M-matrix inverse problem for Sturm-Liouville equations on graphs

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Consider a Sturm-Liouville boundary value problem on a graph with formally self-adjoint boundary conditions at the nodes.

From the M-matrix associated with such a problem we recover, up to a unitary equivalence, the boundary conditions and the potential.
Procedure

- Find asymptotics for the M-matrix as the eigenparameter tends to negative infinity.

- Recover the boundary conditions up to a unitary equivalence from the M-matrix.

- Show the M-matrix is a Herglotz function.

- Prove that the poles of the M-matrix are
  - at the eigenvalues of the associated BVP
  - simple
  - located on the real axis and
  - the residue at a pole is a negative semi-definite matrix of rank equal to the multiplicity of the eigenvalue.

- Recover the potential.
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- For a survey of the physical systems giving rise to boundary value problems on graphs see P. Kuchment, Graph models for waves in thin structures, *Waves in Random Media*, 12 (2002) R1 -R24 and the bibliography thereof.
$G$ denotes a directed graph with a finite number of edges, say $K$, each of finite length and having the path length metric. Each edge $e_i$ of length $l_i$ can thus be considered as the interval $[0, l_i]$. Eg.
The differential equation

With the above identification we can now consider

$$ly := -\frac{d^2y}{dx^2} + q(x)y = \lambda y, \quad (1)$$

where $q$ is real valued and essentially bounded on the graph $G$, to be the system of differential equations

$$-\frac{d^2y_i}{dx^2} + q_i(x)y_i = \lambda y_i \quad (2)$$

for $x \in [0, l_i]$ and $i = 1, \ldots, K$.

Here $q_i$ and $y_i$ are $q$ and $y$ restricted to $e_i$. 

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Next we specify the boundary conditions at each node $\nu$ say. These will depend on the values of $y$ and $y'$ at $\nu$ on each of the incident edges and ultimately we find that the bc’s can be written in the form

$$\sum_{j=1}^{K} [\alpha_{ij}y_j + \beta_{ij}y'_j](0) + \sum_{j=1}^{K} [\gamma_{ij}y_j + \delta_{ij}y'_j](l_j) = 0,$$

(3)

$i = 1 \ldots 2K$, $2K = \text{the total no. of linearly independent bc’s}.$

Note that we only consider boundary conditions which are self-adjoint w.r.t. $l$ and $L^2(G)$.  

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The above BVP on G can be reformulated as an operator eigenvalue problem by setting

\[ Lf := -f'' + qf \]

with domain

\[ D(L) = \{ f \mid f, f' \in AC, L(f) \in \mathcal{L}^2(G), f \text{ obeys } (3) \}. \]
The above BVP on G is also equivalent to the formally self-adjoint system

\[-MT'' + PT = \lambda T\]  \hspace{1cm} (4)

with separated boundary conditions

\[A^*T(0) - B^*T'(0) = 0,\] \hspace{1cm} (5)
\[\Gamma^*T(1) - \Delta^*T'(1) = 0\] \hspace{1cm} (6)
where

\[ M = 4 \text{diag} \left[ \frac{1}{l_1^2}, \ldots, \frac{1}{l_k^2}, \ldots, \frac{1}{l_1^2}, \ldots, \frac{1}{l_k^2} \right], \]

\( P \) is a diagonal, \( 2K \times 2K \), matrix dependent on the potential on each edge of the graph,

\[ A^* = \begin{bmatrix} I & -I \\ 0 & 0 \end{bmatrix}, \quad -B^* = \begin{bmatrix} 0 & 0 \\ I & I \end{bmatrix}, \]

and 

\( \Gamma, \Delta \) are real, constant \( 2K \times 2K \) matrices.
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The second order equation (4) can be rewritten as the following first order system

\[ Y' = Z \quad \text{and} \quad Z' = -G(x, \lambda)Y \]

where \( G(x, \lambda) = M^{-1}(\lambda - P) \).

