic, then, with $p(\xi) := p_{f_M}(\xi)$ we have

$$M(\xi) = \frac{p(2\xi)}{p(\xi) + p(\xi + \frac{1}{2})}$$

and

$$M_{ORT}(\xi) = \frac{p(\xi)}{p(\xi) + p(\xi + \frac{1}{2})}.$$

For example, if $M(\xi) = \cos^2(2\pi\xi)$, then

$$M_{ORT}(\xi) = \frac{\sum_{k\in\mathbb{Z}}(\xi + k)^{-2}}{\sum_{k\in\mathbb{Z}}(\xi + k)^{-2} + \sum_{k\in\mathbb{Z}}(\xi + \frac{1}{2} + k)^{-2}}.$$

■

6. Structure of $D(\langle\varphi\rangle)$: FO Case

In this section we consider a pair of functions (φ, m) such that they satisfy (2.4), φ is non-trivial, and $m(\xi) \neq 0$ for almost every $\xi \in \mathbb{R}$. By Proposition 2.3 and Remark 2.33ii we conclude that $\langle\varphi\rangle$ is FO and $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$. As before, we denote $|\widehat{\varphi}|^2$ by Φ and $|m|^2$ by M. Since Φ is non-trivial, we have $|A_\Phi^{(2)}| > 0$ and Lemma 2.31 implies that $p_\varphi(\xi) > 0$ for almost every $\xi \in \mathbb{R}$, i.e.

$$\langle\varphi\rangle \text{ is a maximal principal shift-invariant space.} \tag{6.1}$$

Recall Corollary 2.16 which shows that in this case there is exactly one m such that (φ, m) satisfies (2.4). We can actually improve upon the argument via Lemma 2.31. Observe first that in Lemma 2.1 (see [**BRS01**]) we can replace 1-periodicity with u-periodicity, for any $u > 0$, and, using essentially the same proof, we reach the same conclusion. Therefore, we can extend Lemma 2.31 in the same way. It follows that for $u = n \in \mathbb{N} \setminus \{1\}$ we conclude that $p_{\varphi, \frac{1}{n}\mathbb{Z}}(\xi) > 0$ for almost every $\xi \in \mathbb{R}$; see (1.3.11) for the notation. Using (1.3.13) it follows that

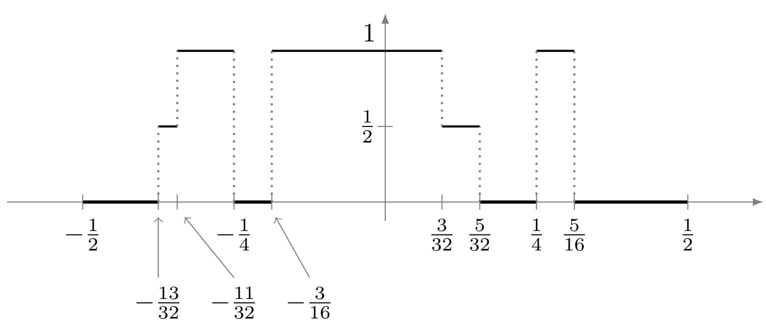
$$\langle\varphi\rangle \text{ is of Type 1.} \tag{6.2}$$

In other words, for every $n \in \mathbb{N} \setminus \{1\}$, we have

$$\langle\varphi\rangle \subsetneq \langle\varphi\rangle_{\frac{1}{n}\mathbb{Z}}. \tag{6.3}$$

One may think that among D^{-1}-invariant, maximal principal shift-invariant spaces, Type 1 occurs only in the FO case. The following examples shows that this is not so.

Example 6.4. Consider the 1-periodic function M given by the following graph.



Observe that M satisfies the Smith–Barnwell condition and is not an FO. It is not difficult to check that, among others, the corresponding Φ will have the following properties:

$$\Phi|_{[3/16,5/16)} \equiv \frac{1}{2}$$
$$\Phi|_{[19/16,5/4)} \equiv \frac{1}{4};$$

since $[3/16, 1/4) + 1 = [19/16, 5/4)$ we establish the Type-1 property by the same argument as in Example 1.43. Moreover,

$$\begin{aligned}\Phi|_{[-3/8,3/16)} &\equiv 1\\ \Phi|_{[-11/16,-1/2)} &\equiv 1\\ \Phi|_{[-3/4,-11/16)} &\equiv \frac{1}{2}\\ \Phi|_{[3/16,5/16)} &\equiv \frac{1}{2}\\ \Phi|_{[1/2,5/8)} &\equiv \frac{1}{2}.\end{aligned}$$

In other words, $\Phi > 0$ on $([-3/4, 1/4) \setminus [-1/2, -3/8)) \cup [1/2, 5/8)$, which guarantees that any corresponding principal shift-invariant space must necessarily be a maximal one. ♢

Going back to our $\langle\varphi\rangle$ which is a D^{-1}-invariant, maximal principal shift-invariant space of Type-1, we recall that, by (1.21) and Corollary 1.49,

$$\langle D\varphi\rangle \text{ is a maximal principal shift-invariant space of Type-1,} \tag{6.5}$$

and, by an induction argument, the same holds for $\langle D_j\varphi\rangle$ for any $j \in \mathbb{N}$.

Remark 6.6. Observe that, in principle, (6.5) does not extend to the property of D^{-1}-invariance. By (1.20) we have $\widehat{D\varphi}(2\xi) = m(\xi/2)\widehat{D\varphi}(\xi)$ and, in most case, (for example, when M satisfies the Smith–Barnwell condition) $\xi \mapsto m(\xi/2)$ would not be 1-periodic. ■

Using Corollary 1.14 and (1.41), we obtain, for every $j \in \mathbb{N} \cup \{0\}$, that

$$D_j(\langle\varphi\rangle) = \langle D_j\varphi\rangle_{\frac{1}{2^j}\mathbb{Z}} \text{ and } \dim_{D_j(\langle\varphi\rangle)} \equiv 2^j. \tag{6.7}$$

Hence the family

$$D_j(\langle\varphi\rangle) \text{ such that } j \in \mathbb{N} \cup \{0\} \tag{6.8}$$

forms a strictly increasing (with respect to j) "cascade" of shift-invariant spaces. It is natural to ask "how big" the space

$$D_\infty(\langle\varphi\rangle) := \overline{\bigcup_{j\in\mathbb{N}\cup\{0\}} D_j(\langle\varphi\rangle)} \tag{6.9}$$

is. The answer to this question is essentially already given in [**Rze00**]; see also [**BR05**] for more precise statements. The spectral function is a very useful tool for answering this question. Consider a shift-invariant space given by $L^2(E)^\vee$ (see Example 1.1.5), where $E = \operatorname{ssupp} \widehat{\varphi}$; observe that E is precisely the union of full orbits of $\widehat{\varphi}$ (or, equivalently, of $\Phi = |\widehat{\varphi}|^2$). It is then straightforward to check that, for every $j \in \mathbb{N} \cup \{0\}$,

$$D_j(\langle\varphi\rangle) \subseteq L^2(E)^\vee; \tag{6.10}$$

with $E = \text{ssupp}\ \widehat{\varphi}$. Since $L^2(E)^\vee$ is a shift-invariant space, we obtain

$$D_\infty(\langle\varphi\rangle) \subseteq L^2(E)^\vee. \tag{6.11}$$

Recall (1.1.23) that the spectral function of $\langle\varphi\rangle$ is

$$\sigma_{\langle\varphi\rangle}(\xi) = \frac{|\widehat{\varphi}(\xi)|^2}{p_\varphi(\xi)};$$

since $p_\varphi > 0$ almost everywhere. It is well known (see [**BR05**]) that

$$\sigma_{D_j(\langle\varphi\rangle)}(\xi) = \sigma_{\langle\varphi\rangle}\left(\frac{\xi}{2^j}\right) = \frac{|\widehat{\varphi}(\xi/2^j)|^2}{p_\varphi(\xi/2^j)}. \tag{6.12}$$

for almost every $\xi \in \mathbb{R}$. Furthermore (see [**BR05**] as well), we have

$$\sigma_{L^2(E)^\vee} = \chi_E \text{ almost everywhere.} \tag{6.13}$$

If $\text{orb}(\xi)$ is a zero-orbit for $\widehat{\varphi}|^2$, then $\xi \notin E$ and

$$\sigma_{D_j(\langle\varphi\rangle)}(\xi) = \sigma_{D_\infty(\langle\varphi\rangle)}(\xi) = \sigma_{L^2(E)^\vee}(\xi) = 0. \tag{6.14}$$

If $\text{orb}(\xi)$ is a full-orbit for $|\widehat{\varphi}|^2$, then $\xi \in E$ and we know from [**PŠW99**] that

$$\sigma_{D_\infty(\langle\varphi\rangle)}(\xi) \geq \lim_{j\to\infty} \sigma_{D_j(\langle\varphi\rangle)}(\xi) = 1. \tag{6.15}$$

It follows that $\sigma_{L^2(E)^\vee} = \sigma_{D_\infty(\langle\varphi\rangle)}$ almost everywhere, which, together with (6.11) implies the answer to our question, namely

$$D_\infty(\langle\varphi\rangle) = L^2(\text{ssupp}\ \widehat{\varphi})^\vee. \tag{6.16}$$

It is now very easy to see that the following statement holds.

Corollary 6.17. *Suppose that $\varphi \in L^2(\mathbb{R})$ is such that $\langle\varphi\rangle$ is FO and $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$. Denote $|\widehat{\varphi}|^2$ by Φ. Then the following are equivalent.*

(a) $L^2(\mathbb{R}) = \overline{\bigcup_{j\in\mathbb{N}\cup\{0\}} D_j(\langle\varphi\rangle)}$;

(b) $\Phi > 0$ *almost everywhere;*

(c) $A_\Phi^{(2)} = I$ *almost everywhere;*

(d) For almost every $\xi \in \mathbb{R}$, $\lim_{j\to\infty} \frac{\Phi(2^{-j}\xi)}{p_\Phi(2^{-j}\xi)} = 1$.

Using the results of our filter analysis, it is now easy to establish the following result (recall our notations from (2.35) and (3.8)).

Corollary 6.18. *Let $m : \mathbb{R} \to \mathbb{C}$ be measurable, 1-periodic with $M := |m^2|$ an FO (recall that in particular this means that $Sol_M \neq \{0\}$). Let $\varphi_0 \in L^2(\mathbb{R})$ be such that (φ_0, m) satisfies (2.4) and $|\widehat{\varphi_0}|^2 = \Phi_{0,M} = \Phi_0$. Then the following are equivalent:*

(a) $L^2(\mathbb{R}) = \overline{\bigcup_{j\in\mathbb{N}\cup\{0\}} D_j(\langle\varphi_0\rangle)}$;

(b) $|A_M^{(2)}| = 1$;

(c) $T_{M,-} < \infty$ *almost everywhere.*

(Recall Lemma 3.5 in connection with (c)).

We also have the following special case.

COROLLARY 6.19. *Let $m : \mathbb{R} \to \mathbb{C}$ be a measurable, 1-periodic function such that $M := |m|^2$ is an FO which satisfies the Smith–Barnwell condition and is dyadically continuous at zero. If $\varphi_0 \in L^2(\mathbb{R})$ is such that (φ_0, m) satisfies (2.4) and $|\widehat{\varphi_0}|^2 = \Phi_{0,M}$, then*

$$L^2(\mathbb{R}) = \overline{\bigcup_{j \in \mathbb{N} \cup \{0\}} D_j(\langle \varphi_0 \rangle)}$$

If $\varphi \in L^2(\mathbb{R})$ is such that (φ, m) satisfies (2.4) and $|\widehat{\varphi}|^2 = f_M$, then

$$L^2(\mathbb{R}) = \overline{\bigcup_{j \in \mathbb{N} \cup \{0\}} D_j(\langle \varphi \rangle)} \Leftrightarrow f_M > 0 \text{ almost everywhere.}$$

REMARK 6.20. Observe that considering various pairs (φ, m) is somewhat subtle. There are at least three possible issues. The first issue is whether, for a given φ, we can consider different corresponding filters. In this section we consider the FO case which implies that m is uniquely determined by φ, so we do not have to worry this issue here. The second issue is whether, for a given m, we can consider different "scaling functions". Indeed, we can, and we shall describe the consequences later.

The third issue is whether, for a given pair (φ, m), we can consider another pair (φ_1, m_1) such that $\langle \varphi \rangle = \langle \varphi_1 \rangle$. Obviously, in such a case, the "cascades" of spaces in (6.8) will be the same. What changes is the "relative position" of $\langle D\varphi \rangle$ (or $\langle D\varphi_1 \rangle$) within $D(\langle \varphi \rangle)$. We turn our attention to this issue now. $\diamondsuit$

As before, we start with a pair (φ, m) that satisfies (2.4), with φ non-trivial, and $\langle \varphi \rangle$ is an FO. Consider another generator of $\langle \varphi \rangle$, i.e. a function $\varphi_1 \in L^2(\mathbb{R})$ such that $\langle \varphi_1 \rangle = \langle \varphi \rangle$. By Proposition 2.3 and Corollary 2.16 there is exactly one m_1 such that (φ_1, m_1) satisfies (2.4). By Remark 2.33ii $A^{(2)}_{\Phi} = A^{(2)}_{\Phi_1}$, $\Phi_1 := |\widehat{\varphi_1}|^2$. Therefore, $M_1 := |m_1|^2$ is an FO, as well. What is the relationship between m and m_1? Is it possible that $m = m_1$?

Observe that, since $\langle \varphi \rangle = \langle \varphi_1 \rangle$ and $U_{\langle \varphi \rangle} = \mathbb{R}$ Corollary 1.1.21b implies that there is exactly one 1-periodic, measurable function ν such that, for almost every $\xi \in \mathbb{R}$, $\nu(\xi) \neq 0$ and $\widehat{\varphi_1}(\xi) = \nu(\xi)\widehat{\varphi}(\xi)$. Hence, from (2.4) we obtain that, for almost every $\xi \in \mathbb{R}$,

$$m_1(\xi) = \frac{\nu(2\xi)}{\nu(\xi)} m(\xi). \tag{6.21}$$

Observe that a "filter multiplier" $\mu(\xi) := \nu(2\xi)/\nu(\xi)$ is 1-periodic as well. Hence, it is not to difficult to describe the entire family of generators which "builds" the same space $\langle \varphi \rangle$. It consists of all the functions of the form $(\nu(\xi)\widehat{\varphi}(\xi))^\vee$, where $\nu : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ is 1-periodic and

$$\nu \in L^2(\mathbb{T}, p_\varphi). \tag{6.22}$$

REMARK 6.23. Observe that the problem is more challenging if we begin from a "filter multiplier" μ. Given 1-periodic, measurable μ, it is not difficult to find all solutions of the equation $\nu(2\xi)/\nu(\xi) = \mu(\xi)$ (see [**Con98**] for details). However, if we add a requirement that ν is also 1-periodic, then one needs to work harder. We shall not dwell on this problem here, though, since (6.22) (and then (6.21)) provides a direct method to find all φ_1. $\blacksquare$

As we shall see, the case $m_1 = m$ is possible only in the trivial case. The following lemma is well known and can be proved in various ways. We provide a proof via Lemma 2.31 for the reader's convenience.

LEMMA 6.24. *If $\nu : \mathbb{R} \to \mathbb{C}$ is a function which satisfies $\nu(2\xi) = \nu(\xi) = \nu(\xi+1)$ for almost every $\xi \in \mathbb{R}$, then ν is an almost everywhere-constant function.*

PROOF. It is enough to prove that, for every $H \subseteq \mathbb{C}$,

$$\{\xi : \nu(\xi) \in H\} = \emptyset \text{ or } \mathbb{R} \text{ almost everywhere.}$$

Observe also that the range of ν is (almost everywhere) equal to the range of $\nu|_I$. Given $H \subseteq \mathbb{C}$, define the set $A := \{\xi \in I : \nu(\xi) \in H\}$. Hence, either $|A| = 0$ or $|A| > 0$. Using the notation from Lemma 2.31, observe that $\{\xi \in \mathbb{R} : \nu(\xi) \in H\} = E_A$ almost everywhere. Applying Lemma 2.31 now completes the proof. □

Using (6.21) and Lemma 6.24, it is obvious that

$$m_1 = m \Leftrightarrow \varphi_1 = \text{const.} \cdot \varphi. \tag{6.25}$$

Hence, the same filter occurs only in a trivial way. Observe that Sol_M is, in principle, an infinite set. It follows from (6.25) that two solutions that differ by more than a constant multiple will generate different principal shift-invariant spaces, i.e., different "cascades" of spaces in (6.8). In other words, different generators of the same principal shift-invariant space must be associated with different filters. As already mentioned in Remark 6.20, what changes when we take a different generator φ_1 of $\langle\varphi\rangle$ is the "relative position" of $\langle D\varphi_1\rangle$ within $D(\langle\varphi_1\rangle)$. In order to analyze this, we start with a specific generator,

$$\varphi_0 := \frac{1}{\sqrt{p_\varphi}} \bullet \varphi;$$

(recall Remark 1.5.36 here) $\mathcal{B}_{\varphi_0}$ is an orthonormal basis for $\langle\varphi\rangle$. Observe that φ_0 is paired with a filter m_0, which is different from the original filter of φ. Since $p_{\varphi_0} \equiv 1$, we obtain that m_0 satisfies the Smith–Barnwell condition. Therefore, we also have a function f_{M_0}, where $M_0 = |m_0|^2$. There is a natural question here as to whether f_{M_0} is equal to $|\widehat{\varphi_0}|^2$ or not. The following short argument gives a positive answer to this question. Since $M_0 \le 1$ and $|\widehat{\varphi_0}|^2 \le 1$, we obtain that, for almost every $\xi \in \mathbb{R}$, the following limit exists: $h(\xi) := \lim_{n\to\infty} |\widehat{\varphi_0}(2^{-n}\xi)|^2$. It is obvious that $0 \le h(\xi) \le 1$ and that $h(2\xi) = h(\xi)$ for almost every $\xi \in \mathbb{R}$. Furthermore, $h(\xi) = 0$ if and only if $\mathrm{orb}(\xi)$ is a zero-orbit (recall Remark 2.17). Observe also that for almost every $\xi \in \mathbb{R}$ we have $|\widehat{\varphi_0}(\xi)|^2 = h(\xi) f_{M_0}(\xi)$. Using (4.14) we obtain

$$1 = \int_I p_{\varphi_0}(\xi)d\xi = \int_{\mathbb{R}} |\widehat{\varphi_0}(\xi)|^2 d\xi = \int_{\mathbb{R}} f_{M_0}(\xi)d\xi.$$

The same derivation holds when the last two integrals are over the set of non-zero orbits (instead of $\mathbb{R}$). Hence, we obtain that

$$\int_{\mathbb{R}} f_{M_0}(\xi)d\xi = 1, \tag{6.26}$$

$$f_{M_0}(\xi) = 0 \text{ whenever } \mathrm{orb}(\xi) \text{ is a zero-orbit for } \varphi_0, \tag{6.27}$$

and

$$|\widehat{\varphi_0}|^2 = f_{M_0}. \tag{6.28}$$

Using the scaling function characterization from [**HW96**] and our results in this section and Section 4, it is easy to see that the following result holds.

COROLLARY 6.29. *Suppose that* $\varphi \in L^2(\mathbb{R})$ *is such that* $\langle\varphi\rangle$ *is FO and* $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$. *Consider* $\varphi_0 := \frac{1}{\sqrt{p_\varphi}} \bullet \varphi$, *and denote the corresponding (unique) filter by* m_0 *and* $M_0 := |m_0|^2$. *Then the following are equivalent:*

(a) φ_0 *is a scaling function of an orthonormal wavelet MRA structure* $(D_j(\langle\varphi\rangle) : j \in \mathbb{Z})$*;*
(b) $L^2(\mathbb{R}) = D_\infty(\langle\varphi\rangle)$*;*
(c) $f_{M_0} > 0$ *almost everywhere;*
(d) $\widehat{\varphi_0}(\xi) \neq 0$ *for almost every* $\xi \in \mathbb{R}$.
(e) M_0 *is associated with a low-pass filter for an MRA orthonormal wavelet (and this low-pass filter is* m_0*).*

REMARK 6.30. Given $\varphi \in L^2(\mathbb{R})$ such that $\langle\varphi\rangle$ is FO and $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$, the family of spaces $D_j(\langle\varphi\rangle)$, for $j \in \mathbb{Z}$, does not change with a different choice of generator, say φ_1, such that $\langle\varphi_1\rangle = \langle\varphi\rangle$. Observe that the spectral function of $\langle\varphi\rangle$ belongs to $L^1(\mathbb{R})$, and it is then well known (see [**Rze00**], [**BR05**], [**BR03**] for more details) that

$$\bigcap_{j\in\mathbb{Z}} D_j(\langle\varphi\rangle) = \{0\}. \tag{6.31}$$

Consider $D^{-1}(\langle\varphi\rangle) \subseteq \langle\varphi\rangle$. Since $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$, but $\langle\varphi\rangle \neq D(\langle\varphi\rangle)$, it follows that $D^{-1}(\langle\varphi\rangle) \neq \langle\varphi\rangle$. We claim that $D^{-1}(\langle\varphi\rangle)$ is not a shift-invariant space. Suppose, to the contrary, that it is. Then $\langle D^{-1}\varphi\rangle \subseteq D^{-1}(\langle\varphi\rangle) \subseteq \langle\varphi\rangle$. If ξ is such that $\mathrm{orb}(\xi)$ is a non-zero orbit for $\widehat{\varphi}$, then it is obvious that it is also a non-zero (and full) orbit for $\widehat{D^{-1}\varphi}$. The consequence of this is that $p_{D^{-1}\varphi} > 0$ almost everywhere. Hence, $\langle D^{-1}\varphi\rangle \subseteq \langle\varphi\rangle$ and $U_{\langle\varphi\rangle} = \mathbb{R} = U_{\langle D^{-1}\varphi\rangle}$. By (1.1.28), we obtain $\langle D^{-1}\varphi\rangle = \langle\varphi\rangle$, i.e. $D^{-1}(\langle\varphi\rangle) = \langle\varphi\rangle$, a contradiction.

