MUCKENHOUPT CLASS WEIGHT DECOMPOSITION AND BMO DISTANCE TO BOUNDED FUNCTIONS

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Abstract. We study the connection between the Muckenhoupt A_p weights and BMO for general bases for $\mathbb{R}^d$. New classes of bases are introduced that allow for several deep results on the Muckenhoupt weights - BMO connection to hold in a very general form. The John-Nirenberg type inequality and its consequences are valid for the new class of Calderón-Zygmund bases which includes cubes in $\mathbb{R}^d$, but also the basis of rectangles in $\mathbb{R}^d$. Of particular interest to us is the Garnett-Jones theorem on the BMO distance, which is valid for cubes. We prove that the theorem is equivalent to the newly introduced A_2-decomposition property of bases. Several sufficient conditions for the theorem to hold are analyzed, as well. However, the question whether the theorem fully holds for rectangles remains open.

1. Introduction

It is well-known for decades that Fourier analysis of several parameters presents numerous challenges and that it is highly non-trivial to extend to this case various important results from the one-parameter setting. We refer our readers to the seminal paper of Chang and Fefferman [1], which even thirty years later illustrates this point very well. Here, we attempt to deal with the connection between the space BMO and the class of so-called Muckenhoupt A_p weights. Although this connection is of general mathematical interest, we are also partially motivated by recent studies that connect various weight properties with Hilbert space basis properties of reproducing function systems (like wavelets, Gabor systems, and alike); see [8], [13], [14], [15], for more details. For example, the BMO distance theorem of Garnett and Jones (see [6]) gives an interesting framework to study "how far" are certain Schauder bases, formed within reproducing function systems, from the Riesz basis property; see, in particular, [15] for some initial results in this direction. Our interest in such matters guided us to the following general question. How dependent is the Garnett-Jones distance theorem on the underlying covering? Here we offer answers to this question that relate the Garnett-Jones theorem to the decomposition properties of Muckenhoupt weights within given covering system.

Our studies in this matter provided a realization that there is an interesting, and it seems very difficult, related question. Is the Garnett-Jones distance theorem valid for "rectangle covering", as well? This question requires an additional explanation. The original proof of the theorem (for "cubes") was a demanding one, and several authors

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studied the proof extensively. As it turned out, the theorem can be proved via a factorization result by Jones [11] and an (almost "miraculous") property of the maximal function vs the weight, that was discovered by Coifman and Rochberg in [2]. The proof of the factorization result by Jones was later simplified significantly by Rubio de Francia [17], see also [3], who further proved that such factorization results hold in a much more general setup, including for "rectangles". Therefore, it was natural to ask whether the Coifman-Rochberg property holds for "rectangles". This question was answered in negative (via a counterexample) by Soria [18]. In addition, it was shown that several consequences of the Coifman-Rochberg property are not valid for "rectangles". There is a subtle detail, though. Soria's results do not claim that the Garnett-Jones theorem is not valid for "rectangles". In this paper we present a general theorem providing necessary and sufficient conditions for the Garnett-Jones theorem to hold. The conditions are, however, quite complex and there is to our knowledge no general mathematical approach that allows one to easily handle them.

This paper is structured as follows. In the next section we introduce basic notions and follow a very general path that includes both "cubes" and "rectangles". The key new property is the one we name the Calderón-Zygmund decomposition property. We prove the John-Nirenberg type results in this general framework and prepare several technical results for later sections. This section is very much in the abstract spirit typical for Jawerth's papers (see, for example, [10]). In the third section we develop notions of A_p-decomposition (and other related) properies. We prove that within very general class of bases the Garnett-Jones theorem is equivalent to the A_2-decomposition property. In the last two sections we explore some sufficient conditions that imply the A_2-decomposition property.

2. Notation and Results

Definition 2.1. A basis for $\mathbb{R}^d$ is any collection $\mathcal{B}$ of measurable, bounded subsets of $\mathbb{R}^d$ with non-empty interior. A structured basis for $\mathbb{R}^d$ is a basis such that there exists a a sequence $B_j \in \mathcal{B}$, $j \in \mathbb{N}$, with $\cup_j \mathring{B}_j = \mathbb{R}^d$ and $B_j \subseteq B_{j+1}$, $j \in \mathbb{N}$.

It is trivial to verify that standard collections such as the Euclidean balls $\mathcal{E}$, cubes $\mathcal{Q}$, dyadic cubes $\mathcal{D}$, and the collection of rectangles $\mathcal{R}$ (aligned with the coordinate axes in $\mathbb{R}^d$) are all structured bases in the sense of Definition 2.1. The technical criteria related to the family $\{B_j\}_j$ in Definition 2.1 are needed to ensure completeness of the space of functions of bounded mean oscillation (BMO) that we are about to introduce.

For any measurable subset $E \subset \mathbb{R}^d$ of positive measure, we define

$$f_E := \frac{1}{|E|} \int_E f(x)\, dx,$$

and, when convenient, we will also use the notation

$$\fint_E f(x)\, dx := \frac{1}{|E|} \int_E f(x)\, dx.$$

Next we define the class of functions of bounded mean oscillation. Traditionally this space has been considered for means taken over Euclidean balls or cubes, but it may be considered for any structured basis.

