Overview

The paremetrix method first appears in the work of E. Levi [64] in the elliptic setting. Independently, this idea was proposed by Feller [23] in the probabilistic setting for an operator which is a sum of an elliptic operator and a bounded non-local operator. The idea of Levi was developed (independently from Feller) in the work of Dressel [16] for a parabolic setting, Ilyin, Kalashnikov and Oleinik [31], and was extended and described in the monograph of Friedman [24]. See also McKean, Singer [69] and Minakshisundaram [70] for a different modification of the Levi's method in the parabolic setting.

For the fractional Laplacian the FS solution exists and is the transition probability density of a symmetric stable process; its asymptotic behaviour was first investigated by Blumenthal and Getoor [6]. This probabilistic paper (as well as a still earlier paper by Polya [81] where the case $n = 1$ was considered) was noticed by specialists in partial differential equations only much later; see the monograph of Eidelman, Ivasyshen and Kochubei [22] for references to other early publications (Ya. M. Drin', S. D. Eidelman, M. V. Fedoryuk) containing the asymptotics or estimates. In particular, the parametrix construction of the solution to the Cauchy problem for hyper-singular integrals was developed in Kochubei [44], see also Kolokoltsov [45], and also the reference list and an extensive overview in the monograph [22]. For the related fractional calculus needed in the method see the monograph of Samko, Kilbas and Marichev [86], ans Samko [84], [85].

The parametrix approach investigated in [44], [22] was further developed in several directions. Singular delta-function type gradient perturbation of the principal operator of order α are studied in Portenko [82, 83] (for $d = 1$) and further in Podolynny, Portenko [80] (for $d \geq 1$). Under the assumption on the drift $b(x)$ that $\|b\|_p < \infty$, where $p > d/(\alpha - 1)$, the existence and uniqueness of a continuous solution is proved. The approach from [82, 83], [80] was extended recently in the works by Osypchuk, Portenko [77, 78]. The effect of p-dependence (already in a different form appearing in [47]) was rediscovered in Bogdan, Jakubowski [7], see also Bogdan, Jakubowski, Sydor [8], and Bogdan, Sydor [9]. In Kim, Song [37] similar problem was investigated in the framework of the existence and uniqueness of the weak solution to the corresponding SDE with b being a generalized function with derivative from a certain Kato class. See also Chen, Wang [14], for the approach which relies on [7] and the martingale problem method from Bass, Chen [4].

The approach of [44] was developed in Knopova, Kulik [41] and Ganychenko, Knopova, Kulik [25], Kulik [57] for (gradient) perturbations of fractional Laplacian with the principal part less than 1. Further extension to more general integro-differential operators with non-homogeneous symbol was done in Knopova, Kulik [42, 43]. This approach requites different type of the estimates on the parametrix, and relies on the earlier works [39], [40] in this direction. On the other hand, in Chen and Zhang [15], Kim, Song and Vondraček [38] the approach from [44] was developed and generalized for (isotropic) perturbation of α-stable operators along a different line. In the recent works of Kühn [54]–[56] the approach of [44] combined with the verification procedure from [41] was developed using the complex analysis technique in order to provide the kernel estimates. Also, quite recently the Cauchy problem with homogeneous symbols of positive order was studied in Litovchenko [66, 67] by analytic methods in a bit different context: it is shown that this Cauchy problem is correctly solvable in the class of generalized functions, and belongs to certain functions function space.

Completely different approach to construction of the parametrix solution was proposed in the works of Ch. Iwasaki (Tsutsumi) and N. Iwasaki [92], [32]–[34], and Kumano-go [58]–[60]; see the monograph by Kumano-go [61] for details and more references. The approach relies on the symbolic calculus technique, which allows to prove the existence of the fundamental solution to the Cauchy problem for an operator L, provided that its symbol is smooth enough (i.e. belongs to the so-called "Hörmander class of symbols"). Hoh [29], [30] constructed the generalized Hörmander classes of symbols and developed the respective symbolic calculus ("Hoh's symbolic calculus", see also the monograph of Jacob [35]), which in turn allowed to develop the parametrix method; this method was further extended to evolution equations in the papers of Böttcher [12], [13].

For second-order elliptic differential operators the well-posedness of the martingale problem was studied in Stroock and Varadhan papers [88, 89]. It was Grigelionis [26] who first gave the martingale formulation of a Markov process associated with certain integro-differential operator. Well-posedness of the martingale problem for a Lévy-type generator with a diffusion part was proved by using purely probabilistic approach in Stroock [87]. Perturbations of α-stable operator were first studied by Tsuchiya in [90, 91] by constructing the perturbation of the resolvent operator in L_p and then showing the well-posedness of the martingale problem. The ideas from [90, 91] were developed and improved by Komatsu in [46, 47], where he investigated space-dependent perturbations of a (non-isotropic) α-stable operator, and proved that the parametrix expansion for the resolvent is well defined, and provides the unique solution to the Cauchy problem. Mikulevicius and Pragarauskas [71]–[75] studied the existence and uniqueness in Hölder and Sobolev spaces of classical solutions to the Cauchy problem (in $\mathbb{R}^d$) to parabolic integro-differential equation of the order $\alpha \in (0, 2)$, where the kernel of the principal part of the operator is a perturbation of radially symmetric α-stable Leśy measure. Then the analytical results are applied for the proof of the uniqueness of the corresponding martingale problem. In Bass [2], [3] similar resolvent approach as in [47] is used, i.e. the resolvent is constructed in the form of convergent series, but in contrast to [47] the series are cont constructed from the resolvent of the non-perturbed process. The resolvent approach for the well-posedness of the martingale problem for Lévy-type pseudo-differential operators in the weak L_2–setting was developed in Hoh [27], [28], see also the monograph of Jacob [36].

Let us discuss some applications of the parametrix construction. Bally, Kohatsu-Higa [1] provide for the diffusion case a probabilistic interpretation of the asymptotic expansion of the transition densities, which enables to find probabilistic representations that may lead to exact Monte Carlo type simulation methods in order to estimate the transition densities. This work was developed further in Li, Kohazu-Higa [65], where an SDE with Hölder coefficients driven by a "stable-type" subordinator is Lévy-driven SDE with Hölder coefficients is considered. Another version of the parametrix, relied on the parametrix representation of of McKean, Singer [69], was developed in Konakov, Mammen [50], [51] in order to derive the rate of convergence of the transition probability density of the respective Markov chain to the transition probability density of a diffusion process, see also the earlier paper by Konakov and Molchanov [49]. This methodology was developed in further publications by Konakov, Menozzi and Molchanov [52], in order to get the estimate on the error in the diffusion approximation, and further in Konakov, Menozzi [53] in order to get the transition probability density approximation (and the respective error) in the model where the SDE is driven by a symmetric stable process without a drift and a Gaussian component. In the recent paper of Mikulevicius and Zhang [76] the earlier results on the uniqueness of the solution to the Cauchy problem in Hölder spaces [72], [73] are used to show that the Euler scheme for a Lévy-driven SDE yields positive weak order of convergence.

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