Therefore, given $\varphi \in L^2(\mathbb{R})$ such that $\langle\varphi\rangle$ is FO and $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$, we always obtain a family $(D_j(\langle\varphi\rangle) : j \in \mathbb{Z})$ which satisfies (6.31), $D_j(\langle\varphi\rangle) \subseteq D_{j+1}(\langle\varphi\rangle)$, satisfies (6.16), and

$$D_j(\langle\varphi\rangle) \text{ is a shift-invariant space } \Leftrightarrow j \geq 0. \tag{6.32}$$

■

As mentioned in the previous remark, the "cascade" of spaces $D_j(\langle\varphi\rangle)$ for $j \in \mathbb{Z}$ does not change with different choice of generators for $\langle\varphi\rangle$. What does change, however, is the interior structure of these spaces. We consider the choice of $\varphi_0 = \frac{1}{\sqrt{p_\varphi}} \bullet \varphi$ first. By Corollary 1.27, we know that

$$D(\langle\varphi_0\rangle) = \langle D\varphi_0\rangle \oplus \langle DT\varphi_0\rangle, \tag{6.33}$$

and both $\langle D\varphi_0\rangle$ and $\langle DT\varphi_0\rangle$ are maximal principal shift-invariant spaces. Since we also have $\langle\varphi\rangle = \langle\varphi_0\rangle \subseteq D(\langle\varphi_0\rangle) = D(\langle\varphi\rangle)$, a natural question to ask is what the position of $\langle\varphi_0\rangle$ is with respect to $\langle D\varphi_0\rangle$ and $\langle DT\varphi_0\rangle$. Without loss of generality, we treat the case of $\langle D\varphi_0\rangle$. We obtain by a simple calculation that

$$\begin{aligned}[][\varphi_0, D\varphi_0](\xi) &= \frac{1}{\sqrt{2}}\left(m_0(\xi/2)p_{\varphi_0}(\xi/2) + m_0(\xi/2+1/2)p_{\varphi_0}(\xi/2+1/2)\right)\\ &= \frac{1}{\sqrt{2}}(m_0(\xi/2) + m_0(\xi/2+1/2)).\end{aligned} \tag{6.34}$$

PROPOSITION 6.35. *Suppose that $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$ is such that $\langle\varphi\rangle$ is FO and $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$, and consider $\varphi_0 := \frac{1}{\sqrt{p_\varphi}} \bullet \varphi$. Then $\langle\varphi_0\rangle$ is neither orthogonal to $\langle D\varphi_0\rangle$ nor to $\langle DT\varphi_0\rangle$, and $\langle\varphi_0\rangle$ does not coincide with either $\langle D\varphi_0\rangle$ or $\langle DT\varphi_0\rangle$*

PROOF. Observe that the second statement follows from the first. Without loss of generality, we prove the first statement in the case of $\langle D\varphi_0\rangle$. Suppose, to the contrary, that $\langle\varphi_0\rangle \perp \langle D\varphi_0\rangle$. By (6.34), it follows that $m_0(\xi/2+1/2) = -m_0(\xi/2)$, i.e. $M_0(\xi/2) = M_0(\xi/2+1/2)$, for almost every $\xi \in \mathbb{R}$. Since M_0 satisfies the Smith–Barnwell condition, we obtain $M_0 \equiv 1/2$. This is in contradiction with φ, and therefore φ_0, being non-trivial. □

Using Corollary 1.27 and Lemma 1.33, we obtain the following result.

COROLLARY 6.36. *Suppose that $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$ is such that $\langle\varphi\rangle$ is FO and $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$. Consider $\varphi_0 := \frac{1}{\sqrt{p_\varphi}} \bullet \varphi$. If $\varphi_1 \in L^2(\mathbb{R})$ is such that $\langle\varphi\rangle = \langle\varphi_1\rangle$, then $\langle D\varphi_1\rangle = \langle D\varphi_0\rangle$ and $\langle DT\varphi_1\rangle = \langle DT\varphi_0\rangle$ if and only if p_{φ_1} is $1/2$-periodic.*

This corollary describes precisely the family of generators of $\langle\varphi\rangle$ which induce the same "inner structure" into $D(\langle\varphi\rangle)$ as φ_0 does. The following example shows that there are many such generators. Observe that one should not carry the idea of "inner structure" too far; for example, there is only one generator which produces an orthonormal basis — namely, φ_0.

EXAMPLE 6.37. Consider φ and φ_0 as in the previous corollary. Consider a family of functions $\nu : \mathbb{R} \to \mathbb{C}$ such that ν is measurable, 1-periodic, $\nu \in L^2(\mathbb{T})$, $\nu(\xi) \neq 0$ almost everywhere, and $|\nu|^2$ is $1/2$-periodic. Obviously, there are many such functions, and it is not difficult to construct them. Consider the family of functions $\nu \bullet \varphi_0$, where ν belongs to the family we just described. It follows that $\langle\nu \bullet \varphi_0\rangle = \langle\varphi_0\rangle = \langle\varphi\rangle$, and $p_{\nu\bullet\varphi_0} = |\nu|^2 \cdot p_{\varphi_0} = |\nu|^2$. It is not difficult to see that in this way we have described the entire family of functions φ_1 which satisfy the conditions of Corollary 6.36. If we denote the filter corresponding to φ_1 by m_1 and denote $M_1 := |m_1|^2$, then we obtain $M_1(\xi) = \left(|\nu(2\xi)|^2/|\nu(\xi)|^2\right) M_0(\xi)$. Hence

$$M_0(\xi) + M_1(\xi + 1/2) = \frac{|\nu(2\xi)|^2}{|\nu(\xi)|^2}.$$

By Lemma 6.24, it follows that m_1 satisfies the Smith–Barnwell condition if and only if $|\nu(\xi)|$ is a constant function. In particular, observe that, for this reason, the class of our examples given here would not even be "detected" via the theory developed in [**PŠW99**], [**PŠWX01**], and [**PŠWX03**]. ♢

Let us now get into more details about the structure of $D(\langle\varphi\rangle)$ with respect to φ_0. Formula (6.34), in connection with orthogonality, leads to the notion of $1/2$-antiperiodicity. We shall say that a function $\mu : \mathbb{R} \to \mathbb{C}$ is $1/2$-*antiperiodic* if, for almost every $\xi \in \mathbb{R}$, $\mu(\xi + 1/2) = -\mu(\xi)$. Two typical examples are the functions $\xi \mapsto \exp(\pm 2\pi i\xi)$. Observe that every $1/2$-antiperiodic function must also be 1-periodic. Furthermore, every 1-periodic function can be decomposed into its $1/2$-periodic and $1/2$-antiperiodic parts. More precisely, if m is 1-periodic, then

$$m(\xi) = a(\xi) + b(\xi)e^{-2\pi i\xi}, \tag{6.38}$$

where a and b are 1/2-periodic; in particular, they are given by

$$a(\xi) := \frac{1}{2}\left(m(\xi) + m(\xi + 1/2)\right)$$

$$b(\xi) := \frac{e^{2\pi i\xi}}{2}\left(m(\xi) - m(\xi + 1/2)\right). \tag{6.39}$$

Observe that

$$m \text{ is } 1/2\text{-periodic} \Leftrightarrow b \equiv 0 \tag{6.40}$$

and

$$m \text{ is } 1/2\text{-antiperiodic} \Leftrightarrow a \equiv 0. \tag{6.41}$$

It is also straightforward to check the following results, which are valid in general (not just in the FO case).

LEMMA 6.42. *If $f, g \in L^2(\mathbb{R})$ are a, b are functions given by (6.38) and (6.39) as the decomposition of $[f, g](\xi)$, then $[Df, Dg](\xi) = a(\xi/2)$ and $[Df, DTg](\xi) = b(\xi/2)$.*

LEMMA 6.43. *If $\varphi \in L^2(\mathbb{R})$ and $m : \mathbb{R} \to \mathbb{C}$, measurable and 1-periodic, satisfy (2.4), and a, b are functions given by (6.38) and (6.39) as the decomposition of $m(\xi)$, then*

$$\varphi = \alpha \bullet D\varphi + \beta \bullet DT\varphi,$$

where $\alpha(\xi) = \sqrt{2}a(\xi/2)$ and $\beta(\xi) = \sqrt{2}b(\xi/2)$.

Going back to our FO case, consider $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$ such that $\langle\varphi\rangle$ is FO and $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$. As before, take $\varphi_0 := \frac{1}{\sqrt{p_\varphi}} \bullet \varphi$ and denote the corresponding filter by m_0. Denote by α_0 and β_0 the functions given in Lemma 6.43 generated by m_0. We have $\langle\varphi_0\rangle \subseteq D(\langle\varphi_0\rangle) = \langle D\varphi_0\rangle \oplus \langle DT\varphi_0\rangle$, and it is natural to look into the orthogonal complement of $\langle\varphi_0\rangle$ within $D(\langle\varphi_0\rangle)$. We focus on the function $\psi \in L^2(\mathbb{R})$ given by

$$\psi := -\overline{\beta_0} \bullet D\varphi_0 + \overline{\alpha_0} \bullet DT\varphi_0. \tag{6.44}$$

Observe first that, by its very definition, $\psi \in D(\langle\varphi_0\rangle)$. Secondly, $\langle\psi\rangle \perp \langle\varphi_0\rangle$ since

$$[\psi, \varphi_0](\xi) = -\overline{\beta_0(\xi)} \cdot \overline{\alpha_0(\xi)} + \overline{\alpha_0(\xi)} \cdot \overline{\beta_0(\xi)} = 0. \tag{6.45}$$

Thirdly, and directly from (6.44), we have that, for almost every $\xi \in \mathbb{R}$,

$$\widehat{\psi}(\xi) = e^{-\pi i\xi}\overline{m_0(\xi/2 + 1/2)}\widehat{\varphi_0}(\xi/2). \tag{6.46}$$

In particular, for every ξ such that $\mathrm{orb}(\xi)$ is FO with respect to φ_0 (or, equivalently, φ), we have

$$\widehat{\psi}|_{\mathrm{orb}(\xi)} \neq 0. \tag{6.47}$$

Furthermore,

$$[\psi, \psi](\xi) = |\beta_0|^2 + |\alpha_0|^2 = |m_0(\xi/2)|^2 + |m_0(\xi/2 + 1/2)|^2 = 1; \tag{6.48}$$

i.e., $\mathcal{B}_\psi$ is an orthonormal basis for the maximal principal shift-invariant space $\langle\psi\rangle$. In particular,

$$D(\langle\varphi_0\rangle) = D(\langle\varphi\rangle) = \langle\varphi_0\rangle \oplus \langle\psi\rangle; \tag{6.49}$$

both the spectral function argument and the dimension function argument lead to this conclusion. It follows now, by induction, that, for every $j \in \mathbb{N}$,

$$D_j(\langle\varphi\rangle) = D_j(\langle\varphi_0\rangle) = \langle\varphi_0\rangle \oplus \langle\psi\rangle \oplus D(\langle\psi\rangle) \oplus \ldots \oplus D_{j-1}(\langle\psi\rangle). \tag{6.50}$$

In addition, for every $j \in \mathbb{N} \cup \{0\}$, $D_j(\mathcal{B}_\psi)$ is an orthonormal basis for $D_j(\langle\psi\rangle)$. Observe also that $D_j(\langle\psi\rangle)$ is a shift-invariant space for every $j \in \mathbb{N} \cup \{0\}$.

For $j \in \mathbb{Z}$ with $j < 0$, observe first that D_j is also a unitary operator, so $D_j(\mathcal{B}_\psi)$ remains an orthonormal basis for the subspace $D_j(\langle\psi\rangle)$. However, $D_j(\langle\psi\rangle)$ is not a shift-invariant space for negative j. Nevertheless, from (6.49) we do get

$$D^{-1}(\langle\psi\rangle) \subsetneqq \langle\varphi_0\rangle, \tag{6.51}$$

and

$$\langle\varphi_0\rangle = D^{-1}(\langle\varphi_0\rangle) \oplus D^{-1}(\langle\psi\rangle). \tag{6.52}$$

Using (6.31), this implies that

$$\langle\varphi_0\rangle = \bigoplus_{j<0} D_j(\langle\psi\rangle). \tag{6.53}$$

Observe also that (2.4) and (6.46) imply that

$$|\widehat{\psi}(\xi)|^2 + |\widehat{\varphi_0}(\xi)|^2 = |\widehat{\varphi_0}(\xi/2)|^2, \tag{6.54}$$

for almost every $\xi \in \mathbb{R}$. It then follows that (recall that $\lim_{n\to\infty} |\widehat{\varphi_0}(2^n\xi)|^2 = 0$; see Remark 2.17ii)

$$|\widehat{\varphi_0}(\xi)|^2 = \sum_{j\in\mathbb{N}} |\widehat{\psi}(2^j\xi)|^2, \tag{6.55}$$

for almost every $\xi \in \mathbb{R}$, and

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = \begin{cases} 1 & \text{if } \mathrm{orb}(\xi) \text{ is FO for } \varphi \\ 0 & \text{if } \mathrm{orb}(\xi) \text{ is not FO for } \varphi \end{cases} \tag{6.56}$$

for almost every $\xi \in \mathbb{R}$. Combining all these results provides us with the following theorem.

THEOREM 6.57. *Suppose that $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$ is such that $\langle\varphi\rangle$ is FO, $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$. Consider $\varphi_0 := \frac{1}{\sqrt{p_\varphi}} \bullet \varphi$ and ψ defined by (6.44). Then $\{\psi_{jk} : j,k \in \mathbb{Z}\}$ is an orthonormal basis for*

$$L^2(\mathit{ssupp}\ \widehat{\varphi})^\vee,$$

and

$$L^2(\mathit{ssupp}\ \widehat{\varphi})^\vee = \bigoplus_{j\in\mathbb{Z}} D_j(\langle\psi\rangle).$$

Furthermore, the following are equivalent:

(a) ψ is an MRA orthonormal wavelet;

(b) $\mathit{ssupp}\ \widehat{\psi} = \mathbb{R}$;

(c) $\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1$ for almost every $\xi \in \mathbb{R}$;

(d) $\mathit{ssupp}\ \widehat{\varphi} = \mathbb{R}$;

(e) $\lim_{n\to\infty} \dfrac{|\widehat{\varphi}(2^{-n}\xi)|^2}{p_\varphi(2^{-n}\xi)} = 1$ for almost every $\xi \in \mathbb{R}$.

REMARK 6.58. We assume that our reader is familiar with the basic definitions from the theory of orthonormal wavelets (as outlined, for example, in [**HW96**]). Here we have shown that, if we have

(i) a D^{-1}-invariant $\langle\varphi\rangle$ such that $\langle\varphi\rangle$ is FO; and
(ii) we select (which one can always do in this case) an "orthonormal basis generator" φ_0 of $\langle\varphi\rangle$,

we obtain a theory completely analogous to the standard theory of MRA orthonormal wavelets with the caveat that our systems do not necessarily span the entirety of $L^2(\mathbb{R})$ but rather the "infinite-dimensional" shift-invariant space $L^2(\text{ssupp }\widehat{\varphi})^\vee$.

Furthermore, this theory captures, as a special case (completely characterized by the previous theorem), a certain part of the standard MRA orthonormal wavelet theory. More precisely, this "certain part" consists precisely of the MRA orthonormal wavelets such that

$$\text{ssupp }\widehat{\psi} = \mathbb{R};$$

this includes, for example, the Haar wavelet and the family of Daubechies wavelets.

Observe also that we have shown that, as soon as we have (i), we always generate an MRA-type structure — namely the family $D_j(\langle\varphi\rangle)$ for $j \in \mathbb{Z}$. Moreover, if we additionally have that ssupp $\widehat{\psi} = \mathbb{R}$, this family is indeed an MRA structure, and, in fact, it must be a family which comes from an orthonormal wavelet. ■

Going back to the original generator, consider $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$ such that $\langle\varphi\rangle$ is FO and $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$. Observe that the formula $\varphi = \alpha \bullet D\varphi + \beta \bullet D\varphi$, given in Lemma 6.43, is still valid. Corollary 1.14(b) holds as well, i.e. $\dim_{D_j(\langle\varphi\rangle)} \equiv 2^j$, for every $j \in \mathbb{N} \cup \{0\}$. What changes are the relative positions of $\langle\varphi\rangle$, $\langle D\varphi\rangle$, and $\langle DT\varphi\rangle$ within $D(\langle\varphi\rangle)$. The relationship between $\langle D\varphi\rangle$ and $\langle DT\varphi\rangle$ is described in Corollary 1.25; they do not intersect, their "parts are orthogonal" at points ξ where $p_\varphi(\xi/2) = p_\varphi(\xi/2 + 1/2)$ and are "at an angle" at other points. We have seen examples where only orthogonality occurs (where p_φ is 1/2-periodic) and it is not difficult to construct examples where the two spaces are only "at an angle".

It is interesting to add $\langle\varphi\rangle$ into this relationship. A typical relationship (with $\varphi_0 = p_\varphi^{-1/2} \bullet \varphi$) is given in (6.33), (6.34), and Proposition 6.35. Observe that it does not allow for an extreme case where $\langle\varphi_0\rangle$ coincides with $\langle D\varphi_0\rangle$. What about general φ? To start, consider the following example which demonstrates a nice pattern.

EXAMPLE 6.59. In order to have $\langle\varphi\rangle = \langle D\varphi\rangle$ with $\langle\varphi\rangle$ being FO, it is enough to find a "multiplier" m which is never zero and such that $D\varphi = m \bullet \varphi$. By taking the square in this equation on the Fourier transform side, it is enough to find a pair of functions $f : \mathbb{R} \to [0, \infty)$ and $\nu : \mathbb{R} \to [0, \infty)$ such that both are measurable, $\nu > 0$ almost everywhere, ν 1-periodic, $f \in L^1(\mathbb{R})$ with $\sum_{k\in\mathbb{Z}} f(\xi + k) > 0$ almost everywhere, and such that $\nu(\xi)f(\xi) = f(\xi/2)$ almost everywhere. Observe that we shall fulfill these requirements if we can find $\nu : (0, \infty) \to (0, \infty)$ measurable and 1-periodic (i.e. it is enough to give ν on $(0, 1]$) and $f : (0, \infty) \to (0, \infty)$ such that $f \in L^1(\mathbb{R})$ and for every $\xi > 0$ we have $\nu(\xi)f(\xi) = f(\xi/2)$.

Consider the numbers $a, b \in \mathbb{R}$ such that $0 < a < 2$, $a < b < \infty$ and $1/a+1/b < 1$. Define ν on $(0, 1]$ by

$$\nu(\xi) := \begin{cases} a & \text{if } 0 < \xi \leq 1/4 \\ b & \text{if } 1/4 < \xi \leq 1 \end{cases}$$

and extend it 1-periodically to $(0,\infty)$. Define f to be identically 1 on $(1/2,1]$ and extend it dyadically to $(0,\infty)$ via the equation $\nu(\xi)f(\xi)=f(\xi/2)$. It is not difficult to see that extending it into "low frequencies" leads to

$$
\begin{aligned}
&f\equiv b \text{ on } (1/4,1/2]\\
&f\equiv b^2 \text{ on } (1/8,1/4]\\
&f\equiv a^n b^2 \text{ on } \left(\frac{1}{2^{n+3}},\frac{1}{2^{n+2}}\right] \text{ for } n\in\mathbb{N}.
\end{aligned}
$$

It follows that

$$\int_0^1 f(\xi)d\xi=\frac{1}{2}+\frac{b}{4}+b^2\sum_{n=0}^{\infty}\frac{a^n}{2^{n+3}}=\frac{1}{2}+\frac{b}{4}+\frac{b^2}{8}\frac{1}{1-\frac{a}{2}}.$$

Since $a<2$, this integral is finite.

The pattern for "high frequencies" is a bit more complex. Observe first that the value of f changes between two consecutive even integers according to the pattern of "3 subintervals + 2 subintervals". For example, on $(2,4]$ we have

$$
\begin{aligned}
&f\equiv\frac{1}{a^2} \text{ on } (2,2+1/4]\\
&f\equiv\frac{1}{ab} \text{ on } (2+1/4,2+1/2]\\
&f\equiv\frac{1}{b^2} \text{ on } (2+1/2,3]\\
\\
&f\equiv\frac{1}{ab} \text{ on } (3,3+1/4]\\
&f\equiv\frac{1}{b^2} \text{ on } (3+1/4,4]
\end{aligned}
$$

Let us check one more iteration, i.e. on $(4, 8]$, where we have

$$f \equiv \frac{1}{a^3} \text{ on } (4, 4 + 1/4]$$
$$f \equiv \frac{1}{a^2 b} \text{ on } (4 + 1/4, 4 + 1/2]$$
$$f \equiv \frac{1}{ab^2} \text{ on } (4 + 1/2, 5]$$

$$f \equiv \frac{1}{ab^2} \text{ on } (5, 5 + 1/4]$$
$$f \equiv \frac{1}{b^3} \text{ on } (5 + 1/4, 6]$$

$$f \equiv \frac{1}{a^2 b} \text{ on } (6, 6 + 1/4]$$
$$f \equiv \frac{1}{ab^2} \text{ on } (6 + 1/4, 6 + 1/2]$$
$$f \equiv \frac{1}{b^3} \text{ on } (6 + 1/2, 7]$$

$$f \equiv \frac{1}{ab^2} \text{ on } (7, 7 + 1/4]$$
$$f \equiv \frac{1}{b^3} \text{ on } (7 + 1/4, 8]$$

One can now approach this pattern in various ways. Here is one we find somewhat elegant. Observe that for any “3+2” pattern, it is enough to determine “the last power of a^{-1}” in order to know exactly how the pattern looks for all five subintervals. So each “3+2” pattern can be written as a pair of numbers, n, m, with $n, m \in \mathbb{N} \cup \{0\}$. We will write them vertically:

$$\begin{matrix} n \\ m \end{matrix}$$

For example, the “starting” pattern, i.e. for $(2, 4]$, is

$$\begin{matrix} 0 \\ 0 \end{matrix}$$

while the “next” pattern, i.e. for $[4, 8)$ is

$$\begin{matrix} 1 \\ 0 \\ 0 \\ 0. \end{matrix}$$

Observe that each “3+2” block determines two new “3+2” blocks within the next dyadic iteration and that it is obtained from the previous one according to the transformation

$$\begin{array}{ccc} & & n+1 \\ & \nearrow & \\ n & \longrightarrow & n \\ m & \longrightarrow & m \\ & \searrow & \\ & & m \end{array} \tag{6.60}$$

This rule obviously holds from $(2,4]$ to $(4,8]$; from $(4,8]$ to $(8,16]$ we have

$$\begin{array}{cc} & 2 \\ 1 & 1 \\ 0 & 0 \\ & 0 \\ & 1 \\ 0 & 0 \\ 0 & 0 \\ & 0 \end{array}$$

One can check inductively that (6.60) holds for all "steps" from $(2^k, 2^{k+1}]$ to $(2^{k+1}, 2^{k+2}]$.

Additionally, observe that pattern (6.60) means that, if we denote $\int_{(2^k, 2^k+2^{k-1}]} f(\xi)d\xi$ by A and $\int_{(2^k+2^{k-1}, 2^{k+1}]} f(\xi)d\xi$ by B, then

$$\begin{aligned} \int_{(2^{k+1}, 2^{k+2}]} f(\xi)d\xi &= \frac{A}{a} + \frac{B}{b} + \frac{A}{b} + \frac{B}{b} \\ &< \frac{A}{a} + \frac{B}{a} + \frac{A}{b} + \frac{B}{b} \\ &= (A+B)\left(\frac{1}{a} + \frac{1}{b}\right). \end{aligned}$$

Hence

$$\int_1^\infty f(\xi)d\xi = \sum_{n=0}^\infty \int_{2^n}^{2^{n+1}} f(\xi)d\xi \leq \text{ const.} \sum_{n=0}^\infty \left(\frac{1}{a} + \frac{1}{b}\right)^n < \infty.$$

It is perhaps somewhat intriguing that such an example exists. We decided to devote some time to describe it rather completely. $\diamondsuit$

Consider the following problem. Find all functions $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$ such that

$$\langle\varphi\rangle = \langle D\varphi\rangle. \tag{6.61}$$

Observe first the following (seemingly weaker) condition

$$\langle D\varphi\rangle \subseteq \langle\varphi\rangle, \varphi \neq 0. \tag{6.62}$$

By Proposition 1.1.20, we have $U_{\langle D\varphi\rangle} \subseteq U_{\langle\varphi\rangle}$ and $|U_{\langle\varphi\rangle}| > 0$. Using (1.21) we obstain

$$\xi \in 2U_{\langle\varphi\rangle} \Rightarrow \xi = 2u, p_\varphi(u) > 0 \Rightarrow p_{D\varphi}(2u) > 0 \Rightarrow 2u \in U_{\langle D\varphi\rangle} \Rightarrow \xi \in U_{\langle\varphi\rangle}.$$

Lemma 2.1 and $|U_{\langle\varphi\rangle}| > 0$ leads to the following result.