Without loss of generality we may then assume that the following three properties hold:

- \( G(x, \lambda) \) must be continuous and symmetric.
- \( A^*B = B^*A \) and \( \Gamma^*\Delta = \Delta^*\Gamma \).
- \( A^*A + B^*B = I \) and \( \Gamma^*\Gamma + \Delta^*\Delta = I \).
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- \( A^*A + B^*B = I \) and \( \Gamma^*\Gamma + \Delta^*\Delta = I \).
The matrix Prüfer angle corresponding to the system with separated bc’s, (4)-(6), is then given by

\[ F(x, \lambda) = (V - iU)^{-1}(V + iU) \]

where

\[ U(x, \lambda) = S(x, \lambda)\Gamma - C(x, \lambda)\Delta, \quad V(x, \lambda) = C(x, \lambda)\Gamma + S(x, \lambda)\Delta \]

and where \( \{S(x, \lambda), C(x, \lambda)\} \) is the solution of

\[ S' = H(x, \lambda)C, \quad C' = -H(x, \lambda)S, \]

\[ S(0, \lambda) = B^*, \quad C(0, \lambda) = A^*. \]

and

\[ H(x, \lambda) = CC^* + SGS^*. \]
Let $W_1$ be a solution of (4) satisfying, at 0, the boundary conditions

$$W_1(0, \lambda) = B \quad \text{and} \quad W_1'(0, \lambda) = A$$

that is $A^* W_1(0, \lambda) - B^* W_1'(0, \lambda) = 0$,

and let

$$\chi(x, \lambda) = W_1(x, \lambda)[\Delta^* W_1'(1, \lambda) - \Gamma^* W_1(1, \lambda)]^{-1},$$

then set

$$R = \begin{bmatrix} -\Gamma & \Delta \\ \Delta & \Gamma \end{bmatrix} \quad \text{and} \quad W(x, \lambda) = \begin{bmatrix} W_2(x, \lambda) & W_3(x, \lambda) \\ W_2'(x, \lambda) & W_3'(x, \lambda) \end{bmatrix}$$

where $W_2$ and $W_3$ are solutions of (4) obeying the terminal conditions $W(1, \lambda) = R$. 

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where $W_2$ and $W_3$ are solutions of (4) obeying the terminal conditions $W(1, \lambda) = R$. 
We define the Weyl M-matrix, $\mathcal{M}(\lambda)$, of (4)-(6) to be given by

$$\Psi = W_2 + W_3\mathcal{M}(\lambda).$$

(7)

where $\Psi$ obeys (5).
The M-matrix defined in this way exists and is well-defined for $\lambda$ not an eigenvalue of (4)-(6).

This is a generalisation of the m-function for the classical scalar Sturm-Liouville operator and is consistant with the M-matrix for general systems given by Krall and the M-vector for trees given by Yurko in

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In the above reference Yurko solves the inverse problem of recovering the operator from the M-vector for Sturm-Liouville operators on trees with continuity and Kirchhoff bc’s at the nodes.

Recently Yurko has also considered inverse problems for graphs which may contain a cycle but still with continuity and Kirchhoff bc’s at the nodes, see V. Yurko, Inverse problems for Sturm-Liouville operators on graphs with a cycle, *Operators and Matrices*, Vol. 2 no. 4 (2008), 543-553.

The problem we wish to solve is thus a generalisation of Yurko’s results since we consider general self-adjoint bc’s for the recovery of the bc’s and general co-normal bc’s for the recovery of the potential.
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Theorem

The M-matrix satisfies

\[ \mathcal{M}^*(\lambda) = i(F^*(1, \lambda) - I)^{-1}(F^*(1, \lambda) + I). \]

The poles of the determinant of \( \mathcal{M}^* \) are precisely the eigenvalues of (4)-(6).

Corollary

The matrix Prüfer angle \( F(1, \lambda) \) determines \( \mathcal{M}(\lambda) \) and vice-versa.
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The matrix Prüfer angle \( F(1, \lambda) \) determines \( \mathcal{M}(\lambda) \) and vice-versa.
**Theorem**

Let $\lambda = -\sigma^2$ then the matrix Prüfer angle, $F$, obeys the following asymptotic approximation as $\sigma \to \infty$

$$F^*(1, -\sigma^2) = (\Gamma^* + i\Delta^*) H (\Gamma^* - i\Delta^*)^{-1},$$

where $H = \text{diag}[1 + O\left(\frac{1}{\sigma}\right), \ldots, 1 + O\left(\frac{1}{\sigma}\right)]$.

