Weak Solutions for a Degenerate Elliptic Dirichlet Problem

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Branko Najman (1946–1996)

Picture taken by G.M. Bergmann at Oberwolfach in 1980.

http://owpdb.mfo.de/detail?photo_id=5675
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Closely Embedded Hilbert Spaces

Let $\mathcal{H}$ and $\mathcal{H}_+$ be two Hilbert spaces. The Hilbert space $\mathcal{H}_+$ is called **closely embedded** in $\mathcal{H}$ if:

1. **(ce1)** There exists a linear manifold $\mathcal{D} \subseteq \mathcal{H}_+ \cap \mathcal{H}$ that is dense in $\mathcal{H}_+$.
2. **(ce2)** The embedding operator $j_+$ with domain $\mathcal{D}$ is closed, as an operator $\mathcal{H}_+ \to \mathcal{H}$.

Axiom (ce1) means that on $\mathcal{D}$ the algebraic structures of $\mathcal{H}_+$ and $\mathcal{H}$ agree.

Axiom (ce2) means that the operator $j_+$ with $\text{Dom}(j_+) = \mathcal{D} \subseteq \mathcal{H}_+$ defined by $j_+ x = x \in \mathcal{H}$, for all $x \in \mathcal{D}$, is closed.
The Kernel Operator

Let $\mathcal{H}_+$ be a Hilbert space that is closely embedded in $\mathcal{H}$, and let $j_+$ denote the corresponding closed embedding. Then $A = j_+j^* \in C(\mathcal{H})^+$ and

$$\langle j_+ h, k \rangle = \langle h, Ak \rangle_+, \quad h \in \text{Dom}(j_+), \; k \in \text{Dom}(A), \quad (2.1)$$

more precisely, $A$ has the range in $\mathcal{H}_+$ and it can also be viewed as the adjoint of the embedding $j_+$. The operator $A$ is called the kernel operator associated to the closed embedding of $\mathcal{H}_+$ in $\mathcal{H}$.

L. Schwartz — for continuous embeddings
Let $T \in \mathcal{C}(\mathcal{G}, \mathcal{H})$ be a closed and densely defined linear operator, where $\mathcal{G}$ is another Hilbert space. On $\text{Ran}(T)$ we consider a new inner product

$$\langle Tu, Tv \rangle_T = \langle u, v \rangle_{\mathcal{G}},$$

(2.2)

where $u, v \in \text{Dom}(T) \ominus \text{Ker}(T)$. With respect to this new inner product $\text{Ran}(T)$ can be completed to a Hilbert space that we denote by $\mathcal{R}(T)$, closely embedded in $\mathcal{H}$, and in such a way that $j_T: \mathcal{R}(T) \rightarrow \mathcal{H}$ has the property that $j_T j_T^* = TT^*$. 
The Space $\mathcal{D}(T)$

Let $T \in \mathcal{C}(\mathcal{H}, \mathcal{G})$ with $\text{Ker}(T)$ a closed subspace of $\mathcal{H}$. Define the norm

$$|x|_T := \|Tx\|_G, \quad x \in \text{Dom}(T) \ominus \text{Ker}(T),$$

and let $\mathcal{D}(T)$ be the Hilbert space completion of the pre-Hilbert space $\text{Dom}(T) \ominus \text{Ker}(T)$ with respect to the norm $|\cdot|_T$ associated the inner product $(\cdot, \cdot)_T$

$$(x, y)_T = \langle Tx, Ty \rangle_G, \quad x, y \in \text{Dom}(T) \ominus \text{Ker}(T).$$

Define $i_T$ from $\mathcal{D}(T)$ and valued in $\mathcal{H}$ by

$$i_Tx := x, \quad x \in \text{Dom}(i_T) = \text{Dom}(T) \ominus \text{Ker}(T).$$

The operator $i_T$ is closed and $\mathcal{D}(T)$ is closely embedded in $\mathcal{H}$, with the underlying closed embedding $i_T$.