LEMMA 6.63. *If $f \in L^2(\mathbb{R}) \setminus \{0\}$ satisfies (6.62), then*

$$U_{\langle\varphi\rangle} = U_{\langle D\varphi\rangle} = \mathbb{R}.$$

Now it is easy to prove the following set of equivalent characterizations.

THEOREM 6.64. *If $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$, then the following are equivalent:*

(a) $\langle\varphi\rangle = \langle D\varphi\rangle$;
(b) $\langle D\varphi\rangle \subseteq \langle\varphi\rangle$;
(c) There exist 1-periodic, measurable functions μ_0 and μ_1 such that $\widehat{\varphi}(\xi/2) = \mu_0(\xi)\widehat{\varphi}(\xi)$ and $\widehat{\varphi}(\xi) = \mu_1(\xi)\widehat{\varphi}(\xi/2)$;
(d) There exists a 1-periodic, measurable function μ such that, for almost every $\xi \in \mathbb{R}$, $\mu(\xi) \neq 0$ and $\widehat{\varphi}(\xi/2) = \mu(\xi)\widehat{\varphi}(\xi)$;
(e) There exists a 1-periodic measurable function μ such that $\widehat{\varphi}(\xi/2) = \mu(\xi)\widehat{\varphi}(\xi)$.

PROOF. It is obvious that $(a) \Rightarrow (b)$ and $(d) \Rightarrow (e)$. By taking $\mu_0 = \mu$ and $\mu_1 = 1/\mu$, it is obvious that $(d) \Rightarrow (c)$. By Proposition 1.1.20 we have $(b) \Leftrightarrow (e)$, and by Corollary 1.1.21 we have $(a) \Leftrightarrow (c)$. Hence it is enough to prove that $(b) \Rightarrow (d)$. If (b) holds, then Lemma 6.63 shows that $p_\varphi > 0$ almost everywhere and $p_{D\varphi} > 0$ almost everywhere. As we already mentioned, $(b) \Rightarrow (e)$, so $D\varphi = (\mu/\sqrt{2}) \bullet \varphi$. By (1.1.8) and (1.1.9) we obtain

$$p_{D\varphi} = |\mu|^2 \frac{1}{2} p_\varphi \text{ almost everywhere,}$$

which implies that $|\mu|^2 > 0$ almost everywhere. □

REMARK 6.65. It is perhaps useful to compare (6.61) to (2.4). Observe that by taking $m(\xi) := \mu_1(2\xi)$ in the previous theorem, we obtain that (6.61) is equivalent to the existence of a measurable

$$\frac{1}{2}\text{-periodic function } m \tag{6.66}$$

such that $\widehat{\varphi}(2\xi) = m(\xi)\widehat{\varphi}(\xi)$. This is not surprising since it is indeed obvious that (6.61) implies $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$.

This also shows that it is not difficult to dyadically adjust the phase. Hence, analogous to the relationship between the (φ, m)-Problem and the the (Φ, M)-Problem (see Remark 2.11), it is not difficult to show that (6.61) boils down to the following problem. Find all pairs (Φ, ν) such that $\Phi : \mathbb{R} \to [0, \infty)$ with $\Phi \in L^1(\mathbb{R}) \setminus \{0\}$, $\nu : \mathbb{R} \to (0, \infty)$ measurable, 1-periodic, and, for almost every $\xi \in \mathbb{R}$,

$$\Phi(\xi/2) = \nu(\xi)\Phi(\xi). \tag{6.67}$$

■

Observe also that our results imply that we are always within the FO case.

COROLLARY 6.68. *If $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$ satisfies (6.61), then $\langle\varphi\rangle$ is a maximal, FO, principal shift-invariant space and, furthermore, for almost every $\xi \in \mathbb{R}$,*

$$M(\xi) = \frac{1}{\nu(2\xi)}.$$

It is now not surprising that the construction of all solutions of (6.67) will follow from the Tauberian-type construction described in (2.34) and (2.35). As it turns out, the connection with FO principal shift-invariant spaces is deeper than that. The following theorem shows that there is a natural one-to-one correspondence between the two.

THEOREM 6.69. *We have the following.*

(a) *If a pair (φ, m) satisfies (2.4) and $\langle\varphi\rangle$ is a FO principal shift-invariant space, then $\varphi_1 := m \bullet \varphi$ satisfies (6.61).*

(b) *If $\varphi_1 \in L^2(\mathbb{R}) \setminus \{0\}$ satisfies (6.61) and μ is a function described in Theorem 6.64(d), then the pair (φ, μ), where $\varphi := \frac{1}{\mu} \bullet \varphi_1$, satisfies (2.4) and $\langle\varphi\rangle$ is a FO principal shift-invariant space.*

In both cases, $\langle\varphi\rangle = \langle\varphi_1\rangle$.

PROOF. (a) Since $\langle\varphi\rangle$ is FO, we have $m(\xi) \neq 0$ for almost every $\xi \in \mathbb{R}$. It follows that $\varphi_1 \in L^2(\mathbb{R}) \setminus \{0\}$ (recall that the definition of an FO postulates that φ is non-trivial). For almost every $\xi \in \mathbb{R}$ we obtain

$$\begin{aligned}\widehat{\varphi_1}(\xi) &= m(\xi)\widehat{\varphi}(\xi)\\ &= m(\xi)m(\xi/2)\widehat{\varphi}(\xi/2)\\ &= m(\xi)m(\xi/2)m(\xi/2)^{-1}\widehat{\varphi_1}(\xi/2)\\ &= m(\xi)\widehat{\varphi_1}(\xi/2).\end{aligned}$$

The desired conclusion now follows from Theorem 6.64, and $\langle\varphi_1\rangle = \langle\varphi\rangle$ is obvious.

(b) Observe that φ must be non-trivial. Now use the same calculation as in (a), i.e. for almost every $\xi \in \mathbb{R}$ we obtain

$$\begin{aligned}\widehat{\varphi}(2\xi) &= \frac{1}{\mu(2\xi)}\widehat{\varphi_1}(2\xi)\\ &= \frac{1}{\mu(2\xi)}\mu(2\xi)\widehat{\varphi_1}(\xi)\\ &= \widehat{\varphi_1}(\xi)\\ &= \mu(\xi)\widehat{\varphi}(\xi).\end{aligned}$$

□

REMARK 6.70. (i) Observe that, up to a constant, the φ_1 in the previous theorem is actually $D^{-1}\varphi$. Indeed, $D^{-1}\varphi \in \langle\varphi\rangle$ if and only if $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$. Hence, we will have, in this case, that $\langle D^{-1}\varphi\rangle = \langle\varphi\rangle$ if and only if $\langle D^{-1}\varphi\rangle$ is a maximal principal shift-invariant space. This, in turn, is equivalent to $p_{D^{-1}\varphi} > 0$ almost everywhere. Since we have $\widehat{\varphi}(2\xi) = m(\xi)\widehat{\varphi}(\xi)$, we obtain $(D^{-1}\varphi)^\wedge(\xi) = \sqrt{2}m(\xi)\widehat{\varphi}(\xi)$ and

$$p_{D^{-1}\varphi}(\xi) = 2|m(\xi)|^2 p_\varphi(\xi). \tag{6.71}$$

Therefore, we obtain the following characterization of the FO principal shift-invariant spaces. If $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$ is such that $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$, then

$$\langle D^{-1}\varphi\rangle = \langle\varphi\rangle \Leftrightarrow \langle\varphi\rangle \text{ is an FO principal shift-invariant space.} \tag{6.72}$$

Observe that this approach also provides a complete solution of the problem (6.61).

(ii) Observe that for φ_1 in the previous theorem, or for $D^{-1}\varphi$ (which is essentially the same) we have $\widehat{\varphi_1}(2\xi) = m(2\xi)\widehat{\varphi_1}(\xi)$, and, since $\xi \mapsto m(2\xi)$ is $\frac{1}{2}$-periodic, it can never satisfy the Smith–Barnwell condition. This is the main reason why the theory developed in [**PŠWX01**], [**PŠWX03**], and [**ŠSW08**] failed to detect such examples.

(iii) If $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$ is such that $\langle\varphi\rangle$ is an FO principal shift-invariant space, then observe that, for every $n \in \mathbb{N}$ we have

$$\langle D^{-n}\varphi\rangle = \langle D^{-(n-1)}\varphi\rangle = \ldots = \langle D^{-1}\varphi\rangle = \langle\varphi\rangle, \tag{6.73}$$

i.e. $\widetilde{\varphi} := D^{-n}\varphi$ satisfies $\langle D^n\widetilde{\varphi}\rangle = \langle\widetilde{\varphi}\rangle$. Observe also that, for almost every $\xi \in \mathbb{R}$,

$$(\widetilde{\varphi})^\wedge(2\xi) = m(2^n\xi)(\widetilde{\varphi})^\wedge(\xi); \tag{6.74}$$

this is the "even ray" part of the sequence described in Remark 5.11iv.

■

In general, we have $\langle\varphi\rangle$ positioned within $D(\langle\varphi\rangle)$ so that it intersects with both $\langle D\varphi\rangle$ and $\langle DT\varphi\rangle$. By taking $\varphi_1 := D^{-1}\varphi$, we obtain one "extreme" case, i.e. $\langle\varphi_1\rangle = \langle D\varphi_1\rangle$. Observe that it is not difficult to get the "other extreme". Consider $\varphi_2 := T_{-1}D^{-1}\varphi$. Obviously, in the FO case we obtain

$$\langle\varphi_2\rangle = \langle\varphi_1\rangle = \langle D\varphi_1\rangle = \langle\varphi\rangle = \langle DT\varphi_2\rangle. \tag{6.75}$$

7. Pre-GMRA

The notion of a generalized multi-resolution analysis (GMRA, for short) has been introduced by L.W. Baggett, H.A. Medina, and K.D. Merrill in [**BMM99**]. It has been studied extensively by many authors; among early references, we mention [**BM99**], [**BR05**], [**BR03**], [**BRS01**], [**BL98**], and [**Pap00**] since their discussions are similar in approach to what we present in this section. Consider a slightly more general structure; we shall say that a family $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ is a *pre-GMRA* if

$$V_j \leq V_{j+1} = D(V_j), \text{ for every } j \in \mathbb{Z} \tag{7.1}$$

and

$$V_0 \text{ is a shift-invariant space.} \tag{7.2}$$

As usual, we refer to V_0 as the *core space* of the family $\{V_j : j \in \mathbb{Z}\}$. Obviously, every shift-invariant space V which is D^{-1}-invariant (or, equivalently, refineable), i.e.

$$V \leq D(V) \tag{7.3}$$

serves as the core space for some pre-GMRA. For every pre-GMRA $\{V_j : j \in \mathbb{Z}\}$, it is natural to consider the following three closed subspaces of $L^2(\mathbb{R})$:

$$V_{-\infty} := \bigcap_{j\in\mathbb{Z}} V_j, \tag{7.4}$$

$$V_\infty := \overline{\bigcup_{j\in\mathbb{Z}} V_j}, \tag{7.5}$$

and

$$W_0 := V_1 \cap V_0^\perp. \tag{7.6}$$

Observe that W_0 and V_∞ are always shift-invariant spaces, while $V_{-\infty}$ may or may not be a shift-invariant space. Let us first recall some standard facts about W_0. It is well-known (consult, for example, [**BR03**] for historical details) that, for almost every $\xi \in \mathbb{R}$,

$$\dim_{V_0}(\xi) + \dim_{W_0}(\xi) = \dim_{V_1}(\xi) = \dim_{V_0}(\xi/2) + \dim_{V_0}(\xi/2 + 1/2), \tag{7.7}$$

and

$$\sigma_{V_0}(\xi) + \sigma_{W_0}(\xi) = \sigma_{V_1}(\xi) = \sigma_{V_0}(\xi/2). \tag{7.8}$$

It is actually possible for W_0 to be an infinitely generated shift-invariant space. Consider the following list of examples.

EXAMPLE 7.9. (i) Consider, for every $j \in \mathbb{Z}$, the space (recall Example 1.1.5)

$$V_j := L^2((-\infty, 2^j])^\vee.$$

It is not difficult to check that $\{V_j : j \in \mathbb{Z}\}$ is a pre-GMRA with the following properties:

$V_0 \neq V_1$ and $\dim_{W_0} \equiv 1$;
V_j is a principal shift-invariant space for every $j \in \mathbb{Z}$;
$V_{-\infty} = L^2((-\infty, 0])^\vee, V_{-\infty}$ is a principal shift-invariant space;
$\dim_{V_{-\infty}} \equiv \infty$;
$V_\infty = L^2(\mathbb{R})^\vee = L^2(\mathbb{R})$.

(ii) Consider $\varepsilon \in (0, 1/6)$ and the set $E \subseteq \mathbb{R}$ defined by

$$E := (-\varepsilon, 0] \cup \left(\bigcup_{n=0}^{\infty} (2^n, 2^n + \varepsilon] \right) \cup (0, 1/2 + \varepsilon].$$

Take $V_0 := L^2(E)^\vee$. Then it is not too difficult to check that V_0 satisfies (7.3) and the corresponding pre-GMRA has the following properties:

$V_0 \neq V_1$ and $|\{\xi : \dim_{W_0}(\xi) = \infty\}| > 0$;

$$\dim_{V_0}(\xi) = \begin{cases} 1 & \text{if } \xi \in (-\varepsilon, 0] \\ \infty & \text{if } \xi \in (0, \varepsilon] \\ 1 & \text{if } \xi \in (\varepsilon, 1/2 + \varepsilon] \\ 0 & \text{if } \xi \in (1/2 + \varepsilon, 1 - \varepsilon] \end{cases};$$

$V_{-\infty} = \{0\}$;
$V_\infty = L^2(\mathbb{R})^\vee = L^2(\mathbb{R})$.

The last two properties show that this is, in fact, a GMRA. Observe also that in this case, V_j is a shift-invariant space for every $j \in \mathbb{Z}$. This example is a slight change from Example 3.7 in [**BR03**].

(iii) Consider Example 3.1 in [**Bow09**]. One can check that M. Bownik constructed there an elaborate example of a pre-GMRA with the following properties:

$\dim_{V_0} \equiv \infty$;
$V_{-\infty} \neq \{0\}$
$V_{-\infty}$ is not a shift-invariant space;

indeed, if it were, one could check that $\sigma_{V_{-\infty}}$ would be identically 0, contradicting the non-triviality of $V_{-\infty}$. It follows that it is not possible for all the V_j spaces to be shift-invariant.

$\diamondsuit$

Using (7.7) and applying Lemma 2.1 directly to the set $I_{V_0} = \{\xi \in \mathbb{R} : \dim_{V_0}(\xi) = \infty\}$, we observe that "mixed quantities" as in Example 7.9ii are not possible if $\dim_{W_0}$ is almost everywhere finite.

PROPOSITION 7.10. *If* $\{V_j : j \in \mathbb{Z}\}$ *is a pre-GMRA such that*

$$\dim_{W_0} < \infty \text{ almost everywhere,}$$

then either $\dim_{V_0} \equiv \infty$ *or* $\dim_{V_0} < \infty$ *almost everywhere.*

Observe that there are examples of pre-GMRAs for which $W_0 = \{0\}$, which is equivalent to

$$D(V_0) = V_0, \tag{7.11}$$

which is equivalent to

$$V_{-\infty} = V_0 = V_j = V_\infty \text{ for every } j \in \mathbb{Z}. \tag{7.12}$$

EXAMPLE 7.13. Take any measurable subset $A \subseteq I$ and define $E := \bigcup_{k\in\mathbb{Z}} 2^k A$. Consider $V_0 := L^2(E)^\vee$. This is a typical example of a shift-invariant space which satisfies (7.11). If $|A| = 0$, then $V_0 = \{0\}$. If $|A| = 1$, then $V_0 = L^2(\mathbb{R})$. If $0 < |A| < 1$, then we have a non-trivial V_0 which satisfies (7.11).

By Proposition 7.10, every $V_0 \neq \{0\}$ which satisfies (7.11) must have that $\dim_{V_0} \equiv \infty$. Observe that $V := L^2((1,\infty))^\vee$ is an example of a shift-invariant space such that $\dim_V \equiv \infty$, $D(V) \leq V$, and $D(V) \neq V$.

$\diamondsuit$

Consider V_∞. It is always a shift-invariant space, and since D^i is continuous for both $i = \pm 1$, we obtain

$$D^i(V_\infty) \subseteq \overline{D^i\left(\bigcup_{j\in\mathbb{Z}} V_j\right)} = \overline{\bigcup_{j\in\mathbb{Z}} D^i(V_j)} = \overline{\bigcup_{j\in\mathbb{Z}} V_{j+i}} = V_\infty.$$

Hence $D(V_\infty) = V_\infty$.

PROPOSITION 7.14. *If* $\{V_j : j \in \mathbb{Z}\}$ *is a pre-GMRA, then* V_∞ *is a shift-invariant space and* $D(V_\infty) = V_\infty$.

We have seen examples of pre-GMRAs for which $V_\infty \neq L^2(\mathbb{R})$ in a previous section (recall Remark 6.30).

The issues surrounding $V_{-\infty}$ are more subtle. As we have seen, $V_{-\infty}$ may or may not be a shift-invariant space. Furthermore, if $\dim_{V_0} \equiv \infty$, then it is possible to have either $V_{-\infty} = \{0\}$ or $V_{-\infty} \neq \{0\}$. As far as we know, the strongest result which appears in the literature is due to M. Bownik in [**Bow09**]. Using our terminology, the Bownik theorem says that if $\{V_j : j \in \mathbb{Z}\}$ is a pre-GMRA such that

$$|\{\xi \in \mathbb{R} : \dim_{V_0}(\xi) < \infty\}| > 0, \tag{7.15}$$

then $V_{-\infty} = \{0\}$. Even though $V_{-\infty}$ may not be a shift-invariant space, we can still apply D^i, $i = \pm 1$, to it and obtain

$$D^i(V_{-\infty}) \subseteq \bigcap_{j\in\mathbb{Z}} D^i(V_j) = \bigcap_{j\in\mathbb{Z}} V_{j+i} = V_{-\infty};$$

i.e. $V_{-\infty}$ satisfies $D(V_{-\infty}) = V_{-\infty}$.

GMRA terminology has been fairly standardized at present; see, for example, [**Bow09**]. In our set up, we have that $\{V_j : j \in \mathbb{Z}\}$ is a GMRA if and only if it is a pre-GMRA and $V_{-\infty} = \{0\}$, $V_\infty = L^2(\mathbb{R})$. If, in addition to being a GMRA, we have $\dim_{V_0} \equiv 1$, then $\{V_j : j \in \mathbb{Z}\}$ is the classical MRA structure of an orthonormal wavelet (see, for example, [**HW96**] for details and standard terminology). Indeed, $\dim_{V_0} \equiv 1$ means that there exists $\varphi \in L^2(|R)$ with $p_\varphi > 0$ almost everywhere and $V_0 = \langle\varphi\rangle$. By taking $\varphi_0 := \frac{1}{\sqrt{p_\varphi}} \bullet \varphi$, we obtain an orthonormal basis $\mathcal{B}_{\varphi_0}$ for V_0; the standard nomenclature is that φ_0 is the *scaling function* of the MRA $\{V_j : j \in \mathbb{Z}\}$. Let us also mention that $\dim_{V_0}$ has been completely characterized for both pre-GMRA and GMRA in [**BR03**].

REMARK 7.16. One should be somewhat careful applying characterization theorems. For example, if $\dim_{V_0} \equiv \infty$, then such a function will satisfy all the characterization properties for both pre-GMRA and GMRA. That does not mean that a particular V_0 will give us the desired structure — it only means that there exists a shift-invariant space U_0 with the same dimension function which generates the desired structure. For example, taking $V_0 := L^2([0,\infty))^\vee$ satisfies (7.11), so it generates a pre-GMRA with $\{0\} \neq V_{-\infty} = V_0 = V_\infty \neq L^2(\mathbb{R})$. Its dimension function is identically ∞, so it satisfies all the characterization results. We can conclude the following, though, based on the results from [**BR03**] and [**Bow09**].

If $\{V_j : j \in \mathbb{Z}\}$ is a pre-GMRA which satisfies (7.15), then $V_{-\infty} = \{0\}$. If $\{V_j : j \in \mathbb{Z}\}$ is a pre-GMRA such that $V_{-\infty} = \{0\}$ we may or may not have (7.15) (see, for example, the construction in the proof of Lemma 3.3 in [**BR03**]).

If $\{V_j : j \in \mathbb{Z}\}$ is a GMRA (in particular, $V_\infty = L^2(\mathbb{R})$), then (see [**BR03**]), for almost every $\xi \in \mathbb{R}$,

$$\liminf_{n\to\infty} \dim_{V_0}(\xi/2^n) \geq 1. \tag{7.17}$$

If $V_j : j \in \mathbb{Z}\}$ is a pre-GMRA which satisfies (7.17), then V_∞ may or may not be $L^2(\mathbb{R})$ (see our example above). However, we could, in this situation, employ the following result, which is essentially in [**BR03**], too. If $\{V_j : j \in \mathbb{Z}\}$ is a pre-GMRA such that, for almost every $\xi \in \mathbb{R}$,

$$\lim_{n\to\infty} \sigma_{V_0}(\xi/2^n) = 1, \tag{7.18}$$

then $V_\infty = L^2(\mathbb{R})$. ■

The class of all pre-GMRAs splits according to the following property:

$$V_j \text{ is a shift-invariant space for every } j \in \mathbb{Z}. \tag{7.19}$$

There are non-trivial examples where (7.19) holds (see Example 7.9i and ii) and non-trivial examples where it does not hold (see Example 7.9iii and all examples in the previous section, recalling Remark 6.30, in particular). Let us first consider the former class. The following result could be considered "folklore", but since we are not aware of a precise reference, we provide details for our readers' convenience.

THEOREM 7.20. *If $\{V_j : j \in \mathbb{Z}\}$ is a pre-GMRA which satisfies (7.19), then $V_{-\infty}$ is a shift-invariant space which satisfies (7.11), and the family $\{U_j : j \in \mathbb{Z}\}$, where $U_j := V_j \cap V_{-\infty}^{\perp}$ for $j \in \mathbb{Z}$, is a pre-GMRA which satisfies (7.19) and $U_{-\infty} = \{0\}$. Furthermore, $V_\infty = V_{-\infty} \oplus U_\infty$ and either $V_{-\infty} = \{0\}$ or $\dim_{V_{-\infty}} \equiv \infty$.*

PROOF. It is obvious that (7.19) implies that $V_{-\infty}$ is a shift-invariant space. We have seen already that $V_{-\infty}$ satisfies (7.11). By (7.7), (7.8), and Proposition 7.10, it now follows easily that either $V_{-\infty} = \{0\}$ or $\dim_{V_{-\infty}} \equiv \infty$.

