1. Uvod u ZVB i CGT.

TM (SZVB) → 1. verzija

$(X_n)_n$ n.j.d. s konačnom varijancom. $\mu = EX_1$, $\operatorname{Var} X_1 < \infty$, $S_n = \sum_1^n X_k$

Vrijedi $\frac{S_n}{n} \xrightarrow{\mathbb{P}} \mu$

Primjer 1.1. (Bernoullijev SZVB)

$(Z_n)_n$ nez. sl. var., $Z_n \sim B(n,p)$, $p \in \langle 0,1 \rangle$

Vrijedi $\frac{Z_n}{n} \xrightarrow{\mathbb{P}} p$ $(*)$

Kakve veze ima sa SZVB?

Za niz $(Z_n)_n$ postoji vjer. prostor i sl. var. $\underbrace{(X_n)_n}_{\text{nezavisne}}$ na tom prostoru t.d.

$$X_i \sim B(p) \quad ; \quad Z_n = \sum_1^n X_i$$

$$(*) \iff \frac{1}{n} \cdot \sum_1^n X_i \xrightarrow{\mathbb{P}} p = EX_i$$

Dokaz → iz čeb. nej. $\varphi(x) = x^2$

$$P\left(\left|\frac{1}{n}\sum_1^n X_i - p\right| > \varepsilon\right) \le \frac{E\left(\left(\frac{1}{n}\sum_1^n X_i - p\right)^2\right)}{\varepsilon^2} = \frac{1}{n^2\varepsilon^2} E\Big(\sum_1^n X_i - \underbrace{np}_{E\sum_1^n X_i}\Big)^2$$

$$= \frac{1}{n^2\varepsilon^2} \underbrace{E(Z_n - EZ_n)^2}_{\operatorname{Var} Z_n} = \frac{1}{n^2\varepsilon^2} \underbrace{\operatorname{Var} Z_n}_{\cancel{n}pq} = \frac{pq}{n\varepsilon^2} \xrightarrow[n\to\infty]{} 0$$

$Z_n = \sum_1^n X_i$

■

Nap Dokaz može i direktno iz tm SZVB → (X_n) n.j.d. s kon. varijancom.

ZAD 1.1.

$(X_n)_n$, za $k \ge 2$ X_k ovisi o X_{k-1} i X_{k+1}, ali nez. sa ostalima.

$+ \operatorname{Var} X_i \le M < \infty$

D: Vrijedi SZVB.

$\underline{Rj}$ Nisu n.j.d. $\rightarrow$ ne mora postojati zaj. oč. $\rightarrow$ što je SZVB u ovom slučaju?

$$\frac{S_n - ES_n}{n} \xrightarrow{\mathbb{P}} 0$$

$$\left(\begin{array}{l}\text{ako n.j.d.} \\ \frac{ES_n}{n} = \frac{n\cdot\mu}{n} = \mu\end{array}\right)$$

Ponovno koristimo Čeb. nej.

$$P\left(\left|\frac{S_n - ES_n}{n}\right| > \varepsilon\right) \leq \frac{E\left(\frac{S_n - ES_n}{n}\right)^2}{\varepsilon^2} = \frac{1}{n^2\varepsilon^2} \operatorname{Var} S_n = (D)$$

Ocijenimo $\operatorname{Var} S_n$

$$\operatorname{Var} S_n = E\left[\left(\sum_{k=1}^{n}(X_k - EX_k)\right)^2\right] = E\left(\sum_{i,j=1}^{n}(X_i - EX_i)(X_j - EX_j)\right)$$

množimo "svaki sa svakim"

$$= \sum_{k=1}^{n} \underbrace{E(X_k - EX_k)^2}_{\substack{\operatorname{Var} X_k \\ \leq M}} + 2\sum_{i=1}^{n-1} \underbrace{E(X_i - EX_i)(X_{i+1} - EX_{i+1})}_{\operatorname{cov}(X_i, X_{i+1})} \leq nM + 2(n-1)M$$

ostanu samo isti ili susjedni i,j indeksi

$$\operatorname{cov}(X_i, X_{i+1}) \underset{CS}{\leq} \sqrt{\operatorname{Var} X_i}\sqrt{\operatorname{Var} X_{i+1}} \leq M$$

Uvrstimo u (D)

$$\hookrightarrow \leq \frac{3n-2}{n^2\varepsilon^2} M \xrightarrow[n\to\infty]{} 0$$

■

TM (JZVB - Kolmogorovljev)

$(X_n)_n$ n.j.d. sa kon. očekivanjem μ.

$$\frac{S_n}{n} \xrightarrow{g.s.} \mu$$

ZAD 1.2.

$(X_n)_n$ n.j.d. sl. var. t.d. $EX_1 = \mu \in \langle 0, +\infty\rangle$.

Dokažite:

(i) $P(\sup_n S_n = +\infty) = 1$ (DZ) (ii) $P(\inf_n S_n > -\infty) = 1$

Rj Zadovoljeni uvjeti za Kolmogorovljev JZVB $\Rightarrow \frac{S_n}{n} \xrightarrow{g.s.} \mu$

(i) Neka je $\Omega_0 := \{\omega : \frac{S_n(\omega)}{n} \xrightarrow[n\to\infty]{} \mu\}$ i

$$\Omega_1 := \{\omega : \sup_n S_n(\omega) \neq +\infty\} = \{\omega : \sup_n S_n(\omega) = K(\omega) < +\infty\}$$

zbog JZVB $P(\Omega_0) = 1$, želimo dokazati $P(\Omega_1) = 0 \rightsquigarrow P(\underbrace{\Omega_1^c}_{\sup = +\infty}) = 1$

za $\omega \in \Omega_1$ $\frac{S_n(\omega)}{n} \leq \frac{K(\omega)}{n} \xrightarrow[n\to\infty]{} 0 < \mu$

$\Rightarrow \omega \notin \Omega_0 \longrightarrow \Rightarrow \underbrace{\Omega_1 \subseteq \Omega_0^c}$,

$P(\Omega_0^c) = 1 - P(\Omega_0) = 0 \Rightarrow$ monotonost vjer. +

TM (CGT)

$(X_n)_n$ n.j.d. s konačnom varijancom: $EX_i = \mu$, $\mathrm{Var}X_i = \sigma^2 \in \langle 0, +\infty\rangle$

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} Z, \qquad Z \sim N(0,1)$$

ZAD 1.3.

$(X_n)_n$ n.j.d., $EX_1 = 0$, $\mathrm{Var}\, X_1 = \sigma^2 > 0$ konačna, ali ne znamo ju.

Znamo $\lim_{n\to\infty} P\left(\frac{S_n}{\sqrt{n}} > 1\right) = \frac{1}{3}$. $\sigma^2 = ?$

Rj Zadovoljeni uvjeti za CGT, tj. $\lim_{n\to\infty} P\left(\frac{S_n}{\sigma\sqrt{n}} < x\right) = \Phi(x)$, $x \in \mathbb{R}$ (Fja distr. od $N(0,1)$)

$$\underset{\substack{\downarrow\\ \text{uvjet iz zad}}}{\frac{1}{3} =} \lim_{n\to\infty} P\left(\frac{S_n}{\sqrt{n}} > 1\right) = \lim_{n\to\infty} P\left(\frac{S_n}{\sigma\sqrt{n}} > \frac{1}{\sigma}\right) \underset{CGT}{=} 1 - \Phi\left(\frac{1}{\sigma}\right)$$

zbog jedinstvenosti limesa $1 - \Phi\left(\frac{1}{\sigma}\right) = \frac{1}{3} \Rightarrow \Phi\left(\frac{1}{\sigma}\right) = \frac{2}{3}$

Iz tablice $\frac{1}{\sigma} \approx 0.43 \rightarrow \sigma^2 \approx 5.41$.

ZAD 1.4.

$(X_n)_n$ n.j.d. s kon. varijancom >0. Znamo da zadovoljeni uvjeti za CGT.

Pokažite da iz CGT slijedi SZVB.