The operator $Ti_T$ admits a unique isometric extension $\widehat{T}: \mathcal{D}(T) \to \mathcal{G}$. 
By definition, \((\mathcal{H}_+; \mathcal{H}; \mathcal{H}_-)\) is called a triplet of closely embedded Hilbert spaces if:

1. \(\mathcal{H}_+\) is a Hilbert space closely embedded in the Hilbert space \(\mathcal{H}\), with the closed embedding denoted by \(j_+\), and such that \(\text{Ran}(j_+)\) is dense in \(\mathcal{H}\).

2. \(\mathcal{H}\) is closely embedded in the Hilbert space \(\mathcal{H}_-\), with the closed embedding denoted by \(j_-\), and such that \(\text{Ran}(j_-)\) is dense in \(\mathcal{H}_-\).

3. \(\text{Dom}(j_+^*) \subseteq \text{Dom}(j_-)\) and for every vector \(y \in \text{Dom}(j_-) \subseteq \mathcal{H}\) we have

\[
\|y\|_- = \sup\left\{ \frac{|\langle x, y \rangle_{\mathcal{H}}|}{\|x\|_+} \mid x \in \text{Dom}(j_+), x \neq 0 \right\}.
\]

The kernel operator \(A = j_+ j_+^*\) is a positive selfadjoint operator in \(\mathcal{H}\) that is one-to-one. Then, \(H = A^{-1}\) is a positive selfadjoint operator in \(\mathcal{H}\) and it is called the Hamiltonian of the triplet. Note that, as a consequence of (th3), we actually have \(\text{Dom}(j_+^*) = \text{Dom}(j_-)\).
Theorem

Let $H$ be a \textit{positive selfadjoint operator} in the Hilbert space $\mathcal{H}$, that admits an \textit{inverse} $A = H^{-1}$, possibly unbounded. Then there exists $T \in \mathcal{C}(\mathcal{H}, \mathcal{G})$, with Ran$(T)$ dense in $\mathcal{G}$ and $H = T^* T$. In addition, let $S = T^{-1} \in \mathcal{C}(\mathcal{G}, \mathcal{H})$. Then:

(i) The Hilbert space $\mathcal{H}_+ := \mathcal{D}(T) := \text{Ran}(S)$ is closely embedded in $\mathcal{H}$ with its embedding $i_T$ having \textit{range dense} in $\mathcal{H}$, and its \textit{kernel operator} $A = i_T i_T^*$ coincides with $H^{-1}$.

(ii) $\mathcal{H}$ is closely embedded in the Hilbert space $\mathcal{H}_- = \mathcal{R}(T^*)$ with its embedding $j_{T^*}$ having \textit{range dense} in $\mathcal{R}(T^*)$. The kernel operator $B = j_{T^*} j_{T^*}^{-1}$ of this embedding is \textit{unitary equivalent} with $A = H^{-1}$. 
Theorem (continued)

(iii) The operator $V = i_T^* | \text{Ran}(T^*)$, that is,

$$
\langle i_T x, y \rangle_H = (x, Vy)_T, \quad x \in \text{Dom}(T), \ y \in \text{Ran}(T^*), \quad (2.6)
$$

extends uniquely to a unitary operator $\tilde{V}$ between the Hilbert spaces $\mathcal{R}(T^*)$ and $\mathcal{D}(T)$.

