Analytic construction of Markov processes

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Introduction

Lévy process: stationary indep. increm.: $\forall t_0 < t_1 < \cdots < t_n$

$$X_{t_1} - X_{t_0}, \quad X_{t_2} - X_{t_1}, \quad X_{t_n} - X_{t_{n-1}} \quad \text{indep.}$$

$$X_t - X_s \doteq X_{t-s}. \quad \text{stat.}$$

Characteristic function: $\mathbb{E}e^{i\xi X_t} = e^{-t\psi(\xi)}$.
Lévy-Khinchin repr.: for ψ:

$$\psi(\xi) = -ib \cdot \xi + \frac{1}{2}\xi Q \xi + \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{i\xi u} + i\xi u 1_{\{|u| \le 1\}}\right) \mu(du),$$

$b \in \mathbb{R}^d$, $Q : \mathbb{R}^n \to \mathbb{R}^d \times \mathbb{R}^d$, pos. semi-def, μ- Lévy meas.:

$$\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |u|^2) \mu(du) < \infty.$$

Transition probab. dens. (if exists): $\mathbb{P}^x(X_t \in dy) = p_t(y - x)dy$.

Question: what happens, if we loose indep. and stat. of increm.?

- still might have the Markov property
- $p_t(x,y) \neq p_t(y-x)$

Examples:

1. SDE, driven by a Lévy noise:

$$dX_t = b(X_t)dt + \sigma(X_{t-})dZ_t, \quad X_0 = x.$$

2. Let $\psi(\xi)$ be the characteristic exponent of a LP Z.
"Lévy-type":

$$\psi(x,\xi) \leftrightsquigarrow X_t,\, X_0 = x \quad \text{where } X_t\text{- Markov proc.?}$$

$e^{-t\psi(x,\xi)}$ – it is NOT clear why it is a charact. func.!!!

Analytic approach, "recipe"

• Brownian motion, $B_0 = x$. Trans. probab. dens.:
$p_t(y-x) = (2\pi t)^{-d/2} e^{-|y-x|^2/(2t)}$.

$p_t(x,y) \equiv p_t(y-x)$ is the fund. solution to the Cauchy Problem

$$\partial_t p_t(x,y) = \frac{1}{2}\Delta_x p_t(x,y), \quad y \in \mathbb{R}^d,$$

and

$$\lim_{t\to 0} \int f(y) p_t(x,y) dy = f(x), \quad f \in C_b(\mathbb{R}^d). \tag{1}$$

(**Recall:** $p_t(x,y)$ is the *fundamental sol.* to a Cauchy Problem for $\partial_t - L$, if $(\partial_t - L_x)p_t(x,y) = 0$, and (1) holds true.)

Characteristic function for B_t: $\mathbb{E}e^{i\xi B_t} = e^{-\frac{t}{2}|\xi|^2}$.

$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{i\xi x} f(x)dx$, $\mathcal{F}^{-1}g(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi x} g(\xi)dx$.

Note that:

$$-\mathcal{F}(\Delta f)(\xi) = |\xi|^2 \mathcal{F}f(\xi),$$

$|\xi|^2$ is called the *symbol* of $-\Delta$.

Taking $\mathcal{F}$ of

$$\partial_t p_t(x) = \frac{1}{2}\Delta p_t(x)$$

we get

$$\partial_t \mathcal{F}p_t(\xi) = -\frac{1}{2}|\xi|^2 \mathcal{F}p_t(\xi) \quad \Rightarrow \quad \mathcal{F}p_t(\xi) = e^{-\frac{t}{2}|\xi|^2}$$

$$p_t(x) = \mathcal{F}^{-1}\left(e^{-\frac{t}{2}|\cdot|^2}\right) = (2\pi t)^{-d} e^{-\frac{|x|^2}{2t}}.$$

Lévy process: The same!
$p_t(x)$ (if exists) – the unique sol. to a Cauchy problem for

$$\partial_t - L, \quad symb(L)(\xi) = -\psi(\xi).$$

i.e. for $f \in S(\mathbb{R}^d)$ define

$$Lf(x) := \mathcal{F}^{-1}\Big(- \psi(\xi)\mathcal{F}f(\xi)\Big)$$

Use the Lévy-Khinchin representation for ψ, and that

$$\mathcal{F}^{-1}\big(i\xi\mathcal{F}f(\xi)\big) = \mathcal{F}^{-1}\big(\mathcal{F}(\nabla f(\xi))\big) = \nabla f(x),$$

$$\mathcal{F}^{-1}\big(e^{i\xi u}\mathcal{F}f(\xi)\big) = f(x+u).$$

Integral representation for L, Lévy case:

- For $f \in S(\mathbb{R}^d)$:

$$
\begin{aligned}
Lf(x) &= \mathcal{F}^{-1}(-\psi f)(x) \\
&= b\nabla f(x) + \frac{1}{2} div\Big(Q\nabla f(x)\Big) \\
&\quad + \int_{\mathbb{R}^n \setminus \{0\}} \left(f(x+u) - f(x) - \nabla f(x) u 1_{|u|\leq 1}\right) \mu(du).
\end{aligned}
$$

- extends up to $f \in C^2_\infty(\mathbb{R}^d)$ ($C^k_\infty(\mathbb{R}^d)$: k-times diff., vanish at ∞)

Cauchy problem: $\forall y \in \mathbb{R}^d$

$$
\begin{aligned}
\partial_t p_t(x,y) &= L_x p_t(x,y), \quad t > 0, \, x \in \mathbb{R}^d, \\
p_t(x,y) &\Rightarrow \delta_x(y), \quad t \to 0.
\end{aligned}
$$

Feller semigroup:

C^k_∞: k times differentiable, with derivatives vanishing at ∞.

$\|\cdot\|_\infty$: norm in C_∞.

- $P_t : C_\infty(\mathbb{R}^d) \to C_\infty(\mathbb{R}^d)$, $P_0 = I$, $P_t \circ P_s = P_{t+s}$.
- Strongly continuous: $\lim_{t\to 0} \|P_t f - f\|_\infty = 0$, $f \in C_\infty$.
- Contraction: $\|P_t\|_\infty \leq 1$.
- Positivity preserving: $f \geq 0$, then $P_t f \geq 0$.

Recall that an oper. $(A, D(A))$ is a **generator** of a Feller semigr. $(P_t)_{t\geq 0}$ if

$$Af = C_\infty\!-\!\lim_{t\to 0} \frac{P_t f - f}{t} \quad D(A) := \{f \in C_\infty : \exists C_\infty\!-\!\lim_{t\to 0} \frac{P_t f - f}{t}\}.$$

Semigroup version of the sol. to a Cauchy Problem:

$$P_t f(x) = \mathbb{E}^x f(Z_t) = \int_{\mathbb{R}^d} f(y) p_t(x,y) dy,$$

C_∞-closure $(A, D(A))$ of $(L, C_\infty^2(\mathbb{R}^d))$ is a generator of $(P_t)_{t\geq 0}$, and

$$\partial_t P_t f(x) = A P_t f(x) = P_t A f(x).$$

$P_t f(x)$ need not be in $C_\infty(\mathbb{R}^d)$!

What if $b = b(x)$, $Q = Q(x)$, $\mu = \mu(x, du)$?

Char. func. $\neq e^{-t\psi(x,\xi)}$!

Still: define

$$Lf(x) := \mathcal{F}^{-1}\big(-\psi(x,\cdot)f\big)(x).$$

$-\psi(x,\xi)$ is called a **symbol** of the operator L.

We can try to solve the Cauchy problem, and show that the solution gives rise to a Markov process!

Notation: L: "Lévy-type" integro-diff. operator.

Why such a generalization? In fact, this is all we expect from a generator of a Feller semigroup!