**Corollary**

Asymptotically as $\sigma \to \infty$, $M^*(\lambda)$ takes the form

$$M^*(\lambda) = i \left[2i\Delta + O\left(\frac{1}{\sigma}\right)\right]^{-1} \left[2\Gamma + O\left(\frac{1}{\sigma}\right)\right].$$
Asymptotics

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\]
Recovery of the boundary conditions

**Theorem**

Let \((\Gamma^*, \Delta^*, P)\) denote the BVP (4)-(6) and \((\tilde{\Gamma}^*, \tilde{\Delta}^*, \tilde{P})\) the BVP (4)-(6) but with \(\Gamma\) replaced by \(\tilde{\Gamma}\), \(\Delta\) by \(\tilde{\Delta}\) and \(P\) by \(\tilde{P}\).

If the problems \((\Gamma^*, \Delta^*, P)\) and \((\tilde{\Gamma}^*, \tilde{\Delta}^*, \tilde{P})\) have the same \(M\)-matrix, \(\mathcal{M}(\lambda)\), where \(\tilde{\Gamma}^* \tilde{\Delta} = \tilde{\Delta}^* \tilde{\Gamma}\) and \(\tilde{\Gamma}^* \tilde{\Gamma} + \tilde{\Delta}^* \tilde{\Delta} = I\). Then \(\Delta = U \tilde{\Delta}\) and \(\Gamma = U \tilde{\Gamma}\) where \(U = \Gamma \tilde{\Gamma}^* + \Delta \tilde{\Delta}^*\) is unitary.
Theorem

*The M-matrix, \( \mathcal{M}(\lambda) \), is a Herglotz function of rank \( 2K \).*

Lemma

For \( \lambda \in \mathbb{R} \), the following Wronskian identities hold:

\[
\begin{align*}
W'_3(x)W_2(x) - W_3^*(x)W'_2(x) &= -I, \\
W_2^*(x)W'_3(x) - W'_2^*(x)W_3(x) &= -I, \\
W'_2(x)W_3(x) - W_2^*(x)W'_3(x) &= I, \\
W_3^*(x)W'_2(x) - W'_3^*(x)W_2(x) &= I, \\
W'_2(x)W_2(x) - W_2^*(x)W'_2(x) &= 0, \\
W'_3(x)W_3(x) - W_3^*(x)W'_3(x) &= 0.
\end{align*}
\]
Green's function

Lemma

The Green's function for the boundary value problem (4)-(6) can be represented as

\[ G(x, t) = \begin{cases} 
W_3(x)\Psi^*(t)M^{-1}, & t < x \\
\Psi(x)W_3^*(t)M^{-1}, & t > x
\end{cases} \]  

(8)
As the M-matrix defined in (7) is a Herglotz function it admits the following representation,

$$M(\lambda) = C + D\lambda + \sum_{\lambda_n} M_n \left( \frac{1}{\lambda_n - \lambda} - \frac{\lambda_n}{1 + \lambda_n^2} \right),$$

where $C = \text{Re}(M(i))$ and $D = \lim_{\eta \to \infty} \left( \frac{1}{i\eta} M(i\eta) \right) \geq 0$. Thus

$$\lim_{\lambda \to \lambda_n} (\lambda - \lambda_n) M(\lambda) = -M_n,$$

i.e. $-M_n$ is the residue of the pole of $M(\lambda)$ at $\lambda_n$. 

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The poles of the M-matrix are simple, located on the real axis and are the eigenvalues of (4)-(6). The residue at a pole is a negative semi-definite matrix of rank equal to the multiplicity of the eigenvalue.
Since $\mathcal{M}(\lambda)$ is a matrix-valued Herglotz function the poles of the M-matrix are simple and located on the real axis and the residue at a pole is a negative semi-definite matrix - F. Gesztesy, E. Tsekanovskii, On matrix valued Herglotz functions, *Math. Nachr.*, **218** (2000), 61-138.

By the definition of the M-matrix all the poles of $\mathcal{M}$ are eigenvalues of (4)-(6). At an eigenvalue of (4)-(6) the Green’s function has a pole, giving that if $\lambda$ is an eigenvalue then $\lambda$ is a pole of $G(x, t)$ and thus, by (8), $\lambda$ is a pole of $\Psi$ and hence, is a pole of $\mathcal{M}$.
Proof

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Proof

- If $F_{n,j}$, $j = 1, \ldots, \nu_n$, is an orthonormal sequence of eigenfunctions corresponding to $\lambda_n$ ($\lambda_n$ not repeated according to multiplicity) where $\nu_n$ is the multiplicity of the eigenvalue $\lambda_n$ then $F_{n,j}$ can be written as

$$F_{n,j}(x) = W_3(\lambda_n, x)c_{n,j}$$

where $c_{n,j}$ is a column vector.