By its definition, every U_j is a shift-invariant space and, for every $j \in \mathbb{Z}$, $V_j = V_{-\infty} \oplus U_j$. Since D is unitary, we obtain

$$\begin{aligned} V_{-\infty} \oplus U_{j+1} &= V_{j+1} \\ &= D(V_j) \\ &= D(V_{-\infty} \oplus U_j) \\ &= D(V_{-\infty}) \oplus D(U_j) \\ &= V_{-\infty} \oplus D(U_j), \end{aligned}$$

which guarantees $U_{j+1} = D(U_j)$. Since $V_{j+1} \geq V_j$, we obtain $U_{j+1} \geq U_j$, so $\{U_j : j \in \mathbb{Z}\}$ is a pre-GMRA which satisfies (7.19). It follows that $U_{-\infty}$ is a shift-invariant space and $V_{-\infty} \oplus U_{-\infty} \leq V_{-\infty}$, hence $U_{-\infty} = \{0\}$. Since $V_{-\infty}$ is a closed space, we obtain

$$V_\infty = \overline{\bigcup_{j\in\mathbb{Z}} V_{-\infty} \oplus U_j} = \overline{V_{-\infty} \oplus \bigcup_{j\in\mathbb{Z}} U_j} = \overline{V_{-\infty}} \oplus \overline{\bigcup_{j\in\mathbb{Z}} U_j} = V_{-\infty} \oplus U_\infty.$$

$\square$

At this point we are interested in the class of pre-GMRAs which do not satisfy (7.19). Since, for a shift-invariant space V we always have that $D(V)$ is a shift-invariant space also, we conclude that, for a pre-GMRA $\{V_j : j \in \mathbb{Z}\}$ which does not satisfy (7.19) there exists exactly one $\ell \in \mathbb{Z}$ such that V_j is a shift-invariant space if and only if $j \geq \ell$. Observe that it must be the case that $\ell \leq 0$. We shall say that the space V_ℓ is the *natural core space* of $\{V_j : j \in \mathbb{Z}\}$. Observe that in all examples described in Remark 6.30 the natural core is identical to the "ordinary" core V_0.

A natural question for us is whether an "FO type wavelet" (described in the previous section) can be developed in some way different than the one described in Theorem 6.57, i.e. from some other pre-GMRA structure. Because an "FO type wavelet" (see (6.46)) satisfies $\widehat{\psi_0}(\xi) \neq 0$ for almost every $\xi \in \mathbb{R}$, our question is really in the spirit of the Lemarié theorem (see [**LR92**]) which shows that all compactly supported wavelets (in the orthonormal case) arise from an MRA structure. For the development of such a line of theorems, consult [**Aus95**] and, ultimately, [**BR05**]. We have the following result which is closely related to the corresponding theorem in [**BR05**].

THEOREM 7.21. *Let $\psi \in L^2(\mathbb{R})$ be such that, for almost every $\xi \in \mathbb{R}$, $\widehat{\psi}(\xi) \neq 0$. If $\{V_j : j \in \mathbb{Z}\}$ is a pre-GMRA such that*
(a) $V_1 = \langle\psi\rangle \oplus V_0$, and
(b) $|\{\xi \in \mathbb{R} : \dim_{V_0}(\xi) < \infty\}| > 0$,
then V_0 is an FO principal shift-invariant space and $\{V_j : j \in \mathbb{Z}\}$ is the MRA structure associated with an orthonormal wavelet $\psi_0 := \frac{1}{\sqrt{p_\psi}} \bullet \psi$.

We will prove this theorem shortly. First, we observe that, regarding the proof of the theorem, one can attack from several directions. One way is to show that V_0 is spanned by the negative dilates of ψ_0 and then apply powerful theorems from [**BR05**] and [**Bow09**]. However, in this case one can produce a significantly more elementary proof, which we will do here. First, a lemma.

LEMMA 7.22. *If V is a shift-invariant space such that*
(a) $|\{\xi \in \mathbb{R} : \dim_V(\xi) < \infty\}| > 0$*;*
(b) For almost every $\xi \in \mathbb{R}$, $\dim_V(\xi) > 0$; and
(c) There exists a 1-periodic, measurable set $U \subseteq \mathbb{R}$ such that, for almost every $\xi \in \mathbb{R}$,

$$\dim_V(2\xi) + \chi_U(2\xi) = \dim_V(\xi) + \dim_V(\xi + 1/2),$$

then $\dim_V \equiv 1$ almost everywhere (in particular, $U = \mathbb{R}$ almost everywhere).

PROOF OF LEMMA 7.22. Consider I_V first. It is a 1-periodic, measurable subset of $\mathbb{R}$. If $u \in 2I_V$, then $u = 2\xi$ with $\dim_V(\xi) = \infty$. By (c), we obtain $\dim_V(u) = \infty$ as well. Hence $2I_V \subseteq I_V$ and, by Lemma 2.1, we have either $|I_V| = 0$ or $I_V = \mathbb{R}$ almost everywhere. By (a), we must have $|I_V| = 0$, i.e.

$$\dim_V(\xi) < \infty \text{ for almost every } \xi \in \mathbb{R}. \tag{7.23}$$

Observe that (b) actually means that

$$\dim_V(\xi) \geq 1 \text{ for almost every } \xi \in \mathbb{R}. \tag{7.24}$$

For $n \in \mathbb{N}$, let us denote $U_n := \{\xi \in \mathbb{R} : \dim_V(\xi) \geq n\}$. Hence every U_n is a measurable, 1-periodic subset of $\mathbb{R}$. By (7.24), $U_1 = \mathbb{R}$ almost everywhere.

We claim that $|\mathbb{R} \setminus U_2| > 0$. Suppose to the contrary that $U_2 = \mathbb{R}$ almost everywhere. For almost every $\xi \in \mathbb{R}$, we obtain that

$$\infty > \dim_V(2\xi) + \chi_U(2\xi) = \dim_V(\xi) + \dim_V(\xi + 1/2) \geq \dim_V(\xi) + 2.$$

Hence for almost every ξ and for every $n \in \mathbb{N}$ we obtain

$$\infty > \dim_V(\xi) \geq \dim_V(\xi/2) + 1 \geq \dim_V(\xi/4) + 2 \geq ... \geq \dim_V(\xi/2^{n-1}) + n - 1 \geq n,$$

which clearly produces a contradiction. Hence

$$|\mathbb{R} \setminus U_2| > 0. \tag{7.25}$$

Consider now $u \in U_2$, i.e. $u = 2\xi$ and $\dim_V(\xi) \geq 2$. By (c), we obtain

$$\dim_V(u) + \chi_U(u) = \dim_V(\xi) + \dim_V(\xi + 1/2) \geq 2_1.$$

Since this implies $\dim_V(u) \geq 2$, i.e. $u \in U_2$, we can employ Lemma 2.1 again to show that either $|U_2| = 0$ or $U_2 = \mathbb{R}$. By (7.25) we must have that $|U_2| = 0$, i.e. $\dim_V \equiv 1$ almost everywhere. Notice that (c) is then only possible with $U = \mathbb{R}$ almost everywhere. $\square$

Notice that if we have the stronger assumption that $\dim_V$ is integrable on $[0, 1]$, then integrating (c) over $[0, 1]$ and using (b) provides the same conclusion even more directly.

PROOF. Proof of Theorem 7.21 Since $\widehat{\psi}(\xi) \neq 0$ almost everywhere, we obtain $p_\psi > 0$ almost everywhere. Hence the definition of ψ_0 makes sense and $\mathcal{B}_{\psi_0}$ is an orthonormal basis for $\langle \psi_0 \rangle = \langle \psi \rangle$. Furthermore, for almost every $\xi \in \mathbb{R}$,

$$\sigma_{\langle \psi \rangle}(\xi) = |\widehat{\psi_0}(\xi)|^2 > 0. \tag{7.26}$$

By (7.8) and (a), we obtain, for almost every $\xi \in \mathbb{R}$, $\sigma_{V_0}(\xi) = \sigma_{V_0}(2\xi) + \sigma_{\langle\psi\rangle}(2\xi) > 0$. Hence

$$\dim_{V_0} > 0 \text{ almost everywhere.}$$

Observe that $\dim_{\langle\psi\rangle} \equiv 1$, so by (7.7) we obtain, for almost every $\xi \in \mathbb{R}$,

$$\dim_{V_0}(2\xi) + 1 = \dim_{V_0}(\xi) + \dim_{V_0}(\xi + 1/2).$$

Therefore, V_0 satisfies the conditions of Lemma 7.22, and we conclude that $\dim_{V_0} \equiv 1$ almost everywhere. It follows that there exists $\varphi_0 \in L^2(\mathbb{R})$ such that $V_0 = \langle\varphi_0\rangle$ and $\mathcal{B}_{\varphi_0}$ is an orthonormal basis for V_0. Since $\sigma_{V_0} > 0$ almost everywhere, it follows that $\widehat{\varphi_0}(\xi) \neq 0$ for almost every $\xi \in \mathbb{R}$. Since $\langle\varphi_0\rangle \leq D(\langle\varphi_0\rangle) = \langle\varphi_0\rangle \oplus \langle\psi_0\rangle$, there exists a filter m_0 such that $\widehat{\varphi_0}(2\xi) = m_0(\xi)\widehat{\varphi_0}(\xi)$ and we must have $m_0(\xi) \neq 0$ for almost every $\xi \in \mathbb{R}$. It follows that $\langle\varphi_0\rangle$ is an FO with ssupp $\widehat{\varphi_0} =$ ssupp $\widehat{\psi_0} = \mathbb{R}$. Hence by Theorem 6.57 we conclude that $\{V_j : j \in \mathbb{Z}\}$ is an MRA structure for an orthonormal wavelet ψ_1 defined by (6.44) from φ_0. It follows that $\langle\psi_0\rangle = \langle\psi_1\rangle$, which implies $\psi_1 = \mu \bullet \psi_0$ for some 1-periodic, measurable function μ with $\mu \neq 0$ almost everywhere. Since $p_{\psi_0} \equiv 1 \equiv p_{\psi_1}$ almost everywhere, it follows that $|\mu|^2 = 1$ almost everywhere. Therefore, ψ_0 is an orthonormal wavelet with the same MRA structure as ψ_1. $\square$

We shall return to the ideas related to questions about GMRA and pre-GMRA structures later. For now, we turn our attention to the non-FO case in the next section.

8. Filter Analysis, Non-FO Case

In this section we consider $M : \mathbb{R} \to [0, \infty)$ which is measurable and 1-periodic such that (see (2.40))

$$|Z_M| > 0. \tag{8.1}$$

For such an M, almost every orbit orb(ξ) either intersects the horizon of M, i.e. H_M or the set $A_M^{(1)}$ (see Remark 2.42). If $|H_M| = 0$, then Sol_M is trivial (again, see Remark 2.42). Hence we also assume that

$$|H_M| > 0. \tag{8.2}$$

Let us begin by exploring some simple relationships among $Z_M, H_M, A_M^{(1)}$. It is obvious that the sets

$$A_M^{(1)} \text{ and } \{\xi \in I : \mathrm{orb}(\xi) \cap H_M \neq \emptyset\} \tag{8.3}$$

form an almost everywhere-partition of I. Furthermore,

$$A_M^{(1)} = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} \left(\frac{1}{2^k} I \cap Z_M\right) \tag{8.4}$$

Hence from (8.3) and (8.4), we obtain

$$\{\xi \in I : \mathrm{orb}(\xi) \cap H_M \neq \emptyset\} = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} \left(\frac{1}{2^k} I \cap Z_M\right)^c. \tag{8.5}$$

As mentioned in (2.47), we have $H_M \subseteq Z_M$, but it is also easy to see that Z_M determines H_M completely:

$$H_M^c \cap Z_M = \bigcup_{n=1}^{\infty} (Z_M \cap 2^n Z_M) = \bigcup_{n=1}^{\infty} 2^n (Z_M \cap \frac{1}{2^n} Z_M). \tag{8.6}$$

It is natural to ask whether H_M determines Z_M completely. As the following example shows, this is not the case.

EXAMPLE 8.7. Take any $M : \mathbb{R} \to [0, \infty)$, measurable and 1-periodic such that

$$M(\xi) > 0 \text{ if } \xi \in [-1/8, 1/8) \text{ and}$$
$$M(\xi) = 0 \text{ if } \xi \in [-1/4, 1/4) \setminus [-1/8, 1/8);$$

observe that we can define M freely on $[-1/2, 1/2) \setminus [-1/4, 1/4)$, i.e. we can add any set of zero points to M there. However, every such M will have the property that

$$H_M = [-1/4, 1/4) \setminus [-1/8, 1/8)$$

and $|A_M^{(1)}| = 0$. ♢

Let us add a few more remarks and examples in order to understand better the notion of the horizon of M. The following examples illustrate that H_M may or may not be either bounded or bounded away from 0.

EXAMPLE 8.8. (i) Consider $M = |m|^2$, where m is the low-pass filter for the Shannon wavelet (see, for example, [**HW96**] for details). This means that M is 1-periodic and $M|_{[-1/2,1/2)} = \chi_{[-1/4,1/4)}$. It is obvious, then, that $H_M = \frac{1}{2} I \subseteq [-1/2, 1/2)$. Observe also that $\Phi = \chi_{[-1/2,1/2)} = 2\Phi_{0,M}$ (see Theorem 2.52) prevents an example of a solution which is maximal in $(\mathrm{Sol}_M, \prec_M)$.

(ii) We introduce the following notation: for each $n \in \mathbb{N}$,

$$A_0 := [\frac{3}{4}, 1)$$
$$A_n := \left[\frac{2^{n+1}+1}{2^{n+2}}, \frac{2^{n+1}+1}{2^{n+1}}\right)$$
$$B_n := \left[\frac{2^{n+1}-1}{2^{n+1}}, \frac{2^{n+2}-1}{2^{n+2}}\right).$$

Observe that $\{A_n : n \in \mathbb{N} \cup \{0\}\}$ forms a partition of $[1/2, 1)$. Define M to be 1-periodic and such that $M|_{[0,1)}$ is given by

$$M(\xi) = \begin{cases} 1 & \text{if } \xi \in A_0 \\ 0 & \text{if } \xi \in [1/2, 3/4) \\ 1 & \text{if } \xi \in 2^{-j} A_n \text{ and } n < j, n \in \mathbb{N} \cup \{0\}, j \in \mathbb{N} \\ 0 & \text{if } \xi \in 2^{-j} A_n \text{ and } n \geq j, n \in \mathbb{N} \cup \{0\}, j \in \mathbb{N}. \end{cases}$$

It is a simple exercise to check that $|A_M^{(1)}| = 0$ and

$$H_M = [-1/2, -1/4) \cup \left(\bigcup_{j \in \mathbb{N}} 2^{-j} A_j\right) \cup \left(\bigcup_{k \in \mathbb{N}} 2^k B_k\right).$$

Hence $H_M \cap (0, \infty)$ is neither bounded nor bounded away from 0.

♢

Is it possible to have $H_M \cap [-1/2, 1/2) = \emptyset$? Yes, but only at the expense of having non-trivial $A_M^{(1)}$.

EXAMPLE 8.9. Take a 1-periodic function M such that $M|_{[-1/2,1/2)}$ equals $\chi_{[0,1/2)}$. Then we have $A_M^{(1)} = [-1, -1/2)$ and $H_M = [1/2, 1)$. $\diamondsuit$

LEMMA 8.10. *If $M : \mathbb{R} \to [0, \infty)$ is measurable, 1-periodic, and such that $|Z_M| > 0$ with $|A_M^{(1)}| = 0$, then*

$$|H_M \cap [-1/2, 1/2)| > 0.$$

PROOF. Suppose to the contrary that $|H_M \cap [-1/2, 1/2)| = 0$. By our assumptions, for almost every $\xi \in \mathbb{R}$, $\mathrm{orb}(\xi) \cap H_M = \{\nu\}$ and $\nu = \nu(\xi) \in \mathbb{R} \setminus [-1/2, 1/2)$. Hence for every such ν and for every $j \in \mathbb{N}$, we have that $M(2^{-j}\nu) > 0$. It follows that $M > 0$ almost everywhere, contrary to the assumption that $|Z_M| > 0$. $\square$

Using essentially the same argument, we also have the following result:

COROLLARY 8.11. *Let $M : \mathbb{R} \to [0, \infty)$ be measurable, 1-periodic, and such that $|Z_M| > 0$ and $|A_M^{(1)}| = 0$. If there exists $0 < \varepsilon < 1/2$ such that $|H_M \cap [-\varepsilon, \varepsilon)| = 0$, then $|Z_M \cap [-\varepsilon, \varepsilon)| = 0$.*

We turn our attention now to the elements of Sol_M. Observe that the analysis of the Tauberian function T_M, in this non-FO case, is similar to the analysis of $T_{M,-}$ (the "low frequency" part) in the FO case. Hence there is no effect of "high frequencies" in the sense that for every $\xi \in H_M$ we can treat $\mathrm{orb}(\xi)$ "individually".

There are several consequences in the non-FO case. If M is non-FO and $\Phi \in \mathrm{Sol}_M$, we cannot expect Φ to have full support; more precisely, we must have

$$\text{ssupp } \Phi \subseteq \bigcup_{n=0}^{\infty} 2^{-n} H_M. \tag{8.12}$$

Maximal solutions in $(\mathrm{Sol}_M, \prec_M)$ have the property that there is an equality in (8.12). Despite that, it is possible to have an almost everywhere strictly positive periodization of Φ (take Example 8.8i, for instance). As before, we abuse notation slightly, and for $\Phi \in \mathrm{Sol}_M$, we denote its periodization by p_Φ, i.e.

$$p_\Phi(\xi) = \sum_{k \in \mathbb{Z}} \Phi(\xi + k), \xi \in \mathbb{R}. \tag{8.13}$$

Observe that for almost every $\xi \in \mathbb{R}$,

$$p_\Phi(2\xi) = M(\xi) p_\Phi(\xi) + M(\xi + 1/2) p_\Phi(\xi + 1/2). \tag{8.14}$$

The lemma below follows directly:

LEMMA 8.15. *If $M : \mathbb{R} \to [0, \infty)$ is measurable, 1-periodic, with $|Z_M \cap (1/2 + Z_M)| > 0$, then for every $\Phi \in Sol_M$,*

$$|\{\xi \in \mathbb{R} : p_\Phi(\xi) = 0\}| > 0.$$

In particular, this holds if $|Z_M \cap [0, 1)| > 1/2$.

Observe that Example 8.8i has the properties that $p_\Phi > 0$ almost everywhere and $|Z_M \cap [0, 1)| = 1/2$. Obviously, the condition in the previous lemma is not a necessary one. For example, even a "small" set in Z_M which contains a neighborhood of 0 will make everything trivial. More precisely, if there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq Z_M$, then

$$A_M^{(1)} = I \text{ and } \mathrm{Sol}_M = \{0\}. \tag{8.16}$$

Regarding the existence of non-trivial solutions, we follow an approach similar to (3.3). We define a function $r_M : H_M \to [0, \infty]$ by considering the power series

$$z \mapsto \sum_{j=1}^{\infty} z^j \left(\prod_{k=1}^{j} \frac{1}{M(2^{-k}\xi)} \right), \tag{8.17}$$

for every $\xi \in H_M$ (observe that the definition makes sense and that r_M is analogous to $r_{M,-}$ in (3.3)); then $r_M(\xi)$ is defined as the radius of convergence of the power series given in (8.17). Exactly as in Section 3, we obtain the following results. If $M : \mathbb{R} \to [0, \infty)$ is measurable, 1-periodic, and satisfies (8.1) and (8.2), then, for almost every $\xi \in H_M$,

$$\begin{aligned} \liminf_{n\to\infty} M(2^{-n}\xi) > 1/2 &\Rightarrow \liminf_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} M(2^{-k}\xi)} > 1/2 \\ &\Rightarrow T_M(\xi) < \infty \\ &\Rightarrow \liminf_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} M(2^{-k}\xi)} \geq 1/2 \end{aligned} \tag{8.18}$$

Using Theorem 2.52 and (8.18), we obtain directly a result which is analogous to Corollary 3.54.

COROLLARY 8.19. *Let $M : \mathbb{R} \to [0, \infty)$ be measurable, 1-periodic, with $|Z_M| > 0$ and $|H_M| > 0$. If, for almost every $\xi \in H_M$, $\liminf_{n\to\infty} M(2^{-n}\xi) > 1/2$, then, for almost every $\xi \in H_M$, $T_M(\xi) < \infty$ and $\Phi_{0,M}(2^{-j}\xi) > 0$ for every $j \in \mathbb{N} \bigcup \{0\}$. If $\liminf_{n\to\infty} M(2^{-n}\xi) > 1/2$, for almost every ξ, then, in addition, $|A_M^{(1)}| = 0$.*

REMARK 8.20. Observe that in the non-FO case, large values of M are not an obstacle to the existence of non-trivial solutions (unlike in the FO case). For example, if there exists $a > 1/2$ such that for almost every $\xi \in H_M$, and for every $k \in \mathbb{N}$, $M(2^{-k}\xi) \geq a$, then, for almost every $\xi \in H_M$,

$$T_M(\xi) \leq \frac{1}{2a - 1} \tag{8.21}$$

Observe that, due to the lack of $T_{M,+}$ in the non-FO case, T_M does not exhibit fractal properties either. By adjusting M on a part of an orbit $\mathrm{orb}(\xi)$ near 0, we completely determine $T_M(\xi)$, and this has no effect on other orbits.

Observe that we still have formula (3.7), i.e. for almost every $\xi \in H_M$, and, for every $n \in \mathbb{N}$,

$$\Phi(\xi) = \left(\prod_{k=1}^{n} M(2^{-k}\xi) \right) \Phi(2^{-n}\xi) \text{ and } \prod_{k=1}^{n} M(2^{-k}\xi) > 0, \tag{8.22}$$

whenever $\Phi \in \mathrm{Sol}_M$. Hence, a statement analogous to Lemma 3.10 holds (we use $\xi \in H_M$ such that $\Phi(\xi) > 0$ instead of $\xi \in A_M^{(2)}$, as was used in Lemma 3.10).