Courége-Waldenfels Theorem

Let $(A, D(A))$ be a gener. of a Feller proc., $C_c^\infty(\mathbb{R}^d) \subset D(A)$. Then $A\Big|_{C_c^\infty(\mathbb{R}^d)}$ is a **pseudo-differential operator** with the symbol $-q(x,\xi)$,

$$q(x,\xi) = q(x,0) - i\ell(x)\cdot\xi + \frac{1}{2}\xi\cdot Q(x)\xi + \int_{\mathbb{R}^d\setminus\{0\}} \left[1 - e^{i\xi u} + i\xi u 1_{(0,1)}|u|\right]\mu(x,du),$$

i.e.

$$Af(x) = -\int_{\mathbb{R}^d} e^{i\xi x} q(x,\xi)\mathcal{F}f(\xi)d\xi.$$

History

- **Elliptic** L: E. Levi (1907), Dressel (1940), Friedman (1964)

McKean, Singer (1967) – a bit different version, suitable for the discrete setting

- **Elliptic + bounded integral op.**: Feller (1936)

- **Extended to some integro-diff. oper.** Drin'(1977), Eidelman, Drin' (1981), Kochubei (1989)

Eidelman, Ivasishyn, Kochubei (2004)

Analytic context (except Kochubei) of Hypersingular integrals (cf. Samko et al.)

Kolokoltsov (2000) : perturbation of an α stable Lévy measure

Important: $\psi^{princilal}(x, c\xi) = c^{\alpha}\psi(x, \xi)$

• $(-\Delta)^{\alpha/2} + b(x)\nabla$, $\alpha \in (1,2)$: irregular (!!) drift
Portenko (1994), Portenko, Podolynny (1995)
Warning! We do not expect a Feller process, if, say, $b(\cdot) \in L_p$ (above papers)!

Bogdan, Jakubowski (2012), Bogdan, Sydor (Prepr.2013) Chen, Zhang (Prepr. 2013), Kim, Song (Prepr. 2014)

• **Symbolic calculus:** Tsutsumi (1974), (1977), Kumano-go (1981), Hoh (1998), Jacob (2002), Boettcher (2005, 2008)
i.e., smoothness conditions on the symbol $\psi(x,\xi)$

Notation: L^x: coefficients are x-dependent

Idea of the construction of the solution to a CP for

$$\partial_t - L^x$$

$\star$ **guess:** constr. of the "candidate for being a fund. solution";

$\star$ **verification;**

$\star$ **uniqueness**

I. Construction: the parametrix method (Levi)

Let $L = L^x \sim symb(-\psi(x,\xi))$, $Q \equiv 0$.
Suppose that $\exists p_t^z(y-x)$ is the fund. sol. to the CP for

$$\partial_t - L^z, \quad z - \text{fixed}.$$

and that we can write this solution as

$$p_t^z(x,y) := \mathcal{F}^{-1}\left(e^{-tq(z,\cdot)}\right)(y-x).$$

This function is called the **parametrix** for the CP for $\partial_t - L^x$.
Take a zero-order appr. of a fund. solution to $\partial_t - L^x$ as

$$p_t^0(x,y) := p_t^y(y-x).$$

Write $p_t(x,y)$ as **parametrix solution**

$$p_t(x,y) = p_t^0(x,y) + r_t(x,y). \tag{2}$$

Assume that we know (!!!) that $p_t(x, y)$ is the fund. sol.
Then applying $\partial_t - L^x$ to $p = p^0 + r$ we get

$$(\partial_t - L^x) r_t(x, y) = -(\partial_t - L^x) p_t^0(x, y) =: \Phi_t(x, y),$$

$$\begin{aligned} r_t(x, y) &= (p \star \Phi)_t(x, y) = (p^0 \star \Phi)_t(x, y) + (r \star \Phi)_t(x, y) \\ &= (p^0 \star \Phi)_t(x, y) + (p^0 \star \Phi^{\star 2})_t(x, y) + (r \star \Phi^{\star 2})_t(x, y) = \dots \end{aligned}$$

($\star$: space and time convolution)
Hence,

$$p_t(x, y) = p_t^0(x, y) + r_t(x, y),$$

$$r_t(x, y) = (p^0 \star \Psi)_t(x, y), \quad \Psi_t(x, y) = \sum_{k=1}^{\infty} \Phi_t^{\star k}(x, y),$$

is the solution to the Volterra-type eq.

$$p = p^0 + p \star \Phi.$$

Problem: (pointwise) convergence of the series.

Convergence of the series

Aim: result of type

Theorem 1

Suppose that the "coefficients" are Hölder cont. Then the series in the repr.

$$p_t(x, y) = p_t^0(x, y) + r_t(x, y)$$

converge in $C_\infty(\mathbb{R}^d)$.

We specify in some cases the precise conditions below.

First goal: estimate $\Phi_t(x, y)$ from above in a "proper way", $t \in (0, T]$, $x, y \in \mathbb{R}^d$.

Auxiliary

$*$: space convolution

Definition 1

$\{H_t(x,y), t>0, x,y\in\mathbb{R}^d\}$, $H_t(x,y)\geq 0$,
has a **sub-convolution property,** if $\forall T>0\ \exists C_{H,T}>0$:

$$(H_{t-s}*H_s)(x,y)\leq C_{H,T}H_t(x,y), \quad 0<s<t\leq T,\ x,y\in\mathbb{R}^d. \tag{3}$$

Super-convolution: "$\geq$".

Example: 1. $g_t^{(\alpha)}(x)$– rot. symm. α-stable dens., $\alpha=2$ -Gaussian.
Chapman-Kolmogorov $\Rightarrow$ sub/super-conv. prop.

2. $G(x)=(1+|x|)^{-d-\alpha}$, $G_t^{(\alpha)}(x)=t^{-d/\alpha}G^{(\alpha)}(x/t^{1/\alpha})$.
$G_t^{(\alpha)}(x)$: the sub/super-conv. prop. Note that $g_t^{(\alpha)}(x)\asymp G_t^{(\alpha)}(x)$.

Lemma 1

Suppose that for $t \in (0, T]$, $x, y \in \mathbb{R}^d$,

$$\left|\Phi_t(x,y)\right| \le C_{\Phi,T}\, t^{-1}\Big(t^{\delta_1} H^1_t(x,y) + t^{\delta_2} H^2_t(x,y)\Big), \qquad (4)$$

$\delta_1,\ \delta_2 \in (0,1)$, $H^1_t(x,y), H^2_t(x,y) \ge 0$ satisfy the sub-conv. prop.,

$$H^1_t(x,y) \ge H^2_t(x,y). \qquad (5)$$

$\zeta = \delta_1 \wedge \delta_2$. Then for every $t \in (0, T]$, $x, y \in \mathbb{R}^d$:

$$\left|\Phi^{\star k}_t(x,y)\right| \le \frac{C_1 C_2^k}{\Gamma(k\zeta)}\, t^{-1+(k-1)\zeta}\Big(t^{\delta_1} H^1_t(x,y) + t^{\delta_2} H^2_t(x,y)\Big), \qquad (6)$$

$$\left|\sum_{k=1}^{\infty} \Phi^{\star k}_t(x,y)\right| \le C t^{-1}\Big(t^{\delta_1} H^1_t(x,y) + t^{\delta_2} H^2_t(x,y)\Big). \qquad (7)$$

Corollary 1

If

$$p_t^0(x,y) \le H_t^1(x,y),$$

then

$$\left|\left(H^1 \star \sum_{k=1}^{\infty} \Phi^{\star k}\right)_t (x,y)\right| \le C t^{\zeta} H_t^1(x,y) \tag{8}$$

yields

$$r_t(x,y) \le C t^{\zeta} H_t^1(x,y), \quad 0 < t < T,\ x,y \in \mathbb{R}^d,$$

and

$$p_t(x,y) \le H_t^1(x,y) + t^{\zeta} H_t^1(x,y), \quad 0 < t < T,\ x,y \in \mathbb{R}^d. \tag{9}$$