Rj. Definiramo $\mu := EX_1$, $\sigma^2 := \operatorname{Var} X_1$, $\sigma^2 < \infty$

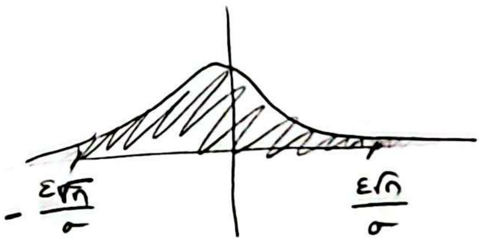
CGT $\rightarrow$ $\dfrac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)$

tj. $P\left(\dfrac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right) = \phi(x)$

Zanima nas $P\left(\left|\dfrac{S_n - n\mu}{n}\right| > \varepsilon\right) = 1 - P\left(\left|\dfrac{S_n - n\mu}{n}\right| \le \varepsilon\right)$

$$P\left(\left|\frac{S_n - n\mu}{n}\right| \le \varepsilon\right) = P\left(\left|\frac{S_n - n\mu}{\sigma\sqrt{n}}\right| \le \frac{\varepsilon\sqrt{n}}{\sigma}\right) \underset{CGT}{\approx} \phi\left(\frac{\varepsilon\sqrt{n}}{\sigma}\right) - \phi\left(-\frac{\varepsilon\sqrt{n}}{\sigma}\right)$$

$$= \phi\left(\frac{\varepsilon\sqrt{n}}{\sigma}\right) - \left(1 - \phi\left(\frac{\varepsilon\sqrt{n}}{\sigma}\right)\right)$$



$$= 2\underbrace{\phi\left(\frac{\varepsilon\sqrt{n}}{\sigma}\right)}_{\xrightarrow{n\to\infty} 1} - 1$$

(ϕ f-ja distr. pa $\phi(+\infty) = 1$)

$$\xrightarrow[n\to\infty]{} 2\cdot 1 - 1 = 1$$

$$P\left(\left|\frac{S_n - n\mu}{n}\right| > \varepsilon\right) \xrightarrow[n\to\infty]{} 1 - 1 = 0$$

■

11.3.2024.

2. SZVB.

$(X_n)_n$ ... niz sl. var., $S_n := \sum_{k=1}^{n} X_k$

Zanimat će nas kvg. po vjerojatnosti od $\frac{S_n - ES_n}{b_n}$, b_n - "slični" kao n

ZAD 1.7. (iz zad. za vježbu od prošlog sata)

$(X_n)_n$ niz sl. var., $S_n = \sum_1^n X_i$, $\sigma_n^2 := \operatorname{Var} S_n$

i $(b_n)_n$ niz neneg. br. t.d. $b_n \xrightarrow[n]{} +\infty$

Ako vrijedi $\frac{\sigma_n^2}{b_n^2} \longrightarrow 0$, tada $\frac{S_n - ES_n}{b_n} \xrightarrow{\mathbb{P}} 0$

koristit ćemo za današnje zadatke

Dokaz (DZ)

UPUTA: Čeb. nej.

NAP 2.1.

Za $\alpha > -1$ vrijedi $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^\alpha}{n^{\alpha+1}} = \frac{1}{\alpha+1}$ $\rightarrow$ oznaka $\boxed{\sum_{k=1}^{n} k^\alpha \sim n^{\alpha+1}}$

$\alpha = -1$ $\boxed{\sum_{k=1}^{n} k^{-1} \sim \log n}$

$\alpha < -1$ red je kvg $\rightarrow \boxed{\sum_{k=1}^{n} k^\alpha \sim 1}$

PRIMJER 2.3. (Coupon collector's problem)

$(X_k)_k$ n.j.d. i uniformne na $\{1, 2, \dots, n\}$

interpretacija $\rightarrow$ sakupljamo sličice/kupone. k-ti objekt koji "sakupimo" je odabran na slučajan način iz skupa svih mogućnosti (objekata) i nezavisan je od prethodnih odabira.

Definiramo

$$\tau_k^n := \inf\{m : |\{X_1, X_2, \dots, X_m\}| = k\}$$ $\Rightarrow$ prvi trenutak u kojem imamo k različitih objekata

Zanima nas granično ponašanje $T_n := J_n^n$, tj. vremena potrebnog da upotpunimo kolekciju.

$J_1^n = 1$ i dodefiniramo $J_0^n = 0$.

Za $1 \le k \le n$, $X_{n,k} := J_k^n - J_{k-1}^n$ → vrijeme potrebno da dobijemo objekt različiti od $k-1$ koje do sada imamo

Koja je distribucija od $X_{n,k}$?

$$X_{n,k} \sim G\left(1 - \frac{(k-1)}{n}\right)$$

ZAŠTO? u svakom novom skupljanju vjer. da smo dobili nešto novo je $\frac{n-(k-1)}{n} = 1 - \frac{(k-1)}{n}$.

Sakupljamo (tj. ponavljamo) sve dok ne dobijemo nešto novo (do J_k^n)

⇒ geom. distr.

$X \sim G(p)$

$EX = \frac{1}{p}$

$\mathrm{Var}\, X = \frac{1-p}{p^2} \le \frac{1}{p^2}$

$$EX_{n,k} = \frac{1}{1 - \frac{(k-1)}{n}} = \frac{n}{n-k+1}$$

$$\mathrm{Var}\, X_{n,k} \le \frac{1}{\left(1 - \frac{(k-1)}{n}\right)^2} = \frac{n^2}{(n-k-1)^2}$$

te $X_{n,k}$ nezavisna od $X_{n,j}$ za $j<k$.

$$ET_n = E\big(\underbrace{J_n^n - J_{n-1}^n}_{X_{n,n}} + \underbrace{J_{n-1}^n - J_{n-2}^n}_{X_{n,n-1}} + \dots + \underbrace{J_2^n - J_1^n}_{X_{n,2}} + \underbrace{J_1^n - J_0^n}_{X_{n,1}}\big)$$

$$= \sum_{k=1}^{n} EX_{n,k} = \sum_{k=1}^{n} \frac{n}{(n-k+1)} = n \sum_{m=1}^{n} \frac{1}{m} \underset{\text{nap } 2.1}{\sim} n \log n$$

$$\mathrm{Var}\, T_n \underset{\text{nez.}}{=} \sum_{k=1}^{n} \mathrm{Var}\, X_{n,k} \le \sum_{k=1}^{n} \frac{n^2}{(n-k+1)^2} = n^2 \sum_{m=1}^{n} \frac{1}{m^2} \le n^2 \underbrace{\sum_{m=1}^{\infty} m^{-2}}_{\text{kvg suma}}$$

Usporedimo s iskazom zad 1.7.

T_n je S_n, $\sigma_n^2 = \mathrm{Var}\, T_n \le n^2 \sum_{m=1}^{\infty} m^{-2}$

Uzet ćemo $b_n = n \log n$. $\frac{\sigma_n^2}{b_n^2} \overset{\sim}{=} \frac{\cancel{n^2} \sum_{m=1}^{\infty} m^{-2}}{\cancel{n^2}(\log n)^2} \xrightarrow{n\to\infty} 0$

zadovoljeni su uvjeti tm-a pa slijedi:

$$\frac{T_n - ET_n}{n\log n} \xrightarrow{\mathbb{P}} 0 \quad \text{tj.} \quad \frac{T_n - n\log n}{n\log n} = \frac{T_n}{n\log n} - 1 \xrightarrow{\mathbb{P}} 0$$

$$\frac{T_n}{n\log n} \xrightarrow{\mathbb{P}} 1 \qquad (*)$$

Konkretan primjer $n = 365$ (zanima nas koliko ljudi trebamo upoznati da bismo znali nekoga rođenog na svaki dan u godini)

Iz $(*)$ treba nam $365\log 365 = 2153.46$ „pokušaja da sakupimo cijeli skup"

↓

skoro $6\times$ veće od 365.

Napomena

(.) $X_{n,k}$ → trokutasti niz

$X_{1,1}$

$X_{2,1}$ $X_{2,2}$

$X_{3,1}$ $X_{3,2}$ $X_{3,3}$ ⇝ obično nez. po retcima.

⋮

PRIMJER 2.4.

Raspoređujemo r loptica u n kutija na slučajan način, tj. tako da su svih n^r rasporeda loptica jednakovjerojatni

Definiramo događaje

$n \to$ o (može ići u n kutija) o^n o^n $o^n \Rightarrow n^r$

$r \to$ _ _ _ _ _ _

$A_i = \{ i\text{-ta kutija je prazna} \}$

$N_n = \{\text{broj praznih kutija}\} = \sum_{m=1}^{n} \mathbb{1}_{A_m}$

↳ N_n će biti S_n iz zad 1,7.

$$P(A_i) = \frac{(n-1)^r}{n^r} = \left(1 - \frac{1}{n}\right)^r$$

↓

svakoj loptici možemo pridružiti bilo koju osim i-te kutije.

$$EN_n = E\left(\sum_{m=1}^{n} \mathbb{1}_{A_m}\right) = \sum_{1}^{n} \underbrace{E\mathbb{1}_{A_m}}_{P(A_m)} = \sum_{1}^{n} \left(1 - \frac{1}{n}\right)^r = n\left(1 - \frac{1}{n}\right)^r$$

Pustit ćemo $r, n \longrightarrow +\infty$ i pretpostavimo da vrijedi $\frac{r}{n} \longrightarrow c$.

