ON A MODEL IN PORO-ELASTICITY

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Abstract. A modification of the material law associated with the well-known Biot system as suggested by M. A. Murad and J. H. Cushman 1996 for some types of clay and first investigated by R. E. Showalter in 2000 is re-considered in the light of a new approach to a comprehensive class of evolutionary problems and extended to anisotropic, inhomogeneous media. The results presented are based on joint work with D. McGhee, Glasgow.

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Introduction

We shall investigate a particular model for poro-elastic media in a more general framework. The model under consideration describes the consolidation of soil as water is draining out of it. We begin by noticing, that on closer inspection of initial boundary value problems of mathematical physics, in particular those describing wave propagation phenomena one is inclined to describe their general form as

$$\partial_0 V + AU = f \quad \text{on } \mathbb{R}_{>0}, \quad V(0+) = \Phi,$$

where $A$ is skew-selfadjoint in a suitable Hilbert space setting. We shall indeed prefer to consider this problem on the whole real time-line and to by-pass the full construction of associated Sobolev lattices we shall assume – without loss of generality – that $\Phi = 0$. This turns our problem into

$$\partial_0 V + AU = f \quad \text{on } \mathbb{R}. \quad (1)$$
As typical examples we may consider:

**Elastic Waves:** \( \varrho_0 \partial_0^2 u - \text{Div} \, T = f, \quad T = C \text{ Grad } u, \)

where \( u \) denotes the displacement field, \( T \) the stress tensor, \( \varrho_0 \) mass density, \( C \) the elasticity tensor, which is assumed to be modeled as bounded, self-adjoint, strictly positive definite mappings in a Hilbert space \( H_{\text{sym}} \) of \( L^2 (\Omega) \)-valued, self-adjoint \( 3 \times 3 \)-matrices. This can be transformed into

\[
\begin{align*}
\varrho_0 \partial_0 (\varrho_0 v) - \text{Div} \, T &= f \\
\varrho_0 (C^{-1} T) - \text{Grad } v &= 0
\end{align*}
\]

where \( v := \partial_0 u \) denotes the velocity field of the displacement field \( u \).
Electro-Magnetic Waves:

\[ \partial_0 \left( D + \partial_0^{-1} \sigma E \right) - \text{curl} \ H + J = 0, \ \partial_0 B + \text{curl} \ E = 0. \]

Here \( E \) denotes the electric and \( H \) the magnetic field, whereas \( D \) is known as the displacement current density and \( B \) as the magnetic induction.

Material relation:

\[ D = \varepsilon E, \ B = \mu H. \]
Following the lead of these examples, the abstract evolutionary problem

$$\partial_0 V + AU = f \quad \text{on } \mathbb{R}. $$

is now completed by an additional rule frequently referred to as a "material law", which for simplicity we assume to be time-translation invariant and more precisely of the form

$$ V = M \left( \partial_0^{-1} \right) U, \quad (2) $$

where $z \mapsto M(z)$ is bounded-operator-valued and analytic in an open ball $B_\mathbb{C}(r, r)$ with some positive radius $r$ centered at $r$. 
The Time Derivative

It is well-known that \( \frac{1}{i} \partial_0 \) can be established as a selfadjoint operator in the space \( L^2 (\mathbb{R}) \) of equivalence classes of square-integrable complex-valued functions on \( \mathbb{R} \). The space \( \hat{C}_\infty (\mathbb{R}) \) of smooth complex-valued functions with compact support is densely embedded in the domain. Indeed, this case is occasionally used as a simple example for an explicit spectral representation, which here is provided by the Fourier transform \( \mathcal{F} \) given as the unitary extension of

\[
\hat{C}_\infty (\mathbb{R}) \subseteq L^2 (\mathbb{R}) \rightarrow L^2 (\mathbb{R})
\]

\( \varphi \mapsto \hat{\varphi} \)

with

\[
\hat{\varphi} (x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp (-ix \cdot t) \varphi (t) \, dt, \ x \in \mathbb{R}.
\]
As a spectral representation the Fourier transform makes $\frac{1}{i} \partial_0$ unitarily equivalent to the multiplication by the argument operator $m$ given by $(m\varphi)(x) = x\varphi(x)$ for $x \in \mathbb{R}$ and $\varphi \in \check{C}_\infty(\mathbb{R})$:

$$\frac{1}{i} \partial_0 = \mathcal{F}^* m \mathcal{F}.$$
We introduce an exponential weight function \( t \mapsto \exp(-\nu t) \), \( \nu \in \mathbb{R} \), and consider the weighted \( L^2 \)-space \( H_{\nu,0} \) generated by completion of \( \mathcal{C}_\infty(\mathbb{R}) \) with respect to the inner product \( \langle \cdot | \cdot \rangle_{\nu,0} \)

\[
(\varphi, \psi) \mapsto \int_{\mathbb{R}} \varphi(t)^* \psi(t) \exp(-2\nu t) \, dt.
\]