Proof of Lemma 1: Induction

$$
\begin{aligned}
&|\Phi_t^{\star(k+1)}(x,y)| = \Big| \int_0^t (\Phi_{t-s}^{\star k} * \Phi_s)(x,y)\, ds \Big| \\
&\le \frac{C_1 C_2^k C_{\Phi,T}}{\Gamma(k\zeta)} \Big\{ \int_0^t (t-s)^{-1+(k-1)\zeta} (t-s)^{\delta_1} s^{-1+\delta_1} (H_{t-s}^1 * H_s^1)(x,y) ds \\
&+ \int_0^t (t-s)^{-1+(k-1)\zeta} (t-s)^{\delta_1} s^{-1+\delta_2} (H_{t-s}^1 * H_s^2)(x,y) ds \\
&+ \int_0^t (t-s)^{-1+(k-1)\zeta} (t-s)^{\delta_2} s^{-1+\delta_1} (H_{t-s}^2 * H_s^1)(x,y) ds \\
&+ \int_0^t (t-s)^{-1+(k-1)\zeta} (t-s)^{\delta_2} s^{-1+\delta_2} (H_{t-s}^2 * H_s^2)(x,y)\, ds \Big\} \\
&\le \frac{(\dots)\Gamma(k\zeta)\Gamma(\zeta)}{\Gamma(k\zeta)\Gamma((k+1)\zeta)} t^{-1+k\zeta} \Big(t^{\delta_1} H_t^1(x,y) + t^{\delta_2} H_t^2(x,y) \Big).
\end{aligned}
$$

1. $L = b(x)\nabla + a(x)\Delta$

$L^{(2)} := \frac{1}{2}\Delta$, $g_t^{(2)}(x) = t^{-d/2} g^{(2)}(xt^{-1/2})$– fund. sol. for $\partial_t - L^{(2)}$.

$G^{(2)}(x) = e^{-|x|^2}$.

Properties:

$$G^{(2)}((1+\epsilon)x) \le g^{(2)}(x) \le G^{(2)}((1-\epsilon)x), \tag{10}$$

$$\left|(\nabla^k g^{(2)})(x)\right| \le CG^{(2)}((1-\epsilon)x), \quad k \ge 1; \tag{11}$$

$$\left|(L^{(2)} g^{(2)})(x)\right| \le CG^{(2)}((1-\epsilon)x). \tag{12}$$

For any $\delta > 0$, there exists $C, \epsilon > 0$ such that

$$|x|^\delta G^{(2)}(x) \le CG^{(2)}((1-\epsilon)x), \quad \epsilon \in (0,1); \tag{13}$$

$$p_t^z(x) := \frac{1}{(ta(z))^{d/2}} g^{(2)}\Big(\frac{x}{(a(z)t)^{1/2}}\Big)$$

is the tr. pr. dens. for a LP with generator $L^z = -\frac{a(z)}{2}\Delta$.
Take $p_t^0(x,y) := p_t^y(y-x)$.

$$\begin{aligned}
\Phi_t(x,y) &= (L_x - \partial_t)p_t^0(x,y) \\
&= \Big[\Big(L_x^y - \partial_t\Big)p_t^0(x,y) + (L_x - L_x^y)p_t^0(x,y)\Big] \\
&= \Big(a(x) - a(y)\Big)L_x^{(2)} p_t^0(x,y) + \Big(b(x), \nabla_x p_t^0(x,y)\Big) \\
&= \Big(a(x) - a(y)\Big)\frac{1}{t^{d/2+1}a^{d/2+1}(y)}(L^{(2)}g^{(2)})\left(\frac{y-x}{t^{1/2}a^{1/2}(y)}\right) \\
&\quad - \frac{1}{t^{(d+1)/2}a^{(d+1)/2}(y)}\left(b(x), (\nabla g^{(2)})\left(\frac{y-x}{t^{1/2}a^{1/2}(y)}\right)\right) \\
&=: \Phi_t^1(x,y) + \Phi_t^2(x,y).
\end{aligned}$$

Assume that $a(\cdot)$ is bounded and Hölder:

$$|a(x) - a(y)| \leq c\big(|x - y|^{\eta} \wedge 1\big). \tag{14}$$

Then by (13)

$$\begin{aligned} |\Phi_t^1(x, y)| &\leq C \frac{|y - x|^{\eta} \wedge 1}{t} \frac{1}{t^{d/2}} G^{(2)} \left(\frac{(1 - \epsilon)(y - x)}{t^{1/2}} \right) \\ &\leq C t^{-1+\eta/2} \frac{1}{t^{d/2}} G^{(2)} \left(\frac{c(y - x)}{t^{1/2}} \right). \end{aligned}$$

From (11) and bdd of $b(x)$ we get

$$|\Phi_t^2(x, y)| \leq C t^{-1/2} \frac{1}{t^{d/2}} G^{(2)} \left(\frac{c(y - x)}{t^{1/2}} \right). \tag{15}$$

We have (4) with $H^1 = H^2 = G^{(2)}$.

2. $L = b(x)\nabla + a(x)(-\Delta)^{\alpha/2}$, $a : \mathbb{R}^d \to \mathbb{R}$.

$L^{(\alpha)} := (-\Delta)^{\alpha/2}$. $g_t^{(\alpha)}(x) = t^{-d/\alpha} g^{(\alpha)}(xt^{-1/\alpha})$– fund. sol. for $\partial_t - L^{(\alpha)}$.
$G^{(\alpha)}(x) = (1 + |x|)^{-d-\alpha}$.
Properties:

$$g^{(\alpha)}(x) \asymp G^{(\alpha)}(x), \tag{16}$$

$$\left|(\nabla^k g^{(\alpha)})(x)\right| \leq C G^{(\alpha+k)}(x), \quad k \geq 1; \tag{17}$$

$$\left|(L^{(\alpha)} g^{(\alpha)})(x)\right| \leq C G^{(\alpha)}(x), \tag{18}$$

For any $\lambda > 0$, $c > 0$ there exists $C > 0$ such that

$$G^{(\lambda)}(cx) \leq C G^{(\lambda)}(x). \tag{19}$$

$$G^{(\lambda_1)}(x) \leq G^{(\lambda_2)}(x), \quad \lambda_1 > \lambda_2; \tag{20}$$

$$|x|^\epsilon G^{(\lambda)}(x) \leq C G^{(\lambda-\epsilon)}(x), \quad \epsilon \in (0, \lambda); \tag{21}$$

$$p_t^z(x) := \frac{1}{(ta(z))^{d/\alpha}} g^{(\alpha)}\Big(\frac{x}{(a(z)t)^{1/\alpha}}\Big)$$

is the tr. pr. dens. for a LP with generator $L^z = a(z)(-\Delta)^{-\alpha/2}$.
Take $p_t^0(x,y) = p_t^y(y-x)$,
use $L^{(\alpha)} g(cx) = c^\alpha (L^{(\alpha)} g)(cx)$ with $c = (a(y)t)^{-1/\alpha}$

$$\begin{aligned}
\Phi_t(x,y) &= \Big[\Big(L_x^y - \partial_t\Big) p_t^0(x,y) + (L_x - L_x^y) p_t^0(x,y)\Big] \\
&= \Big(a(x) - a(y)\Big) L_x^{(\alpha)} p_t^0(x,y) + \Big(b(x), \nabla_x p_t^0(x,y)\Big) \\
&= \Big(a(x) - a(y)\Big) \frac{1}{(ta(y))^{d/\alpha+1}} (L^{(\alpha)} g^{(\alpha)}) \left(\frac{y-x}{t^{1/\alpha} a^{1/\alpha}(y)}\right) \\
&\quad - \frac{1}{(ta(y))^{(d+1)/\alpha}} \left(b(x), (\nabla g^{(\alpha)}) \left(\frac{y-x}{t^{1/\alpha} a^{1/\alpha}(y)}\right)\right) \\
&=: \Phi_t^1(x,y) + \Phi_t^2(x,y).
\end{aligned}$$