Iz toga slijedi $\frac{EN_n}{n} \longrightarrow e^{-c}$

$$\left(\begin{array}{l} \frac{r}{n} \longrightarrow c \Rightarrow r \sim cn \\ \frac{EN_n}{n} = \frac{\cancel{n}(1-\frac{1}{n})^r}{\cancel{n}} \sim \Big(\underbrace{\left(1+\frac{-1}{n}\right)^n}_{e^{-1}}\Big)^c \longrightarrow (e^{-1})^c = e^{-c} \end{array} \right)$$

Još moramo izračunati varijancu od N_n. $\operatorname{Var} N_n = EN_n^2 - (EN_n)^2$ ✓

$$E[N_n^2] = E\left(\sum_{m=1}^{n} 1_{A_m}\right)^2 = E\left(\sum_{1\le k,m\le n} 1_{A_k} 1_{A_m}\right) = \sum_{1\le k,m\le n} E\underbrace{1_{A_k} 1_{A_m}}_{1_{A_k \cap A_m}} =$$

$$= \sum_{1\le k,m\le n} P(A_k \cap A_m)$$

$$\operatorname{Var} N_n = E[N_n^2] - (EN_n)^2 = \sum_{1\le k,m\le n} \left[P(A_k \cap A_m) - P(A_k)P(A_m)\right]$$

$$(EN_n)^2 = \left(\sum_{1}^{n} P(A_i)\right)^2 = \sum_{1\le k,m\le n} P(A_k)P(A_m)$$

razbijemo na 2 sl. 1. $k\ne m$, 2. $k=m$

odabir k i m

$$= \underbrace{n(n-1)\left[\left(1-\frac{2}{n}\right)^r - \left(1-\frac{1}{n}\right)^{2r}\right]}_{\text{sl. } k\ne m} + \underbrace{n\left[\left(1-\frac{1}{n}\right)^r - \left(1-\frac{1}{n}\right)^{2r}\right]}_{\text{sl. } k=m}$$

Za b_n ćemo uzeti baš n $(b_n = n)$

$$\frac{\sigma_n^2}{b_n^2} = \frac{\operatorname{Var} N_n}{n^2} \sim \frac{\cancel{n^2}\left(e^{-2c} - e^{-2c}\right)}{\cancel{n^2}} + \cancel{\frac{n\left(e^{-c} - e^{-2c}\right)}{n^2}}^{\,0} \longrightarrow 0 + 0 = 0$$ ✓

$$\frac{N_n - EN_n}{n} \xrightarrow{\mathbb{P}} 0, \text{ tj. } \frac{N_n - n(1-\frac{1}{n})^r}{n} = \frac{N_n}{n} - \underbrace{\left(1-\frac{1}{n}\right)^r}_{\to e^{-c}} \xrightarrow{\mathbb{P}} 0$$

$$\boxed{\frac{N_n}{n} \xrightarrow{\mathbb{P}} e^{-c}}$$

∎

ZAD 2.5.

X sl. var. s očekivanjem 0 i varijancom 1.

$(X_n)_n$ nez. t.d. $X_k \overset{d}{=} k \cdot X$, za $\forall k$

D: $\frac{S_n}{n^2} \xrightarrow{\mathbb{P}} 0$

Rj: $\sigma_n^2 = \mathrm{Var}\, S_n = \mathrm{Var}\left(\sum_{k=1}^{n} X_k\right) \underset{\text{nez.}}{=} \sum_{k=1}^{n} \mathrm{Var}\, X_k = (\square)$

$\mathrm{Var}\, X_k = E(X_k^2) - (EX_k)^2 = E(X_k^2) = E(k^2 X^2) = k^2 \underbrace{EX^2}_{=\mathrm{Var}\, X \text{ jer } EX=0} = k^2 \cdot 1 = k^2$

$EX_k = E(k \cdot X) = k \overset{0}{EX} = 0$

$(\square)$ $\sigma_n^2 = \sum_{k=1}^{n} k^2$

$b_n = n^2$

$\frac{\sigma_n^2}{b_n^2} = \frac{\sum_{k=1}^{n} k^2}{(n^2)^2} \sim \frac{n^3}{n^4} \xrightarrow[n\to\infty]{} 0$ ($\sum_{k=1}^{n} k^2 \sim n^3$)

pa po zad 1.7. $\frac{S_n - ES_n}{n^2} \xrightarrow{\mathbb{P}} 0$, a lako se vidi $ES_n = 0$

$\Rightarrow \frac{S_n}{n^2} \xrightarrow{\mathbb{P}} 0$

■

SZVB - nastavak (18.3.)

2.6. (X_n) n.j.d.
(Y_n) n.j.d. i $EX_1 = EY_1 = \mu < \infty$

Def $(Z_n)_n$ t.d. $Z_{2n} = Y_n$
$Z_{2n-1} = X_n$

D: Za $(Z_n)_n$ vrijedi SZVB.

Rj: Iz Hinčinovog SZVB. $(EX_1, EY_1 < \infty$ i n.j.d.$)$

$$\frac{X_1 + \dots + X_n}{n} \xrightarrow{\mathbb{P}} \mu \quad i \quad \frac{Y_1 + \dots + Y_n}{n} \xrightarrow{\mathbb{P}} \mu$$

$$\frac{Z_1 + \dots + Z_n}{n} - \mu = \frac{X_1 + \dots + X_{\lfloor \frac{n+1}{2} \rfloor}}{n} + \frac{Y_1 + \dots + Y_{\lfloor \frac{n}{2} \rfloor}}{n} - \mu$$

bit će $\lfloor \frac{n+1}{2} \rfloor$ nep. indeksa
i $\lfloor \frac{n}{2} \rfloor$ parnih

$$= \frac{\lfloor \frac{n+1}{2} \rfloor}{n} \left(\frac{X_1 + \dots + X_{\lfloor \frac{n+1}{2} \rfloor}}{\lfloor \frac{n+1}{2} \rfloor} - \mu \right) + \frac{\lfloor \frac{n}{2} \rfloor}{n} \cdot \left(\frac{Y_1 + \dots + Y_{\lfloor \frac{n}{2} \rfloor}}{\lfloor \frac{n}{2} \rfloor} - \mu \right)$$

Za proizv. $\varepsilon > 0$ vrijedi

$$P\left(\left| \frac{Z_1 + \dots + Z_n}{n} - \mu \right| > \varepsilon \right)$$

$$\le P\left(\left\{ \frac{\lfloor \frac{n+1}{2} \rfloor}{n} \left| \frac{X_1 + \dots + X_{\lfloor \frac{n+1}{2} \rfloor}}{\lfloor \frac{n+1}{2} \rfloor} - \mu \right| > \frac{\varepsilon}{2} \right\} \cup \left\{ \frac{\lfloor \frac{n}{2} \rfloor}{n} \left| \frac{Y_1 + \dots + Y_{\lfloor \frac{n}{2} \rfloor}}{\lfloor \frac{n}{2} \rfloor} - \mu \right| > \frac{\varepsilon}{2} \right\} \right)$$

nej. Δ

$$\le P\left(\underbrace{\frac{\lfloor \frac{n+1}{2} \rfloor}{n}}_{\text{između } \frac{1}{2} \text{ i } 1} \left| \frac{X_1 + \dots + X_{\lfloor \frac{n+1}{2} \rfloor}}{\lfloor \frac{n+1}{2} \rfloor} - \mu \right| > \frac{\varepsilon}{2} \right) + P\left(\underbrace{\frac{\lfloor \frac{n}{2} \rfloor}{n}}_{\text{između } \frac{1}{2} \text{ i } 1} \left| \frac{Y_1 + \dots + Y_{\lfloor \frac{n}{2} \rfloor}}{\lfloor \frac{n}{2} \rfloor} - \mu \right| > \frac{\varepsilon}{2} \right)$$

$$\le P\left(\left| \frac{X_1 + \dots + X_{\lfloor \frac{n+1}{2} \rfloor}}{\lfloor \frac{n+1}{2} \rfloor} - \mu \right| > \frac{\varepsilon}{2} \right) + P\left(\left| \frac{Y_1 + \dots + Y_{\lfloor \frac{n}{2} \rfloor}}{\lfloor \frac{n}{2} \rfloor} - \mu \right| > \frac{\varepsilon}{2} \right) \xrightarrow[n \to \infty]{} 0$$

ZAD 2.7.