(iv) The operator $H$, when viewed as a linear operator with domain dense in $\mathcal{D}(T)$ and range in $\mathcal{R}(T^*)$, extends uniquely to a unitary operator $\tilde{H}: \mathcal{D}(T) \rightarrow \mathcal{R}(T^*)$, and $\tilde{H} = \tilde{V}^{-1}$. 
Theorem (continued)

(v) The operator \( \Theta : \mathcal{R}(T^*) \to \mathcal{D}(T)^* \) defined by

\[
(\Theta \alpha)(x) := (\tilde{V}\alpha, x)_T, \quad \alpha \in \mathcal{R}(T^*), \ x \in \mathcal{D}(T),
\]

provides a canonical and unitary identification of the Hilbert space \( \mathcal{R}(T^*) \) with the conjugate space \( \mathcal{D}(T)^* \), in particular, for all \( y \in \text{Dom}(T^*) \)

\[
\|y\|_{T^*} = \sup\left\{ \frac{|\langle y, x \rangle_{\mathcal{H}}|}{|x|_T} \mid x \in \text{Dom}(T) \setminus \{0\} \right\}.
\]
Generation of Triplets of Hilbert Spaces: The General Picture

\[ \mathcal{H}_+ = \mathcal{D}(T) \xrightarrow{i_T} \mathcal{H} \xrightarrow{i_T^*} \mathcal{H}_- \]

\[ \mathcal{D}(T) \xrightarrow{i_T} \mathcal{H} \xleftarrow{i_T} \mathcal{D}(T) = \mathcal{H}_+ \]

\[ \mathcal{H}_- = \mathcal{R}(T^*) = \mathcal{H}_+ \]

\[ \mathcal{H} = A^{-1} \]

\[ \mathcal{V} = \tilde{A} \]

\[ \mathcal{H} = \tilde{A}^{-1} \]

\[ H \quad \text{Hamiltonian} \quad \text{Berezansky — continuous embeddings} \]

\[ A = H^{-1} \quad \text{Kernel Operator} \]

\[ H = T^* T \quad \text{Factor Operator} \]

\[ A = SS^* \quad \text{Factor Operator} \]
The Gradient

Let $\Omega$ be an open (nonempty) set of the $\mathbb{R}^N$. Let $D_j = i \frac{\partial}{\partial x_j}$, $(j = 1, \ldots, N)$ be the operators of differentiation with respect to the coordinates of points $x = (x_1, \ldots, x_N)$ in $\mathbb{R}^N$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$, let $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$, $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$. $\nabla_l = (D^\alpha)|_{|\alpha|=l}$ denotes the gradient of order $l$, where $l$ is a fixed nonnegative integer. Letting $m = m(N, l)$ denote the number of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_N = l$, $\nabla_l$ can be viewed as an operator acting from $L_2(\Omega)$ into $L_2(\Omega; \mathbb{C}^m)$ defined on its maximal domain, the Sobolev space $W_2^l(\Omega)$, by

$$\nabla_l u = (D^\alpha u)|_{|\alpha|=l}, \quad u \in W_2^l(\Omega).$$
The Underlying Spaces

\( W^l_2(\Omega) \) consists of those functions \( u \in L_2(\Omega) \) whose distributional derivatives \( D^\alpha u \) belong to \( L_2(\Omega) \) for all \( \alpha \in \mathbb{Z}_+^N, |\alpha| \leq l \) and with norm

\[
\| u \|_{W^l_2(\Omega)} = \left( \sum_{|\alpha| \leq m} \| D^\alpha u \|^2_{L_2(\Omega)} \right)^{1/2},
\]

(3.1)

\( W^l_2(\Omega) \) becomes a Hilbert space that is continuously embedded in \( L_2(\Omega) \).

\( \overset{\circ}{W}^l_2(\Omega) \) denotes the closure of \( C_0^\infty(\Omega) \) in the space \( W^l_2(\Omega) \).
The space $\mathcal{L}_p^l(Ω)$, $(1 \leq p < \infty)$ is defined as the completion of $C^\infty_0(Ω)$ under the metric corresponding to

$$\|u\|_{p,l} := \|
abla_l u\|_{L^p(Ω)} = \left( \int_{Ω} \left( \sum_{|α|=l} |D^α u(x)|^2 \right)^{p/2} \, dx \right)^{1/p}, \quad u \in C^\infty_0(Ω).$$