The associated norm will be denoted by \( | \cdot |_{\nu,0} \). The multiplication operator

\[
\mathcal{C}_\infty(\mathbb{R}) \subseteq H_{\nu,0} \rightarrow \mathcal{C}_\infty(\mathbb{R}) \subseteq H_{0,0} = L^2(\mathbb{R})
\]

\[
\varphi \mapsto \exp(-\nu m) \varphi
\]

with

\[
(\exp(-\nu m) \varphi)(x) = \exp(-\nu x) \varphi(x), \quad x \in \mathbb{R},
\]

clearly has a unitary extension, which we shall denote by \( \exp(-\nu m) \), where the \( m \) serves as a reminder for 'multiplication'.
Its inverse will be denoted by \( \exp (\nu m) \). Thus, the operator

\[
\frac{1}{i} \partial_\nu := \exp (\nu m) \frac{1}{i} \partial_0 \exp (-\nu m)
\]

defines a unitarily equivalent operator \( \frac{1}{i} \partial_\nu \), which is now selfadjoint in \( H_{\nu,0} \).

We shall use again the notation \( \partial_0 \) for the normal operator \( \partial_\nu + \nu \), which is justified since indeed

\[
(\partial_\nu + \nu) \varphi = \partial_0 \varphi
\]

for \( \varphi \in \mathcal{C}_\infty (\mathbb{R}) \).

Obviously we have that the spectrum of \( \partial_\nu \) is purely imaginary. In fact, the spectrum \( \sigma (\partial_\nu) \) is also purely continuous spectrum:

\[
\sigma (\partial_\nu) = \sigma_c (\partial_\nu) = i \mathbb{R}.
\]
Thus, in particular for $\nu \in \mathbb{R}\setminus\{0\}$ we have the bounded invertibility of $\partial_0 = \partial_\nu + \nu$.

With $\mathcal{L}_\nu := \mathcal{F} \exp(-\nu m)$

$$\frac{1}{i} \partial_\nu = \mathcal{L}_\nu^* m \mathcal{L}_\nu.$$
Evolutionary Dynamics and Material Laws

General Linear Material Laws

We shall now consider the initially stated evolutionary problem in precise terms. For this we need to extend the operators $\partial_0, A$ to the tensor product spaces $H_{\nu,0} \otimes H$ by interpreting $A$ as $1_{H_{\nu,0}} \otimes A$ with $1_{H_{\nu,0}} : H_{\nu,0} \to H_{\nu,0}$ as the identity operator in $H_{\nu,0}$ and the time derivative $\partial_0$ as $\partial_0 \otimes 1_H$, where $1_H : H \to H$ denotes the identity operator in $H$.

In this sense, our aim is to be able to find $U \in H_{\nu,0} \otimes H$ such that for a given $f \in H_{\nu,0} \otimes H$ we have

$$\left( \partial_0 M \left( \partial_0^{-1} \right) + A \right) U = f. \quad (3)$$

The operator $M \left( \partial_0^{-1} \right)$ will be referred to as the material operator.
Here \((M(z))_{z \in B_C(r,r)}\) is a uniformly bounded, holomorphic family of linear operators in \(H\) with \(r \leq \frac{1}{2\nu}\). It is

\[
M \left( \partial_0^{-1} \right) := \mathcal{L}_\nu^* M \left( \frac{1}{i m + \nu} \right) \mathcal{L}_\nu.
\]

To warrant a solution theory we require an additional constraint on such causal materials: There should be a constant \(c \in \mathbb{R}_{>0}\) such that

\[
\Re \left( z^{-1} M(z) \right) \geq c > 0. \quad \text{(posdef)}
\]

for all \(z \in B_C(r, r)\).
Solution Theory

**Theorem 1:** Let \((M(z))_{z \in B_{\mathbb{C}}(r,r)}\) be a holomorphic family of uniformly bounded linear operators on \(H\), \(\nu \geq \frac{1}{2r}\), satisfying our definiteness condition (posdef) and \(A\) skew-selfadjoint in \(H\), then we have for every \(f \in H_{\nu,0} \otimes H\) a unique solution \(U \in H_{\nu,0} \otimes H\) of the problem

\[
(\partial_0 M(\partial_0^{-1}) + A) \ U = f.
\]