$(X_n)_n$ n.j.d. takvih da $X_1 \sim U(0,1)$.

$Y_n := \cos \frac{X_n}{X_{n+1}}$. Dokažite da $(Y_n)_n$ zadovoljava SZVB.

Rj $(Y_n)_n$ nije niz nezavisnih sl. var., ali ako promatramo

$(Y_{2k})_k$ i $(Y_{2k-1})_k$ svaki niz jest niz n.j.d.

$\left((Y_{2k})_k \rightarrow \cos \frac{X_2}{X_3}, \cos \frac{X_4}{X_5}, \cos \frac{X_6}{X_7}, \dots \right)$

↓ koristimo zad 2.6. → treba još samo pokazati da postoji EY_1,

ali to vrijedi jer je Y_1 ograničena $(|Y_1| \leq 1)$.

$$\underset{\text{ZAD 2.6.}}{\Rightarrow} \quad \frac{Y_1 + \dots + Y_n}{n} \xrightarrow{\mathbb{P}} \mu := EY_1 = \dots = \int_0^1 y \sin \frac{1}{y} \, dy$$

2. način → iz Čeb. nej. (DZ)

ZAD 2.8. $(X_n)_n$ n.j.d. $P(X_1 = (-1)^k k) = \frac{c}{k^2 \log k}$, $k \geq 2$

$c > 0$ t.d. distribucija vjerojatnosna.

Vrijedi li SZVB?

Rj Pokušajmo iskoristiti Hinčinov SZVB. Treba pokazati $E|X_1| < \infty$.

za $k = 2l$ $\quad P(X_1 = (-1)^{2l} 2l) = P(X_1 = \underbrace{2l}_{k}) = \frac{c}{k^2 \log k}$

za $k = 2l-1$ $\quad P(X_1 = (-1)^{2l-1} \cdot (2l-1)) = P(X_1 = -\underbrace{(2l-1)}_{k}) = \frac{c}{k^2 \log k}$

→ za $k \geq 2$ $\quad P(|X_1| = k) = \frac{c}{k^2 \log k}$

pa je $\quad E|X_1| = \sum_{k=2}^{\infty} k \cdot \frac{c}{k^2 \log k} = c \sum_{k=2}^{\infty} \frac{1}{k \log k} \geq c \int_2^{\infty} \frac{dx}{x \log x} = +\infty$

$\Rightarrow$ ne možemo iskoristiti Hinčinov SZVB.

Pokušajmo sa Fellerovim SZVB. Treba pokazati $\lim_{x\to\infty} x P(|X_1|>x) = 0$

$$n P(|X_1|>n) \doteq n \cdot \sum_{k=n+1}^{\infty} \underbrace{P(|X_1|=k)}_{\frac{C}{k^2 \log k}} = Cn \sum_{k=n+1}^{\infty} \frac{1}{k^2 \log k} \leq Cn \int_n^\infty \frac{dx}{x^2 \log x}$$

$\leq \frac{1}{\log n}$

$$\leq \frac{Cn}{\log n} \underbrace{\int_n^\infty \frac{dx}{x^2}}_{-x^{-1}\big|_n^\infty} = \frac{C\cancel{n}}{\log n} \cdot \frac{1}{\cancel{n}} \xrightarrow[n\to\infty]{} 0$$

Po Fellerovom SZVB slijedi $\frac{S_n}{n} - \mu_n \xrightarrow{\mathbb{P}} 0$ uz

$\underbrace{\mu_n}$ $:= E(X_1 \mathbb{1}_{|X_1|\leq n})$

S obzirom da je $\mu_n = E(X_1 \mathbb{1}_{|X_1|\leq n}) = \sum_{k=2}^{n} \cancel{k}(-1)^k \frac{C}{k^{\cancel{2}} \log k}$

$$= \sum_{k=2}^{n} (-1)^k \frac{C}{k \log k} \xrightarrow[n\to\infty]{} \sum_{2}^{\infty} (-1)^k \frac{C}{k\log k} \underbrace{<\infty}$$

$\overset{\shortparallel}{\mu}$ alternirajući red sa opadajućim el.

Dakle, $\frac{S_n}{n} \xrightarrow{\mathbb{P}} \mu$

Primjer 2.9.

$$p_k = \frac{1}{2^k k(k+1)} \quad , \quad p_0 = 1 - \sum_{k=1}^{\infty} p_k$$

Definiramo $(X_n)_n$ n.j.d. sa $P(X_1 = -1) = p_0$,

$$P(X_1 = 2^k - 1) = p_k \, , \quad k \geq 1$$

X_n = osvojeni iznos tijekom n-te igre.

$$EX_n = \left(\sum_{k=1}^{\infty} (2^k - 1) p_k\right) + (-1) p_0 = \sum_{k=1}^{\infty} \frac{2^k - 1}{2^k k(k+1)} - p_0$$

$$= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} - \cancel{\sum_{k=1}^{\infty} \frac{1}{2^k k(k+1)}} - \left(1 - \cancel{\sum_{k=1}^{\infty} \frac{1}{2^k k(k+1)}}\right)$$

$$= \underbrace{\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)}_{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cancel{\dots}} - 1 = 1 - 1 = 0$$

Iz Hinčinovog SZVB $\frac{S_n}{n} \xrightarrow{\mathbb{P}} 0$

Međutim, uzmimo $b_n = 2^{m(n)}$, $m(n) = \min \left\{ m : \frac{1}{2^m m^{\frac{3}{2}}} \leq \frac{1}{n} \right\}$.

Koristimo SZVB za trokutaste nizove $\rightsquigarrow$ uzmemo $X_{n,k} = X_k$.

$b_n > 0$, $b_n \longrightarrow +\infty$ ✓ jer $m(n) \xrightarrow[n]{} +\infty$

(i) $$\sum_{k=1}^{n} P(|X_{n,k}| > b_n) = \sum_{k=1}^{n} P(|X_k| > b_n) \underset{\text{j.d.}}{=} \sum_{k=1}^{n} P(|X_1| > b_n)$$

$$= n P(|X_1| > 2^{m(n)}) = n \sum_{j=m(n)+1}^{\infty} p_j = n \sum_{j=m(n)+1}^{\infty} \frac{1}{2^j j(j+1)} \leq$$

$$\frac{1}{j(j+1)} \leq \frac{1}{m(n)^2} \quad \text{(} j \geq m(n)+1 \text{)}$$

$$\leq n \sum_{j=m(n)+1}^{\infty} \frac{1}{2^j m(n)^2} = \frac{n}{m(n)^2} \sum_{j=1}^{\infty} \frac{1}{2^{m(n)}} \cdot \frac{1}{2^j}$$

$$= \frac{n}{m(n)^2 2^{m(n)}} \cdot 1 \underset{\frac{1}{m(n)^{\frac{3}{2}} 2^{m(n)}} \leq \frac{1}{n}}{\leq} \frac{1}{m(n)^{\frac{1}{2}}} \xrightarrow[n \to \infty]{} 0 \quad ✓$$

(ii) $\underbrace{b_n^{-2}\sum_{k=1}^{n} E\tilde{X}_{n,k}^2}_{\|} \longrightarrow 0$ $\qquad (\tilde{X}_{n,k} = X_{n,k}\mathbb{1}_{|X_{n,k}|\le b_n})$

$$\frac{1}{2^{2m(n)}}\cdot n\cdot E\left(X_1^2\mathbb{1}_{|X_1|\le 2^{m(n)}}\right) \le \frac{m(n)^{\frac{3}{2}}}{2^{m(n)}}\underbrace{EX_1^2\mathbb{1}_{|X_1|\le 2^{m(n)}}}_{(1)} \qquad (\square)$$

$$\frac{n}{2^{m(n)}} \le m(n)^{\frac{3}{2}}$$

$$(1) = \sum_{k=1}^{m(n)}(2^k-1)^2 p_k + (-1)^2 p_0 = \sum_{k=1}^{m(n)}(2^{2k}-2\cdot 2^k+1)p_k + p_0$$

$$= \sum_{k=1}^{m(n)} 2^{2k}p_k - 2\underbrace{\left(\sum_{k=1}^{m(n)}[(2^k-1)p_k]-p_0\right)}_{\le 1} + 3\underbrace{\sum p_k}_{\le 3} - p_0$$