The elements of $\mathcal{L}_p^l(Ω)$ can be realized as locally integrable functions on $Ω$ vanishing at the boundary $∂Ω$ and having distributional derivatives of order $l$ in $L^p(Ω)$.
Moreover, these functions, after modification on a set of zero measure, are absolutely continuous on every line which is parallel to the coordinate axes.
On $\Omega$ there is defined an $m \times m$ matrix valued measurable function $a$, more precisely, $a(x) = [a_{\alpha\beta}(x)]$, $|\alpha|, |\beta| = l$, $x \in \Omega$, where the scalar valued functions $a_{\alpha,\beta}$ are measurable on $\Omega$ for all multi-indices $|\alpha|, |\beta| = l$.

(C1) For almost all (with respect to the n-dimensional standard Lebesgue measure) $x \in \Omega$, the matrix $a(x)$ is nonnegative (positive semidefinite), that is,

$$\sum_{|\alpha|,|\beta|=l} a_{\alpha\beta}(x) \overline{\eta}_{\beta} \eta_{\alpha} \geq 0, \text{ for all } \eta = (\eta_{\alpha})_{|\alpha|=l} \in \mathbb{C}^m.$$ 

According to the condition (C1), there exists an $m \times m$ matrix valued measurable function $b$ on $\Omega$, such that

$$a(x) = b(x)^* b(x), \text{ for almost all } x \in \Omega,$$

where $b(x)^*$ denotes the Hermitian conjugate matrix of the matrix $b(x)$. 

The Assumptions
Conditions

(C2) There is a nonnegative measurable function $c$ on $\Omega$ such that, for almost all $x \in \Omega$ and all $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{C}^N$, 

$$|b(x)\tilde{\xi}| \geq c(x)|\tilde{\xi}|,$$

where $\tilde{\xi} = (\xi^\alpha)|_{|\alpha|=1}$ is the vector in $\mathbb{C}^m$ with $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_N^{\alpha_N}$.

(C3) All the entries $b_{\alpha\beta}$ of the $m \times m$ matrix valued function $b$ are functions in $L_{1,\text{loc}}(\Omega)$.

(C4) The function $c$ in (C2) has the property that $1/c \in L_2(\Omega)$. 
The Operator $T$

Under the conditions (C1)–(C4), we consider the operator $T$ acting from $L_2(\Omega)$ to $L_2(\Omega; \mathbb{C}^m)$ and defined by

$$(Tu)(x) = b(x)\nabla_l u(x), \quad \text{for almost all } x \in \Omega, \quad (3.2)$$

on its domain

$$\text{Dom}(T) = \{ u \in W_2^l(\Omega) \mid b\nabla_l u \in L_2(\Omega; \mathbb{C}^m) \}. \quad (3.3)$$
The Problem

Our aim is to describe, in view of the abstract model, the triplet of closely embedded Hilbert spaces \((D(T); L_2(\Omega); \mathcal{R}(T^*))\) associated with the operator \(T\) defined at (3.2) and (3.3).

In terms of these results, we obtain information about weak solutions for the corresponding operator equation involving the Hamiltonian operator \(H = T^* T\) of the triplet, which in fact is a Dirichlet boundary value problem in \(L_2(\Omega)\) with homogeneous boundary values.
The Problem

This problem is associated to the differential sesqui-linear form

\[ a[u, v] = \int_{\Omega} \langle a(x) \nabla I(x), \nabla I(x) \rangle \, dx \]

\[ = \sum_{|\alpha| = |\beta| = l} \int_{\Omega} a_{\alpha\beta}(x) D^\beta u(x) D^\alpha v(x) \, dx, \quad u, v \in C_0^\infty(\Omega), \]  

which, as will be seen, can be extended up to elements of \( \mathcal{D}(T) \).