Moreover, the solution depends continuously on the data in \(H_{\nu,0} \otimes H\) and is causal.
A modified Biot system

We shall now establish that the Biot system is of the stated form, where the skew-selfadjointness of $A$ stems from the structure of $A$ as a block operator matrix of the form

$$A = \begin{pmatrix} 0 & B^* \\ -B & 0 \end{pmatrix} \quad (((H)))$$

with $B : D(B) \subseteq H_1 \rightarrow H_0$ a closed, densely defined operator between Hilbert spaces $H_1$ and $H_0$. We shall specify the material law later.
For simplicity and definiteness we focus here on the case of Dirichlet boundary conditions. By suitably establishing $\mathring{\text{grad}}$ as the closure of the classical gradient applied to smooth functions with compact support in a non-empty, open subset $\Omega \subseteq \mathbb{R}^3$ as a mapping from the Hilbert space $L^2(\Omega)$ of equivalence classes (with respect to a.e.-equality as equivalence relation) of square-integrable functions to the 3-component $L^2$-type space $L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega)$, a generalized Dirichlet boundary condition can be formulated as being in the domain of $\mathring{\text{grad}}$. With $-\text{div}$ as the adjoint of $\mathring{\text{grad}}$ we obtain a skew-selfadjoint operator

$$S = \begin{pmatrix} 0 & \text{div} \\ \mathring{\text{grad}} & 0 \end{pmatrix}.$$
Given the definition

\[ \mathcal{E} := \text{Grad} \, u, \]

where \( u \) is the displacement and \( \text{Grad} \, u := \frac{1}{2} \left( \partial \otimes u + (\partial \otimes u)^	op \right) \)
denotes the symmetric part of the Jacobi matrix \( \partial \otimes u \), another first order dynamic equation

\[ \partial_0 \mathcal{E} = \text{Grad} \, v \]

can be constructed, where \( v = \partial_0 u \) describes the velocity of the deformation. Let \( H_{\text{sym}} \) denote the Hilbert space of \( L^2(\Omega) \)-valued, selfadjoint \( 3 \times 3 \)-matrices, with the inner product induced by the Frobenius norm

\[
(\Phi, \Psi) \mapsto \int_{\Omega} \text{trace} \left( \Phi(x)^* \, \Psi(x) \right) \, dx.
\]
By a suitable choice of boundary condition, i.e. domain, for
\[
\begin{pmatrix}
0 & -\text{Div} \\
-\text{Grad} & 0
\end{pmatrix}
\]
we can enforce skew-selfadjointness in the Hilbert space
\[
H := \left( L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega) \right) \oplus H_{\text{sym}}.
\]
For sake of definiteness, let us again implement a Dirichlet boundary condition by considering the closure \(\hat{\text{Grad}}\) of the restriction of \(\text{Grad}\) to smooth vector fields with compact support in \(\Omega\) as an operator from \(L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega)\) to \(H_{\text{sym}}\) and \(-\text{Div}\) as its adjoint. Then
\[
E := \begin{pmatrix}
0 & -\text{Div} \\
-\hat{\text{Grad}} & 0
\end{pmatrix}
\]
is of the form \(((H))\) and therefore skew-selfadjoint in \(H\) (and so also in \(H_{\nu,0} \otimes H\)).

The classical Biot system has been modified for certain types of clays by Murad and Cushman 1996. The mathematical model has been discussed by R.E. Showalter 2000. At the heart of the
Then assuming that equation (4) can be solved for $\mathcal{E}$, we have

$$\mathcal{E} = (C + \text{trace}^* \lambda \text{trace} \, \partial_0)^{-1} \mathcal{T} +$$

$$+ (C + \text{trace}^* \lambda \text{trace} \, \partial_0)^{-1} \text{trace}^* \alpha \, p, \quad (5)$$

and so we arrive at again at a system operator

$$\partial_0 \mathcal{M} \left( \partial_0^{-1} \right) - \begin{pmatrix}
0 & 0 & \text{Div} & 0 \\
0 & 0 & 0 & -\text{div} \\
\text{Grad} & 0 & 0 & 0 \\
0 & -\text{grad} & 0 & 0 \\
\end{pmatrix}. \quad (6)$$
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Here $M \left( \partial_0^{-1} \right)$ is given by