Definition 2.2. Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, and let $\mathcal{B}$ be a structured basis for $\mathbb{R}^d$. We say that $f \in BMO(\mathcal{B})$ provided that

$$\|f\|_{BMO(\mathcal{B})} := \sup_{B\in\mathcal{B}} \frac{1}{|B|}\int_B |f(x) - f_B|\,dx < \infty. \tag{2.1}$$

To verify that BMO is a Banach space when we factorize over the constant functions, one can simply use a similar argument as in the classical case. One verifies that for the sets $B_j \in \mathcal{B}$, $j \in \mathbb{N}$, given by Definition 2.1,

$$\begin{aligned}
|f_{B_k} - f_{B_{k+1}}| &= \frac{1}{|B_k|}\left|\int_{B_k} (f(x) - f_{B_{k+1}})dx\right| \\
&\le \frac{|B_{k+1}|}{|B_k|}\frac{1}{|B_{k+1}|}\int_{B_{k+1}} |f(x) - f_{B_{k+1}}|dx \\
&\le C_k \|f\|_{BMO(\mathcal{B})},
\end{aligned}$$

from which we deduce that for $r \in \mathbb{N}$,

$$\int_{B_r} |f(x) - f_{B_1}|\,dx \le C'_r \|f\|_{BMO(\mathcal{B})}.$$

It follows from this estimate that any Cauchy-sequence $\{f_j\}$ in $BMO(\mathcal{B})$ has a corresponding sequence of representatives $\{f_j - (f_j)_{B_1}\}_j$ which is Cauchy in L^1 on any compact subset of $\mathbb{R}^d$. Hence $\{f_j - (f_j)_{B_1}\}$ has a limit g in $L^1_{\text{loc}}(\mathbb{R}^d)$, and an application of Fatou's lemma shows that $g \in BMO(\mathcal{B})$ with $f_j \to g$ in $BMO(\mathcal{B})$.

Definition 2.3. Let $w : \mathbb{R}^d \to (0, \infty)$, and let $\mathcal{B}$ be a basis for $\mathbb{R}^d$.

(i) For $1 < p < \infty$, we say that $f \in A_p(\mathcal{B})$ provided that

$$[w]_{A_p(\mathcal{B})} := \sup_{E\in\mathcal{B}} \fint_E w(x)\,dx \cdot \left[\fint_E w^{-\frac{1}{p-1}}(x)\,dx\right]^{p-1} < \infty. \tag{2.2}$$

(ii) We say that $w \in A_1(\mathcal{B})$ provided there exists a finite constant c such that for every $x \in \mathbb{R}^d$ and every $E \in \mathcal{B}$ with $x \in E$,

$$\fint_E w(y)\,dy \le c w(x).$$

The smallest such c is denoted by $[w]_{A_1(\mathcal{B})}$.

(iii) Finally, we say that $w \in A_\infty(\mathcal{B})$ provided

$$[w]_{A_\infty(\mathcal{B})} := \sup_{E\in\mathcal{B}} \left(\fint_E w\right) \exp\left(\fint_E \log w^{-1}\right) < \infty.$$

Definition 2.4. Let $\mathcal{B}$ be a basis for $\mathbb{R}^d$. We say that $\mathcal{B}$ has the Calderón-Zygmund (C-Z) decomposition property provided that there exists a constant C such that for any $B_0 \in \mathcal{B}$, any $f \in L^1(B_0)$, and any $\alpha > 0$,

- $B_0 = F \cup G$ with $F \cap G = \emptyset$
- $|f(x)| \le \alpha$ almost everywhere for $x \in F$

- G is a union $G = \cup_k B_k$ of sets $B_k \in \mathcal{B}$, whose interiors are mutually disjoint so that for each B_k,
$$\alpha \leq \frac{1}{|B_k|} \int_{B_k} |f(x)|\, dx \leq C\alpha.$$

Remark 2.5. It is known that the family of cubes $\mathcal{Q}$ and the family $\mathcal{R}$ of rectangles aligned with the coordinate axes in $\mathbb{R}^d$ both satisfy the C-Z decomposition property, this can be derived directly from results in [5] and [12], respectively.

Proposition 2.6. *Suppose $\mathcal{B}$ is a structured basis for $\mathbb{R}^d$ satisfying the C-Z decomposition property. Then for $f \in BMO(\mathcal{B})$, $B_0 \in \mathcal{B}$, and for $\lambda > \frac{3C}{2}\|f\|_{BMO(\mathcal{B})}$,*

$$|\{x \in B_0 : |f(x) - f_{B_0}| > \lambda\}| \leq \exp\left\{ - \frac{b\lambda}{\|f\|_{BMO(\mathcal{B})}} \right\} |B_0|, \tag{2.3}$$

where $b := \log \frac{3}{2}/3C$.

Proof. Let $f \in BMO(\mathcal{B})$ and let $B_0 \in \mathcal{B}$. We may assume that $\|f\|_{BMO(\mathcal{R})} = 1$. We apply the C-Z decomposition property to $|f - f_{B_0}|\chi_{B_0}$ at level $\alpha = \frac{3}{2}$. We obtain a pairwise disjoint collection $\{B_j^{(1)}\}_j$ such that
$$\frac{3}{2} \leq \frac{1}{|B_j^{(1)}|} \int_{B_j^{(1)}} |f(x) - f_{B_0}|\, dx \leq \frac{3C}{2},$$
and $|f(x) - f_{B_j^{(1)}}| \leq 3/2$ on $B_0 \backslash \bigcup_j B_j^{(1)}$. We notice that
$$\sum_j |B_j^{(1)}| \leq \frac{2}{3} \sum_j \int_{B_j^{(1)}} |f(x) - f_{B_0}|\, dx \leq \frac{2}{3} \int_{B_0} |f(x) - f_{B_0}|\, dx \leq \frac{2}{3} |B_0| \|f\|_{BMO(\mathcal{B})} = \frac{2}{3}|B_0|.$$
Also notice that
$$|f_{B_j^{(1)}} - f_{B_0}| = \left| \frac{1}{|B_j^{(1)}|} \int_{B_j^{(1)}} (f(x) - f_{B_0})\, dx \right| \leq \frac{1}{|B_j^{(1)}|} \int_{B_j^{(1)}} |f(x) - f_{B_0}|\, dx \leq \frac{3C}{2}.$$