- The form which the negative semi-definite matrix $-M_n$ takes is then

$$-\sum_{j=1}^{\nu_n} c_{n,j}c_{n,j}^* = -M_n.$$
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-\sum_{j=1}^{\nu_n} c_{n,j}c^*_{n,j} = -M_n.
\]
Define $C_n := [c_{n,1}, \ldots, c_{n,\nu_n}, 0, \ldots, 0]$. Then the rank of $C_n$ is $\nu_n$. Also the rank of $C_n$ is equal to the number of non-zero eigenvalues of $C_n$, counted by multiplicity. Denote these eigenvalues by $\mu_1, \ldots, \mu_{\nu_n}$.

Then $C_nC_n^*$ has non-zero eigenvalues $|\mu_1|^2, \ldots, |\mu_{\nu_n}|^2$. Thus $C_nC_n^*$ has rank $\nu_n$ and $\text{Rank}(-M_n) = \nu_n$. \qed
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Co-normal boundary conditions

**Definition**

For the system formulation (4)-(6) the bc’s at $x = 1$, are co-normal if and only if $\Gamma$ and $\Delta$ are such that, when

$$S = \left\{ \begin{pmatrix} u \\ u' \end{pmatrix} \in \bigoplus_{\mathbb{C}^{2K}} | \Gamma^* u - \Delta^* u' = 0 \right\},$$

is such that there exists a subspace $N$, of dimension $n$, of $\mathbb{C}^{2K}$ so that \( \begin{pmatrix} u \\ 0 \end{pmatrix} \in S \) for all $u \in N$ and there exists a real diagonal matrix $D = \text{diag}\{d_1, \ldots, d_{2K}\}$ such that $\begin{pmatrix} u \\ u' \end{pmatrix} \in S$ if and only if $u \in N$ and $(Du - u') \cdot v = 0$ for all $v \in N$. 

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Lemma

Suppose that the boundary conditions at $x = 1$, are co-normal and that $S$, $N$, $D$ are as given in the definition of co-normal bc’s. Then there exists an orthonormal basis $w_1, \ldots, w_{2K}$ for $\mathbb{C}^{2K}$ and real numbers $\mu_1, \ldots, \mu_n$ such that, without loss of generality, $\Delta$ and $\Gamma$ may be written as

$$\Delta = \begin{bmatrix} \frac{w_1}{\sqrt{1 + |\mu_1|^2}}, & \ldots, & \frac{w_n}{\sqrt{1 + |\mu_n|^2}}, & 0, & \ldots, & 0 \end{bmatrix}$$

and

$$\Gamma = \begin{bmatrix} \frac{\mu_1 w_1}{\sqrt{1 + |\mu_1|^2}}, & \ldots, & \frac{\mu_n w_n}{\sqrt{1 + |\mu_n|^2}}, & w_{n+1}, & \ldots, & w_{2K} \end{bmatrix}.$$
Co-normal boundary conditions on a graph correspond in nature to co-normal (non-oblique) boundary conditions for elliptic partial differential operators.

Most physically interesting boundary conditions on graphs fall into the co-normal category. In particular, ‘Kirchhoff’, Dirichlet, Neumann and periodic boundary conditions are all co-normal, but this class does not include all self-adjoint boundary-value problems on graphs.
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Most physically interesting boundary conditions on graphs fall into the co-normal category. In particular, ‘Kirchhoff’, Dirichlet, Neumann and periodic boundary conditions are all co-normal, but this class does not include all self-adjoint boundary-value problems on graphs.
Marčenko’s result