$$\le 1 + \sum_{k=1}^{m(n)} 2^{\cancel{2}k}\cdot\frac{1}{\cancel{2^k}\,k(k+1)} = 1 + \sum_{k=1}^{m(n)}\frac{2^k}{k(k+1)}$$

$$= 1 + \sum_{k=1}^{\frac{m(n)}{2}} 2^k\underbrace{\frac{1}{k(k+1)}}_{\le 1} + \sum_{\frac{m(n)}{2}}^{m(n)} 2^k\frac{1}{k(k+1)} \qquad \frac{1}{k(k+1)} \le \frac{1}{\frac{m(n)}{2}\left(\frac{m(n)}{2}+1\right)}$$

$$\le 1 + \sum_{k=1}^{\frac{m(n)}{2}} 2^k + \frac{4}{m(n)\,(m(n)+1)}\sum_{\frac{m(n)}{2}}^{m(n)} 2^k$$

$$= 1 + 2^{\frac{m(n)}{2}+1} - 2 + \frac{4}{m(n)^2+m(n)}\left(2^{m(n)+1} - 2^{\frac{m(n)}{2}+1}\right)$$

$$\le \frac{2^{m(n)}}{m(n)^2}\,C \quad \text{za velike } n.$$

$$(\square) \le \frac{m(n)^{\frac{3}{2}}}{\cancel{2^{m(n)}}}\cdot\frac{\cancel{2^{m(n)}}}{m(n)^2}\,C = C\cdot\frac{1}{m(n)^{\frac{1}{2}}} \xrightarrow[n\to\infty]{} 0$$

$$\overset{\text{TM}}{\Longrightarrow} \quad \frac{S_n-a_n}{b_n} \xrightarrow{\mathbb{P}} 0 \qquad \text{uz} \quad a_n = \sum_{k=1}^{n} E\tilde{X}_{n,k}$$

$$a_n = nEX_1\mathbb{1}_{|X_1|\le 2^{m(n)}} = n\left(\sum_{k=1}^{m(n)}(2^k-1)p_k - p_0\right)$$

$$= n\left(\underbrace{\sum_{k=1}^{m(n)} 2^k \frac{1}{2^k k(k+1)}}_{\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\dots+\frac{1}{m(n)+1}} - \sum_{k=1}^{m(n)} p_k - \left(1-\sum_{k=1}^{\infty} p_k\right)\right)$$

$$= n\left(\cancel{1} - \frac{1}{m(n)+1} - \cancel{1} + \sum_{k=m(n)+1}^{\infty} p_k\right) = n\left(-\frac{1}{m(n)+1} + \sum_{k=m(n)+1}^{\infty} \frac{1}{2^k \underbrace{k(k+1)}_{\geq}}\right) \quad \leq \frac{1}{m(n)^2}$$

$$\leq n\left(-\frac{1}{m(n)+1} + \frac{1}{m(n)^2}\underbrace{\sum_{k=m(n)+1}^{\infty}\frac{1}{2^k}}_{\to 0 \text{ (ostatak konv. reda)}}\right) \sim -\frac{n}{m(n)}$$

S obzirom da je $\underset{\text{iz def.}}{n \leq} 2^{m(n)} m(n)^{\frac{3}{2}} \leq Cn \quad / \log_2$

$$\Rightarrow \underline{\log_2 n \sim m(n)} \qquad a_n = -\frac{n}{\log_2 n}$$

$$b_n = 2^{m(n)} \sim n(m(n))^{-\frac{3}{2}} \sim n(\log_2 n)^{-\frac{3}{2}}$$

$$\Rightarrow \frac{S_n - a_n}{b_n} \sim \frac{S_n + \frac{n}{\log n}}{n/(\log_2 n)^{\frac{3}{2}}} \xrightarrow{\mathbb{P}} 0$$

$$\frac{1}{(\log_2 n)^{\frac{1}{2}}} \longrightarrow 0$$

$$\Rightarrow \underbrace{\frac{1}{(\log_2 n)^{\frac{1}{2}}} \cdot \frac{S_n + \frac{n}{\log_2 n}}{\frac{n}{(\log_2 n)^{\frac{3}{2}}}}} \xrightarrow{\mathbb{P}} 0 \quad \text{(zbog konačnosti mjere } \mathbb{P}\text{)}$$

$$\frac{S_n + \frac{n}{\log_2 n}}{n/\log_2 n} \xrightarrow{\mathbb{P}} 0$$

$$\text{tj. } \frac{S_n}{\frac{n}{\log_2 n}} \xrightarrow{\mathbb{P}} -1$$

3. KONVERGENCIJA REDOVA

ZAD 3.1.

$(X_n)_n$ n.j.d. $X_k \sim N(0,1)$.

D: $\sum_{n=1}^{\infty} X_n \frac{\sin(n\pi t)}{n}$ kvg. g.s. za $\forall t \in \mathbb{R}$

Rj želimo iskoristiti tm s predavanja $\sum_1^{\infty} Var(Y_n) < \infty \rightarrow \sum_1^{\infty} (Y_n - EY_n) < \infty$ g.s.

(tj. koristit ćemo korolar koji kaže ako $\sum Var Y_n < \infty$ i $\sum EY_n < \infty \Rightarrow \sum_1^{\infty} Y_n < \infty$ g.s.)

Neka $t \in \mathbb{R}$ proizvoljan

Za tm nam ne treba n.j.d. samo nezavisnost

$$Y_n := X_n \frac{\sin(n\pi t)}{n}$$

$$EY_n = \underbrace{(EX_n)}_{=0} \frac{\sin(n\pi t)}{n} = 0$$

$$\sum_1^{\infty} Var Y_n = \sum_1^{\infty} \frac{\overbrace{\sin^2(n\pi t)}^{\leq 1}}{n^2} \underbrace{Var(X_n)}_{=1} \leq \sum_1^{\infty} \frac{1}{n^2} < \infty$$

$$\underset{TM}{\Rightarrow} \sum_1^{\infty} Y_n < \infty \quad g.s.$$

ZAD 3.2.

$(X_n)_n$ n.j.d., $X_n \sim N(1,1)$, $a > 1$

D: $\sum_{n=1}^{\infty} \frac{X_n}{a^n} < \infty$ g.s.

Rj slično kao u prethodnom zadatku promatramo varijancu, tj.

$$\sum_{n=1}^{\infty} Var\left(\frac{X_n}{a^n}\right) = \sum_{n=1}^{\infty} \frac{1}{a^{2n}} \cdot \underbrace{Var(X_n)}_{=1} = \sum_{n=1}^{\infty} \frac{1}{a^{2n}} < \infty, \quad a > 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{X_n}{a^n} - E\frac{X_n}{a^n}\right) < \infty \quad g.s.$$

$$\sum_{n=1}^{\infty} \frac{\overbrace{EX_n}^{=1}}{a^n} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{a^n}}_{\text{geom. red.}} < \infty, \quad \forall a>1$$

$\Rightarrow$ (korolar koji spomenuli u zad 3.1) $\Rightarrow \sum_{n=1}^{\infty} \frac{X_n}{a^n} < \infty$ g.s.

ZAD 3.3.

$(X_n)_n$ nez., $EX_n = 0$, $Var X_n = \sigma_n^2$

i) Ako je $\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} < \infty$ D: $\frac{X_n}{n} \xrightarrow{g.s.} 0$ (i $\frac{S_n}{n} \xrightarrow{g.s.} 0$)

Rj

Definiramo $Y_n := \frac{X_n}{n} \rightsquigarrow Var Y_n = \frac{1}{n^2} Var X_n = \frac{\sigma_n^2}{n^2}$

po pretp. $\sum_1^{\infty} Var Y_n = \sum_1^{\infty} \frac{\sigma_n^2}{n^2} < \infty$, Y_n nez. i $EY_n = \frac{1}{n} EX_n = 0$

$\Rightarrow$ [TM] $\Rightarrow \sum_1^{\infty} Y_n < \infty$ g.s.

$= \sum_1^{\infty} \frac{X_n}{n} < \infty$ g.s. $\underset{\text{red kvg. pa članovi moraju ići u 0}}{\Longrightarrow} \frac{X_n}{n} \xrightarrow{g.s.} 0$

Iz KRONECKEROVE LEME

$(a_n)_n$ niz poz. br., $a_n \nearrow +\infty$, $(x_n)_n$ niz realnih br.

Ako $\sum_1^{\infty} \frac{x_n}{a_n}$ kvg $\Rightarrow \frac{1}{a_n} \sum_{m=1}^{n} x_m \to 0$, $n \to \infty$

$a_n := n$ $(a_n \nearrow +\infty)$ i $\sum_1^{\infty} \frac{X_n(\omega)}{n}$ kvg. za g.s. ω ($X_n(\omega)$ je broj!)