Next we apply the C-Z decomposition property to each set $B_j^{(1)}$ and function $|f - f_{B_j^{(1)}}|\chi_{B_j^{(1)}}$ at level $\alpha = 3/2$. After k iterations of this process, we apply the C-Z property to each set $B_j^{(k-1)}$ and function $|f - f_{B_j^{(k-1)}}|\chi_{B_j^{(k-1)}}$ at level $\alpha = 3/2$. By repeating the estimates above, we obtain pairwise disjoint sets $\{B_{i,j}^{(k)}\}_i \subseteq B_j^{(k-1)}$ satisfying
$$\frac{3}{2} \leq \frac{1}{|B_{i,j}^{(k)}|} \int_{B_{i,j}^{(k)}} |f - f_{B_j^{(k-1)}}|\, dx \leq \frac{3C}{2},$$
and $|f(x) - f_{B_j^{(k-1)}}| \leq \frac{3}{2}$ on $B_j^{(k-1)} \backslash \bigcup_i B_{i,j}^{(k)}$. Moreover, $|f_{B_{i,j}}^{(k)} - f_{B_j}^{(k-1)}| \leq \frac{3}{2}C$, and
$$\sum_i |B_{i,j}^{(k)}| = \frac{2}{3} \sum_i \int_{B_{i,j}^{(k)}} |f(x) - f_{B_0}|\, dx \leq \frac{2}{3} \int_{B_j^{(k-1)}} |f(x) - f_{B_j^{(k-1)}}|\, dx$$
$$\leq \frac{2}{3} |B_k^{(k-1)}| \leq \cdots \leq \left(\frac{2}{3}\right)^k |B_0|.$$

For the sake of convenience, we relabel the sets $\{B_{i,j}^{(k)}\}_{i,j\in\mathbb{N}}$ as $\{B_j^{(k)}\}_{j\in\mathbb{N}}$. Now consider $x \in \bigcup_j B_j^{(k-1)} \backslash \bigcup_j B_j^{(k)}$ (if this set is empty, we simply skip the following estimates). Suppose $x \in B_{j_{k_{k-1}}}^{(k-1)}$. By the construction of $B_{j_{k_{k-1}}}^{(k-1)}$, we can find a chain

$$B_0 \supseteq B_{j_{k_1}}^{(1)} \supseteq B_{j_{k_2}}^{(2)} \supseteq \cdots \supseteq B_{j_{k_{k-1}}}^{(k-1)}.$$

Notice that

$$\begin{aligned}|f(x) - f_{B_0}| &\leq |f(x) - f_{B_{j_{k_{k-1}}}^{(k-1)}}| + \sum_{i=1}^{k-2} |f_{B_{j_{k_{k-i}}}^{(k-i)}} - f_{B_{j_{k_{k-i-1}}}^{(k-i-1)}}| + |f_{B_{j_1}^{(1)}} - f_{B_0}| \\ &\leq \frac{3C}{2} \cdot k,\end{aligned}$$

so, in particular, we may conclude that

$$\left\{x \in B_0 : |f(x) - f_{B_0}| > \frac{3Ck}{2}\right\} \subseteq \bigcup_j B_j^{(k)}.$$

Now given $\lambda > \frac{3C}{2}$, we pick $k \in \mathbb{N}$ such that $\frac{3Ck}{2} < \lambda \leq \frac{3C(k+1)}{2}$. Notice that

$$\begin{aligned}|\{x \in B_0 : |f(x) - f_{B_0}| > \lambda\}| &\leq \left|\left\{x \in B_0 : |f(x) - f_{B_0}| > \frac{3Ck}{2}\right\}\right| \\ &\leq \left|\bigcup_j B_j^{(k)}\right| \\ &\leq \left(\frac{2}{3}\right)^k |B_0| \\ &\leq |B_0| e^{-\lambda b},\end{aligned}$$

for $b := \frac{\log \frac{3}{2}}{3C}$. The case of a general $f \in BMO(\mathcal{B})$ is obtained by applying the previous result to $g = f/\|f\|_{BMO(\mathcal{B})}$ with threshold $\lambda \to \lambda/\|f\|_{BMO(\mathcal{B})}$. □

Proposition 2.7. *Let $\mathcal{B}$ be a structured basis that satisfies the C-Z decomposition property. A measurable function $f : \mathbb{R}^d \to \mathbb{C}$ belongs to $BMO(\mathcal{B})$ if and only if there exists $\mu > 0$, such that*

$$\frac{1}{|E|}\int_E e^{\mu|f-f_E|}\,dx \leq C(\mu, f) < \infty, \qquad E \in \mathcal{B}. \tag{2.4}$$

Proof. First we notice by Jensen's inequality, and any $E \in \mathcal{B}$,

$$\frac{\mu}{|E|}\int_E |f - f_E|\,dx \leq \log \frac{1}{|E|}\int_E e^{\mu|f-f_E|}\,dx.$$