Assume that $a(\cdot)$ is bounded and Hölder:

$$|a(x) - a(y)| \le c\big(|x-y|^\eta \wedge 1\big). \tag{22}$$

Then

$$\begin{aligned} |\Phi_t^1(x,y)| &\le C\frac{|y-x|^\kappa \wedge 1}{t}\frac{1}{t^{d/\alpha}}G^{(\alpha)}\left(\frac{y-x}{t^{1/\alpha}}\right) \\ &\le Ct^{-1+\kappa/\alpha}Q_t^{(\kappa)}(x,y), \quad \kappa < \eta \wedge 1. \end{aligned}$$

From (17) (19) and (20) we have

$$|\Phi_t^2(x,y)| \le Ct^{-1/\alpha}\frac{1}{t^{d/\alpha}}G^{(\alpha+1)}\left(\frac{y-x}{t^{1/\alpha}}\right) \le Ct^{-1/\alpha}Q_t^{(0)}\left(\frac{y-x}{t^{1/\alpha}}\right), \tag{23}$$

where

$$Q_t^{(\lambda)}(x,y) := \left(\left|\frac{y-x}{t^{1/\alpha}}\right|^\lambda \wedge \frac{1}{t^{\lambda/\alpha}}\right)\frac{1}{t^{d/\alpha}}G^{(\alpha)}\left(\frac{y-x}{t^{1/\alpha}}\right).$$

$$Q_t^{(\lambda)}(x,y) := \left(\left| \tfrac{y-x}{t^{1/\alpha}} \right|^{\lambda} \wedge \tfrac{1}{t^{\lambda/\alpha}} \right) \tfrac{1}{t^{d/\alpha}} G^{(\alpha)} \left(\tfrac{y-x}{t^{1/\alpha}} \right).$$

We have:

$$\left| \Phi_t(x,y) \right| \le C \Big(t^{-1+\kappa/\alpha} Q_t^{(\kappa)}(x,y) + t^{-1+(1-1/\alpha)} Q_t^{(0)}(x,y) \Big).$$

$\delta := 1 - 1/\alpha$.

1. Kernels $Q^{(\lambda)}$ do not satisfy the sub-conv. property, because $Q_t^{(\lambda)}(x,x) = 0$. But modified kernels do:

$$H_t^{(\lambda)}(x,y) := \left(\left(\left| \frac{y-x}{t^{1/\alpha}} \right|^{\lambda} \vee 1 \right) \wedge t^{-\lambda/\alpha} \right) \frac{1}{t^{d/\alpha}} G^{(\alpha)} \left(\frac{y-x}{t^{1/\alpha}} \right). \tag{24}$$

Clearly,

$$Q_t^{(\lambda)}(x,y) \le H_t^{(\lambda)}(x,y).$$

2. $\delta = 1 - 1/\alpha > 0$ only if $\alpha > 1$ (the integral part dominates the drift)

what happens if $\alpha \in (0,1)$?

Suppose that b is Lipschitz (can be relaxed up to Hölder)

χ_t: solution to the Cauchy problem

$$\partial_t \chi_t(x) = b(\chi_t(x)), \quad \chi_0(x) = x.$$

$\theta_t(y)$: inverse flow, i.e. a solution to

$$\partial_t \theta_t(y) = -b(\theta_t(y)), \quad \theta_0(y) = y.$$

Choice of p^0:

$$p^0_t(x,y) = p^y_t(\theta_t(y) - x).$$

Not a sol. to a CP with frozen coeff.!

We have

$$\partial_t g_t^{(\alpha)}(x) = L^{(\alpha)} g_t^{(\alpha)}(x), \quad L^{(\alpha)} g(cx) = c^\alpha (L^{(\alpha)} g)(cx).$$

Therefore (take $c = (a(y)t)^{-1/\alpha}$)

$$\begin{aligned}\partial_t p_t^y(\theta_t(y) - x) = a(y)\frac{1}{(a(y)t)^{d/\alpha+1}}(L^{(\alpha)} g^{(\alpha)})\Big(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)}\Big) \\ + \Big(\partial_t \theta_t(y), \frac{1}{(ta(y))^{(d+1)/\alpha}}(\nabla g^{(\alpha)})\Big(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)}\Big)\Big),\end{aligned}$$

$$\begin{aligned}L_x p_t^0(x, y) = a(x)\frac{1}{(a(y)t)^{d/\alpha+1}}(L^{(\alpha)} g^{(\alpha)})\Big(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)}\Big) \\ - \Big(b(x), \frac{1}{(ta(y))^{(d+1)/\alpha}}(\nabla g^{(\alpha)})\Big(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)}\Big)\Big).\end{aligned}$$

Since $\partial_t \theta_t(y) = -b(\theta_t(y))$, we finally get

$$
\begin{aligned}
\Phi_t(x,y) &= \big(L_x - \partial_t\big) p_t^0(x,y) \\
&= \Big(a(x) - a(y)\Big) \frac{1}{(a(y)t)^{d/\alpha+1}} (L^{(\alpha)} g^{(\alpha)}) \Big(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \Big) \\
&+ \frac{1}{(a(y)t)^{(d+1)/\alpha}} \Big(b(\theta_t(y)) - b(x), (\nabla g^{(\alpha)}) \Big(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \Big) \Big) \\
&=: \Phi_t^1(x,y) + \Phi_t^2(x,y).
\end{aligned}
$$

b is bounded $\Rightarrow |\theta_t(y) - y| \le c \int_0^t |b(\theta_s(y)| ds \le Ct. \Rightarrow$

$$
\begin{aligned}
|a(x) - a(y)| &\le c(|\theta_t(y) - x|^\kappa \wedge 1) + c|\theta_t(y) - y|^\kappa \\
&\le c(|\theta_t(y) - x|^\kappa \wedge 1) + ct^\kappa.
\end{aligned}
$$

$$
\begin{aligned}
|\Phi_t^1(x,y)| &\le Ct^{-1+\kappa/\alpha}\Big(\Big|\frac{\theta_t(y)-x}{t^{1/\alpha}}\Big|^{\kappa}\wedge t^{-\kappa/\alpha}\Big)\frac{1}{t^{d/\alpha}}G^{(\alpha)}\Big(\frac{\theta_t(y)-x}{t^{1/\alpha}}\Big)\\
&\quad + Ct^{-1+\kappa}\frac{1}{t^{d/\alpha}}G^{(\alpha)}\Big(\frac{\theta_t(y)-x}{t^{1/\alpha}}\Big)\\
&= Ct^{-1+\kappa/\alpha}Q_t^{(\kappa)}(x,y)+Ct^{-1+\kappa}Q_t^{(0)}(x,y).
\end{aligned}
$$

$b(x)$ is Lipschitz $\Rightarrow$

$$
\begin{aligned}
|\Phi_t^2(x,y)| &\le Ct^{-d/\alpha}\Big|\frac{\theta_t(y)-x}{t^{1/\alpha}}\Big|\Big|(\nabla g^{(\alpha)})\Big(\frac{\theta_t(y)-x}{t^{1/\alpha}a^{1/\alpha}(y)}\Big)\Big|\\
&\le Ct^{-d/\alpha}\Big|\frac{\theta_t(y)-x}{t^{1/\alpha}}\Big|G^{(\alpha+1)}\Big(\frac{\theta_t(y)-x}{t^{1/\alpha}a^{1/\alpha}(y)}\Big)
\end{aligned}
$$

$\Rightarrow |\Phi_t^2(x,y)| \le CQ_t^{(0)}(x,y)$.
Thus, we have (4).