Finally, let us observe that in the non-FO case there is no ergodic behavior either. It also means that the Smith–Barnwell type condition does not play the same role in the FO case. Let us illuminate this in the following simple example. Take an $a \in \mathbb{R}$ with $a \geq 0$, and define $M_a : \mathbb{R} \to [0, \infty)$ to be 1-periodic and such that $M_a|_{[-1/2,1/2)}$ is given by

$$M_a(\xi) := \begin{cases} 1 & \text{if } \xi \in [-1/8, 1/8) \\ 0 & \text{if } \xi \in [-1/4, -1/8) \cup [1/8, 1/4) \\ a & \text{if } \xi \in [-1/2, -3/8) \cup [3/8, 1/2) \\ a+1 & \text{if } \xi \in [-3/8, -1/4) \cup [1/4, 3/8). \end{cases} \tag{8.23}$$

Observe that, for every $a \geq 0$, $\Phi = \chi_{[-1/4,1/4)}$ is a maximal solution in Sol_{M_a} (obviously, Φ does not depend on a). On the other hand, for almost every $\xi \in \mathbb{R}$,

$$M_a(\xi) + M_a(\xi + 1/2) = 1 + a. \tag{8.24}$$

Despite this, one should not completely disregard the Smith–Barnwell condition in the non-FO case, either. For example, for $0 < b \leq 1/2$, the condition

$$M(\xi) + M(\xi + 1/2) = b \text{ almost everywhere}$$

would imply $\mathrm{Sol}_M = \{0\}$; but this would simply be a consequence of $M \leq 1/2$. However, there are other issues, like zero-sets and Parseval frame representation which bring the Smith–Barnwell condition into focus. ■

Let us observe first that some zeroes are unavoidable and play an essential role, while the others can be replaced with positive values.

EXAMPLE 8.25. Consider a 1-periodic M such that $M|_{[-1/4,1/8)} \equiv 1$ and $M|_{[-1/2,-1/4) \cup [1/8,1/4)} \equiv 0$. It follows that the maximal solution is $\Phi = \chi_{[-1/2,1/4)}$, and this holds irrespective of the values of M on $[1/4, 1/2)$. Hence, the zeroes on $[-1/2, -1/4) \cup [1/8, 1/4)$ cannot be changed without altering Sol_M. On the other hand, possible zeroes within $[1/4, 1/2)$ could be changed arbitrarily without affecting Sol_M. Observe also that for $\xi \in [1/8, 1/4)$, we must have $M(\xi)+M(\xi+1/2) = 0$. ♢

Consider $M : \mathbb{R} \to [0, \infty)$ which is measurable, 1-periodic, such that (8.1) and (8.2) hold, and such that Sol_M is non-trivial. Let Φ be a maximal solution in $(\mathrm{Sol}_M, \prec_M)$ and consider its periodization p_Φ (see (8.13)). Consider $U := \{\xi \in \mathbb{R} : p_\Phi(2\xi) = 0\}$. It is fairly obvious that U does not depend on the choice of maximal solution in $(\mathrm{Sol}_M, \prec_M)$ and that U is measurable and 1/2-periodic. If $|U| = 0$, then we leave M as it is. Observe that, in this case, using (8.14) we must have the property that for almost every $\xi \in \mathbb{R}$,

$$M(\xi) + M(\xi + 1/2) > 0. \tag{8.26}$$

If $|U| > 0$, then for $\xi \in U$ we have two possibilities; either $p_\Phi(\xi)+p_\Phi(\xi+1/2) = 0$ or $p_\Phi(\xi)+p_\Phi(\xi+1/2) > 0$. In the second case, we must keep $M(\xi)+M(\xi+1/2) = 0$; otherwise Sol_M would change. However, in the first case, we can change the values of M freely without any effect on Sol_M — say, we put $M(\xi) = M(\xi + 1/2) = 1/2$.

This short analysis shows that in the study of the non-FO case, we can, without loss of generality, restrict ourselves to the class of M with the property that for every maximal solution Φ in $(\mathrm{Sol}_M, \prec_M)$, we have, for almost every $\xi \in \mathbb{R}$,

$$\text{either } M(\xi) + M(\xi + 1/2) > 0 \text{ or } p_\Phi(\xi) + p_\Phi(\xi + 1/2) > 0. \tag{8.27}$$

Following the well-known process of semiorthogonalization (as we did in the FO case), it is not difficult to restrict our analysis onto the class of the "Smith–Barnwell"-type filters. We start with $\widetilde{M} : \mathbb{R} \to [0,\infty)$ which is measurable, 1-periodic, satisfies (8.1) and (8.2), and $\mathrm{Sol}_{\widetilde{M}}$ is non-trivial. Furthermore, we also require (8.27). Take any maximal solution $\widetilde{\Phi}$ in $(\mathrm{Sol}_{\widetilde{M}}, \prec_{\widetilde{M}})$. Consider a function $M : \mathbb{R} \to [0,\infty)$ defined by

$$M(\xi) := \begin{cases} \frac{p_{\widetilde{\Phi}}(\xi)}{p_{\widetilde{\Phi}}(2\xi)} \widetilde{M}(\xi) & \text{if } p_{\widetilde{\Phi}}(2\xi) > 0 \\ 1/2 & \text{if } p_{\widetilde{\Phi}}(2\xi) = p_{\widetilde{\Phi}}(\xi) = p_{\widetilde{\Phi}}(\xi + 1/2) = 0 \\ 0 & \text{if } p_{\widetilde{\Phi}}(2\xi) = 0 < p_{\widetilde{\Phi}}(\xi) + p_{\widetilde{\Phi}}(\xi + 1/2). \end{cases} \tag{8.28}$$

It is easy to check that Φ, defined by

$$\Phi(\xi) := \begin{cases} \frac{\widetilde{\Phi}(\xi)}{p_{\widetilde{\Phi}}(\xi)} & \text{if } p_{\widetilde{\Phi}}(\xi) > 0 \\ 0 & \text{if } p_{\widetilde{\Phi}}(\xi) = 0, \end{cases} \tag{8.29}$$

satisfies $\Phi(2\xi) = \Phi(\xi)M(\xi)$, with M 1-periodic and measurable, such that $Z_M = Z_{\widetilde{M}}$, $H_M = H_{\widetilde{M}}$, and the elements in $(\mathrm{Sol}_M, \prec_M)$ can be described, by the same transformations as the one given in (8.27), from the elements of $(\mathrm{Sol}_{\widetilde{M}}, \prec_{\widetilde{M}})$. In other words, we can reduce the study of $\widetilde{M}$ to the study of M. Observe, however, that M also has the following properties: for almost every $\xi \in \mathbb{R}$,

$$M(\xi) + M(\xi + 1/2) = 0 \text{ or } 1, \tag{8.30}$$

and

$$M(\xi) + M(\xi + 1/2) = 0 \Leftrightarrow p_\Phi(\xi) + p_\Phi(\xi + 1/2) > 0. \tag{8.31}$$

REMARK 8.32. (i) Consider $M : \mathbb{R} \to][0,1]$, measurable, 1-periodic, such that (8.30) holds and $|Z_M| > 0$ and $|H_M| > 0$. Since ergodic properties do not play a role in this case, observe that in order to have non-trivial Sol_M, it is sufficient to have

$$|\{\xi \in H_M : \liminf_{n\to\infty} M(2^{-n}\xi) > 1/2\}| > 0.$$

(ii) Suppose that $\varphi \in L^2(\mathbb{R})$ is such that $\langle\varphi\rangle \subseteq D(\langle\varphi\rangle)$, $\varphi \neq 0$, and the corresponding m has the property that $|\{\xi : m(\xi) = 0\}| > 0$. We can still observe that $\varphi_0 := \left(p_\varphi^{-1/2} \chi_{p_\varphi > 0}\right) \bullet \varphi$, and it is well-known that $\langle\varphi_0\rangle = \langle\varphi\rangle$, with $\mathcal{B}_{\varphi_0}$ forming a Parseval frame for $\langle\varphi\rangle$. Let us denote the corresponding filter by m_0 and $M := |m_0|^2$, $\Phi := |\widehat{\varphi_0}|^2$. It follows then that M is non-FO, that $\Phi \in \mathrm{Sol}_M$, but also that M satisfies (8.30) and that, for almost every $\xi \in \mathbb{R}$, $p_\Phi(\xi) = 0$ or 1. As before, we can remove "non-essential" zeroes from M, and obtain (8.31). Hence, from the point of view of non-FO principal shift-invariant spaces $\langle\varphi\rangle$, our reduction to (8.30) and (8.31) allows us still to capture all principal shift-invariant spaces of interest.

■

Consider now a 1-periodic, measurable $M : \mathbb{R} \to [0,1]$ such that (8.30) holds. As in the FO case (see (4.10)) we can define $f_M : \mathbb{R} \to [0,1]$ by

$$f_M(\xi) := \prod_{n=1}^{\infty} M(2^{-n}\xi) \text{ for } \xi \in \mathbb{R}.$$

Again, f_M is measurable and satisfies $f_M(2\xi) = M(\xi) f_M(\xi)$. Observe that we can again apply a partial version of the "peeling-off argument"; only the part which applies to the estimate of the norm of f_M (since (8.30) does not allow the orthogonality property, but it does give $M(\xi) + M(\xi + 1/2) \leq 1$). Hence, we do obtain (4.14) in this case, as well:

$$\|f_M\|_{L^1(\mathbb{R})} \leq 1. \tag{8.33}$$

Therefore, in this case we also have (see Section 4 for comparison)

$$f_M \in \mathrm{Sol}_M. \tag{8.34}$$

Again, it is possible to have trivial f_M and yet have non-trivial Sol_M. Similarly as before, we have, in this case, that f_M is a maximal solution in $(\mathrm{Sol}_M, \prec_M)$ if and only if

$$f_M(\xi) > 0 \text{ for almost every } \xi \in H_M. \tag{8.35}$$

Again, positivity leads to dyadic continuity at zero, i.e. for $\xi \in H_M$, we have

$$\begin{aligned} f_M(\xi) > 0 &\Rightarrow \lim_{n\to\infty} M(2^{-n}\xi) = 1 \\ f_M(\xi) > 0 &\Rightarrow \lim_{n\to\infty} f_M(2^{-n}\xi) = 1, \end{aligned} \tag{8.36}$$

where the second limit is an increasing one. The following examples (rather well-known ones) emphasize a few points about positivity and H_M, $A_M^{(1)}$, (8.30), and (8.31).

EXAMPLE 8.37. (i) Consider a 1-periodic function M such that $M|_{[-1/2,1/2)} = \chi_{[0,1/2)}$. In this case, we have $|Z_M| > 0$, $H_M = [1/2, 1)$, $A_M^{(1)} = [-1, -1/2)$, $f_M = \chi_{[0,1)}$, $p_{f_M} \equiv 1$, and, for almost every $\xi \in \mathbb{R}$, $M(\xi) + M(\xi + 1/2) = 1$.

(ii) Consider a filter associated to the Shannon wavelet, i.e. 1-periodic M with $M|_{[-1/2,1/2)} = \chi_{[-1/4,1/4)}$. In this case, we have $|Z_M| > 0$, $H_M = \frac{1}{2}I$, $A_M^{(1)} = \emptyset$, $f_M = \chi_{[-1/2,1/2)}$, $p_{f_M} \equiv 1$, and, for almost every $\xi \in \mathbb{R}$, $M(\xi) + M(\xi + 1/2) = 1$.

$\diamondsuit$

It is also not difficult to see that a theorem analogous to Theorem 4.16 holds in this case, as well. Likewise, the proof of Proposition 4.24 also goes through in this case. Hence we can similarly develop a remark analogous to Remark 4.26. Observe, however, that all these results hold within a special subclass of non-FO filters. This is the content of the following result.

PROPOSITION 8.38. *If $M : \mathbb{R} \to [0, \infty)$ is measurable, 1-periodic, with $|Z_M| > 0$ and such that M is dyadically continuous at zero, then $|A_M^{(1)}| = 0$.*

PROOF. Obviously, for almost every $\xi \in \mathbb{R}$, we have $\lim_{n\to\infty} M(2^{-n}\xi) = 1$. This implies that $M(2^{-j}\xi) \neq 0$ for j large enough. Hence $\mathrm{orb}(\xi) \cap H_M \neq \emptyset$. $\square$

Observe that in the previous statement (unlike in Corollary 8.19) we did not require any assumptions on H_M, but we do get "maximal horizon" as a consequence.

EXAMPLE 8.39. Consider $0 < \varepsilon < 1/4$ and define M to be 1-periodic and such that $M|_{[-1/2,1/2)} = \chi_{[-\varepsilon,\varepsilon)}$. Then M is dyadically continuous at zero, $M(\xi) + M(\xi + 1/2) = 1$ for $\xi \in [-\varepsilon, \varepsilon)$, $M(\xi) + M(\xi + 1/2) = 0$ for $\xi \in [-1/4, 1/4) \setminus [-\varepsilon, \varepsilon)$,

$f_M = \chi_{[-2\varepsilon,2\varepsilon)}$, f_M is dyadically continuous at zero, $p_{f_M}|_{[-1/2,1/2)} = \chi_{[-2\varepsilon,2\varepsilon)}$, $|A_M^{(1)}| = 0$, and $H_M = [-2\varepsilon, 2\varepsilon) \setminus [-\varepsilon, \varepsilon)$. ♢

REMARK 8.40. The previous example provides an illustration of the generality of our method. For $0 < \varepsilon \leq 1/4$, denote φ_ε the function in $L^2(\mathbb{R})$ such that $\widehat{\varphi_\varepsilon} = \chi_{[-2\varepsilon,2\varepsilon)}$. Obviously, $\langle\varphi_\varepsilon\rangle \subseteq D(\langle\varphi_\varepsilon\rangle)$, and let us introduce

$$W_\varepsilon := D(\langle\varphi_\varepsilon\rangle) \cap \langle\varphi_\varepsilon\rangle^\perp,$$

which, of course, means W_ε is a shift-invariant space. Consider the dimension functions, $\dim_{W_\varepsilon}$ of W_ε. It is not difficult to check that the following identities hold:

$$0 < \varepsilon \leq 1/8 \Rightarrow \dim_{W_\varepsilon}(\xi) = \begin{cases} 0 & \text{if } \xi \in [-2\varepsilon, 2\varepsilon) \\ 1 & \text{if } \xi \in [-4\varepsilon, 4\varepsilon) \setminus [-2\varepsilon, 2\varepsilon) \\ 0 & \text{if } \xi \in [-1/2, 1/2) \setminus [-4\varepsilon, 4\varepsilon), \end{cases}$$

$$1/8 < \varepsilon < 1/6 \Rightarrow \dim_{W_\varepsilon}(\xi) = \begin{cases} 0 & \text{if } \xi \in [-2\varepsilon, 2\varepsilon) \\ 1 & \text{if } \xi \in [-1+4\varepsilon, -2\varepsilon) \cup [2\varepsilon, 1-4\varepsilon) \\ 2 & \text{if } \xi \in [-1/2, -1+4\varepsilon) \cup [1-4\varepsilon, 1/2), \end{cases}$$

$$\varepsilon = 1/6 \Rightarrow \dim_{W_\varepsilon}(\xi) = \begin{cases} 0 & \text{if } \xi \in [-1/3, 1/3) \\ 2 & \text{if } \xi \in [-1/2, 1/2) \setminus [-1/3, 1/3), \end{cases}$$

$$1/6 < \varepsilon < 1/4 \Rightarrow \dim_{W_\varepsilon}(\xi) = \begin{cases} 0 & \text{if } \xi \in [-1+4\varepsilon, 1-4\varepsilon) \\ 1 & \text{if } \xi \in [-2\varepsilon, 2\varepsilon) \setminus [-1+4\varepsilon, 1-4\varepsilon) \\ 2 & \text{if } \xi \in [-1/2, 1/2) \setminus [-2\varepsilon, 2\varepsilon), \end{cases}$$

$$\varepsilon = 1/4 \Rightarrow \dim_{W_\varepsilon} \equiv 1.$$

We would like to emphasize a few points here (in what follows, we take M to be as it was in Example 8.39, i.e. $M|_{[-1/2,1/2)} = \chi_{[-\varepsilon,\varepsilon)}$.

(i) In the case $\varepsilon \in [0, 1/8] \cup \{1/4\}$, $\dim_{W_\varepsilon} \leq 1$, i.e. W_ε is a principal shift-invariant space. We should approach this in the spirit of the discussion of (8.26) and (8.27). For some ε, there will be ξ such that $M(\xi)+M(\xi+1/2) = 0$. Of course, for $\varepsilon = 1/4$, $M(\xi) + M(\xi + 1/2) = 1$ almost everywhere. For $\varepsilon \in (0, 1/8]$, one has that $M(\xi) + M(\xi + 1/2) = 0$ for any $\xi \in [\varepsilon, 1/2 - \varepsilon)$, but this is a non-essential issue in the sense that we can replace M by M' given by

$$M'|_{[-1/2,1/2)} := \chi_{[-\varepsilon,\varepsilon)\cup[-1/2+\varepsilon,-1/2+2\varepsilon)\cup[2\varepsilon,1/2-\varepsilon)},$$

which satisfies $\mathrm{Sol}_M = \mathrm{Sol}_{M'}$ and, for almost every ξ,

$$M'(\xi) + M'(\xi + 1/2) = 1.$$

In this sense, the values of $\varepsilon \in (0, 1/8]$ and $\varepsilon = 1/4$ are "related".

For $\varepsilon \in (1/8, 1/4)$, however, it is not possible to perform a similar replacement, and that $M(\xi) + M(\xi + 1/2)$ must equal zero for $\xi \in [\varepsilon, 1/2 - \varepsilon)$ since any modification otherwise would force Sol_M to change — for example, $\Phi := \widehat{\varphi_\varepsilon}$ would be "pushed out" of Sol_M.

(ii) The case $\varepsilon \in (0, 1/8] \cup \{1/4\}$ actually belongs to the theory of MRA Parseval frame wavelets, as developed in [**PŠWX01**], [**PŠWX03**], and [**ŠSW08**]. We just need to replace M with M' as above, and M' is a *generalized low-pass filter* (in the terminology of the aforementioned papers); the corresponding *pseudo-scaling function* is φ_ε. However, when $\varphi \in (1/8, 1/4)$, we have filters which cannot be captured by the theory of MRA Parseval frame wavelets: observe that W_ε is a "2-dimensional" shift-invariant space and cannot be spanned by a single generating wavelet ψ.

(iii) As we have just seen, this approach to filters is more comprehensive than previous approaches. It is interesting, though, that from the point of view of the corresponding pre-GMRA structures we do not obtain anything new. More precisely, if $\varepsilon \in (1/8, 1/4)$, then $\varepsilon/2 < 1/8$ and

$$D(\langle \varphi_{\varepsilon/2} \rangle) = \langle \varphi_\varepsilon \rangle.$$

It is not difficult to check that in both cases we do get a GMRA, and the two structures consist of the same spaces — the only difference is that the core space is "shifted". For the ε-case, $V_0 = \langle \varphi_\varepsilon \rangle$, while for the $\varepsilon/2$-case, $V_0 = \langle \varphi_{\varepsilon/2} \rangle = D^{-1}(\langle \varphi_\varepsilon \rangle)$. Observe that $\langle \varphi_{\varepsilon/2} \rangle$ is the core space of an MRA PFW structure as described in [**ŠSW08**].

■

It is useful to compare the previous Remark with Remark 4.18, in particular with respect to Theorem 4.16. As we mentioned earlier, the theorem analogous to Theorem 4.16 holds in the non-FO case, as well. However, it does not characterize all the non-FO filters induced by generalized low-pass filters (in the terminology of Remark 4.26). Using (8.26), (8.27), and (8.31), it is not difficult to check that the following result holds.

Proposition 8.41. *Let $M : \mathbb{R} \to [0,1]$ be measurable and 1-periodic, such that $|Z_M| > 0$ and (8.30) holds. If $f_M(\xi) > 0$ for almost every $\xi \in H_M$, then the following are equivalent:*

(a) M is induced by a generalized low-pass filter (i.e. there exists a generalized low-pass filter m so that $f_M = f_{|m|^2}$);

(b) $|A_M^{(1)}| = 0$ and

$$|\{\xi \in \mathbb{R} : p_{f_M}(2\xi) = 0 \text{ and } p_{f_M}(\xi) + p_{f_M}(\xi + 1/2) > 0\}| = 0.$$

Let us now turn our attention to the filters induced by low-pass filters of MRA orthonormal wavelets. Observe that Proposition 8.41 shows that in this case we can focus our attention, without loss of generality, on the class

(8.42)
$$\mathcal{M}_{SB}^{non\text{-}FO} := \{M : \mathbb{R} \to [0,1] : M \text{ is measurable, 1-periodic, } |Z_M| > 0, |A_M^{(1)}| = 0, \\ \mathrm{Sol}_M \neq \{0\}, M(\xi) + M(\xi + 1/2) = 1 \text{ almost everywhere}\}.$$

Again, as in the FO case, the main insight is essentially given in Theorem 3.17 from [**PŠW99**]:

Theorem 8.43 ([**PŠW99**]). *Let $m : \mathbb{R} \to \mathbb{C}$ be a measurable, 1-periodic function such that $M := |m|^2 \in \mathcal{M}_{SB}^{non\text{-}FO}$. Then m is a low-pass filter (for an MRA orthonormal wavelet) if and only if $f_M(\xi) > 0$ for almost every $\xi \in H_M$, and*

$$\int_{\mathbb{R}} f_M(\xi) d\xi = 1.$$

Remark 8.44. Observe that Example 8.37i provides an M which is *not* induced by an MRA orthonormal wavelet filter, but it does satisfy the conditions that $f_M(\xi) > 0$ for almost every $\xi \in H_M$ and $\int_{\mathbb{R}} f_M(\xi) d\xi = 1$. Observe, however, that the condition $|A_M^{(1)}| = 0$ is not satisfied.

■

The analysis of the condition (8.35) is analogous to the FO case and we leave the details to the interested reader. Let us explore the condition

$$\int_{\mathbb{R}} f_M(\xi)d\xi = 1 \tag{8.45}$$

some more. First of all, using (8.12) and $f_M \leq 1$, we obtain

$$\int_{\mathbb{R}} f_M(\xi)d\xi = \int_{H_M} \left(\sum_{n=0}^{\infty} \frac{f_M(\xi/2^n)}{2^n} \right) d\xi \leq 2|H_M|. \tag{8.46}$$

Hence if (8.45) holds, then we have

$$|H_M| \geq 1/2. \tag{8.47}$$

Proposition 8.48. *If $M \in \mathcal{M}_{SB}^{non\text{-}FO}$ and f_M satisfies (8.45) and $|H_M| = 1/2$, then*

$$f_M = \chi_{\bigcup_{n=0}^{\infty} 2^{-n} H_M}.$$

Proof. Directly from (8.45) and (8.46). □

The following list of examples covers various aspects related to conditions (8.35) and (8.45).

Example 8.49. Let $0 < \varepsilon < 1/2$. Consider the 1-periodic function M such that $M|_{[-1/2,1/2)} = \chi_{[\varepsilon-1/2,\varepsilon)}$. Then $M \in \mathcal{M}_{SB}^{non\text{-}FO}$, $H_M = [2\varepsilon-1, \varepsilon-1/2) \cup [\varepsilon, 2\varepsilon)$, $|H_M| = 1/2$, $f_M = \chi_{[2\varepsilon-1,2\varepsilon)}$, and f_M satisfies both (8.35) and (8.45). This example is well-known (see, for example, [**HW96**]).

♢

Consider M' from Remark 8.40 with $\varepsilon \in (0, 1/8]$. Then $M' \in \mathcal{M}_{SB}^{non\text{-}FO}$, $|H_{M'}| = 2\varepsilon \leq 1/4 < 1/2$ (i.e. (8.45) does not hold), and $f_{M'} = \chi_{[-2\varepsilon,2\varepsilon)}$ satisfies (8.35).