Hence, (2.4) clearly imples that $f \in BMO(\mathcal{B})$. Conversely, suppose that $f \in BMO(\mathcal{B})$. Then (2.3) holds for some $b > 0$, and it is easy to verify that (2.4) holds for any $\mu < b$. □

Lemma 2.8. *Let $\mathcal{B}$ be a basis for $\mathbb{R}^d$. For a measurable function $\varphi : \mathbb{R}^d \to \mathbb{R}$, we have $w := e^{\varphi} \in A_p(\mathcal{B})$, $1 < p < \infty$, if and only if there exist two constants $C_1, C_2 < \infty$ such that*

$$\sup_{E\in\mathcal{B}} \frac{1}{|E|}\int_E e^{\varphi(x)-\varphi_E}\,dx \le C_1, \qquad \text{and} \qquad \sup_{E\in\mathcal{B}} \frac{1}{|E|}\int_E e^{-\frac{\varphi(x)-\varphi_E}{p-1}}\,dx \le C_2. \tag{2.5}$$

Proof. First, suppose the two conditions in (2.5) hold. Then for any $E \in \mathcal{B}$,

$$\begin{aligned}
&\frac{1}{|E|}\int_E e^{\varphi(x)}\,dx \cdot \left[\frac{1}{|E|}\int_E e^{-\frac{\varphi(x)}{p-1}}\,dx\right]^{p-1}\\
&\qquad= \frac{1}{|E|}\int_E e^{\varphi(x)-\varphi_E}\,dx \cdot \left[\frac{1}{|E|}\int_E e^{-\frac{\varphi(x)-\varphi_E}{p-1}}\,dx\right]^{p-1}\\
&\qquad\le \sup_{E\in\mathcal{B}}\frac{1}{|E|}\int_E e^{\varphi(x)-\varphi_E}\,dx \cdot \left[\sup_{E\in\mathcal{B}}\frac{1}{|E|}\int_E e^{-\frac{\varphi(x)-\varphi_E}{p-1}}\,dx\right]^{p-1}\\
&\qquad\le C_1C_2
\end{aligned}$$

Hence, e^{φ} satisfies the A_p condition with A_p-constant at most C_1C_2. For the converse statement, we take $e^{\varphi} \in A_p(\mathcal{B})$. By Jensen's inequality, for any $\alpha > 0$ and $E \in \mathcal{B}$, $-\frac{1}{|E|}\int_E \varphi\,dx \le \frac{1}{\alpha}\log\frac{1}{|E|}\int_E e^{-\alpha\varphi}\,dx$. Hence,

$$\begin{aligned}
\frac{1}{|E|}\int_E e^{\varphi-\varphi_E}\,dx &= \frac{1}{|E|}\int_E e^{\varphi}\,dx \cdot e^{-\frac{1}{|E|}\int_E \varphi\,dx}\\
&\le \frac{1}{|E|}\int_E e^{\varphi}\,dx \cdot \left[\frac{1}{|E|}\int_E e^{-\frac{\varphi(x)}{p-1}}\,dx\right]^{p/p'}\\
&= [e^{\varphi}]_{A_p(\mathcal{B})} < \infty.
\end{aligned}$$

The other estimate in (2.5) follows along similar lines. □

Proposition 2.9. *Let $w : \mathbb{R}^d \to \mathbb{C}$ be a positive measurable function, and define $\varphi := \log(w)$. Then*

a. *For any structured basis $\mathcal{B}$ for $\mathbb{R}^d$ and $1 < p < \infty$, $w \in A_p(\mathcal{B})$ implies that $\varphi \in BMO(\mathcal{B})$.*
b. *For any structured basis $\mathcal{B}$ for $\mathbb{R}^d$ and $w = e^{\lambda\varphi} \in A_2$ with A_2-constant C and $\lambda > 0$, it follows that $\varphi \in BMO(\mathcal{B})$ with $\|\varphi\|_{BMO(\mathcal{B})} \le \frac{\log C}{\lambda}$.*
c. *Suppose $\mathcal{B}$ is a structured basis and satisfies the C-Z decomposition property. Let $\varphi \in BMO(\mathcal{B})$. Then for any $1 < p < \infty$, there exists $\lambda > 0$ such that $e^{\lambda\varphi} \in A_p(\mathcal{B})$.*

Proof. First, suppose that $w \in A_p(\mathcal{B})$. Notice that by Jensen's inequality,

$$\frac{1}{|E|}\int_E [\varphi(x) - \varphi_E]^+ dx \le \log\left\{\frac{1}{|E|}\int_E e^{\varphi(x)-\varphi_E}\,dx\right\},$$

and

$$\frac{1}{|E|}\int_E [\varphi(x) - \varphi_E]^- dx \le \frac{1}{p-1}\log\left\{\frac{1}{|E|}\int_E e^{-\frac{\varphi(x)-\varphi_E}{p-1}}\,dx\right\}.$$

Hence,

$$\frac{1}{|E|}\int_E |\varphi(x)-\varphi_E|dx \le \log\left\{\frac{1}{|E|}\int_E e^{\varphi(x)-\varphi_E}\,dx\right\} + \frac{1}{p-1}\log\left\{\frac{1}{|E|}\int_E e^{-\frac{\varphi(x)-\varphi_E}{p-1}}\,dx\right\},$$

which is bounded independent of $E \in \mathcal{B}$ by (2.5). Hence $\varphi \in BMO(\mathcal{B})$, which proves (a).