Some discussion

As a Corollary of the method, we get upper estimates on

$$p_t^0(x, y), \quad r_t(x, y),$$
$$p_t(x, y), \quad \left|\partial_t p_t(x, y)\right|$$

for $t \in (0, T]$, $x, y \in \mathbb{R}^d$.

Important: parametrix gives the estimates for $t \in (0, T]$. Constants in the series $\sum_{k=1}^{\infty} \Phi_t^{\star k}(x, y)$ explode at the rate e^{cT}!

(no hope to get the estimates for $t > 0$!)

What if $b(x)$ is γ-Hölder continuous?

Recipe: to "smoothify", and introduce the flow for the smoothified drift:

χ_t: solution to

$$\partial_t \chi_t(x) = \tilde{b}_t(\chi_t(x)), \quad \chi_0(x) = x.$$

$\theta_t(y)$: inverse flow, i.e. a solution to

$$\partial_t \theta_t(y) = -\tilde{b}_t(\theta_t(y)), \quad \theta_0(y) = y.$$

Still

$$\Big|\Phi_t(x,y)\Big| \le Ct^{-1}\Big(t^{\delta_1} H_t^1(x,y) + t^{\delta_2} H_t^2(x,y)\Big),$$

where $\delta_1, \delta_2 > 0$ under the balance condition

$$\alpha + \gamma > 1.$$

α -stable, non-rot. symm.

$(-\Delta)^{\alpha/2} \leftrightsquigarrow \mu-$ symm. rot. invar.: $\mu(du) = c_\alpha |u|^{-d-\alpha} du$
What if $\mu-$ symm., but not rot. inv.?

$$\mu(A) = \int_0^\infty \int_{S^d} 1_{\{r\ell \in A\}} r^{-1-\alpha} \Lambda(d\ell) dr.$$

$g_t^{(\alpha)}(x)$ -trans. prob. dens., $g_t^{(\alpha)}(x) = t^{-d/\alpha} g^{(\alpha)}(xt^{-1/\alpha})$.

Now:

If Λ is "bad", we do not have $g^{(\alpha)}(x) \sim G^{(\alpha)}(x)$!!!

If $\mu(B(\theta, r) \cap S^d) \leq Cr^\gamma$, $\theta \in S^d$, r small, then

$$g^{(\alpha)}(x) \leq \frac{C}{t^{d/\alpha}} \frac{1}{(1 + |x| t^{-1/\alpha})^{\gamma+\alpha}}.$$

Not even integrable for $\gamma + \alpha < d$...

Example. $\tilde{\mu}(du) = \mu(du_1) + \mu(du_2)$, i.e.,
$supp\mu = \{u \in \mathbb{R}^2 : u_1 = 0 \text{ or } u_2 = 0\}$.

$\tilde{p}(x) = p(x_1)p(x_2) \asymp \frac{1}{(1+|x_1|)^{1+\alpha}} \frac{1}{(1+|x_2|)^{1+\alpha}}$

Sofar we had $\sigma(x) \in \mathbb{R}$ in $\sigma(x)L^{(\alpha)}$.
What happens if $\sigma(x)$: $d \times d$ matrix?

$\tilde{p}^z \leftrightsquigarrow \sigma(z)\tilde{\mu}(du)$,

$$\tilde{p}^z(x) = |\det \sigma(z)|^{-1}\tilde{p}(\sigma^{-1}(z)x), \quad \tilde{p}_t(x) = t^{-2/\alpha}\tilde{p}(xt^{-1/\alpha}).$$

Take $\sigma(z)$ as a rotation, s.t. $\sigma(z)x = (x_1, 0)$. Then

$$\tilde{p}^z_t(x) \asymp \frac{1}{t^{2/\alpha}} \frac{1}{(1 + |x_1|t^{-1/\alpha})} \quad \text{in non-integrable!.}$$

Problem:

In the Lévy case:

• Upper bound in the form

$$p_t(x) \le C\rho_t^d f(x\rho_t), \quad t \in (0, T], x \in \mathbb{R}^d,$$

where $f \in L_1(\mathbb{R}^d)$, ρ_t- scaling, is impossible;

• Even if $p_t(x)$ has good upper bound, the change $\mu(du) \rightsquigarrow \mu(x, du)$ can ruin the "good estimate"!

To estimate the prmx series, we first need meaningful upper bound on $|\nabla^k p|$

A bit away from α-stable: one more example.

$d = 1$, "Discretized"α-stable Lévy measure:

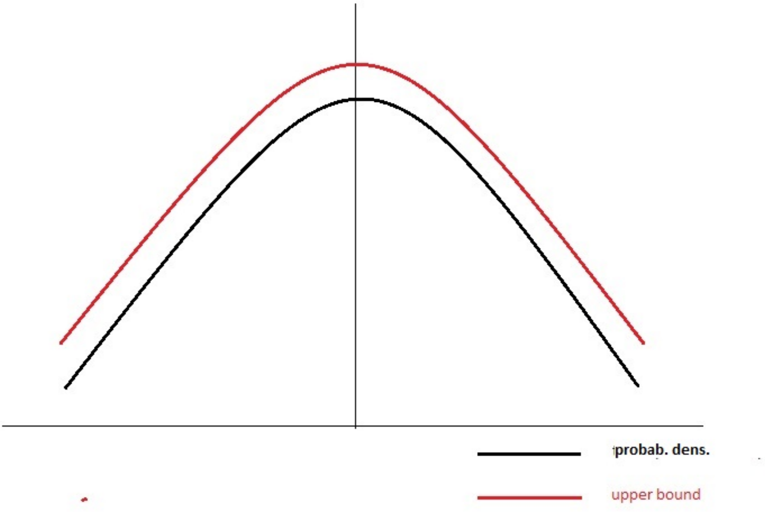
$$\mu(dy) = \sum_{n=-\infty}^{\infty} 2^{n\gamma} \Big(\delta_{2^{-n\upsilon}}(dy) + \delta_{-2^{-n\upsilon}}(dy) \Big), \qquad (25)$$

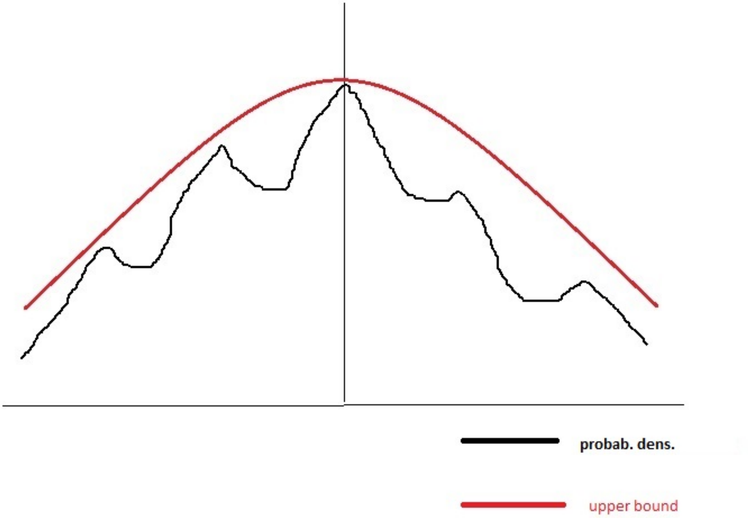
where $\upsilon > 0$, $0 < \gamma < 2\upsilon$. Then $\alpha = \gamma / v \in (0, 2)$, $\psi(\xi) \asymp |\xi|^{\alpha}$, but

$$p(x) \leq (1 + |x|)^{-\alpha}.$$

– integrable only for $\alpha \in (1, 2)$.