Example 8.50. Consider the 1-periodic function M such that

$$M|_{[-1/2,1/2)} = \chi_{[0,1/4)} + \frac{1}{2}\chi_{[-1/4,0)\cup[1/4,1/2)}.$$

It is not difficult to check that $M \in \mathcal{M}_{SB}^{non\text{-}FO}$, but f_M does not satisfy (8.35), since $f_M(\xi) = 0$ for $\xi \in (-\infty, 0)$. Observe also that $H_M \cap (-\infty, 0) = [-1/2, -1/4)$. Considering $[0, \infty)$, observe that $f_M(\xi) = 1$ for $\xi \in [0, 1/2)$. Using induction over $n \in \mathbb{N} \cup \{0\}$, it is not difficult to prove that for $\xi \in [2^n - 1/2, 2^n)$ we have $f_M(\xi) = 2^{-(n+1)}$. Hence

$$\int_{\mathbb{R}} f_M(\xi)d\xi = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot \frac{1}{2} = 1.$$

Using induction we can also show that

$$H_M \cap [0, \infty) = \bigcup_{n=0}^{\infty} [2^n - 1/2, 2^n - 1/4)$$

and

$$|H_M| = \infty.$$

♢

9. Pre-GMRA Core

We begin with an example. It illustrates the point of view we take in this section.

EXAMPLE 9.1. Consider one of the well-known Lemarié wavelets (see [**Lem90**] and [**HW96**]) given by

$$\widehat{\psi} = \chi_{[-4/7,-2/7)\cup[2/7,3/7)\cup[12/7,16/7)}. \tag{9.2}$$

Observe that $T_\alpha\psi \in \langle\psi\rangle$ for every $\alpha \in \mathbb{R}$. Recall that ψ is a non-MRA orthonormal wavelet (and also an MSF wavelet, see [**HW96**]). Nevertheless, there is a GMRA structure associated with ψ; there is a shift-invariant space V such that

$$D(V) = V \oplus \langle\psi\rangle. \tag{9.3}$$

Using standard dimensions function techniques (see, for example, [**HW96**] and [**ŠSW08**]) it is straightforward to show that V is a "two-dimensional" shift-invariant space since

$$\dim_V(\xi) = \begin{cases} 2 & \text{if } \xi \in [-1/7, 1/7) \\ 1 & \text{if } \xi \in [-1/2,-3/7) \cup [-2/7,-1/7) \cup [1/7,2/7) \cup [3/7,1/2) \\ 0 & \text{if } \xi \in [-3/7,-2/7) \cup [2/7,3/7), \end{cases} \tag{9.4}$$

V is then the core of the GMRA of ψ. It may be surprising to see that, despite the fact that $(D_j(V), j \in \mathbb{Z})$ is a GMRA but not an MRA, we can actually consider this same structure as an MRA by "shifting" the core space. More precisely, consider

$$\psi_0 := \frac{1}{\sqrt{2}} D^{-1}\psi,$$

i.e.

$$\widehat{\psi_0} = \chi_{[-2/7,-1/7)\cup[1/7,3/14)\cup[6/7,8/7)}. \tag{9.5}$$

Observe that $\widehat{\psi_0} \in \langle\psi_0\rangle \leq V$ and, since $T_{1/2}\psi \in \langle\psi\rangle$,

$$D^{-1}(\langle\psi\rangle) = \langle\psi_0\rangle. \tag{9.6}$$

Furthermore,

$$\dim_{\langle\psi_0\rangle}|_{[-1/2,1/2)} = \chi_{[-2/7,3/14)}. \tag{9.7}$$

It follows that the space $V_0 := V \cap \langle\psi_0\rangle^\perp$ is a shift-invariant space such that

$$V_0 = D^{-1}(V), \tag{9.8}$$

and

$$\dim_{V_0}|_{[-1/2,1/2)} = \chi_{[-1/2,-3/7)\cup[-1/7,1/7)\cup[3/14,2/7)\cup[3/7,1/2)}. \tag{9.9}$$

We conclude that V_0 generates essentially the same structure as V; more precisely, $D_{j+1}(V_0) = D_j(V)$ for $j \in \mathbb{Z}$. Furthermore, by (9.9) and direct calculation, it follows that ψ_0 is an MRA Parseval frame wavelet (see [**ŠSW08**] for more on Parseval frame wavelets). $\diamondsuit$

The result above is not "an accident", as the following proposition shows.

PROPOSITION 9.10. *If $\psi \in L^2(\mathbb{R})$ is a Parseval frame wavelet such that $\langle\psi\rangle_{\frac{1}{2}\mathbb{Z}} = \langle\psi\rangle_{\mathbb{Z}}$, then $\psi_0 := \frac{1}{\sqrt{2}}D^{-1}(\psi)$ is also a Parseval frame wavelet and $D^{-1}(\langle\psi\rangle) = \langle\psi_0\rangle$.*

PROOF. The last statement holds since $T_{1/2}\psi \in \langle\psi\rangle$. In order to prove the first part, we need to check that two basic equations hold; see [**ŠSW08**] for details. For the first equation, we have

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}_0(2^j\xi)|^2 = \frac{1}{2}\sum_{j\in\mathbb{Z}} |\sqrt{2}\widehat{\psi}(2\cdot 2^j\xi)|^2 = \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^{j+1}\xi)|^2 = \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1.$$

For the second equation, recall Lemma 1.3.7. For $\xi \in \mathbb{R}$ and q an odd integer, we obtain

$$\begin{aligned}
\sum_{j\geq 0} \widehat{\psi_0}(2^j\xi)\overline{\widehat{\psi_0}(2^j(\xi+q))} &= \frac{1}{2}\sum_{j\geq 0} \sqrt{2}\widehat{\psi}(2^{j+1}\xi)\overline{\sqrt{2}\widehat{\psi}(2^{j+1}(\xi+q))} \\
&= \sum_{j\geq 1} \widehat{\psi}(2^j\xi)\overline{\widehat{\psi}(2^j(\xi+q))} \\
&= \left(\sum_{j\geq 0} \widehat{\psi}(2^j\xi)\overline{\widehat{\psi}(2^j(\xi+q))}\right) - \widehat{\psi}(\xi)\overline{\widehat{\psi}(\xi+q)} \\
&= 0 - 0 \\
&= 0.
\end{aligned}$$

□

REMARK 9.11. (i) Even if ψ is an orthonormal wavelet, ψ_0 is not going to be one. Observe that $\|\psi_0\|_2^2 = \frac{1}{2}\|\psi\|_2^2$, i.e. $\|\psi_0\|_2^2 \leq 1/2$, always.

(ii) It is not difficult to extend the above to other powers of 2. However, we will take a somewhat more general approach.

(iii) If, in the result above, we have a shift-invariant space V such that $D(V) = V \oplus \langle\psi\rangle$, then $V_0 := D^{-1}(V)$ satisfies $V = D(V_0) = V_0 \oplus \langle\psi_0\rangle$ and

$$\dim_V(\xi) = \dim_{V_0}(\xi) + \dim_{\langle\psi_0\rangle}(\xi) \leq \dim_{V_0}(\xi) + 1.$$

(iv) If ψ is of Type 1, then the result above, Propsition 9.10, is not going to be satisfied in many cases. However, compare the result with Theorem 7.21.

■

We turn our attention to the Type 3 case now; recall that MSF Parseval frame wavelets (and MSF orthonormal wavelets) are of Type 3 (see [**ŠSW08**] and [**ŠW11**]). We shall consider an even more general situation where V may not be a principal space.

THEOREM 9.12. *A shift invariant space $V \leq L^2(\mathbb{R})$ satisfies the property $T_\alpha(V) = V$ for every $\alpha \in \mathbb{R}$ if and only if there exists a measurable set E such that $V = L^2(E)^\vee$.*

PROOF. The proof of sufficiency follows directly from the fact that the set of support of $\widehat{f}$ equals the set of support of $\widehat{T_\alpha f}$. Let us prove the necessity of the given condition.

Suppose that $V \leq L^2(\mathbb{R})$ is a closed subspace such that $T_\alpha(V) = V$ for every $\alpha \in \mathbb{R}$. If V is trivial, there is nothing to prove. Consider any $\varphi \in V \setminus \{0\}$. Denote by $S_\varphi :=$ ssupp $\widehat{\varphi}$. Consider the space $L^2(\mathbb{R}, |\widehat{\varphi}|^2 d\xi)$. Obviously, $m \in L^2(\mathbb{R}, |\widehat{\varphi}|^2 d\xi)$ if and only if $m \bullet \varphi \in L^2(\mathbb{R})$. Furthermore,

$$L^2(S_\varphi)^\vee = \{m \bullet \varphi : m \in L^2(\mathbb{R}, |\widehat{\varphi}|^2 d\xi)\}.$$

We claim first that $m\bullet\varphi \in V$ for every $m : \mathbb{R} \to \mathbb{C}$ with m continuous with compact support. Take $\varepsilon > 0$. There exists $N > 0$ sufficiently large so that $m \equiv 0$ outside of $[-N/2, N/2]$ and

$$\frac{1}{2\|m\|_\infty} \left(\int_{\mathbb{R}\setminus[-N/2,N/2]} |\widehat{\varphi}(\xi)|^2 d\xi \right)^{1/2} \leq \frac{\varepsilon}{2}. \tag{9.13}$$

By the Stone-Weierstrass theorem, we can choose a trigonometric polynomial μ which is N-periodic and

$$\|\mu|_{[-N/2,N/2]} - m\|_{L^2(\mathbb{R},|\widehat{\varphi}|^2 d\xi)} < \frac{\varepsilon}{2} \tag{9.14}$$

and

$$\|\mu\|_\infty \leq 2\|m\|_\infty. \tag{9.15}$$

From (9.13) through (9.15), it then follows that

$$\|m \bullet \varphi - \mu \bullet \varphi\|_2 = \|\mu - m\|_{L^2(\mathbb{R},|\widehat{\varphi}|^2 d\xi)} < \varepsilon.$$

Recall that μ is a finite linear combination of exponentials, $e_{k/N}$ for $k \in \mathbb{Z}$ which are N-periodic. By our assumption, $e_{k/N} \bullet \varphi \in V$ i.e. $\mu \bullet \varphi \in V$. Since ε was arbitrary and V is closed, we obtain $m \bullet \varphi \in V$.

Since every $m \in L^2(\mathbb{R}, |\widehat{\varphi}|^2 d\xi)$ can be approximated via a sequence of continuous, compactly supported functions, and since V is closed, we conclude that, for every $\varphi \in V \setminus \{0\}$ we have

$$L^2(S_\varphi)^\vee \leq V. \tag{9.16}$$

Finally, since V is a shift-invariant space, we know that there exists a countable family $\{\varphi_i : i \in \mathbb{N}\} \subseteq V$ such that $\varphi_i + (\chi_{S_{\varphi_{i+1}} \setminus S_{\varphi_i}}) \bullet \varphi_{i+1} \in V$ and $\langle \{\varphi_i : i \in \mathbb{N}\} \rangle = V$. It is then not difficult to construct a function $\psi \in V$ such that $S_f \subseteq S_\psi$, modulo null sets, for every $f \in V$. It follows that $V \subseteq L^2(S_\psi)^\vee \subseteq V$, i.e. $V = L^2(S_\psi)^\vee$. □

REMARK 9.17. Consider $V = L^2(S)^\vee$, where $S \subseteq \mathbb{R}$ is measurable. Obviously, for every $j \in \mathbb{Z}$, $V_j := D_j(V) = L^2(2^j S)^\vee$. Hence the property $V_j \subseteq V_{j+1}$ holds for every $j \in \mathbb{Z}$ if and only if $V \subseteq D(V)$ if and only if $S \subseteq 2S$ (modulo null sets).

If S is bounded, then we can choose $j_0 \in \mathbb{Z}$ such that $2^{j_0} S \subseteq [-1/4, 1/4)$. Obviously in such a case, V_{j_0} is a principal shift-invariant space generated by $(\chi_{2^{j_0} S})^\vee$ and this is a core space of an MRA Parseval frame wavelet of MSF-type if and only if

$$\mathbb{R} = \bigcup_{j\in\mathbb{Z}} 2^j S \text{ (modulo null sets).}$$

■

COROLLARY 9.18. *Let $\psi \in L^2(\mathbb{R})$ be an MSF orthonormal wavelet (recall that there is a GMRA structure $\{V_j\}$ associated with ψ). Then, for every $n \in \mathbb{N}$, we have $\psi_n := \frac{1}{2^{n/2}} D_{-n}(\psi)$ is a Parseval frame wavelet associated with the GMRA structure $\{V_j^n\}$, where $V_j^n := V_{j-n}$. Furthermore, if $\widehat{\psi}$ is compactly supported, then there exists $n_0 \in \mathbb{N}$ such that ψ_{n_0} is an MRA Parseval frame wavelet.*

PROOF. Since ψ is an MSF wavelet, it follows that $T_\alpha \psi \in \langle \psi \rangle$ for every $\alpha \in \mathbb{R}$. By Proposition 9.10, ψ_1 is a Parseval frame wavelet and $D_{-1}(\langle \psi \rangle) = \langle \psi_1 \rangle$. Recall that $D(V_0) = V_0 \oplus \langle \psi \rangle$. It follows then, as in Remark 9.11iii, that $V_{-1} \oplus \langle \psi_1 \rangle = V_0 = D(V_{-1})$.

Recall that, for every $\alpha \in \mathbb{R}$, we have that $T_\alpha D_{-1}\psi = D_{-1}T_{\alpha/2}\psi$. Hence, for every $\alpha \in \mathbb{R}$, we have $T_\alpha \psi_1 \in \langle \psi_1 \rangle$. We can now repeat these steps, going from ψ_1 to ψ_2, and so on, so that the first part of the theorem follows by induction on n.

If $\widehat{\psi}$ is compactly supported, then, for $n_0 \in \mathbb{N}$ sufficiently large, the support of $\widehat{\psi_{n_0}}$ is contained in $[-1/2, 1/2)$. It follows that V_{-n_0} is a shift-invariant space contained in $L^2([-1/2,1/2))^\vee$. This implies that $\dim_{V_{-n_0}} \leq 1$, i.e. that V_{-n_0} is a principal shift-invariant space. Hence $D(V_{-n_0}) = V_{-n_0} \oplus \langle \psi_{n_0} \rangle$ and ψ_{n_0} is an MRA Parseval frame wavelet. □

REMARK 9.19. Suppose that we have a GMRA $\{V_j\}$ such that $D(V_0) = V_0\langle\psi\rangle$ and $\psi \in L^2(\mathbb{R})$ is such that

$$T_{1/2}\psi \notin \langle \psi \rangle. \tag{9.20}$$

If this holds, then $V_{-1} = D^{-1}(V_0)$ is not a shift-invariant space, i.e. V_0 is also a natural core of the GMRA. Observe that we must have $V_0 = V_{-1} \oplus D^{-1}(\langle\psi\rangle)$. Hence, it is enough to prove that $D^{-1}(\langle\psi\rangle)$ is not a shift-invariant space. And, indeed, if it were a shift-invariant space, then we would have

$$D^{-1}(T_{1/2}\psi) = TD^{-1}\psi \in D^{-1}(\langle\psi\rangle),$$

i.e. $T_{1/2}\psi \in \langle\psi\rangle$, which clearly contradicts (9.20).

This shows that if we have a Parseval frame wavelet and a shift-invariant space V such that $D(V) = V \oplus \langle\psi\rangle$, then we can apply D_{-1} and obtain the same type of structure if and only if $T_{1/2}\psi \in \langle\psi\rangle$. In other words, we can "shift" a core space "one step down". However, if we encounter (9.20) such a procedure cannot continue and we have reached the natural core of the corresponding GMRA structure. ■

CHAPTER 3

Wavelet Structure

1. The Space of Negative Dilates

Given $\psi \in L^2(\mathbb{R})$, we are interested in the behavior of the "wavelet family", $\{\psi_{jk} : j, k \in \mathbb{Z}\}$; recall the notation from (2.1.2). We begin by introducing some standard notation. For $\psi \in L^2(\mathbb{R})$, we denote the "main resolution level" by

$$W_0 = W_0(\psi) := \langle \psi \rangle. \tag{1.1}$$

For every $j \in \mathbb{Z}$,

$$W_j = W_j(\psi) = D_j(W_0) = \overline{\operatorname{span}}\{\psi_{jk} : k \in \mathbb{Z}\}. \tag{1.2}$$

If $j \in \mathbb{Z}$ with $j \geq 0$, then

$$W_j(\psi) \text{ is a shift-invariant space,} \tag{1.3}$$

and its spectral function is

$$\sigma_{W_j}(\xi) = \begin{cases} \frac{|\widehat{\psi}(\xi/2^j)|^2}{p_\psi(\xi/2^j)} & \text{if } \xi/2^j \in U_{\langle\psi\rangle} \\ 0 & \text{otherwise,} \end{cases} \tag{1.4}$$

and its dimension function is

$$\dim_{W_j}(\xi) = \sum_{k\in\mathbb{Z}} \sigma_{W_j}(\xi + k) = \sum_{\ell=0}^{2^j-1} \chi_{U_{\langle\psi\rangle}}\left(\frac{\xi+\ell}{2^j}\right). \tag{1.5}$$

If $j \in \mathbb{Z}$ and $j < 0$, then $W_j(\psi)$ may or may not be a shift-invariant space. More precisely, and using the notation from (II1.2), we have that, for $j \in \mathbb{Z}$ with $j < 0$, the following equivalencies hold:

$$\begin{aligned} W_j \text{ is a shift-invariant space} &\iff T_{2^j}\psi \in \langle\psi\rangle \\ &\iff W_j(\psi) = \langle D_j\psi \rangle \\ &\iff W_j(\psi) = \overline{\operatorname{span}}\{\psi^{jk} : k \in \mathbb{Z}\} \end{aligned} \tag{1.6}$$

If this is the case, observe that $W_j(\psi)$ is then a principal shift-invariant space. It follows directly from (1.6) that

$$W_j(\psi) \text{ is a shift-invariant space for every } j \in \mathbb{Z} \iff \langle\psi\rangle \text{ is of Type 3.} \tag{1.7}$$

As we have seen, negative dilate spaces for individual dilate levels often do not preserve shift invariance. It is then natural to consider the space of (all) negative dilates,

$$V_0(\psi) := \overline{\operatorname{span}}\{\psi_{jk} : j < 0, k \in \mathbb{Z}\}. \tag{1.8}$$

Again, $V_0(\psi)$ may or may not be a shift-invariant space. Obviously we will have

$$\begin{aligned} V_0(\psi) \text{ is a shift-invariant space } &\iff V_0 = \langle D_j\psi : j < 0\rangle \\ &\iff V_0 = \overline{\operatorname{span}}\{\psi^{jk} : j < 0, k \in \mathbb{Z}\}. \end{aligned} \tag{1.9}$$

REMARK 1.10. The question of the shift-invariance of $V_0(\psi)$ has been studied by many authors. Let us briefly recall some of the main directions.

(i) If $\{\psi_{jk} : j,k \in \mathbb{Z}\}$ is a Riesz basis for $L^2(\mathbb{R})$, then there is a complete characterization in the sense of the dual system; it must be of the same form, $\{\varphi_{jk} : j,k \in \mathbb{Z}\}$ for some $\varphi \in L^2(\mathbb{R})$ (see [**DH02**], [**Zal99**], and [**KKL01**] for more details).

(ii) If $\{\psi_{jk} : j,k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$, then the characterization issue is significantly more complicated. For a nice overview and several important results, see [**BW03**]. Briefly, the existence of the standard dual frame of the same form will ensure the shift-invariance of $V_0(\psi)$. However, this condition is not a necessary one. Furthermore, there are other conditions that may provide (1.9).

This question is also closely related to the question of the period of the frame $\{\psi_{jk} : j,k \in \mathbb{Z}\}$ introduced by I. Daubechies and B. Han; see [**DH02**] and [**BW03**].

(iii) Systems of the form $\{\psi_{jk} : j,k \in \mathbb{Z}\}$ are known as *affine systems*, while systems of the form

$$\{\psi_{jk} : j \geq 0, k \in \mathbb{Z}\} \cup \{\psi^{jk} : j < 0, k \in \mathbb{Z}\} \tag{1.11}$$

are known as *quasi-affine systems*. They were introduced by A. Ron and Z. Shen in [**RS97**] (see also [**CSS98**], [**CS94**], and [**BW03**]). Let us mention that M. Bownik and E. Weber ([**BW03**]) prove that a system of the form (1.11) which is also a frame has canonical dual of the same form (i.e. form (1.11)) if and only if, for every $q \in 2\mathbb{Z}+1$ for almost every $\xi \in \mathbb{R}$,

$$\sum_{j=0}^{\infty} \widehat{\psi}(2^j\xi)\overline{\widehat{\psi}(2^j(\xi+q)} = 0. \tag{1.12}$$

Recall that (1.12) is the "t_q equation" which plays a crucial role in the characterization of orthonormal wavelets and Parseval frame wavelets — see Chapter 7 in [**HW96**] for details.

(iv) If $\psi \in L^2(\mathbb{R})$ is a Parseval frame wavelet, then $V_0(\psi)$ is a shift-invariant space. This is a well-known result of L. Baggett; see the nice survey in [**Bow08**].

■

Let us denote by

$$\mathcal{V}_{\mathrm{SIS}} := \{\psi \in L^2(\mathbb{R}) : V_0(\psi) \text{ is a shift-invariant space}\}. \tag{1.13}$$

Observe that [**ŠSW08**, p.268–269] shows that for $\psi \in L^2(\mathbb{R})$, we have

$$\begin{aligned} V_0(\psi) \text{ is a shift-invariant space} \Rightarrow T_{1/2}D_j\psi \in \langle D_\ell\psi : \ell \leq 0\rangle \\ \text{for every } j \in \mathbb{Z}, j \leq 0. \end{aligned} \tag{1.14}$$

Consider now $\psi \in L^2(\mathbb{R})$ such that $V_0(\psi)$ is a shift-invariant space. It follows then that

$$V_0(\psi) \leq V_0(\psi) + \langle\psi\rangle = D(V_0(\psi)). \tag{1.15}$$

Observe that the sum in (1.15) is not necessarily an orthogonal one. Actually, our functions now split into two very different classes. Consider first $\psi \in \mathcal{V}_{\mathrm{SIS}}$ such that

$$\psi \in V_0(\psi). \tag{1.16}$$

This property is equivalent to

$$D(V_0(\psi)) = V_0(\psi). \tag{1.17}$$

Remark 1.18. Recall (2.7.11), (2.7.12), and Example 2.7.13 for the description of such principal shift-invariant spaces that satisfy (2.7.11). Clearly, our class of spaces satisfying (1.17) is a subclass of those spaces. In particular, if $\psi \neq 0$, then (1.16) implies

$$\dim_{V_0(\psi)} \equiv \infty. \tag{1.19}$$

As it turns out, $\{\psi_{jk} : j,k \in \mathbb{Z}\}$ can have a variety of complicated structures and it may well be that (1.16) could possibly hold. M. Bownik and Z. Rzeszotnik, in [**BR05**], constructed examples of functions $\psi_\delta \in L^2(\mathbb{R})$ with $\delta > 0$ such that $\{(\psi_\delta)_{jk} : j,k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ with frame bounds 1 and $1+\delta$ such that

$$V_0(\psi_\delta) = L^2(\mathbb{R}). \tag{1.20}$$

■

Observe that (1.15) shows that, for every $\psi \in \mathcal{V}_{\mathrm{SIS}}$,

$$\{V_j(\psi) := D_j(V_0(\psi)) : j \in \mathbb{Z}\} \tag{1.21}$$

is a pre-GMRA. If, in addition, ψ satisfies (1.16), then

$$V_{-\infty}(\psi) = V_0(\psi) = V_\infty(\psi); \tag{1.22}$$

observe that even (1.20) is possible. In particular, assuming $\psi \neq 0$, we get from (1.16) that

$$\bigcap_{j\in\mathbb{Z}} D_j(V_0(\psi)) \neq \{0\}. \tag{1.23}$$

Remark 1.24. Given a pre-GMRA (in the general sense) it is natural to ask about the relationship of properties analogous to (1.16), (1.19), and (1.23). Using [**Bow08**], [**Bow09**] (see also Remark 2.7.16) we have, assuming $V_0 \neq \{0\}$,

$$D(V_0) = V_0 \Rightarrow \bigcap_{j\in\mathbb{Z}} D_j(V_0) \neq \{0\} \Rightarrow \dim_{V_0} \equiv \infty,$$

and none of the reversed implications are true.