For (b), we consider $w = e^{\lambda\varphi} \in A_2$ with A_2-constant C. From the proof of Lemma 2.8 it follows directly that for any $E \in \mathcal{B}$,

$$\frac{1}{|E|}\int_E e^{\pm\lambda(\varphi-\varphi_E)}\,dx \le C,$$

which implies that

$$\frac{1}{|E|}\int_E e^{\lambda|\varphi-\varphi_E|}\,dx \le C.$$

It now follows from Jensen's inequality that

$$\frac{1}{|E|}\int_E |\varphi-\varphi_E|\,dx \le \frac{1}{\lambda}\cdot\log\frac{1}{|E|}\int_E e^{\lambda|\varphi-\varphi_E|}\,dx \le \frac{\log C}{\lambda},$$

which implies that $\|\varphi\|_{BMO(\mathcal{B})} \le \frac{\log C}{\lambda}$.

For (c), suppose $\varphi \in BMO(\mathcal{B})$. Since $\mathcal{B}$ satisfies the C-Z decomposition property, (2.4) holds for some $\mu > 0$. Next we notice that both conditions in (2.5) hold by choosing $\lambda = \mu(p-1)$ when $1 < p \le 2$ and $\lambda = \mu$ for $p > 2$. Thus for this choice of λ, $e^{\lambda\varphi} \in A_p(\mathcal{B})$. □

Remark 2.10. An example is given in [4] of a basis $\mathcal{B}$, without the C-Z decomposition property, for which the implication $\varphi \in BMO(\mathcal{B}) \Rightarrow e^{\lambda\varphi} \in A_2(\mathcal{B})$ fails for any $\lambda > 0$, so (c) is not a universal property.

However, by adapting the technique introduced by Hytönen and Pérez [9], one can obtain the following uniform version of (a) for structured bases $\mathcal{B}$ and weights $w \in A_\infty(\mathcal{B})$,

$$\|\log w\|_{BMO(\mathcal{B})} \le \log(2e[w]_{A_\infty(\mathcal{B})}).$$

See also [4, Theorem 8.1].

3. BMO and the distance to L^∞

Let $\mathcal{B}$ be any structured basis. We now consider the distance from an arbitrary function $f \in BMO(\mathcal{B})$ to L^∞,

$$\operatorname{dist}(f, L^\infty) := \inf_{g\in L^\infty} \|f-g\|_{BMO(\mathcal{B})}. \tag{3.1}$$

It is known that, in general, L^∞ is not a closed subset of $BMO(\mathcal{B})$, so $\operatorname{dist}(f, L^\infty) = 0$ does not necessarily imply that $f \in L^\infty$.

For any $f \in L^1_{\text{loc}}$ we define the maximal function $\mathcal{M}_{\mathcal{B}}$ by

$$\mathcal{M}_{\mathcal{B}}f(x) := \sup_{E\in\mathcal{B};x\in E} \frac{1}{|E|}\int_E |f(y)|\,dy.$$

When there is no risk of confusion, we simply denote $\mathcal{M} := \mathcal{M}_{\mathcal{B}}$.

Inspired by the terminology introduced by Pérez [16], we call $\mathcal{B}$ a structured Muckenhoupt basis if $\mathcal{B}$ is structured and for every $w \in A_p(\mathcal{B})$, $1 < p < \infty$, there is a constant $C := C(p, w)$ such that

$$\int_{\mathbb{R}^d} |\mathcal{M}_{\mathcal{B}} f(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^d} |f(x)|^p w(x)\, dx, \qquad f \in L^p(\mathbb{R}^d; w).$$

A structured Muckenhoupt basis $\mathcal{B}$ has the Jones factorisation property, see [3,17]: Every $w \in A_p(\mathcal{B})$, $1 < p < \infty$, can be written $w = w_1 w_2^{1-p}$ for some $w_1, w_2 \in A_1(\mathcal{B})$.

We introduce the following quantity

$$\varepsilon(f) = \inf\{\lambda > 0 : [e^{f/\lambda}]_{A_2(\mathcal{B})} < \infty\}. \tag{3.2}$$

As mentioned in the introduction of this paper, the Coifman-Rochberg property plays an important role in the proof of the standard Garnett-Jones theorem. The decomposition property given in Definition 3.1 below could be considered as a generalized version of the Coifman-Rochberg property (observe the uniformity of the main constant involved in the definition and see also Example 3.3 below). The decomposition property has been used implicitly in similar forms in the literature, see e.g. [5, Section IV.5], but as far as we know have not been explicitly stated as here.

Definition 3.1. Let $p \in [1, \infty)$. We say that a basis $\mathcal{B}$ has the A_p-decomposition property if there exist $\delta := \delta(\mathcal{B}, p)$ with $0 < \delta \leq 1$, and a constant $C := C(\mathcal{B}, p)$, such that for any $w \in A_p(\mathcal{B})$ there exist $\varphi \in L^\infty$, with $1/\varphi \in L^\infty$, and $u \in A_p(\mathcal{B})$ such that

$$w^\delta(x) = u(x)\varphi(x), \tag{3.3}$$

with $[u]_{A_p(\mathcal{B})} \leq C$.

The following proposition shows that the A_1-decomposition property implies the A_p-decomposition property, $1 \leq p < \infty$, for structured Muckenhoupt bases.