$\Rightarrow a_n^{-1} \sum_{m=1}^{n} X_m = n^{-1} S_n \xrightarrow{g.s.} 0$

KRONECKEROVA L.

ii) Ako je $\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} = \infty$, i $\sigma_n^2 \le n^2$ pokažite da postoje sl. var. X_n sa očekivanjem $EX_n = 0$ i $Var X_n \le \sigma_n^2$ t.d. $\frac{X_n}{n}$ ne kvg. prema 0 g.s.

Rj: Definiramo nez. $(X_n)_n$ sa

$$\mathbb{P}(X_n = n) = \mathbb{P}(X_n = -n) = \frac{\sigma_n^2}{2n^2} \quad \left(\leq \frac{1}{2} \text{ jer } \sigma_n^2 \leq n^2 \right)$$

i $\mathbb{P}(X_n = 0) = 1 - \frac{\sigma_n^2}{n^2}$

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n \geq n) = \sum_{n=1}^{\infty} \frac{\sigma_n^2}{2n^2} = \frac{1}{2} \sum \frac{\sigma_n^2}{n^2} = +\infty$$

$\Rightarrow \mathbb{P}(X_n \geq n \text{ b.m.p.}) = 1$ tj. $\mathbb{P}\left(\frac{X_n}{n} \geq 1 \text{ b.m.p.}\right) = 1$

$\downarrow$

Borel-Cantelli 2. lema

$\overline{\lim} \frac{X_n}{n} \geq 1$ g.s.

3.4. $X_n \geq 0$, $n \geq 1$, nezavisne

Ekvivalentno je:

i) $\sum_{n=1}^{\infty} X_n < \infty$ g.s.

ii) $\sum_{n=1}^{\infty} \left(\mathbb{P}(X_n > 1) + \mathbb{E}(X_n \mathbb{1}_{X_n \leq 1}) \right) < \infty$

iii) $\sum_{n=1}^{\infty} \mathbb{E}\left(\frac{X_n}{1+X_n} \right) < \infty$

Rj. Za (i) $\Leftrightarrow$ (ii) koristimo Kolmog. tm o tri reda uz $A=1$

$\boxed{(i) \Rightarrow (ii)}$ $\sum X_n < \infty$ g.s.

TM TRI REDA $\Rightarrow \sum_{n=1}^{\infty} \mathbb{P}(X_n > 1) < \infty$ g.s. i $\sum_1^{\infty} \mathbb{E} X_n \mathbb{1}_{X_n \leq 1} < \infty$

$\Rightarrow$ suma ta dva reda konv ✓

$\boxed{(ii) \Rightarrow (i)}$ $\sum_{n=1}^{\infty} \left(\underbrace{\mathbb{P}(X_n > 1)}_{\geq 0} + \underbrace{\mathbb{E}(X_n \mathbb{1}_{X_n \leq 1})}_{\geq 0} \right) < \infty$

$\Rightarrow \sum_{n=1}^{\infty} \mathbb{P}(X_n > 1) < \infty$ & $\sum_{n=1}^{\infty} \mathbb{E} X_n \mathbb{1}_{X_n \leq 1} < \infty$ g.s.

ako pokažemo $\sum \mathrm{Var}(X_n \mathbb{1}_{X_n \leq 1}) < \infty$ iz obrata tm o tri reda slijedi $\sum_1^{\infty} X_n < \infty$ g.s.

$$\mathrm{Var}\, X_n \mathbb{1}_{X_n \leq 1} \leq \mathbb{E}\left[(X_n \mathbb{1}_{X_n \leq 1})^2 \right] = \mathbb{E}(X_n^2 \underbrace{\mathbb{1}_{X_n \leq 1}}_{X_n^2 \leq X_n \text{ ako } \leq 1}) \leq \mathbb{E}(X_n \mathbb{1}_{X_n \leq 1})$$

$$\leadsto \sum_1^{\infty} \mathrm{Var}(X_n \mathbb{1}_{X_n \leq 1}) \leq \sum_1^{\infty} \mathbb{E}(X_n \mathbb{1}_{X_n \leq 1}) < \infty$$ ✓

$\boxed{(ii) \Leftrightarrow (iii)}$ Tvrdimo:

$$\frac{X_n}{1+X_n} \leq X_n \mathbb{1}_{X_n \leq 1} + \mathbb{1}_{X_n > 1} \leq \frac{2X_n}{1+X_n}$$

iz toga slijedi $/\mathbb{E}$ $\rightarrow \mathbb{E} \frac{X_n}{1+X_n} \leq \mathbb{E}(X_n \mathbb{1}_{X_n \leq 1}) + \mathbb{P}(X_n > 1) \leq 2\mathbb{E} \frac{X_n}{1+X_n}$

tj. $\sum_1^\infty \mathbb{E}\frac{X_n}{1+X_n} \leq \sum_1^\infty \left(\mathbb{E}(X_n \mathbb{1}_{X_n\leq 1}) + \mathbb{P}(X_n>1)\right) \leq 2\sum_1^\infty \mathbb{E}\frac{X_n}{1+X_n}$

iz čega se vidi da su (ii) i (iii) ekvivalentne.

Preostalo je još dokazati tvrdnju; pomnožimo sve sa $1+X_n$, ekvivalentno je pokazati

$$X_n \leq \underbrace{(X_n \mathbb{1}_{X_n\leq 1} + \mathbb{1}_{X_n>1})(1+X_n)} \leq 2X_n$$

$$= \underbrace{\underline{\underline{X_n \mathbb{1}_{X_n\leq 1}}} + \mathbb{1}_{X_n>1} + X_n^2 \mathbb{1}_{X_n\leq 1} + \underline{\underline{X_n \mathbb{1}_{X_n>1}}}}_{X_n}$$

$$= X_n + \underbrace{\mathbb{1}_{X_n>1} + X_n^2 \mathbb{1}_{X_n\leq 1}}_{\geq 0} \geq X_n \quad \text{L.S.} \checkmark$$

a $X_n^2 \mathbb{1}_{X_n\leq 1} \leq X_n \mathbb{1}_{X_n\leq 1}$ i $\mathbb{1}_{X_n>1} \leq X_n \mathbb{1}_{X_n>1}$

pa $\leq X_n + \underbrace{X_n \mathbb{1}_{X_n\leq 1} + X_n \mathbb{1}_{X_n>1}}_{X_n} = \underline{\underline{2X_n}}$. D.S. $\checkmark$

ZAD 3.5.

$$\psi(x) = \begin{cases} x^2, & |x|\leq 1 \\ |x|, & |x|>1 \end{cases}$$

za nez $(X_n)_n$, $\mathbb{E}X_n = 0$,

$\sum_{n=1}^\infty \mathbb{E}\psi(X_n) < \infty$

$\leadsto$ red $\sum_{n=1}^\infty X_n$ kvg. g.s.

Rj Provjeravamo kvg. tri reda iz Kolm. tm o tri reda.

uz $A=1$

(i) $\sum_{n=1}^\infty \mathbb{P}(|X_n|>1) = \sum_{n=1}^\infty \mathbb{E}(\underbrace{\mathbb{1}_{|X_n|>1}}_{\leq |X_n| \mathbb{1}_{|X_n|>1}}) \leq \sum_{n=1}^\infty \mathbb{E}(|X_n| \mathbb{1}_{|X_n|>1})$

$\leq \sum_{n=1}^\infty \mathbb{E}\psi(X_n) < \infty \checkmark$

(ii) Def $Y_n := X_n \mathbb{1}_{|X_n|\le 1}$ .. Iz $\mathbb{E}X_n = 0 \Rightarrow \mathbb{E}Y_n = -\mathbb{E}X_n \mathbb{1}_{|X_n|>1}$

$$\sum_{n=1}^{\infty} |\mathbb{E}Y_n| = \sum_{n=1}^{\infty} |\mathbb{E}X_n \mathbb{1}_{|X_n|>1}| \le \sum_{n=1}^{\infty} \mathbb{E}|X_n| \mathbb{1}_{|X_n|>1} \le \sum \mathbb{E}\Psi(X_n) < \infty$$

(iii) $$\sum_{n=1}^{\infty} \operatorname{Var} Y_n \le \sum_{n=1}^{\infty} \mathbb{E}Y_n^2 = \sum_{n=1}^{\infty} \mathbb{E}(X_n^2 \mathbb{1}_{|X_n|\le 1}) \le \sum_{1}^{\infty} \mathbb{E}\Psi(X_n) < \infty$$

ZAD 3.6. $(X_n)_n$ nez. $\sum_{n=1}^{\infty} \mathbb{E}|X_n|^{p(n)} < \infty$, $0 < p(n) \le 2$, $\forall n$

i $\mathbb{E}X_n = 0$ kada $p(n) > 1$.