What happens if we restrict ourselves to pre-GMRAS $\{V_j : j \in \mathbb{Z}\}$ given in (1.21)? Obviously, the given implications remain true. The first implication cannot be reversed since M. Bownik constructed in [**Bow08**] an example of a $\psi \in L^2(\mathbb{R})$ such that $\psi \notin V_0(\psi)$ such that ψ satisfies (1.23) and $\{\psi_{jk} : j,k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$. The question of whether

$$\dim_{V_0(\psi)} \equiv \infty \Rightarrow \bigcap_{j\in\mathbb{Z}} D_j(V_0(\psi)) \neq \{0\}$$

is, as far as we know, open at this time.

If we restrict ourselves to the case of ψ belonging to the class of Parseval frame wavelets, then it has been shown in [**ŠSW08**] that

$$\psi \in V_0(\psi) \iff V_0(\psi) = L^2(\mathbb{R}). \tag{1.25}$$

Whether any of the implications can be reversed is, as far as we know, an open problem. Actually, even more basic questions remain open as well. Considering all three properties, (1.16), (1.19), and (1.23), is there a Parseval frame wavelet which satisfies at least one of these conditions? This question applied to (1.23) is known as the Baggett problem; see [**Bow08**] for details. ■

Let us turn our attention to the second important class. Consider $\psi_0 \in \mathcal{V}_{\mathrm{SIS}}$ such that

$$\psi_0 \notin V_0(\psi_0). \tag{1.26}$$

Consider the space $U_0 := V_1(\psi_0) \cap V_0(\psi_0)^{\perp}$. By (1.26), U_0 is a non-trivial shift-invariant space, and it is easy to see that $\sigma_{U_0} \leq \sigma_{\langle\psi_0\rangle}$; hence $\dim_{U_0} \leq \dim_{\langle\psi_0\rangle} \leq 1$. Hence, U_0 is a principal shift-invariant space, and there exists a function $\psi \in L^2(\mathbb{R})$ such that

$$U_0 = \langle\psi\rangle \text{ and } \mathcal{B}_\psi \text{ is a Parseval frame for } \langle\psi\rangle. \tag{1.27}$$

Hence

$$V_1(\psi_0) = V_0(\psi_0) \oplus \langle\psi\rangle, \tag{1.28}$$

and $\psi \neq 0$. Since D is a unitary operator, we obtain that, for every $j \in \mathbb{N}$,

$$V_j(\psi_0) = V_0(\psi_0) \oplus \left(\bigoplus_{\ell=1}^{j} W_\ell(\psi) \right). \tag{1.29}$$

Observe that, despite this property, it can happen that $\dim_{V_0(\psi)} \equiv \infty$; we can even have that $\{(\psi_0)_{jk} : j,k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ (see Remark 1.24). However, we have the following result which generalizes slightly some well-known results for GMRAs and Parseval frame wavelets.

Proposition 1.30. *Let $\{V_j : j \in \mathbb{Z}\}$ be a pre-GMRA and $\psi \in L^2(\mathbb{R}) \setminus \{0\}$ such that $\mathcal{B}_\psi$ is a Parseval frame for ψ. If*

(i) $V_1 = V_0 \oplus \langle\psi\rangle$, and

(ii) $|\{\xi \in \mathbb{R} : \dim_{V_0}(\xi) < \infty\}| > 0$,

then $V_{-\infty} = \{0\}$, $\dim_{V_0} < \infty$ almost everywhere, $V_\infty = \bigoplus_{j\in\mathbb{Z}} W_j(\psi)$, and

$$\sigma_{V_0}(\xi) = \sum_{j=1}^{\infty} |\widehat{\psi}(2^j\xi)|^2.$$

If this is the case, then the following are equivalent:

(a) $\{V_j\}$ is a GMRA;

(b) $\{\psi_{jk} : j,k \in \mathbb{Z}\}$ is a semiorthogonal Parseval frame wavelet;

(c) For almost every $\xi \in \mathbb{R}$, $\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1$.

Proof. By Proposition 2.7.10 we conclude that $\dim_{V_0} < \infty$ almost everywhere. By the Bownik theorem [**Bow09**] we have $V_{-\infty} = \{0\}$. This and a property analogous to (1.29) leads to $V_\infty = \bigoplus_{j\in\mathbb{Z}} W_j(\psi)$.

Regarding the spectral function, observe that, for almost every $\xi \in \mathbb{R}$, we have (by a standard argument)

$$\begin{aligned} 1 \geq \sigma_{V_0}(\xi) &= \sigma_{V_1}(2\xi) = \sigma_{V_0}(2\xi) + \sigma_{\langle\psi\rangle}(2\xi) \\ &= \sigma_{V_0}(2\xi) + |\widehat{\psi}(2\xi)|^2. \end{aligned} \tag{1.31}$$

Using the recursion, boundedness, and monotonicity in (1.31), it follows by a standard argument that, for almost every $\xi \in \mathbb{R}$,

$$1 \geq \sigma_{V_0}(\xi) = H(\xi) + \sum_{j=1}^{\infty} |\widehat{\psi}(2^j \xi)|^2,$$

where $H(\xi) = \lim_{N\to\infty} \sigma_{V_0}(2^N \xi)$. In particular, $H : \mathbb{R} \to [0,1]$ is a measurable function such that, for almost every $\xi \in \mathbb{R}$, $H(\xi) = H(2\xi)$. We wish to prove that $H(\xi) = 0$ almost everywhere. Suppose, to the contrary, that $|\{\xi \in \mathbb{R} : H(\xi) > 0\}| > 0$. Then there exists a $\delta > 0$ such that the set $A := \{\xi \in I : H(\xi) \geq \delta\}$ has positive measure. It is then obvious that we can form infinitely many sets $(A_i : i \in \mathbb{N})$ such that they are disjoint, that $\bigcup_{i\in\mathbb{N}} A_i \subseteq A$, and $|A_i| > 0$, for every $i \in \mathbb{N}$. Using now Lemma 2.2.31, we conclude that $E_{A_i} = \mathbb{R}$ almost everywhere for each $i \in \mathbb{N}$. It follows that, for almost every $\xi \in \mathbb{R}$, $H(\xi + k) \geq \delta$ for infinitely many $k \in \mathbb{Z}$. Since

$$\dim_{V_0}(\xi) = \sum_{k\in\mathbb{Z}} \sigma_{V_0}(\xi + k) \geq \sum_{k\in\mathbb{Z}} H(\xi + k) = \infty \text{ almost everywhere,}$$

we obtain a contradiction with (ii). Hence $H \equiv 0$ almost everywhere, which implies that $\sigma_{V_0}(\xi) = \sum_{j=1}^{\infty} |\widehat{\psi}(2^j\xi)|^2$, which leads to

$$\sigma_{V_\infty}(\xi) = \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 \text{ almost everywhere.} \tag{1.32}$$

Hence $V_\infty = L^2(\mathbb{R})$ if and only if

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1 \text{ almost everywhere.}$$

The rest of the proof is now straightforward. □

Hence, if we are in a fairly "non-extreme" case, i.e. $\psi_0 \in \mathcal{V}_{\mathrm{SIS}}$ with $\psi_0 \notin V_0(\psi)$ and $\dim_{V_0(\psi)} \not\equiv \infty$ then, in theory, we obtain a nice pre-GMRA structure in standard form.

2. Orthogonality

Orthogonal structures are perhaps the most thoroughly studied among wavelet structures. We briefly revisit some of the results from a slightly different point of view. For every $\psi \in L^2(\mathbb{R})$ we have (recall also results from [**HW96**])

$$\langle\psi\rangle \perp \langle D\psi\rangle \iff \sum_{k\in\mathbb{Z}} \widehat{\psi}\left(\frac{\xi+k}{2}\right)\overline{\widehat{\psi}(\xi+k)} = 0 \text{ almost everywhere,} \tag{2.1}$$

and

$$\langle\psi\rangle \perp \langle DT\psi\rangle \iff \sum_{k\in\mathbb{Z}} (-1)^k \widehat{\psi}\left(\frac{\xi+k}{2}\right)\overline{\widehat{\psi}(\xi+k)} = 0 \text{ almost everywhere.} \tag{2.2}$$

It is not difficult to check (and it is well-known, for example see [**HW96**]) that

(2.3)
$$\begin{aligned}\langle D\psi\rangle \perp \langle\psi\rangle \perp \langle DT\psi\rangle &\iff \langle\psi\rangle \perp D(\langle\psi\rangle)\\ &\iff \sum_{k\in\mathbb{Z}} \widehat{\psi}(2(\xi+k))\overline{\widehat{\psi}(\xi+k)} = 0 \text{ almost everywhere.}\end{aligned}$$

Since D_j is an orthogonal operator, it is obvious that (2.3) is equivalent to

$$D_j(\langle\psi\rangle) = W_j(\psi) \perp D_{j+1}(\langle\psi\rangle) = W_{j+1}(\psi), \tag{2.4}$$

for every $j \in \mathbb{Z}$. Following (1.7), we examine first the special case where $\langle\psi\rangle$ is of Type 3.

Consider $\psi \in L^2(\mathbb{R})$ such that $\langle\psi\rangle$ is of Type 3. Recall (see (1.3.5)) that in this case there exists an almost everywhere-measurable partition of $[0,1)$, say $\{A_k : k \in \mathbb{Z}\}$ such that

$$\operatorname{ssupp} \widehat{\psi} \subseteq \bigcup_{k\in\mathbb{Z}} (A_k + k). \tag{2.5}$$

If we denote, for every $k \in \mathbb{Z}$, the set B_k by

$$B_k := \operatorname{ssupp} \widehat{\psi} \cap (A_k + k), \tag{2.6}$$

then $B := \bigcup_{k\in\mathbb{Z}} B_k = \operatorname{ssupp} \widehat{\psi}$. In particular, if $g \in L^2(\mathbb{R})$ is such that $\widehat{g} = \chi_B$, then

$$\langle\psi\rangle = \langle g\rangle, \tag{2.7}$$

and $\mathcal{B}_g$ is a Parseval frame for $\langle\psi\rangle$. Observe also that in this case (2.3) is equivalent to

$$\text{For almost every } \xi \in \mathbb{R},\ \text{if } \widehat{\psi}(\xi) \neq 0,\ \text{then } \widehat{\psi}(2\xi) = 0. \tag{2.8}$$

Remark 2.9. Observe that, even in this case, having $W_{-1}(\psi) \perp W_0(\psi) \perp W_1(\psi)$ does not necessarily ensure that $W_{-1}(\psi)$ is orthogonal to $W_1(\psi)$. Indeed, take $\psi \in L^2(\mathbb{R})$ such that

$$\widehat{\psi} = \chi_{[1/8,1/4)\cup[1/2,1)}.$$

It is obvious that $\langle\psi\rangle$ is of Type 3 and that ψ satisfies (2.8). However, $W_{-1}(\psi)$ is not orthogonal to $W_1(\psi)$ since

$$\widehat{D_{-1}\psi} = \sqrt{2}\chi_{[1/16,1/8)\cup[1/4,1/2)} \text{ and } \widehat{D\psi} = \frac{1}{\sqrt{2}}\chi_{[1/4,1/2)\cup[1,2)}.$$

Therefore, even in this case, we need the full scope of the orthogonality property. More precisely, it follows easily from [**HW96**, p. 102] that, for every $\psi \in L^2(\mathbb{R})$,

$$\langle\psi\rangle \perp D_n(\langle\psi\rangle) \text{ for every } n \in \mathbb{N} \tag{2.10}$$

if and only if

$$W_j(\psi) \perp W_k(\psi) \text{ for all } j,k \in \mathbb{Z}, j \neq k \tag{2.11}$$

if and only if, for every $n \in \mathbb{N}$ and for almost every $\xi \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} \widehat{\psi}(2^n(\xi+k))\overline{\widehat{\psi}(\xi+k)} = 0. \tag{2.12}$$

Observe that (2.12) is one of the four characterizing equations for the orthonormal wavelet (see [**HW96**, Ch. 7]). Nevertheless, the analysis of the Type 3 case is simpler, since every $\psi \in L^2(\mathbb{R})$ such that $\langle\psi\rangle$ is of Type 3 satisfies the "t_q-equation" given in (1.12). ■

We shall say that $\psi \in L^2(\mathbb{R})$ is a *semiorthogonal pre-wavelet* if ψ satisfies (2.11). Hence, if $\psi \in L^2(\mathbb{R})$ is a semiorthogonal pre-wavelet and $\langle\psi\rangle$ is of Type 3, then ψ satisfies two of the characterizing equations for orthonormal wavelets. Furthermore, if $\psi = g$ (g as described in (2.7)), then the more general version of the third equation is satisfied as well, i.e., for almost every $\xi \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi + k)|^2 = 0 \text{ or } 1. \tag{2.13}$$

Obviously, the orthogonality properties of $W_j(\psi)$ spaces in the Type 3 case, as they relate to the wavelet structure, are going to depend mainly on the fourth equation. Observe that (1.7) ensures that, in this case, we cannot have $\dim_{V_0(\psi)} \equiv \infty$.

LEMMA 2.14. *If $\psi \in L^2(\mathbb{R})$ is a semiorthogonal pre-wavelet such that $\langle\psi\rangle$ is of Type 3, then $\dim_{V_0} < \infty$ almost everywhere.*

PROOF. By (1.7), $W_j(\psi)$ is a principal shift-invariant space for every $j \in \mathbb{Z}$. By (2.11) we have that

$$\bigoplus_{j<0} W_j(\psi)$$

is a shift-invariant space which contains all $D_j\psi$ for $j < 0$. Using (1.9) we conclude that $V_0(\psi)$ is a shift-invariant space and

$$V_0(\psi) = \bigoplus_{j<0} W_j(\psi). \tag{2.15}$$

By (1.7) again, we have

$$\sigma_{V_0(\psi)} = \sum_{j<0} \sigma_{W_j(\psi)}.$$

By taking $g := \left(\frac{1}{\sqrt{p_\psi}}\chi_{U_{\langle\psi\rangle}}\right) \bullet \psi$, we obtain that, for almost every $\xi \in \mathbb{R}$,

$$\sigma_{V_0(\psi)}(\xi) = \sum_{j=1}^{\infty} |\widehat{g}(2^j\xi)|^2.$$

It follows then that (see [**ŠSW08**, (27)])

$$\int_0^1 \dim_{V_0(\psi)}(\xi)d\xi = \int_0^1 \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} |\widehat{g}(2^j(\xi + k))|^2 d\xi = \|\widehat{g}\|_2^2 < \infty.$$

Hence we conclude that $\dim_{V_0(\psi)}$ must be finite almost everywhere. □

It is now straightforward to describe completely the structure of $\{\psi_{jk} : j, k \in \mathbb{Z}\}$ for functions ψ for which $\langle\psi\rangle$ is of Type 3. Observe that this completely covers the "MSF-case" orthogonality properties.

THEOREM 2.16. *Let $\psi \in L^2(\mathbb{R})$ and let $K :=$ ssupp $(\widehat{\psi})$.*

(a) $\langle\psi\rangle$ is of Type 3 if and only if $\{K + k : k \in \mathbb{Z}\}$ is an almost everywhere-disjoint family. If $\langle\psi\rangle$ is of Type 3, then $\mathcal{B}_\psi$ is a Parseval frame for $\langle\psi\rangle$ if and only if $\widehat{\psi} = \chi_K$.

(b) If $\langle\psi\rangle$ is of Type 3, then ψ is a semiorthogonal pre-wavelet if and only if $\{2^jK : j \in \mathbb{Z}\}$ is an almost everywhere disjoint family. If $\langle\psi\rangle$ is of Type 3 and ψ is a semiorthogonal pre-wavelet, then the following properties hold:

- *$V_0(\psi)$ is a shift-invariant space.*

- $\dim_{V_0(\psi)} < \infty$ *almost everywhere.*
- $\{V_j : j \in \mathbb{Z}\}$ *is a pre-GMRA.*
- $V_j(\psi)$ *is shift-invariant space for every* $j \in \mathbb{Z}$.
- $V_j(\psi) = \displaystyle\bigoplus_{\ell=-\infty}^{j-1} W_\ell(\psi)$ *for every* $j \in \mathbb{Z}$.
- $V_{-\infty}(\psi) = \{0\}$.
- $V_\infty(\psi) = L^2(\displaystyle\bigcup_{j\in\mathbb{Z}} 2^j K)^\vee$.
- $\sigma_{V_j}(\psi) = \displaystyle\sum_{\ell=j+1}^{\infty} \chi_K(2^\ell \boldsymbol{\cdot}) = \chi_{\bigcup_{\ell=j+1}^{\infty} 2^{-\ell}K}$ *for every* $j \in \mathbb{Z}$.
- $\sigma_{V_\infty}(\psi) = \chi_{\bigcup_{j\in\mathbb{Z}} 2^j K}$.
- $\dim_{V_0(\psi)} = \displaystyle\sum_{j<0} \chi_{\bigcup_{k\in\mathbb{Z}}(2^j K + k)}$.

(c) If $\langle\psi\rangle$ is of Type 3 and ψ is a semiorthogonal pre-wavelet, then ψ is a Parseval frame wavelet if and only if, for almost every $\xi \in \mathbb{R}$,

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1. \tag{2.17}$$

If this is the case, then $\widehat{\psi} = \chi_K$ (i.e. ψ is of MSF type), $\{2^j K : j \in \mathbb{Z}\}$ is an almost everywhere partition of $\mathbb{R}$, and $\{V_j(\psi) : j \in \mathbb{Z}\}$ is a GMRA. Furthermore, these last two conditions are equivalent and are equivalent to $(\chi_K)^\vee$ being a Parseval frame wavelet.

(d) If $\langle\psi\rangle$ is of Type 3 and ψ is a semiorthogonal Parseval frame wavelet, then ψ is an orthonormal wavelet if and only if $\{K + k : k \in \mathbb{Z}\}$ is an almost everywhere partition of $\mathbb{R}$. If $\langle\psi\rangle$ is of Type 3 and ψ is a semiorthogonal pre-wavelet, then ψ is an orthonormal wavelet if and only if, for almost every $\xi \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi + k)|^2 = 1 \text{ and } \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1$$

Remark 2.18. If we combine the previous theorem with Proposition 2.9.10, then we conclude that the choice of GMRA structures is perhaps somewhat more limited than one may expect.

First of all, if $\psi \in L^2(\mathbb{R})$ is such that $\widehat{\psi} = \chi_K$ and ψ is an MSF Parseval frame wavelet, then, for every $n \in \mathbb{N}$, $\psi_n := 2^{-n/2} D_{-n}(\psi)$ is an MSF Parseval frame wavelet. Furthermore, they share essentially the same GMRA $\{V_j : j \in \mathbb{Z}\}$, except that the core space for the "ψ_n-GMRA" is "shifted" to $V_{-n}(\psi)$.

Secondly, recall that every orthonormal wavelet, η, is associated with a GMRA $\{V_j(\eta) : j \in \mathbb{Z}\}$. Furthermore, for every such GMRA, there exists an MSF wavelet ψ such that $\dim_{V_0(\eta)} = \dim_{V_0(\psi)}$. If, in addition, $\widehat{\psi}$ is compactly supported, then there exists an MRA Parseval frame wavelet which generates essentially the same GMRA. More precisely, there is an MRA Parseval frame wavelet pseudo-scaling function φ (see [**PŠWX01**], [**PŠWX03**], [**ŠSW08**] for details) and $n \in \mathbb{N}$ such that $V_0(\psi) = D_n(\langle\varphi\rangle)$. ■.

When considering functions $\psi \in L^2(\mathbb{R})$ such that $\langle\psi\rangle$ is of Type 1 or of Type 2, several aspects of the above construction change. Some of the issues are illustrated in the following example.

EXAMPLE 2.19. Consider $\psi_0 \in L^2(\mathbb{R})$ such that

$$\widehat{\psi_0} = \chi_{[-2,-1)} + \chi_{[1,2)},$$

and $\psi_1 \in L^2(\mathbb{R})$ such that

$$\widehat{\psi_1} = -\chi_{[-2,-1)} + \chi_{[1,2)}.$$

Observe that $\mathcal{T}_{\langle\psi_0\rangle} = \mathcal{T}_{\langle\psi_1\rangle} = \{3\}$, i.e. both $\langle\psi_0\rangle$ and $\langle\psi_1\rangle$ are of Type 2. Furthermore, we have that ψ_0 (and ψ_1, for that matter) is a semiorthogonal pre-wavelet. Notice that $\langle\psi_0\rangle \perp \langle\psi_1\rangle$, that, for every $j \in \mathbb{Z}$, $W_j(\psi_0) \perp \langle\psi_1\rangle$, and that $T_{1/2}\psi_0 \in \langle\psi_1\rangle$. To understand this example, it is useful to notice that

$$L^2([-2,-1) \cup [1,2))^\vee = \langle\psi_0\rangle \oplus \langle\psi_1\rangle$$

and

$$\dim_{\langle\psi_0\rangle} \equiv 1 \equiv \dim_{\langle\psi_1\rangle}.$$

It follows that

$$V_0(\psi_0) = \bigoplus_{\ell=-\infty}^{-1} W_\ell(\psi)$$

and

$$T_{1/2}\psi_0 \notin V_0(\psi_0). \tag{2.20}$$

It follows (consult (1.14)) that $V_0(\psi_0)$ is not a shift-invariant space. Moreover, by essentially the same argument, none of the spaces $D_j(V_0(\psi_0))$, for $j \in \mathbb{Z}$ are shift-invariant spaces (despite the fact that $D(V_0(\psi_0)) = V_0(\psi_0) \oplus \langle\psi_0\rangle$). Even the space $\bigoplus_{j\in\mathbb{Z}} W_j(\psi_0)$ is not a shift-invariant space.