Proposition 3.2. *Let $\mathcal{B}$ be a structured Muckenhoupt basis satisfying the A_1-decomposition property. Then $\mathcal{B}$ also satisfies the A_p-decomposition property for any $p \in (1, \infty)$.*

Proof. Let $w \in A_p(\mathcal{B})$, $1 < p < \infty$. We first make a Jones factorisation of w and write

$$w = w_1 \cdot w_2^{1-p},$$

with $w_1, w_2 \in A_1(\mathcal{B})$. We use (3.3) to obtain

$$w_1^\delta = u_1 \cdot \varphi_1, \qquad \text{and} \qquad w_2^\delta = u_1 \cdot \varphi_2$$

with $[u_i]_{A_1(\mathcal{B})} \leq C$, and $\varphi_i, 1/\varphi_i \in L^\infty$, for $i = 1, 2$. Then

$$w^\delta = (u_1 u_2^{1-p})(\varphi_1 \varphi_2^{1-p}),$$

with $\varphi_1 \varphi_2^{p-1} \in L^\infty$, $1/(\varphi_1 \varphi_2^{1-p}) \in L^\infty$,

$$[u_1 u_2^{1-p}]_{A_p(\mathcal{B})} \leq [u_1]_{A_1(\mathcal{B})} \cdot [u_2]_{A_1(\mathcal{B})}^{p-1} \leq C \cdot C^{p-1} = C^p,$$

where the first inequality follows easily by standard arguments adapted to the basis $\mathcal{B}$. We conclude that $\mathcal{B}$ has the A_p-decomposition property. □

Example 3.3. Let $\mathcal{Q}$ be the basis consisting of cubes in $\mathbb{R}^d$. Then $\mathcal{Q}$ has the A_1-decomposition property. To verify this, we let $M := \mathcal{M}_{\mathcal{Q}}$ denote the usual Hardy-Littlewood maximal operator associated with $\mathcal{Q}$. Then any $w \in A_1(\mathcal{Q})$ can be decomposed as

$$w = \varphi M(f)^{\varepsilon},$$

where $\varphi, 1/\varphi \in L^{\infty}$, $0 < \varepsilon \leq 1$, and f is a locally integrable function, see [5]. Hence,

$$w^{1/2} = \varphi^{1/2} M(f)^{\varepsilon/2},$$

and according to the Theorem of Coifman and Rochberg [2], there is a constant $K_{d,\varepsilon}$ independent of f such that $[M(f)^{\varepsilon/2}]_{A_1(\mathcal{Q})} \leq K_{d,\varepsilon}$. It then follows that $\sqrt{\varphi}$ and $u := M(f)^{\varepsilon/2}$ satisfy the conditions in Definition 3.1 with constants $C = K_{d,\varepsilon}$ and $\delta = 1/2$.

We consider the following result a folklore, although we have not found it stated in this form in the literature. We believe that our readers can convince themselves that the statement is valid, by carefully following standard proofs in the literature on weights, see e.g. [7],

Lemma 3.4. *Let $\mathcal{B}$ be a structured Muckenhoupt basis that has the C-Z decomposition property. Then there exists constants $0 < b, B < \infty$ such that*

$$f \in BMO(\mathcal{B}) \text{ with } \|f\|_{BMO(\mathcal{B})} \leq b \Longrightarrow e^f \in A_2(\mathcal{B}) \text{ with } \|e^f\|_{A_2} \leq B.$$

Theorem 3.5. *Let $\mathcal{B}$ be a structured Muckenhoupt basis that has the C-Z decomposition property. Then there exists a positive constant C_1 such that for $f \in BMO(\mathcal{B})$,*

$$C_1 \varepsilon(f) \leq \mathit{dist}(f, L^{\infty}). \tag{3.4}$$

Moreover, the A_2-decomposition property holds for $\mathcal{B}$ if and only if there exists a positive constant C_2 such that for $f \in BMO(\mathcal{B})$,

$$\mathit{dist}(f, L^{\infty}) \leq C_2 \varepsilon(f). \tag{3.5}$$

In particular, if $\mathcal{B}$ has the C-Z decomposition property and satisfies the A_2-decomposition property, then we have the equivalence,

$$C_1 \varepsilon(f) \leq \mathit{dist}(f, L^{\infty}) \leq C_2 \varepsilon(f), \qquad f \in BMO(\mathcal{B}). \tag{3.6}$$

Proof. We claim that there exists C_1 such that for $f \in BMO(\mathcal{B})$, $\varepsilon(f) \leq C_1 \|f\|_{BMO(\mathcal{B})}$. Notice that for any $B \in \mathcal{B}$,

$$\begin{aligned}
\int_B e^{|f - f_B|/\lambda} dx &= \int_0^{\infty} \frac{e^{t/\lambda}}{\lambda} |\{x \in B : |f(x) - f_B| > t\}| \, dt \\
&\leq C \int_0^{\infty} \frac{e^{t/\lambda}}{\lambda} \exp\left\{ -\frac{bt}{\|f\|_{BMO(\mathcal{B})}} \right\} |Q| \, dt \\
&= C \frac{|B|}{\lambda} \int_0^{\infty} \exp\left\{ \frac{t}{\lambda} - \frac{bt}{\|f\|_{BMO(\mathcal{B})}} \right\} dt \\
&= C \frac{|B|}{\lambda} \left(\frac{b}{\|f\|_{BMO(\mathcal{B})}} - \frac{1}{\lambda} \right)^{-1} < \infty,
\end{aligned}$$

whenever $\lambda > \frac{\|f\|_{BMO(\mathcal{B})}}{b}$. Hence by Lemma 2.8, $e^{f/\lambda} \in A_2(\mathcal{B})$ for every $\lambda > \frac{\|f\|_{BMO(\mathcal{B})}}{b}$, and consequently, $\varepsilon(f) \leq \frac{\|f\|_{BMO(\mathcal{B})}}{b}$.