**Lemma**

There exists a kernel, \( k_{h,m,q}(t, y) \), \( (k_{\infty,m,q}(t, y) \) resp.) such that,

\[
\begin{align*}
v_{h(\infty),m,q}[f](t) := \int_t^1 k_{h(\infty),m,q}(t, y)f(y) \, dy,
\end{align*}
\]

defines a continuous linear transformation on \( L^2[0, 1] \), and if \( y_\lambda \) is the solution of \(-my''_\lambda = \lambda y_\lambda \) on \([0, 1]\) with \( y'_\lambda(1) = hy_\lambda(1) \) \( (y_\lambda(1) = 0 \) resp.), for \( m > 0 \) a real constant, then

\[
\begin{align*}
z_\lambda := (I + v_{h,m,q})y_\lambda \quad (z_\lambda := (I + v_{\infty,m,q})y_\lambda \ \text{resp.})
\end{align*}
\]

is the solution of \(-mz''_\lambda + qz_\lambda = \lambda z_\lambda \) on \([0, 1]\) with \( z'_\lambda(1) = hy_\lambda(1) \) and \( z_\lambda(1) = y_\lambda(1) \) \( (z_\lambda(1) = 0 \) and \( z'_\lambda(1) = y'_\lambda(1) \) resp.), for each \( \lambda \in \mathbb{R} \).
Lemma

Let the problems \((\Gamma^*, \Delta^*, P)\) and \((\tilde{\Gamma}^*, \tilde{\Delta}^*, \tilde{P})\) have the same M-matrix. If there exists a linear continuous transformation operator, \(H\), on \(L^2[0, 1]\), independent of \(\lambda\), which maps

\[
H[W_3(\lambda, x)] = \tilde{W}_3(\lambda, x),
\]

where \(\tilde{W}_3(\lambda, x)\) is the solution to \((\tilde{\Gamma}^*, \tilde{\Delta}^*, \tilde{P})\) obeying \(\tilde{W}_3(\lambda, 1) = \tilde{\Delta}\) and \(\tilde{W}_3'(\lambda, 1) = \tilde{\Gamma}\), then \(H\) is unitary.
Recovery of the operator

Theorem

If the problems \((\Gamma^*, \Delta^*, P)\) and \((\tilde{\Gamma}^*, \tilde{\Delta}^*, \tilde{P})\) have the same M-matrix, and if we assume that the boundary conditions at \(x = 1\) are co-normal and that the weight matrix \(M\) commutes with \(\Gamma, \Delta, \tilde{\Gamma}, \tilde{\Delta}\), then \(P = U\tilde{P}U^*\).
Since the bc’s at $x = 1$ are co-normal and $M$ commutes with $\Gamma, \Delta, \tilde{\Gamma}, \tilde{\Delta}$ we can use Marčenko’s result and ultimately show that there exits a Volterra map $V_{P,M}$ such that if $\bar{Y}(t)$ is the solution of $-M\bar{Y}(t)'' = \lambda \bar{Y}(t)$ with $\bar{Y}(1) = \Delta$ and $\bar{Y}'(1) = \Gamma$, then

$$\bar{Z}(t) := (I + V_{P,M})\bar{Y}(t)$$

is the solution of $-M\bar{Z}(t)'' + P\bar{Z}(t) = \lambda \bar{Z}(t)$ with $\bar{Z}(1) = \Delta$ and $\bar{Z}'(1) = \Gamma$. 
Outline of proof

Now \((I + V_{P,M})^{-1} = I + W_{P,M}\), where \(W_{P,M}\) is Volterra-Marčenko.

So if \(Y\) is the solution of \(-MY'' + PY = \lambda Y\) on \([0, 1]\) with \(Y(1) = \Delta\) and \(Y'(1) = \Gamma\) then

\[Z := (I + W_{P,M})Y\]

is the solution of \(-MZ'' = \lambda Z\) on \([0, 1]\) with \(Z(1) = \Delta\) and \(Z'(1) = \Gamma\).

Giving

\[\tilde{Z} := U^*(I + W_{P,M})Y = U^*Z\]

is the solution of \(-U^*MU(U^*Z)'' = \lambda(U^*Z)\) with \(\tilde{Z}(1) = U^*Z(1) = U^*\Delta = \tilde{\Delta}\) and \(\tilde{Z}'(1) = U^*Z'(1) = U^*\Gamma = \tilde{\Gamma}\).
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Outline of proof

Now \((I + V_{P,M})^{-1} = I + WP_M\), where \(WP_M\) is Volterra-Marčenko.

So if \(Y\) is the solution of \(-MY'' + PY = \lambda Y\) on \([0, 1]\) with \(Y(1) = \Delta\) and \(Y'(1) = \Gamma\) then

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\(\tilde{Z}(1) = U^*Z(1) = U^*\Delta = \tilde{\Delta}\) and \(\tilde{Z}'(1) = U^*Z'(1) = U^*\Gamma = \tilde{\Gamma}\).
Since $UM = MU$ we get that $\tilde{Z}$ is the solution of 
$-M\tilde{Z}'' = \lambda \tilde{Z}$ with $\tilde{Z}(1) = \tilde{\Delta}$ and $\tilde{Z}'(1) = \tilde{\Gamma}$.