D: $\sum_{n=1}^{\infty} X_n$ kvg. g.s.

Rj Ponovno koristimo tm o tri reda sa $A = 1$

(i) $$\sum_{n=1}^{\infty} \mathbb{P}(|X_n|>1) = \sum_{n=1}^{\infty} \mathbb{E}\underbrace{\mathbb{1}_{|X_n|>1}}_{\substack{\text{na ovom dog} \\ |X_n|^{p(n)}>1}} \le \sum_{n=1}^{\infty} \mathbb{E}|X_n|^{p(n)} \mathbb{1}_{|X_n|>1} \le \sum_{n=1}^{\infty} \mathbb{E}|X_n|^{p(n)} < \infty$$

(ii) $Y_n := X_n \mathbb{1}_{|X_n|\le 1}$ za $p(n) \le 1$ $|Y_n| = |X_n| \mathbb{1}_{|X_n|\le 1} \le |X_n|^{p(n)} \mathbb{1}_{|X_n|\le 1} \le |X_n|^{p(n)}$

$\Rightarrow |\mathbb{E}Y_n| \le \mathbb{E}|X_n|^{p(n)}$

za $p(n) > 1$ $\mathbb{E}X_n = 0 \Rightarrow \mathbb{E}Y_n = -\mathbb{E}X_n \mathbb{1}_{|X_n|>1}$

$$|\mathbb{E}Y_n| = |\mathbb{E}X_n \mathbb{1}_{|X_n|>1}| \le \mathbb{E}(\underbrace{|X_n| \mathbb{1}_{|X_n|>1}}_{\le |X_n|^{p(n)} \text{ jer } p(n)>1})$$

$$\le \mathbb{E}|X_n|^{p(n)} \mathbb{1}_{|X_n|>1} \le \mathbb{E}|X_n|^{p(n)}$$

u oba slučaja dobijemo

$|\mathbb{E}Y_n| \le \mathbb{E}|X_n|^{p(n)}$

$$\rightsquigarrow \sum_{n=1}^{\infty} |\mathbb{E}Y_n| \le \sum_{n=1}^{\infty} \mathbb{E}|X_n|^{p(n)} < \infty$$

(iii) $$\sum_{1}^{\infty} \operatorname{Var} Y_n \le \sum_{1}^{\infty} \mathbb{E}Y_n^2 = \sum_{1}^{\infty} \mathbb{E}|X_n|^2 \mathbb{1}_{|X_n|\le 1} \underset{p(n)\le 2}{\le} \sum_{1}^{\infty} \mathbb{E}|X_n|^{p(n)} \mathbb{1}_{|X_n|\le 1}$$

$$\le \sum_{1}^{\infty} \mathbb{E}|X_n|^{p(n)} < \infty$$

4. JZVB.

Z 4.1. $(X_n)_n$ nezavisne $X_n \sim \begin{pmatrix} -\sqrt{\ln n} & \sqrt{\ln n} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. JZVB?

Rj Ne možemo donijeti zaključak iz Kolmogorovljevog tm o JZVB.

ZAŠTO?

Koristimo KOLMOGOROVLJEVU DOVOLJNI UVJET

$$\left(\begin{array}{l} (X_n)_n \text{ nez.} \quad (b_n)_n > 0 \quad \text{t.d.} \quad b_n \to +\infty. \\ \sum_{n=1}^{\infty} \frac{\operatorname{var} X_n}{b_n^2} < \infty \Rightarrow \frac{S_n - ES_n}{b_n} \xrightarrow{g.s.} 0 \end{array} \right)$$

$b_n := n$, $(b_n > 0$ i $b_n \to +\infty)$

$$\sum_{n=1}^{\infty} \frac{\operatorname{var} X_n}{n^2} = \left[\operatorname{var} X_n = EX_n^2 - (EX_n)^2 \quad EX_n \overset{\text{sim. sl. var.}}{=} \dots = 0, \quad EX_n^2 = \log n \right] = \sum_{n=1}^{\infty} \frac{\overbrace{\log n}^{\leq \sqrt{n}}}{n^2} \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} n^{-\frac{3}{2}} < \infty$$

$\Rightarrow$ KOLM. DOV. UVJET $\Rightarrow$ $\frac{S_n - ES_n}{n} = \frac{S_n}{n} \xrightarrow{g.s.} 0$

$ES_n = \sum \underbrace{EX_i}_{0} = 0$

ZAD 4.2

$(X_n)_n$ nez. t.d. $X_n \sim \begin{pmatrix} -n\alpha & 0 & n\alpha \\ \frac{1}{2n^2} & 1-\frac{1}{n^2} & \frac{1}{2n^2} \end{pmatrix}$. JZVB?

Rj: $\sum_{n=1}^{\infty} \frac{Var X_n}{n^2} = (*)$

$Var X_n = EX_n^2 - \underbrace{(EX_n)^2}_{=0} \overset{\text{zbog sim}}{=} EX_n^2 = \cancel{2} \cdot \cancel{n^2}\alpha^2 \cdot \frac{1}{\cancel{2n^2}} = \alpha^2$

$(*) = \sum_{n=1}^{\infty} \frac{\alpha^2}{n^2} = \alpha^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

$\Rightarrow$ KOLM. DOV. UVJET $\Rightarrow \frac{S_n - ES_n}{n} = \frac{S_n}{n} \xrightarrow{g.s.} 0$

4.1. ✓

4.2. ✓

$\to$ kolokvij 2022. $\to$ 5. zad

4.3. $(X_n)_n$ nezavisne, $X_1 = \Pi$,

za $n \geq 2$ $\quad \mathbb{P}(X_n = -n) = \mathbb{P}(X_n = n) = \dfrac{1}{4n\log n}$

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{2n\log n}$$

SZVB? JZVB?

Rj Nizovi nisu n.j.d. (nisu j.d.) jer svaki X_n poprima vrijednosti $-n, 0, n$ (ovise o indeksu)

SZVB $\to$ ne možemo koristiti Fellerov ni Hinčinov SZVB.

$\mathbb{E}X_n = 0$ (sim.) $\rightsquigarrow \operatorname{Var} X_n = \mathbb{E}X_n^2 = n^{\cancel{2}} \cdot \dfrac{1}{2\cancel{n}\log n} = \dfrac{n}{2\log n}$

želimo iskoristiti tm da $\dfrac{S_n - \mu_n}{b_n} \xrightarrow{P} 0$ ako $\mu_n = \mathbb{E}S_n$,

$\sigma_n^2 = \operatorname{Var} S_n$ i $\dfrac{\sigma_n^2}{b_n^2} \longrightarrow 0$

Uzmemo $b_n = n$

$$\frac{\sigma_n^2}{b_n^2} = \frac{1}{n^2} \cdot \operatorname{Var} S_n \underset{\text{nez.}}{=} \frac{1}{n^2} \sum_{k=1}^{n} \operatorname{Var} X_k = \frac{1}{n^2} \cdot \sum_{k=1}^{n} \frac{k}{2\log k}$$

$$= \frac{1}{n^2} \sum_{k=1}^{n_0} \frac{k}{2\log k} + \frac{1}{n^2} \sum_{k=n_0+1}^{n} \frac{k}{2\log k}$$

$$\leq \frac{1}{n^2} \sum_{k=1}^{n_0} \frac{k}{2\log k} + \frac{1}{2n^2 \log n_0} \underbrace{\sum_{k=n_0+1}^{n} k}_{\sim n^2}$$

$$\lesssim \underbrace{\frac{1}{n^2}}_{\to 0} \underbrace{\sum_{1}^{n_0} \frac{k}{2\log k}}_{\text{kon.}} + \underbrace{\frac{1}{2\log n_0}}_{\text{proizv. malo}} \leq \varepsilon \quad \text{za dov. veliki } n_0 \text{ i } \forall n \geq n_0$$

$$\Rightarrow \frac{S_n - \mathbb{E}S_n}{n} = \frac{S_n}{n} \xrightarrow{\mathbb{P}} 0$$

JZVB → ne možemo Kolmog. tm, ali (pravo, nije n.j.d.) možemo kolmogorovljev dovoljan uvjet probati.

$$\sum_{n=1}^{\infty} \frac{Var(X_n)}{n^2} = \sum_{n=1}^{\infty} \frac{1}{2n\log n} = \infty$$ (po integralnom kriteriju)

⇝ Kolm. dov. uvjet ne daje zaključak.