Observe also that $\mathcal{B}_\psi$ is an orthonormal basis for $\langle\psi_0\rangle$, where

$$\psi := \frac{1}{\sqrt{2}} \bullet \psi_0.$$

Moreover, $\{\psi_{jk} : j, k \in \mathbb{Z}\}$ is an orthonormal system (see [**HW96**, p.100–103]) and, for almost every $\xi \in \mathbb{R}$,

$$\frac{1}{2} = \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 > 0.$$

Still, $V_0(\psi) = V_0(\psi_0)$ is not a shift-invariant space. Notice also that ψ_0 and ψ do not satisfy the "t_q-equation" for $q = 3$.

$\diamondsuit$

Going beyond the Type 3 case, let us observe a few simple details. Consider a semiorthogonal pre-wavelet ψ_0 and define $\psi \in L^2(\mathbb{R})$ by

$$\psi := \left(\frac{1}{\sqrt{p_{\psi_0}}}\chi_{U_{\langle\psi_0\rangle}}\right) \bullet \psi_0 \tag{2.21}$$

Then ψ is a semiorthogonal pre-wavelet, $\langle\psi\rangle = \langle\psi_0\rangle$, and $\mathcal{B}_\psi$ is a Parseval frame for $\langle\psi_0\rangle$. By a slight abuse of notation, for $j \in \mathbb{Z}$, we define a closed subspace,

$$V_j := \bigoplus_{\ell=-\infty}^{j-1} W_\ell(\psi). \tag{2.22}$$

It follows that $DV_j = V_{j+1}$ for every $j \in \mathbb{Z}$, but V_0 is not(!) necessarily a shift-invariant space. Let us also define two more closed subspaces:

$$V_\infty := \bigoplus_{j\in\mathbb{Z}} W_j(\psi), \text{ and}$$
$$U_0 := \bigoplus_{j=0}^{\infty} W_j(\psi). \tag{2.23}$$

Notice that U_0 is a shift-invariant space (and, therefore, $U_0^\perp$ is also a shift-invariant space), but V_∞ may or may not be a shift-invariant space. Furthermore,

$$V_0 = V_\infty \cap U_0^\perp. \tag{2.24}$$

The following statement is now a straightforward consequence.

LEMMA 2.25. *If $\psi_0 \in L^2(\mathbb{R})$ is a semiorthogonal pre-wavelet and ψ is defined by (2.21), then the following are equivalent.*

(a) V_0 is a shift-invariant space;
(b) V_∞ is a shift-invariant space;
(c) $V_0 = \langle D_j\psi : j \in \mathbb{Z}, j < 0\rangle$;
(d) $\langle D_j\psi : j \in \mathbb{Z}, j < 0\rangle \subseteq V_\infty$.

Observe also that for ψ given by (2.21), we have, for almost every $\xi \in \mathbb{R}$,

$$\sigma_{U_0}(\xi) = \sum_{j=-\infty}^{0} |\widehat{\psi}(2^j\xi)|^2. \tag{2.26}$$

It follows that, for almost every $\xi \in \mathbb{R}$,

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 \leq 1. \tag{2.27}$$

Furthermore, since $\mathcal{B}_\psi$ is a Parseval frame for $\langle\psi\rangle$ and D is a unitary operator, it is not difficult to see that

$$\{\psi_{jk} : j,k \in \mathbb{Z}\} \text{ is a Parseval frame for } V_\infty. \tag{2.28}$$

THEOREM 2.29. *Let $\psi_0 \in L^2(\mathbb{R})$ be a semiorthogonal pre-wavelet and let ψ be defined by (2.21). If ψ satisfies (1.12) — the "t_q equation" — and, for almost every $\xi \in \mathbb{R}$,*

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 0 \text{ or } 1, \tag{2.30}$$

then $V_0 = V_0(\psi) = V_0(\psi_0)$ is a shift-invariant space and, for almost every $\xi \in \mathbb{R}$,

$$\sigma_{V_0(\psi)}(\xi) = \sum_{j=1}^{\infty} |\widehat{\psi}(2^j\xi)|^2;$$

moreover, $V_\infty = V_\infty(\psi) = V_\infty(\psi_0) = L^2(E)^\vee$, where

$$E := \{\xi \in \mathbb{R} : \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1\}.$$

Furthermore, ψ is a semiorthogonal Parseval frame wavelet (for $L^2(\mathbb{R})$) if and only if $E = \mathbb{R}$ up to a null set.

PROOF. Using a standard argument from [**HW96**, p.338–342] we can show that for every f in a dense subset of $L^2(R)^\vee$ we have

$$\begin{aligned}\|f\|_2^2 &= \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \left(\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2\right) d\xi \\ &= \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \left(\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2\right) d\xi + \int_{\mathbb{R}} \overline{\widehat{f}(\xi)} \left(\sum_{p\in\mathbb{Z}} \sum_{q\in 2\mathbb{Z}+1} \widehat{f}(\xi+2^p q) t_q(2^{-p}\xi)\right) d\xi \\ &= \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{jk}\rangle|^2.\end{aligned}$$

Hence $\{\psi_{jk} : j,k \in \mathbb{Z}\}$ is a Parseval frame for $L^2(E)^\vee$. In particular, $L^2(E)^\vee \subseteq V_\infty$, and, obviously, $V_\infty = V_\infty(\psi) = V_\infty(\psi_0)$ is a shift-invariant space. By Lemma 2.25, $V_0 = V_0(\psi) = V_0(\psi_0)$ is a shift-invariant space as well. Using a standard argument (see [**PŠWX01**], [**PŠWX03**], [**ŠSW08**]) we obtain that, for almost every $\xi \in \mathbb{R}$,

$$\dim_{V_0(\psi)}(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^j(\xi+k)|^2. \tag{2.31}$$

Integrating both sides over $[0,1]$ leads to $\dim_{V_0(\psi)}$ being finite almost everywhere. It now follows immediately that, for almost every $\xi \in \mathbb{R}$,

$$\sigma_{V_0(\psi)}(\xi) = \sum_{j=1}^{\infty} |\widehat{\psi}(2^j\xi)|^2.$$

The last statement of this theorem is now obvious. □

A natural question is now whether (1.12) (the "t_q equation") and (2.30) (let us call it the "generalized Calderón condition") are also necessary. In the case of a Parseval frame wavelet, the answer is well known and positive (see [**HW96**, Section 7]). The issue is a bit subtler when we consider it in greater generality. In the Parseval frame wavelet case, the role of V_∞ is played by $L^2(\mathbb{R})$, and it is not immediately clear which of $L^2(\mathbb{R})$'s many properties are essential to prove the desired result. We found it useful to consult the proof for general reproducing function systems given in [**Lab02**] and [**HLW02**]. It turns out that the invariance to all translations plays a crucial role. We begin with the following auxiliary result which, at least partially, would be considered "folklore".

PROPOSITION 2.32. *If $V \leq L^2(\mathbb{R})$ is a shift-invariant space, then the following are equivalent.*

(a) $D(V) = V$.
(b) $T_\alpha(V) = V$ for every $\alpha \in \mathbb{R}$ and $D(V) = V$;
(c) $V = L^2(E)^\vee$, where $E \subset \mathbb{R}$ is measurable and $E = 2E$ up to null sets.

PROOF. It is obvious that $(c) \Rightarrow (b) \Rightarrow (a)$. Let us prove that $(a) \Rightarrow (c)$.

Given (a), recall that, for every $j \in \mathbb{Z}$, we have

$$D_j T_k \psi = T_{k/2^j} D_j \psi. \tag{2.33}$$

Since V is a shift-invariant space, we have $T_k(V) = V$ for every $k \in \mathbb{Z}$. Since $D(V) = V$, we obtain from (2.33) that, for every $k \in \mathbb{Z}$ and every $n \in \mathbb{N}$,

$$T_{k/2^n}(V) = V. \tag{2.34}$$

Since translational invariances form a closed subgroup of $(\mathbb{R}, +)$, we obtain that, for every $\alpha \in \mathbb{R}$,

$$T_\alpha(V) = V. \tag{2.35}$$

By Theorem 2.9.12, we obtain $V = L^2(E)^\vee$ for some measurable set E. Applying (a) again, we conclude that $E = 2E$ almost everywhere. $\square$

THEOREM 2.36. *Let $\psi \in L^2(\mathbb{R})$ be a semi-orthogonal pre-wavelet such that $\mathcal{B}_\psi$ is a Parseval frame for $\langle\psi\rangle$. If $V_0(\psi)$ is a shift-invariant space, then ψ satisfies (1.12) and (2.30).*

PROOF. By Lemma 2.25, $V_0(\psi)$ being a shift-invariant space implies $V_\infty(\psi)$ is a shift-invariant space. By the definition of $V_\infty(\psi)$ we obtain that $D(V_\infty(\psi)) = V_\infty(\psi)$. By Proposition 2.32, there exists $E \subseteq \mathbb{R}$ which is measurable, satisfies $E = 2E$ almost everywhere, and

$$V_\infty(\psi) = L^2(E)^\vee. \tag{2.37}$$

In particular, for every $j, k \in \mathbb{Z}$, ssupp $(\widehat{\psi_{jk}}) \subseteq E$ almost everywhere. Hence, for almost every $\xi \in E^c$,

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 0.$$

Similarly, if at least one of the points ξ or $\xi + q$ belongs to E^c, we have (1.12) for such a ξ. It follows that we need to prove that for almost every $\xi \in E$,

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1,$$

and that for almost every $\xi \in E$, we have (1.12) whenever $\xi + q \in E$, too.

Observe that, since $\sigma_{V_\infty(\psi)} = \chi_E$ for almost every $\xi \in E$, we have

$$1 \geq \sigma_{V_0(\psi)}(\xi) + \sum_{j=0}^{\infty} \sigma_{W_j(\psi)}(\xi).$$

Since $E = 2E$ almost everywhere, we obtain that, for almost every $\xi \in E$,

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 \leq 1;$$

in particular, $\xi \mapsto \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2$ is locally integrable. We can now repeat the standard proof from [**HW96**], Section 7.1, with all functions being adjusted on E versus $\mathbb{R}$, to conclude the proof of the theorem. $\square$

REMARK 2.38. It is worth mentioning that even a more general proof, the one given for the main result in [**HLW02**], applies and will lead to the proof of the theorem above. The significance of Proposition 2.32 may be even more transparent in that context since it allows for a function "$\omega(x) = N^2(T_x f)$" (given in Proposition 2.4 in [**HLW02**]) to be defined for all $x \in \mathbb{R}$, i.e. for $f \in V_\infty(\psi)$ we need that $T_x f \in V_\infty(\psi)$ for every $x \in \mathbb{R}$.

■

It is now straightforward to prove the following results.

Corollary 2.39. *If $\psi \in L^2(\mathbb{R})$ is a semiorthogonal pre-wavelet such that $V_0(\psi)$ is a shift-invariant space, then*

$$\dim_{V_0(\psi)} < \infty \text{ almost everywhere.}$$

For the following result, recall Section 2.7.

Corollary 2.40. *If $\{V_j\}$ is a pre-GMRA (in a general sense) such that (2.7.19) holds, then there exist two measurable sets $E_{-\infty} \subseteq E_\infty \subseteq \mathbb{R}$ such that $2E_{-\infty} = E_{-\infty}$ almost everywhere, $2E_\infty = E_\infty$ almost everywhere, and*

$$V_{-\infty} = L^2(E_{-\infty})^\vee \leq L^2(E_\infty)^\vee = V_\infty.$$

We conclude this section with a remark about coefficients. Consider a semiorthogonal pre-wavelet $\psi \in L^2(\mathbb{R})$ such that $V_0(\psi)$ is a shift-invariant space. Then, for some set $E \subseteq \mathbb{R}$ with $E = 2E$ almost everywhere, we have

$$L^2(E)^\vee = \bigoplus_{j\in\mathbb{Z}} W_j(\psi). \tag{2.41}$$

Some level of the reproducing property is possible even if $\mathcal{B}_\psi$ is not a Parseval frame for $\langle\psi\rangle$. Let us be more precise. Given a function $f \in L^2(E)^\vee$, consider, for every $j \in \mathbb{Z}$, a function $f_j \in W_j(\psi)$ which is the orthogonal projection of f on $W_j(\psi)$. Recall that then the sum

$$\sum_{j\in\mathbb{Z}} \|f_j\|_2^2$$

converges and is equal to $\|f\|_2^2$. Hence, we have unconditional convergence in the following formula:

$$f = \sum_{j\in\mathbb{Z}} f_j. \tag{2.42}$$

Using results from Section 1.6 we obtain that, for every $j \in \mathbb{Z}$,

$$p_\psi \bullet (D_{-j} f_j) = \sum_{k\in\mathbb{Z}} \langle D_{-j} f_j, T_k\psi\rangle T_k\psi; \tag{2.43}$$

notice that $D_{-j} f_j \in \langle\psi\rangle$. Using the notation $\nu_j(\xi) := p_\psi(2^{-j}\xi)$, $j \in \mathbb{Z}$, we apply D_j on (2.43) and, since D_j is unitary, we obtain, for every $j \in \mathbb{Z}$,

$$\begin{aligned} (\nu_j \widehat{f}_j)^\vee &= \sum_{k\in\mathbb{Z}} \langle f_j, \psi_{jk}\rangle \psi_{jk} \\ &= \sum_{k\in\mathbb{Z}} \langle f, \psi_{jk}\rangle \psi_{jk} \end{aligned} \tag{2.44}$$

3. General Case

In this section, we consider $\psi \in L^2(\mathbb{R})$ such that $V_0(\psi)$ is a shift-invariant space; in other words, $\psi \in \mathcal{V}_{\mathrm{SIS}}$ (recall (1.13)). Using (1.16), (1.17), (1.21), and Proposition 2.32, we obtain directly a complete description for the class of functions which satisfy (1.16).

Corollary 3.1. *If $\psi \in \mathcal{V}_{SIS}$ and $\psi \in V_0(\psi)$, then there exists a measurable set $E \subset \mathbb{R}$ such that $E = 2E$ almost everywhere and, for every $j \in \mathbb{Z}$,*

$$L^2(E)^\vee = V_j(\psi) = V_{-\infty}(\psi) = V_\infty(\psi).$$

If, in addition, $|E| > 0$, then

$$\dim_{V_0(\psi)} \equiv \infty.$$

REMARK 3.2. In [**BR05**] the authors constructed examples of functions $\varphi \in L^2(\mathbb{R})$ such that $V_0(\varphi) = L^2(\mathbb{R})$. Consider a measurable set $E \subseteq \mathbb{R}$ with $2E = E$ almost everywhere. Define $\psi \in L^2(\mathbb{R})$ such that $\widehat{\psi} := \widehat{\varphi}\chi_E$. Observe that we have $\widehat{\psi_{jk}} = \widehat{\varphi_{jk}}\chi_E$, since $E = 2E$ almost everywhere. It follows then easily that $V_0(\psi) = L^2(E)^\vee$ and $\psi \in V_0(\psi)$. $\diamondsuit$

We turn our attention now to the more interesting case of $\psi \in \mathcal{V}_{\mathrm{SIS}}$ such that

$$\psi \notin V_0(\psi). \tag{3.3}$$

Let us denote by $\eta = \eta(\psi)$ the orthogonal projection of ψ on (the closed space) $V_0(\psi)$ and denote by φ,

$$\varphi = \varphi(\psi) := \psi - \eta. \tag{3.4}$$

Since $V_0(\psi)$ is a shift-invariant space, it is easy to check (and it is well known) that, for every $k \in \mathbb{Z}$,

$$T_k\eta = \text{ the orthogonal projection of } T_k\psi \text{ on } V_0(\psi). \tag{3.5}$$

Hence, we have the well-known result that

$$D(V_0(\psi)) = V_0(\psi) \oplus \langle\varphi\rangle. \tag{3.6}$$

It follows that for every $n \in \mathbb{N}$,

$$D_{-n}(\langle\varphi\rangle) \leq V_0(\psi) \perp \langle\varphi\rangle. \tag{3.7}$$

Hence φ is a semiorthogonal pre-wavelet and

$$V_0(\varphi) \leq V_0(\psi). \tag{3.8}$$

To understand better what can transpire, it is useful at this point to consider the following example.

EXAMPLE 3.9. Take a function $\eta \in L^2(\mathbb{R})$ such that $V_0(\eta) = L^2((-\infty, 0))^\vee$ (see Example 3.2). Take a function $\varphi \in L^2(\mathbb{R})$ such that $\widehat{\varphi} := \chi_{[1/2,1)}$. Consider $\psi := \eta + \varphi$ and notice that $\eta \perp \varphi$. Observe that, for every $n \in \mathbb{N}$, $\eta \in V_{-\infty}(\eta) \leq V_{-n}(\eta)$ and $\lim_{n\to\infty} 2^{-n/2} D_{-n}\varphi = 0$. It is not difficult to see that

$$V_{-\infty}(\eta) = V_\infty(\eta) = L^2((-\infty, 0))^\vee$$

and

$$V_{-\infty}(\varphi) = \{0\} \leq V_0(\varphi) = L^2([0, 1/2))^\vee \leq V_\infty(\varphi) = L^2([0, \infty))^\vee.$$

It follows that, for every $j \in \mathbb{Z}$,

$$V_j(\psi) = L^2((-\infty, 2^{j-1}))^\vee.$$

In particular, $\varphi = \varphi(\psi)$ and $\eta = \eta(\psi)$.

$\diamondsuit$

It turns out that the key property is the shift-invariant property of $V_0(\varphi)$.

THEOREM 3.10. *Let $\psi \in \mathcal{V}_{SIS}$ and let η be the orthogonal projection of ψ on $V_0(\psi)$. Then $V_\infty(\psi)$ is a shift-invariant space and*

$$V_\infty(\psi) = V_0(\psi) \oplus \left(\bigoplus_{j \geq 0} W_j(\varphi) \right),$$

where $\varphi := \psi - \eta$. Furthermore, $\langle \varphi \rangle = \{0\}$ if and only if $\psi \in V_0(\psi)$. Moreover, if $V_0(\varphi)$ is a shift-invariant space, then $\dim_{V_0(\varphi)} < \infty$ *almost everywhere and $U := V_0(\psi) \cap (V_0(\varphi)^\perp)$ is a shift-invariant space such that $D(U) = U$ and*

$$V_\infty(\psi) = U \oplus \left(\bigoplus_{j \in \mathbb{Z}} W_j(\varphi) \right)$$

PROOF. The first statement follows from Lemma 2.25, (3.6), and (3.7). The second statement is obvious. Let us prove the third statement. If $V_0(\varphi)$ is a shift-invariant space, then $V_\infty(\varphi)$ is a shift-invariant space, too. Obviously then $V_\infty(\varphi) = \bigoplus_{j \in \mathbb{Z}} W_j(\varphi)$ and there exists $E \subseteq \mathbb{R}$, measurable and with $E = 2E$ almost everywhere, such that $V_\infty(\varphi) = L^2(E)^\vee$ and, moreover, $\dim_{V_0(\varphi)} < \infty$ almost everywhere (see Theorem 2.36 and Corollary 2.39). Since both $V_0(\varphi)$ and $V_0(\psi)$ are shift-invariant spaces and (3.8) holds, we obtain that U is shift invariant as well. Observe that

$$V_0(\varphi)^\perp = V_\infty(\varphi)^\perp \oplus \left(\bigoplus_{j \geq 0} W_j(\varphi) \right),$$

$$V_\infty(\varphi)^\perp = L^2(E^c)^\vee,$$

and

$$\bigoplus_{j \geq 0} W_j(\varphi) \perp V_0(\psi).$$

It follows that

$$U = V_0(\psi) \cap L^2(E^c)^\vee. \tag{3.11}$$

Since D is a bijection and $2E^c = E^c$ almost everywhere we obtain

$$\begin{aligned} D(U) &= D(V_0(\psi)) \cap D(L^2(E^c)^\vee) \\ &= (V_0(\psi) \oplus \langle \varphi \rangle) \cap L^2(E^c)^\vee \\ &= V_0(\psi) \cap L^2(E^c)^\vee \\ &= U, \end{aligned}$$

since $\langle \varphi \rangle \subseteq L^2(E)^\vee$. It follows that there exists a measurable set $G \subseteq \mathbb{R}$ such that $G = 2G$ almost everywhere and $U = L^2(G)^\vee$. Observe that $G \subseteq E^c$ almost everywhere and that

$$V_\infty(\psi) = L^2(G \cup E)^\vee. \tag{3.12}$$

The last statement follows easily, and also we have that

$$V_\infty(\psi) = L^2(\mathbb{R}) \Leftrightarrow G = E^c \text{ almost everywhere.} \tag{3.13}$$

□

Recall now the open questions listed in Remark 1.24. As the following result shows, we have actually resolved (in a positive way) one of these questions.

COROLLARY 3.14. *If $\psi \in \mathcal{V}_{SIS}$, then*

$$\bigcap_{j\in\mathbb{Z}} D_j(V_0(\psi)) \neq \{0\} \Leftrightarrow \dim_{V_0(\psi)} \equiv \infty.$$

PROOF. As mentioned in Remark 1.24 the proof of "⇒" is an important theorem by M. Bownik, [**Bow09**]. We need to prove "⇐", i.e.

$$\bigcap_{j\in\mathbb{Z}} D_j(V_0(\psi)) = \{0\} \Rightarrow \dim_{V_0(\psi)} < \infty \text{ almost everywhere.} \tag{3.15}$$

Observe first that if the intersection is trivial, then either $\psi = 0$ or $\psi \notin V_0(\psi)$ (see Corollary 3.1).

Furthermore, we know that both $V_0(\varphi)$ and $V_\infty(\varphi)$ are closed subspaces (not necessarily shift-invariant spaces) and we can define $U := V_0(\psi) \cap (V_0(\varphi)^\perp)$. Then U is a closed subspace of $L^2(\mathbb{R})$ and arguing as in the proof of Theorem 3.10 shows that U satisfies (3.11) in the sense that

$$U = V_0(\psi) \cap V_\infty(\varphi)^\perp$$

and $D(U) = U$ — observe that $D(V_\infty(\varphi)) = V_\infty(\varphi)$ by its definition (irrespective of $V_\infty(\varphi)$ being a shift-invariant space). Since $U \leq V_0(\psi)$ and $D(U) = U$, we obtain

$$U \leq \bigcap_{j\in\mathbb{Z}} D_j(V_0(\psi)) = \{0\}, \tag{3.16}$$

i.e. U is trivial. It follows that $V_0(\psi) = V_0(\varphi)$. In particular, $V_0(\varphi)$ is a shift-invariant space and the corresponding U is trivial. By Theorem 3.10, we obtain that $\dim_{V_0(\psi)} < \infty$ almost everywhere. □

Notice that we have also proved the following result.

COROLLARY 3.17. *Let $\psi \in \mathcal{V}_{SIS}$, let η be the orthogonal projection of ψ on $V_0(\psi)$, and let $\varphi := \psi - \eta$. Then*

$$\bigcap_{j\in\mathbb{Z}} D_j(V_0(\psi)) = \{0\} \Leftrightarrow V_0(\psi) = V_0(\varphi).$$

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