We need some other form of the upper estimate...


probab. dens.
upper bound


probab. dens.
upper bound

Suppose that for the Lévy trans. pr. dens.

$$p_t(x) \le C\rho_t^d \big(f_{up}(\rho_t\cdot) * P_t\big)(x), \quad t\in(0,1], x\in\mathbb{R}^d, \tag{26}$$

where $f_{up}\in L_1(\mathbb{R}^d)$, and P_t is a "compound Poisson type" measure.

Example: doable for μ as in (25). $\psi(\xi)\asymp|\xi|^\alpha$, $|\xi|\ge 1$, $\rho_t=t^{-1/\alpha}$.
R.h.s.- integrable!

If

$$\big|\nabla^k p_t(x)\big| \le C_k\rho_t^{d+k}\big(f_{up}(\rho_t\cdot)*P_t\big)(x), \quad t\in(0,1], x\in\mathbb{R}^d,\ k=0,1,2. \tag{27}$$

($*$ is the space conv.)

$\Rightarrow$ can estimate the conv. of the prmx series in C_∞.

Split (assume no drift and quad. part)

$$\psi(\xi) = \Big(\int_{|u|\le 1/\rho_t} + \int_{|u|>1/\rho_t} \Big) \big(1 - e^{i\xi u} + i\xi u 1_{|u|\le 1}\big)\mu(du)$$
$$= \psi_t^{(1)}(\xi) + \psi_t^{(2)}(\xi).$$

Since $supp \mu_1 \cap supp \mu_2 = \varnothing$,

$$p_t(x) = (\bar{p}_t * P_t)(x)$$

$\bar{p}_t(x) = \mathcal{F}^{-1}(e^{-t\psi_t^{(1)}})(x). \qquad P_t \leftrightsquigarrow e^{-t\psi_t^{(2)}}.$

Estimate

$$\left|\nabla^k \bar{p}_t(x)\right| \le C_k \rho_t^{d+k} f_{up}(\rho_t x), \quad t \in (0,1], x \in \mathbb{R}^d,\ k = 0,1,2. \tag{28}$$

P_t in (28)– as above.

Suppose that we proved (technical!!)

$$\left|\Phi_t(x,y)\right| \le C t^{-1+\delta}\big(\tilde{g}_t * G_t\big)(y-x).$$

$\delta \in (0,1)$, $(G_t)_{t\ge 0}$ comp. Poisson type probab. meas.,

$$\tilde{g}_t(x) := t^{\delta} g_t(x), \quad g_t(x) := \rho_t^d e^{-b\rho_t\|x\|}.$$

$$g_{t-s} * g_s(x) \le C g_t((1-\epsilon)x).$$

Lemma 2

For any $(\theta_k)_{k\ge 1}$ s.t. $\theta_1 = 1$, $\theta_{k+1} < \theta_k$, $\theta_k > 0$, $k \ge 1$,

$$\left|\Phi_t^{\star k}(x,y)\right| \le C_k t^{-1+\delta k}\big(g_t^{(k)} * G_t^{(k)}\big)(y-x), \tag{29}$$

where $(G^{(k)})_{k\ge 1}$ are compound Poiss. type meas.,

$$g_t^{(k)}(x) := t^{\delta k} \rho_t^d e^{-b\theta_k \rho_t \|x\|}. \tag{30}$$

In general the constants C_k explode as $k \to \infty$!

We change the estimation procedure after some $k_0 \geq 1$.

$\exists k_0$: we have $t^{\delta k} \rho_t^n \leq c_1$. Then

$$\left(g_{t-s}^{(\ell)} * g_s^{(1)}\right)(x) \leq c_2(k_0) g_{t,\zeta}(x), \quad \ell \geq k_0,$$

where$g_{t,\zeta}(x) := e^{-\zeta b \rho_t \|x\|}$. This allows to get

$$\left|\Phi_t^{\star \ell}(x,y)\right| \leq D_\ell t^{-1+\delta(\ell)} \left(g_{t,\zeta} * G_t^{(\ell)}\right)(y-x), \quad \ell \geq k_0.$$

PS: Compare with the method in Friedman!

Verification

$$p_t(x,y) = p_t^0(x,y) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}^0(x,z)\Psi_s(z,y)dzds$$

$$|\nabla_x^k p_t^0(x,y)| \le C(t-s)^{-k}(\dots) \implies p_t(\cdot,y) \notin C_b^2(\mathbb{R}^d)!$$

We cannot directly show that $p_t(x,y)$ and solves the CP!

$$P_t f(x) := \int_{\mathbb{R}^d} p_t(x,y)f(y)dy, \quad f \in C_b(\mathbb{R}^d).$$

Theorem 2

$(P_t)_{t\ge 0}$ is the Feller semigroup, i.e. $P_t : C_\infty \to C_\infty$, $P_t f \ge 0$ if $f \ge 0$, and $P_{t+s} = P_t P_s$.

Meaning: $p_t(x,y)$ is the trans. probab. dens. of a Markov proc.

Idea of the proof of Theorem 2

Use the approximate fundamental solution:

$$p_{t,\epsilon}(x,y) := p^0_{t+\epsilon}(x,y) + \int_0^t \int_{\mathbb{R}^n} p^0_{t-s+\epsilon}(x,z)\Psi_s(z,y)dzds.$$

$$P_{t,\epsilon}f(x) = \int_{\mathbb{R}^d} f(y)p_{t,\epsilon}(x,y)dy.$$

For $f \in C_\infty(\mathbb{R}^d)$, $\epsilon > 0$, we have $P_{\cdot,\epsilon}f(\cdot) \in C^{1,2}\big((0,\infty)\times\mathbb{R}^d\big)$.

Properties:

- $$\lim_{\epsilon \to 0} \|P_{t,\epsilon} f - P_t f\|_\infty \to 0, \quad \text{unif. in } t \in [0, t_0] \tag{31}$$
$$P_{t,\epsilon} f(x) \to 0, \quad |x| \to \infty, \text{ unif. in } t \in [0, t_0]. \tag{32}$$

- $$\lim_{t,\epsilon \to 0+} \|P_{t,\epsilon} f - f\|_\infty = 0. \tag{33}$$

$P_{t,\epsilon} f(x)$ is the "Approximate solution":

$$Q_{t,\epsilon} f(x) = \big(\partial_t - L_x\big) P_{t,\epsilon} f(x), \quad f \in C_\infty(\mathbb{R}^d).$$

- $$\lim_{\epsilon \to 0} Q_{t,\epsilon} f(x) = 0, \quad \text{unif. in } (t, x) \in [\tau, t_0] \times \mathbb{R}^d; \tag{34}$$

- $$\lim_{\epsilon \to 0} \int_0^t Q_{s,\epsilon} f(x) ds = 0, \quad \text{unif. in } (t, x) \in [\tau, t_0] \times \mathbb{R}^d \tag{35}$$

L satisfies the positive maximum principle (PMP):

$f \in D(L)$, $f(x_0) \geq 0$, where $x_0 = \arg\max f(x)$, $\Rightarrow Lf(x_0) \leq 0$.

Lévy-type operators satisfy PMP!