$$
\begin{pmatrix}
\rho_0 & 0 & 0 & 0 \\
0 & c_0 + \alpha^{*} \text{trace } C_\lambda^{-1} \text{trace } \alpha & \alpha^{*} \text{trace } C_\lambda^{-1} & 0 \\
0 & C_\lambda^{-1} \text{trace } \alpha & C_\lambda^{-1} & 0 \\
0 & 0 & 0 & \kappa^{-1} \partial_0^{-1}
\end{pmatrix}
$$

with

$$C_\lambda := (C + \text{trace } \lambda \text{ trace } \partial_0).$$

To arrive at (5) it is crucial to discuss the equation

$$(C + \text{trace } \lambda \text{ trace } \partial_0) U = F.$$

Noting that $\text{trace } \text{trace }^{*} = 3$, we can define the orthogonal projector

$$\mathbb{P} := \frac{1}{3} \text{trace }^{*} \text{trace}$$

in $\mathbb{C}^{3 \times 3}$ which extends canonically to $H_{\text{sym}} \subset L^{2}(\Omega)^{3 \times 3}$. 
Defining $\mathbb{P}' := 1 - \mathbb{P}$, we obtain the decomposition

$$
\begin{pmatrix}
C_\mathbb{P} + 3\lambda \partial_0 & \mathbb{P}C\mathbb{P}' \\
\mathbb{P}'C\mathbb{P} & C_{\mathbb{P}'}
\end{pmatrix}
\begin{pmatrix}
\mathbb{P} \\
\mathbb{P}'
\end{pmatrix}
U =
\begin{pmatrix}
\mathbb{P}F \\
\mathbb{P}'F
\end{pmatrix},
$$

where $C_\mathbb{P}$ and $C_{\mathbb{P}'}$ denote the restrictions of $C$ to the subspaces $\mathbb{P}[H_{\text{sym}}]$ and $\mathbb{P}'[H_{\text{sym}}]$, respectively, which clearly retain the property of strict positive-definiteness.

Multiplication of this system of linear equations by

$$
\begin{pmatrix}
1 \\
-\mathbb{P}'C\mathbb{P}(C_\mathbb{P} + 3\lambda \partial_0)^{-1} 0
\end{pmatrix},
$$

i.e. performing an admissible row operation, we obtain the equivalent triangular system

$$
\begin{pmatrix}
C_\mathbb{P} + 3\lambda \partial_0 & \mathbb{P}C\mathbb{P}' \\
0 & C_{\mathbb{P}'} - \mathbb{P}'C\mathbb{P}(C_\mathbb{P} + 3\lambda \partial_0)^{-1} \mathbb{P}C\mathbb{P}'
\end{pmatrix}
\begin{pmatrix}
\mathbb{P} \\
\mathbb{P}'
\end{pmatrix}
U =
\begin{pmatrix}
\mathbb{P}F \\
-\mathbb{P}'C\mathbb{P}(C_\mathbb{P} + 3\lambda \partial_0)^{-1} \mathbb{P}F + \mathbb{P}'F
\end{pmatrix}.
$$
Here \((C_{\mathcal{P}} + 3\lambda \partial_0)^{-1}\) is well-defined simply as the solution operator of the ordinary differential equation

\[ C_{\mathcal{P}} U + 3\lambda \partial_0 U = G \]

in the Hilbert space \(\mathbb{P}[H_{\text{sym}}]\).

Since

\[
\left( C + \text{trace}^* \lambda \text{trace} \partial_0 \right)^{-1} = \\
= C_{\mathcal{P}}^{-1}_{\mathcal{P}'} + \partial_0^{-1} \left( \left( 1 - C_{\mathcal{P}}^{-1}_{\mathcal{P}'} C_{\mathcal{P}} \right) \frac{1}{3} \lambda^{-1} \mathcal{P} + \frac{1}{3} \lambda^{-1} C_{\mathcal{P}}^{-1}_{\mathcal{P}'} \mathcal{P} C_{\mathcal{P}} C_{\mathcal{P}'} \right) + \\
+ \partial_0^{-2} \Theta_0 \left( \partial_0^{-1} \right)
\]

with \(\Theta_0\) analytic at 0 we obtain a 0-analytic material law operator of the form