Ali vrijedi $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq n) = \sum_{n=1}^{\infty} \frac{1}{2n\log n} = \infty$

zbog nez. i BC-2 $\Rightarrow \mathbb{P}(|X_n| \geq n \text{ b.m.p.}) = 1$

slično kao na tm s predavanja sad se pokaže da $\frac{S_n - \mathbb{E}S_n}{n}$ ne kvg. g.s.

$$\left(\left|\frac{S_n - S_{n-1}}{n}\right| = \left|\frac{X_n}{n}\right| \geq 1 \text{ b.m.p.} \right.$$

$$\left. \text{a ako } \frac{S_n - \mathbb{E}S_n}{n} \text{ kvg, razlika bi trebala biti } < \varepsilon \right)$$

4.4. $(X_n)_n$ n.j.d. , $\mathbb{E}X_1 = \mu \neq 0$.

$S_n = \sum_{k=1}^{n} X_k$, $Y_n = \max_{1 \le k \le n} X_k$. D: $\frac{Y_n}{S_n} \xrightarrow{g.s.} 0$

Rj. Iz Kolmog. JZVB $(\mathbb{E}X_1 < \infty) \Rightarrow \frac{S_n}{n} \xrightarrow{g.s.} \mu$,

Promatramo g.s. konv. $\frac{Y_n}{n}$ → onda to možemo podijeliti sa $\frac{S_n}{n}$ da dobijemo traženu tvrdnju.

$$\frac{Y_n}{n} = \frac{\max_{1 \le k \le n} X_k}{n}$$

Prop $(X_n)_n$ n.j.d. i $\mathbb{E}X_1 = \mu < \infty \Rightarrow \frac{X_n}{n} \xrightarrow{g.s.} 0$

Dokaz
$$\frac{X_n}{n} = \frac{S_n - S_{n-1}}{n} = \underbrace{\frac{S_n}{n}}_{\text{JZVB} \downarrow \mu} - \underbrace{\frac{S_{n-1}}{n-1}}_{\text{JZVB} \downarrow \mu} \cdot \underbrace{\frac{n-1}{n}}_{\downarrow 1} \xrightarrow[n]{g.s.} \mu - \mu = 0$$

Dakle, $\boxed{\frac{X_n}{n} \xrightarrow{g.s.} 0}$

Lema $a_n \in \mathbb{R}$, $n \ge 1$. Ako $\frac{a_n}{n} \xrightarrow[n \to \infty]{} 0$ onda i $\frac{\max_{1 \le k \le n} |a_k|}{n} \xrightarrow[n]{} 0$

Dokaz $\varepsilon > 0$ proizv.

$\frac{|a_n|}{n} < \varepsilon$ za $n > n_0(\varepsilon) =: n_0$

$$\frac{\max_{1 \le k \le n} |a_k|}{n} \le \frac{\max_{1 \le k \le n_0} |a_k|}{n} + \frac{\max_{n_0 \le k \le n} |a_k|}{n}$$

$$\le \frac{\overbrace{\max_{1 \le k \le n_0} |a_k|}^{\text{konačno}}}{n} + \max_{n_0 \le k \le n} \frac{|a_k|}{k} \le \frac{C}{n} + \varepsilon \quad / \overline{\lim}$$

$$\overline{\lim} \frac{\max_{k \le n} |a_k|}{n} \le \varepsilon \quad \forall$$

primjenom prethodne leme na $(Y_n)_n \Rightarrow \frac{Y_n}{n} \xrightarrow{g.s.} 0$

$$\Rightarrow \frac{Y_n}{S_n} = \frac{\frac{Y_n}{n}}{\frac{S_n}{n}} \xrightarrow{g.s.} \frac{0}{\mu} = 0$$

■

4.5. $(X_n)_n$ nez.

$$\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = \frac{1}{2}(1 - 2^{-n})$$

$$\mathbb{P}(X_n = -2^n) = \mathbb{P}(X_n = 2^n) = \frac{1}{2^n} \cdot \frac{1}{2}$$

JZVB?

Rj:

$$\underbrace{\mathbb{E}X_n = 0}_{\text{sim.}}, \quad \underset{\sigma_n^2}{\mathrm{Var}X_n} = \mathbb{E}X_n^2 = \cancel{2} \cdot \cancel{\tfrac{1}{2}}(1 - 2^{-n}) + 2^{2n} \cdot 2^{-n-1} \cdot 2 = \underline{\underline{1 - 2^{-n} + 2^n}}$$

Probamo iskoristiti Kolmog. dov. uvjet, $b_n = n$

$$\sum_{n=1}^{\infty} \frac{\sigma_n^2}{b_n^2} = \sum_{n=1}^{\infty} \frac{\overbrace{1 - 2^{-n}}^{\geq 0} + 2^n}{n^2} \geq \sum_{n=1}^{\infty} \frac{2^n}{n^2} = \sum_{n=1}^{\infty} \frac{\frac{2^n}{n}}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

$\frac{2^n}{n} \geq \frac{1}{n} \ \forall n$

$\leadsto$ ne kvg $\Rightarrow$ ne daje zaključak.

Kreiramo novi niz sl. var. $(Y_n)_n$ t.d.

$$Y_n = \begin{cases} X_n, & \text{ako } X_n \in \{-1, 1\} \\ 0, & \text{inače} \end{cases} \Rightarrow \begin{array}{l} \mathbb{P}(Y_n = -1) = \mathbb{P}(Y_n = 1) = \frac{1}{2}(1 - 2^{-n}) \\ \mathbb{P}(Y_n = 0) = \frac{1}{2} \cdot \frac{1}{2^n} \end{array}$$

Vrijedi $\mathbb{P}(Y_n \neq X_n) = \mathbb{P}(Y_n = 0) = \frac{1}{2^{n+1}}$

$$\leadsto \sum_{n=1}^{\infty} \mathbb{P}(Y_n \neq X_n) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} < \infty$$

$$\underset{BC\text{-}1}{\Rightarrow} \mathbb{P}(Y_n \neq X_n \text{ b.m.p.}) = 0 \quad \text{tj.}$$

$$\mathbb{P}(Y_n = X_n \text{ eventually}) = 1$$

tj. g.s. $X_n = Y_n$ za dov. velike n.

ovdje pokazujemo da nizovi sl.v. $(X_n)_n$ i $(Y_n)_n$ <u>ekvivalentni</u>

Za niz $(Y_n)_n$ vrijedi $\mathbb{E}Y_n = 0$, $\operatorname{Var} Y_n = 1 - \frac{1}{2^n}$

i $\displaystyle\sum_{n=1}^{\infty} \frac{\operatorname{Var} Y_n}{n^2} = \sum_{n=1}^{\infty} \frac{1-2^{-n}}{n^2} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{<\infty} - \underbrace{\sum_{n=1}^{\infty} \frac{1}{2^n n^2}}_{<\infty} < \infty$

$\Rightarrow$ Kolm. dov. uvjet $\Rightarrow \dfrac{\sum_1^n Y_k}{n} \xrightarrow{g.s.} 0$

S obzirom da su $(X_n)_n$ i $(Y_n)_n$ ekvivalentni g.s., slijedi

$\dfrac{\sum_{k=1}^n X_k}{n} \xrightarrow{g.s.} 0$

$\dfrac{\sum_{k=1}^n Y_k}{n} \xrightarrow{g.s.} 0$, a za neki $n_0(\omega) = n_0$, $\forall n > n_0$ vrijedi

$X_{n(\omega)} = Y_{n(\omega)} \Rightarrow \dfrac{\sum_{k=1}^n X_k}{n} = \dfrac{\sum_{k=1}^{n_0} X_k + \sum_{k=n_0+1}^{n} X_k}{n}$

$= \dfrac{\sum_{k=1}^{n_0} X_k}{n} + \dfrac{\sum_{n_0+1}^{n} Y_k}{n} \underset{\pm \sum_1^{n_0} Y_k}{=} \dfrac{\overbrace{\sum_{k=1}^{n_0} (X_k - Y_k)}^{C}}{n} + \dfrac{\sum_{k=1}^{n} Y_k}{n} \longrightarrow 0$

$\dfrac{C}{n} \xrightarrow{n\to\infty} 0$, $\dfrac{\sum_{k=1}^{n} Y_k}{n} \xrightarrow{n\to\infty} 0$