Now we notice that $\varepsilon(f) = \varepsilon(f-g)$ whenever $g \in L^\infty$, so by considering a sequence $g_n \in L^\infty$ satisfying $\operatorname{dist}(f, L^\infty) = \lim_{n\to\infty} \|f - g_n\|_{BMO(\mathcal{B})}$, we obtain

$$\varepsilon(f) \leq \frac{\operatorname{dist}(f, L^\infty)}{b},$$

and (3.4) holds with $C_1 := b$.

We now turn to the upper estimate in (3.5). Let us first assume that the A_2-decomposition property holds for $\mathcal{B}$. Let C be the constant given by the A_2-decomposition property of $\mathcal{B}$, see Definition 3.1. Pick $\varepsilon(f) < \lambda \leq 2\varepsilon(f)$. Then $w := e^{f/\lambda} \in A_2(\mathcal{B})$. We now write $w^\delta = u \cdot \varphi$, which implies that

$$\delta \frac{f}{\lambda} = \log u + \log \varphi,$$

with $\log u \in BMO(\mathcal{B})$ and $\log \varphi \in L^\infty$. Hence,

$$\left\| f - \frac{\lambda}{\delta} \log \varphi \right\|_{BMO(\mathcal{B})} = \left\| \frac{\lambda}{\delta} \log(u) \right\|_{BMO(\mathcal{B})} \leq \frac{2\lambda}{\delta} \log C \leq \frac{4 \log C}{\delta} \varepsilon(f),$$

and we conclude that $\operatorname{dist}(f, L^\infty) \leq \frac{4 \log C}{\delta} \varepsilon(f)$.

We now turn to the converse statement. Assume that (3.5) holds. Let $w \in A_2(\mathcal{B})$, and let b, B a be the constants from Lemma 3.4. We first notice that $w \in A_2(\mathcal{B})$ implies that $\varepsilon(\log w) \leq 1$, so by (3.5), $\operatorname{dist}(\log w, L^\infty) \leq C_2$, where we may assume that $C_2 \geq b/2$. Then

$$\operatorname{dist}\left(\frac{b}{2C_2} \log w, L^\infty \right) \leq \frac{b}{2}.$$

We pick $g \in L^\infty$ such that

$$\frac{b}{2C_2} \log w = \left(\frac{b}{2C_2} \log w - g \right) + g,$$

with $\| \frac{b}{2C_2} \log w - g \|_{BMO(\mathcal{B})} \leq b$. Then

$$w^{\frac{b}{2C_2}} = \exp\left(\frac{b}{2C_2} \log w - g \right) \cdot \exp(g) := u \cdot \varphi,$$

where according to Lemma 3.4, $[u]_{A_2(\mathcal{B})} \leq B$, and $\varphi, 1/\varphi \in L^\infty$. We conclude that $\mathcal{B}$ has the A_2-decomposition property with constants B and $\delta := \frac{b}{2C_2} \leq 1$.

□

4. The Δ-condition

In this section we introduce a condition on a structured basis that is equivalent to the A_1-decomposition property. The m'th order iterated maximal function, defined by

$$\mathcal{M}^{(m)} = \underbrace{\mathcal{M} \circ \cdots \circ \mathcal{M}}_{m \text{ copies of } \mathcal{M}},$$

will play a centrole role in the following definition.

Definition 4.1. Let $\mathcal{B}$ be any structured basis. We say that $\mathcal{B}$ satisfy the Δ-condition if there exists a uniform constants $N := N(\mathcal{B}) \in \mathbb{N}$ and $\delta := \delta(\mathcal{B}) \in (0, 1]$ such that for every $w \in A_1(\mathcal{B})$ there is an $K := K(\mathcal{B}, w) \in \mathbb{N}$ such that for $m \geq K$,

$$\mathcal{M}^{(m)} w^\delta(x) \leq N^m w^\delta(x), \qquad a.e. \tag{4.1}$$

Remark 4.2. For $w \in A_1(\mathcal{B})$, it suffices to have one instance of (4.1) satisfied, e.g.,

$$\mathcal{M}^{(K)} w^\delta(x) \leq N^K w^\delta(x), \qquad a.e. \tag{4.2}$$

since for $m > K$, we may write $m = sK + r$ with $0 \leq r < M$, and using (4.2) together with the fact that $\mathcal{M} w^\delta(x) \leq [w]_{A_1}^\delta w^\delta(x)$, one easily obtains

$$\mathcal{M}^{(sK+r)} w^\delta(x) \leq N^{sK} [w]_{A_1}^{\delta r} w^\delta(x), \qquad a.e.$$

so (4.1) holds with

$$\mathcal{M}^{(m)} w^\delta(x) \leq (2N)^m w^\delta(x), \qquad a.e.,$$

for $m \geq \max\{K, \delta M \log_2 [w]_{A_1}\}$.

For a structured basis $\mathcal{B}$ satisfying the Δ-condition, we form the following iterated maximal function for any $w \in A_1(\mathcal{B})$,

$$\widetilde{\mathcal{M}} w^\delta := \sum_{m=1}^\infty \frac{1}{(2N)^m} \mathcal{M}^{(m)}(w^\delta),$$

where we notice that the Δ-condition (4.1) ensures pointwise convergence of $\widetilde{\mathcal{M}} w^\delta$. It follows from the sublinearity of $\mathcal{M}$ that $\widetilde{\mathcal{M}} w^\delta \in A_1(\mathcal{B})$ with $[\widetilde{\mathcal{M}} w^\delta]_{A_1} \leq 2N$. Conversely, notice that $[\widetilde{\mathcal{M}} w^\delta]_{A_1} \leq C$, with C independent of w, implies the Δ-condition (4.1).