\[ M \left( \partial_0^{-1} \right) = M_0 + \partial_0^{-1} M_1 + \partial_0^{-2} M_2 \left( \partial_0^{-1} \right). \quad (7) \]
Here

\[ M_0 := \begin{pmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & c_0 + \alpha^{*} \text{trace} \ C_{P'}^{-1} \text{trace}^{*} \alpha & \alpha^{*} \text{trace} \ C_{P'}^{-1} \text{trace}^{*} \alpha & 0 \\ 0 & C_{P'}^{-1} \text{trace}^{*} \alpha & C_{P'}^{-1} \text{trace}^{*} \alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \]

\[ M_1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha^{*} \text{trace} \ \Xi & \alpha^{*} \text{trace} \ \Xi & 0 \\ 0 & \text{trace}^{*} \alpha & \Xi & 0 \\ 0 & 0 & 0 & \kappa^{-1} \end{pmatrix} , \]

with \[
\Xi := \frac{1}{3} \left( 1 - P' \ C_{P'}^{-1} P' C \ P \right) \ P \ \lambda^{-1} \ P \ \left( 1 - P \ C \ P' \ C_{P'}^{-1} \ P' \right). \]
Moreover,

\[
M_2 \left( \partial_0^{-1} \right) := \begin{pmatrix}
0 & 0 & \alpha \star \text{trace } \Theta_0 \left( \partial_0^{-1} \right) \text{trace*}_\alpha & \alpha \star \text{trace } \Theta_0 \left( \partial_0^{-1} \right) & 0 \\
0 & \Theta_0 \left( \partial_0^{-1} \right) \text{trace*}_\alpha & 0 & \Theta_0 \left( \partial_0^{-1} \right) & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

We observe that \( M_0 \) is symmetric and also strictly positive definite on the range of

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \mathbb{P}' & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Indeed, transforming \( M_0 \) with suitable matching row and column operations, we obtain

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \mathbb{P}' & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}\begin{pmatrix}
\rho_0 & 0 & 0 & 0 \\
0 & c_0 & 0 & 0 \\
0 & 0 & C_{\mathbb{P}'}^{-1} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \mathbb{P}' & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
This is clearly positive definite, since $\varrho_0$, $c_0$ and $C_P$ are selfadjoint, strictly positive definite and bounded on this range. Moreover, the null space of $M_0$ is the range of

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & P & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

and we have

$$
P E P = \frac{1}{3} P \lambda^{-1} P.
$$
Therefore

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & P & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
M_1
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & P & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
=
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & P & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} \lambda^{-1} & 0 \\
0 & 0 & 0 & \kappa^{-1}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & P & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

which shows the positive definiteness of the real part of $M_1$ on null space of $M_0$, i.e. the range of

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & P & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Theorem 2: Let $M\left(\partial_0^{-1}\right)$ be given by (7) with a coupling term $\alpha$ continuous, linear and $\rho_0$, $C$, $\lambda$, $c_0$, $\kappa$ selfadjoint, continuous and strictly positive definite in their appropriate $L^2(\Omega)$-type spaces. Then for every $F \in \mathcal{H}_{\nu,0}$ there exists a unique solution $U \in \mathcal{H}_{\nu,0}$ of the problem

$$(\partial_0 M \left(\partial_0^{-1}\right) - A) U = F,$$

where

$$A := \begin{pmatrix}
0 & 0 & \text{Div} & 0 \\
0 & 0 & 0 & -\text{div} \\
\text{Grad} & 0 & 0 & 0 \\
0 & -\text{grad} & 0 & 0
\end{pmatrix}.$$ 

Moreover, the solution depends continuously on the data in $\mathcal{H}_{\nu,0}$, i.e. 0 is in the resolvent set of the operator $\partial_0 M \left(\partial_0^{-1}\right) - A$. 
Summary

We have presented a solution theory for an evolutionary system describing propagation in poro-elastic media. The problem has been shown to be of the form

\[(\partial_0 M (\partial_0^{-1}) + A) U = f\]

with $A$ skew-selfadjoint in a Hilbert space $H$ and where $(M(z))_{z \in B_C(r,r)}$ is a uniformly bounded, holomorphic family of linear operators in $H$, $r \in \mathbb{R}_{>0}$ small, satisfying a one-sided constraint of the form

\[\forall c \in \mathbb{R}_{>0} \forall z \in B_C(r,r) \Re (z^{-1} M(z)) \geq c.\]

Thus it is covered by a previously developed solution scheme.
Literature