The problem can be reformulated as follows: given \( f \in \mathcal{D}(T)^* \) (which is canonically identified with the \( \mathcal{R}(T^*) \)), find \( v \in \mathcal{D}(T) \) such that

\[ a[u, v] = \langle u, f \rangle \text{ for all } u \in \mathcal{D}(T), \]  

where \( \langle \cdot, \cdot \rangle \) denotes the duality between \( \mathcal{D}(T) \) and \( \mathcal{D}(T)^* \).

The problem in (3.5) can be considered only for \( u \in W_2^2(\Omega) \), or, even more restrictively, only for \( u \in C_0^\infty(\Omega) \).
The Main Result

Theorem

For $\Omega$ a domain in $\mathbb{R}^N$ and $l \in \mathbb{N}$, let $a(x) = [a_{\alpha\beta}(x)] = b(x)^*b(x)$, $|\alpha|, |\beta| = l$, $x \in \Omega$, satisfy the conditions (C1)–(C4), and consider the differential sesqui-linear form

$$a[u, v] = \int_{\Omega} \langle a(x)\nabla_l(x), \nabla_l(x) \rangle \, dx$$

$$= \sum_{|\alpha|=|\beta|=l} \int_{\Omega} a_{\alpha\beta}(x)D^\beta u(x) \overline{D^\alpha v(x)} \, dx, \quad u, v \in C_0^\infty(\Omega),$$
The Main Result

Theorem (Continuation)

Then:
1. The operator $T$ acting from $L^2(\Omega)$ to $L^2(\Omega; \mathbb{C}^m)$ and defined by $(Tu)(x) = b(x)\nabla_I u(x)$ for $x \in \Omega$ and $u \in \text{Dom}(T) = \{ u \in W^1_2(\Omega) | b\nabla_I u \in L^2(\Omega; \mathbb{C}^m) \}$ is closed, densely defined, and injective.

2. The pre-Hilbert space $\text{Dom}(T)$ with norm $|u|_T = \left( \int_{\Omega} |b(x)\nabla_I u(x)|^2 \, dx \right)^{1/2}$, has a unique Hilbert space completion, denoted by $\mathcal{H}^I_a(\Omega)$, that is continuously embedded into $L^1(\Omega)$. 
The Main Result

Theorem (Continuation)

(3) The conjugate space of $\mathcal{H}_a(\Omega)$, denoted by $\mathcal{H}_a^\prime(\Omega)$, can be realized in such a way that, for any $f \in \mathcal{H}_a^\prime(\Omega)$ there exist elements $g \in L_2(\Omega; \mathbb{C}^m)$ such that

$$f(u) = \int_{\Omega} \langle g(x), b(x)\nabla l u(x) \rangle \, dx, \quad u \in \mathcal{W}_2(\Omega),$$

(3.6)

and

$$\| f \|_{\mathcal{H}_a^\prime(\Omega)} = \inf \{ \| g \|_{L_2(\Omega; \mathbb{C}^m)} : g \in L_2(\Omega; \mathbb{C}^m) \text{ such that } (3.6) \text{ holds} \}.$$
The Main Result

Theorem (Continuation)

(4) \((\mathcal{H}_a(\Omega); L_2(\Omega); \mathcal{H}_a^{-1}(\Omega))\) is a triplet of closely embedded Hilbert spaces.

(5) For every \(f \in \mathcal{H}_a^{-1}(\Omega)\) there exists a unique \(v \in \mathcal{H}_a(\Omega)\) that solves the Dirichlet problem associated to the sesquilinear form \(a\), in the sense that

\[ a[u, v] = \langle u, f \rangle \text{ for all } u \in \mathcal{H}_a(\Omega). \]

More precisely, \(v = \tilde{H}^{-1}f\), where \(\tilde{H}\) is the unitary operator acting between \(\mathcal{H}_a(\Omega)\) and \(\mathcal{H}_a^{-1}(\Omega)\) that uniquely extends the positive selfadjoint operator \(H = T^* T\) in \(L_2(\Omega)\).
The Main Result

The Paper