Prove, that $f \geq 0 \Rightarrow P_t f \geq 0$. PMP

Proof. $f \in C_\infty(\mathbb{R}^d)$, $f \geq 0$, and suppose that

$$\inf_{t,x} P_t f(x) < 0. \tag{36}$$

$\Rightarrow \exists t_0 > 0$: $\inf_{t \leq t_0, x \in \mathbb{R}^d} P_t f(x) < 0$.

$$u_\epsilon(t,x) := P_{t,\epsilon} f(x) + \theta t, \quad \theta > 0.$$

By unif. approximation, $\exists \eta > 0, \epsilon_1 > 0$ s.t.

$$\inf_{t \leq t_0, x \in \mathbb{R}^d} u_\epsilon(t,x) < -\eta, \quad \epsilon < \epsilon_1.$$

$P_{t,\epsilon} f(x) \to 0$, $|x| \to \infty$, unif. in $t \in [0, t_0] \Longrightarrow$

$$u_\epsilon(t,x) \to \theta t > 0, \quad |x| \to \infty, \quad \text{unif. in } t \in [0, t_0].$$

$\Rightarrow$

$$(t_\epsilon, x_\epsilon) := \arg\inf u_\epsilon(t,x) \in [0, t_0] \times \mathbb{R}^d.$$

$f(x) \geq 0$, $\lim_{t,\epsilon \to 0+} \|P_{t,\epsilon} f - f\|_\infty = 0 \Rightarrow \exists \epsilon_0 > 0$, $\tau > 0$ such that

$$u_\epsilon(t,x) \geq -\frac{\eta}{2}, \quad t \leq \tau, \quad \epsilon < \epsilon_0, \quad x \in \mathbb{R}^d.$$

Since

$$u_\epsilon(t_\epsilon, x_\epsilon) = \min_{t \in [0,t_0], x \in \mathbb{R}^d} u_\epsilon(t,x) < -\eta < -\frac{\eta}{2},$$

we have $t_\epsilon > \tau$ as soon as $\epsilon < \epsilon_0$.

PMP $\Rightarrow$

$$L_x u_\epsilon(t,x)|_{(t,x)=(t_\epsilon,x_\epsilon)} \geq 0.$$

In addition, for $\epsilon < \epsilon_0$ we always have

$$\partial_t u_\epsilon(t,x)|_{(t,x)=(t_\epsilon,x_\epsilon)} \leq 0,$$

where the sign "$<$" may appear only if $t_\epsilon = T$.
Then

$$(\partial_t - L_x)u_\epsilon(t,x)|_{(t,x)=(t_\epsilon,x_\epsilon)} \leq 0.$$

On the other hand, since $t_\epsilon \in [\tau, T]$, $\epsilon < \epsilon_0, \Rightarrow$

$$(\partial_t - L_x)u_\epsilon(t,x)|_{(t,x)=(t_\epsilon,x_\epsilon)} = \theta + (\partial_t - L_x)P_{t,\epsilon}f(x_\epsilon) \to \theta > 0, \quad \epsilon \to 0.$$

$\Rightarrow$ contradiction to (36).

Other parts of Th.2: follow from PMP

To show that for $f \in C^2_\infty(\mathbb{R}^d)$

$$P_t f(x) - f(x) = \int_0^t P_s L f(x)\, ds, \quad P_t 1 = 1,$$

take

$$u_\epsilon(t,x) = P_{t,\epsilon} f(x) - f(x) - \int_0^t P_{s,\epsilon} L f(x)\, ds + \theta t, \quad \epsilon > 0.$$

To show

$$P_{t+s} f(x) = P_t P_s f(x),$$

take

$$u_\epsilon(t,x) = P_{t+s,\epsilon} f(x) - P_{t,\epsilon} P_s f(x) + \theta t.$$

Uniqueness: analytic proof

$P_t f(x) - f(x) = \int_0^t P_s L f(x)\, ds \Rightarrow Af = Lf$ on C_∞^2.

A – closed operator, $\Rightarrow (L, C_\infty^2(\mathbb{R}^d))$ is closable. Show, that $\overline{L} = A$.

Take $f \in D(A)$, and consider $P_t f$ and $P_{t,\epsilon} f$.

$P_{t,\epsilon} f \in C_\infty^2(\mathbb{R}^d) \in D(A) \Rightarrow AP_{t,\epsilon} f = LP_{t,\epsilon} f = \partial_t P_{t,\epsilon} f$. We have:

$$\lim_{\epsilon \to 0} \|P_{t,\epsilon} f - P_t f\|_\infty = 0, \quad \lim_{\epsilon \to 0} \|(\partial_t - L) P_{t,\epsilon} f\| = 0.$$

Since $f \in D(A)$, then $P_t f \in D(A)$ and $AP_t f = \partial_t P_t f$.

If (!!) $\lim_{\epsilon \to 0} \|\partial_t P_{t,\epsilon} - \partial_t P_t f\|_\infty = 0$, then $\lim_{\epsilon \to 0} \|LP_{t,\epsilon} f - AP_t f\|_\infty = 0$,

$$\Rightarrow \overline{(L, C_\infty^2(\mathbb{R}^d))}^{\|\cdot\|_\infty} = (A, D(A)).$$

Uniqueness: probabilistic proof

In Thm 2 we proved: $\exists\ X_t$, which solves the Martingale problem (MP) for $(L, C^2_\infty(\mathbb{R}^d))$, i.e. $\exists \mathbb{P}$ -probab. meas. on $\mathbb{R}^d$, s.t. $\forall f \in C^2_\infty(\mathbb{R}^d)$

$$Y_t := f(X_t) - \int_0^t Lf(X_s)ds$$

is $\mathbb{P}$-martingale.

Now we prove, that this MP is well posed

i.e. $\forall$ probab. meas. π on $\mathbb{R}^d$ $\exists!$ probab. meas. $\mathbb{P}$ on $\mathbb{R}^d$, s.t. $\mathbb{P}(X_0 \in du) = \pi(du)$, and for $\forall f \in C^2_\infty$ Y_t is a martingale w.r.t. $\mathbb{P}$.

A few results from Ethier, Kurtz

Ethier,Kurtz Cor. 4.4.3: If 2 sol. to the Martingale problem have the same one-dim distr., they coincide.

Ethier,Kurtz, Cor. 4.3.4: $t_2 > t_1$. If X is right cont.,

$$\mathbb{E}\Big[u(t_2, X_{t_1}) - u(t_1, X_{t_1})\Big] = \mathbb{E}\Big[\int_{t_1}^{t_2} v(s, X_{t_1})ds\Big|\mathcal{F}_{t_1}\Big],$$

$$\mathbb{E}\Big[u(t_1, X_{t_2}) - u(t_1, X_{t_1})\Big] = \mathbb{E}\Big[\int_{t_1}^{t_2} w(t_1, s, X_s)ds\Big|\mathcal{F}_{t_1}\Big],$$

$$\lim_{\delta\to 0} \mathbb{E}[w(t-\delta, t, X_t) - w(t, t, X_t)] = 0, \quad t > 0,$$

then

$$u(t, X_t) - \int_0^t \big(v(s, X_s) + w(s, s, X_s)\big)\,ds \quad \text{is } \mathcal{F}_t\text{- mart.}$$

Take $g(t,x)$: $g(\cdot,\cdot) \in C^{1,2}_{b,\infty}((0,\infty)\times\mathbb{R}^d)$, $Lg(t,x) \in C_b(\mathbb{R}^d)$.

Ethier,Kurtz, Cor. 4.3.4: the process

$$g(t,X_t) - \int_0^t \Big(\partial_s g(s,X_s) + L_x g(s,X_s)\Big) ds$$

is a martingale w.r.t. $\mathbb{P}$.

Fix $f \in C_\infty(\mathbb{R}^d)$, $T>0$, and put $g^T(t,x) := P_{T-t}f(x)$. If (!!!) we know, that $g^T(t,x)$ is harmonic for $\partial_t + L$, i.e. if

$$\big(\partial_t + L\big) g^T(t,x) = 0,$$

and $\partial_t g^T(t,x)$, $L_x g^T(t,x)$ are continuous, $\Rightarrow$

$$\mathbb{E}^{\mathbb{P}} f(X_T) = \mathbb{E}^{\mathbb{P}} g^T(T,X_T) \overset{mart.}{=} \mathbb{E}^{\mathbb{P}} g^T(0,X_0) = \mathbb{E}^{\mathbb{P}} P_T f(X_0),$$

$$\int f(y)\mathbb{P}(X_T \in dy) = \int P_T f(z)\pi(dz) = \int f(y)\Big[\int p_T(z,y) d\pi(dz)\Big] dy$$

$\Rightarrow$ uniqueness.