Theorem 4.3. *Let $\mathcal{B}$ be any structured basis. Then $\mathcal{B}$ has the A_1-decomposition property if and only if $\mathcal{B}$ satisfies the Δ-condition.*

Proof. Suppose that $\mathcal{B}$ satisfies the Δ-condition with constants δ and N. Take any $w \in A_1$. We write

$$w^\delta = \frac{w^\delta}{\widetilde{\mathcal{M}} w^\delta} \cdot \widetilde{\mathcal{M}} w^\delta,$$

A brute force estimate yields $\mathcal{M}^{(m)}(w^\delta)(x) \leq [w]_{A_1}^{\delta m} w^\delta(x)$, which together with (4.1) implies the estimate

$$\widetilde{\mathcal{M}} w^\delta(x) = \sum_{m=1}^{M-1} \frac{1}{(2N)^m} \mathcal{M}^{(m)} w^\delta(x) + \sum_{m=M}^\infty \frac{1}{(2N)^m} \mathcal{M}^{(m)} w^\delta(x) \leq C w^\delta(x),$$

for some finite $C := C(w)$ independent of x. Hence,

$$\frac{w^\delta(x)}{2N} \leq \widetilde{\mathcal{M}}(w^\delta)(x) \leq C w^\delta(x). \tag{4.3}$$

Therefore, $\frac{w^\delta}{\widetilde{\mathcal{M}} w^\delta}, \frac{\widetilde{\mathcal{M}} w^\delta}{w^\delta} \in L^\infty$, and this implies the A_1-decomposition property since $[\widetilde{\mathcal{M}} w^\delta]_{A_1} \leq 2N$.

Conversely, suppose that $\mathcal{B}$ has the A_1-decomposition property with constants δ and C. Take any $w \in A_1(\mathcal{B})$. Write $w^\delta = u \cdot b$, with $[u]_{A_1} \le C$. Clearly, $w^\delta(x) \le \|b\|_{L^\infty} u(x)$ and $u(x) \le \|1/b\|_{L^\infty} w^\delta(x)$, so

$$\mathcal{M}^{(m)} w^\delta(x) \le \|b\|_{L^\infty} \mathcal{M}^{(m)} u(x) \le \|b\|_{L^\infty} C^m u(x) \le \|b\|_{L^\infty} \|1/b\|_{L^\infty} C^m w^\delta(x).$$

Hence for $m \ge M$, with M such that $(\|b\|_{L^\infty} \|1/b\|_{L^\infty})^{1/M} \le 2$, we have the Δ-condition satisfied with $N = 2C$ and δ. □

At this point we are not aware of an example of a structured basis that satisfies the A_2-decomposition property, but does not satisfy the A_1-decomposition property, c.f. Proposition 3.2.

5. The reverse Hölder classes

In the following we let $\mathcal{M}$ denote a general non-negative sublinear operator defined on the measurable functions on $\mathbb{R}^d$.

Theorem 5.1. *Let $\mathcal{B}$ be a structured Muckenhoupt basis that satisfies the C-Z decomposition property. Suppose there is a uniform constant C such that the reverse Hölder estimate*

$$\fint_E \mathcal{M} w \, dx \le C \fint_E (\mathcal{M} w)^s dx, \qquad E \in \mathcal{B},\ 0 < s < 1, \tag{5.1}$$

holds for $w \in A_1(\mathcal{B})$, and that $w \asymp \mathcal{M} w$ for $w \in A_1(\mathcal{B})$. Then $\mathcal{B}$ satisfies the A_2-decomposition property, and the distance to $BMO(\mathcal{B})$ formula (3.6) *thus holds for $\mathcal{B}$.*

Proof. Take any $w \in A_2(\mathcal{B})$ and make a Jones factorization

$$w = w_1 w_2^{-1}, \qquad w_1, w_2 \in A_1(\mathcal{B}).$$

We decompose $w_i \in A_1(\mathcal{B})$ as

$$w_i = \frac{w_i}{\mathcal{M} w_i} \mathcal{M} w_i, \qquad i = 1, 2,$$

with $\frac{w_i}{(\mathcal{M} w_i)}$ bounded and bounded from below since w_i is in $A_1(\mathcal{B})$. Hence,

$$\log(w) = \left[\log\left(\frac{w_1}{\mathcal{M} w_1}\right) - \log\left(\frac{w_2}{\mathcal{M} w_2}\right)\right] + \left[\log(\mathcal{M} w_1) - \log(\mathcal{M} w_2)\right].$$

We now use [4, Theorem 8.1] to obtain

$$\|\log(\mathcal{M} w_1) - \log(\mathcal{M} w_2)\|_{BMO(\mathcal{B})} \le 2C,$$

with C the constant from the reverse Hölder inequality (5.1). Thus, for the constant b from Lemma 3.4,

$$\frac{b}{4C} \log w = \log \varphi + \log u,$$

with $\log \varphi \in L^\infty$, and $\|\log u\|_{BMO(\mathcal{B})} \le b$. The A_2-decomposition property now follows directly from Lemma 3.4 with $\delta = \min\{1, \frac{b}{4C}\}$.

The claim about the distance formula (3.6) now follows directly from this A_2-decomposition property and the fact that $\mathcal{B}$ satisfies the C-Z decomposition property. □

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