$g^T(t,x)$ is NOT harmonic for $\partial_t + L$, but we can approximate.

Define

$$g_\epsilon^T(t,x) := P_{T-t,\epsilon} f(x).$$

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}} P_{0,\epsilon} f(X_T) - \mathbb{E}^{\mathbb{P}} P_{T,\epsilon} f(X_0) \\
&\qquad = \mathbb{E}^{\mathbb{P}} g_\epsilon^T(T, X_T) - \mathbb{E}^{\mathbb{P}} g_\epsilon^T(0, X_0) \\
&\qquad = \mathbb{E}^{\mathbb{P}} \int_0^T (\partial_t + L_x) g_\epsilon^T(t, X_t) dt \to 0, \epsilon \to 0.
\end{aligned}$$

I.e., in the limit we get

$$\mathbb{E}^{\mathbb{P}} f(X_T) = \mathbb{E}^{\mathbb{P}} P_T f(X_0).$$

$p_t(x,y) = p_t^0(x,y) + r_t^0(x,y).$

We arrive at:

Theorem 3

$p_t(x,y)$ is a fund. sol. to the CP for $\partial_t - A$:

$p_\cdot(x,y) \in C^1(0,\infty)$, $p_t(\cdot,y) \in D(A)$, and

$$(\partial_t - A_x)p_t(x,y) = 0, \quad p_t(x,y) \to \delta_x(y),\ t \to 0.$$

Some other versions in the prmx

Komatsu: $L = B^{(\alpha)} + A^{(\alpha)}$, where $A^{(\alpha)}$ is a (non-isotropic) α-stable operator defined on the test functions in $L_p(\mathbb{R}^d)$, $\alpha \in (0,2)$, $B^{(\alpha)}$ is the x-dependent perturbation of $A^{(\alpha)}$.

For the potential operator R_λ of the extension $\tilde{L}$ of L

$$R_\lambda = G_\lambda^{(\alpha)}(I - B^{(\alpha)} G_\lambda^{(\alpha)})^{-1} = G_\lambda^{(\alpha)} \sum_{k=0}^{\infty} \left[B^{(\alpha)} G_\lambda^{(\alpha)}\right]^k,$$

(resolvent formulation of the parametrix method.)

$G_\lambda^{(\alpha)}$ is the λ-potential for $A^{(\alpha)}$
$B^{(\alpha)} G_\lambda^{(\alpha)}$ is L_p -bounded (Calderon-Zygmund theory).

L_p–estimates for $G_\lambda^{(\alpha)} \Rightarrow$ uniqueness of the sol. to the MP for L.

$p = p(\alpha)$.

Kumano-go, Iwasaki, Jacob:

Take the symbol $q(x,\xi)$ of L, is smooth enough in x and ξ (i.e. belongs to the so-called "Hörmander class of symbols") .

Parametrix: Construct a semigroup whose symbol is a sum of the main term $e^{-tq(x,\xi)}$ ("parametrix") and the remainder $r(t,x,\xi)$ which is proved to belong to certain Hörmander class.

Check, that it is a Feller semirg., and the fund. sol. to $\partial_t - L$.

Applications

Consider an integral functional:

$$I_T(h) = \int_0^T h(X_t)\,dt, \tag{37}$$

where $h : \mathbb{R}^d \to \mathbb{R}$ is a given measurable function.
Examples in fin. math.:

$$\mathbb{E} I_T(h), \quad \mathbb{E} e^{-\rho I_T(h)}(X_s - K)_+$$

How to calculate?

Approximate $I_T(h)$ by the Riemannian sums

$$I_{T,n}(h) = \frac{T}{n} \sum_{k=0}^{n-1} h(X_{(kT)/n}), \quad n \geq 1.$$

Simulate

$$\mathbb{E}^x\Big[\frac{T}{n}\sum_{k=0}^{n-1} h(X_{kT/n})\Big]$$

- Monte-Carlo method, need to know $X_{kT/n} \sim p_{kT/n}(x,y)dy$.

Convergence? Accuracy?

Strong rate: $\mathbb{E}^x\Big|I_T(h) - I_{T,n}(h)\Big|^p$

Weak rate: $\Big|\mathbb{E}^x(I_T(h))^k f(X_T) - \mathbb{E}^x(I_{T,n}(h))^k f(X_T)\Big|$

$\exists \partial_t p_t(x,y)$, and

$$\Big|\partial_t p_t(x,y)\Big| \le B_{T,X} t^{-\beta} q_{t,x}(y), \quad t \le T,$$

$\beta \ge 1$, and for fixed t, x $q_{t,x}(\cdot)$ is a distr. dens.

- **discrete approximation** of X?

Consider **[Konakov et al]** a Lévy driven SDE: $b, \sigma : \mathbb{R}^d \to \mathbb{R}$,

$$X_t = x + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dZ_s,$$

Euler scheme: Grid: $\Lambda = \{t_i | \, t_i = ih, \, 0 \le i \le N\}$,
$\phi(t) := \inf\{t_i : \; t_i \le t < t_{i+1}\}$,

$$X_t^N = x + \int_0^t b(X^N_{\phi(s)})ds + \int_0^t \sigma(X^N_{\phi(s)})dZ_s,$$

$$X_t^{i+1} = X_{t_i} + (t - t_i)b(X_{t_i}) + \sigma(X_{t_i})(Z_t - Z_{t_i}), \quad t \in (t_i, t_{i+1}).$$

Weak error $\left|\mathbb{E}^x g(X_T) - \mathbb{E}^x g(X_T^N)\right|$ –?
Need to estimate $p - p^N$.

"Frozen process", with tr.pr.dens. $\tilde{p}(t,x,y)$:

$$\tilde{X}_t = x + \int_0^t b(y)du + \int_0^t \sigma(y)dZ_u = x + tb(y) + \sigma(y)Z_t.$$

Consider a "frozen" Markov chain: $h = T/N$,

$$\tilde{X}^N_{t_j} = x, \quad \tilde{X}^N_{t_{i+1}} = \tilde{X}^N_{t_i} + b(y)h + \sigma(y)\big(Z_{t_{i+1}} - Z_{t_i}\big), \quad i \in [j, k-1].$$

$p^N(t_k - t_j, x, y) \leftrightsquigarrow X^N$ E. sch., $\tilde{p}^N(t_k - t_j, x, y) \leftrightsquigarrow \tilde{X}^N$ fr.ch.

$$p^N(t_k - t_j, x, y) = \tilde{p}^N(t_k - t_j, x, y) + \sum_{r=1}^{k-j} \big(\tilde{p}^N \otimes_N H_N^{(r)}\big)(t_k - t_j, x, y),$$

$0 \le j < k \le N$, where

$$f \otimes_N g(t,x,y) = \int_0^t \int_{\mathbb{R}^d} f(\phi(u), x, z) g(t - \phi(u), z, y), \forall t \in \{(t_i)_{i \in [1,N]}\},$$

$$p^N(0,x,y) = \tilde{p}^N(0,x,y) = \delta(x-y).$$

Then– series expansion of $p - p^N$.

in $p(t,x,y)$: "usual" time/space conv., in p^N: $\otimes_N$ conv.

Other applications

where else do we need the estimates on $p_t(x, y)$:

• In statistics: e.g. to estimate the likehood relation
(cf. Kohatsu-Higa et al, also Ivanenko, Kulik)

• Adding a potential: need to describe a Kato class
(recent: Q.-Z. Chen, P. Kim, R. Song, K.)

Thank you for attention!