Let 
$$\tilde{Y} := (I + V_{\bar{P},M})\tilde{Z} = (I + V_{\bar{P},M})U^*(I + W_{P,M})Y,$$

then $\tilde{Y}$ is the solution of $-M\tilde{Y}'' + \tilde{P}\tilde{Y} = \lambda \tilde{Y}$ with $\tilde{Y}(1) = \tilde{\Delta}$ and $\tilde{Y}'(1) = \tilde{\Gamma}$.

Let $HY := (I + V_{\bar{P},M})U^*(I + W_{P,M})Y$ for $Y \in L^2[0, 1]$. If $Y$ is any solution of $-MY'' + PY = \lambda Y$, then $\tilde{Y} := HY$ is the solution of $-M\tilde{Y}'' + \tilde{P}\tilde{Y} = \lambda \tilde{Y}$ with $\tilde{Y}(1) = U^*Y(1)$ and $\tilde{Y}'(1) = U^*Y'(1)$. 
Outline of proof

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Let $HY := (I + V_{\tilde{P},M})U^*(I + W_{P,M})Y$ for $Y \in L^2[0, 1]$. If $Y$ is any solution of $-MY'' + PY = \lambda Y$, then $\tilde{Y} := HY$ is the solution of $-M\tilde{Y}'' + \tilde{P}\tilde{Y} = \lambda \tilde{Y}$ with $\tilde{Y}(1) = U^*Y(1)$ and $\tilde{Y}'(1) = U^*Y'(1)$. 
Outline of proof

- In particular $HW_3 = \tilde{W}_3$ and $HW_2 = \tilde{W}_2$, and hence $H$ is unitary.

- So (it can be shown that)

$$U^* = H = (I + V_{P,M})U^*(I + W_{P,M}).$$

- Therefore $\tilde{W}_3 = U^*W_3$ and $\tilde{W}_2 = U^*W_2$, and

$$- M(U^*W_j)'' + \tilde{P}U^*W_j = \lambda U^*W_j, \quad (12)$$

for $j = 2, 3$. 
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Therefore $\tilde{W}_3 = U^*W_3$ and $\tilde{W}_2 = U^*W_2$, and

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for $j = 2, 3$. 

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Premultiplying equation (12) by $U$ and noting that $MU = UM$, gives

$$-MW_j'' + U\tilde{P}U^*W_j = \lambda W_j,$$

(13)

for $j = 2, 3$.

We also have that

$$-MW_j'' + PW_j = \lambda W_j,$$

(14)

for $j = 2, 3$.

So

$$(U\tilde{P}U^* - P)W_j = 0,$$

for $j = 2, 3$.

Since the column space of $[W_2, W_3]$ spans the solution space of $-MY'' + PY = \lambda Y$, we can show that $P = U\tilde{P}U^*$.
Outline of proof

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- Since the column space of $[W_2, W_3]$ spans the solution space of $-MY'' + PY = \lambda Y$, we can show that $P = U	ilde{P}U^*$.
For the above result, we only require $M_n = \tilde{M}_n$ and not that the entire M-matrices are equal.
Recovery of the operator

Corollary

If all the edges of the graph $G$ have the same length, $l$, and the systems problems $(\Gamma^*, \Delta^*, P)$ and $(\tilde{\Gamma}^*, \tilde{\Delta}^*, \tilde{P})$ have the same $M$-matrix, then $\Delta = U\tilde{\Delta}$, $\Gamma = U\tilde{\Gamma}$ and $P = U\tilde{P}U^*$ where $U = \Gamma\tilde{\Gamma}^* + \Delta\tilde{\Delta}^*$ is a unitary matrix.

Corollary

If the problems $(\Gamma^*, \Delta^*, P)$ and $(\Gamma^*, \Delta^*, \tilde{P})$ have the same $M$-matrix and $M$ commutes with $\Gamma$, $\Delta$ then $P = \tilde{P}$.
Corollary

If all the edges of the graph $G$ have the same length, $l$, and the systems problems $(\Gamma^*, \Delta^*, P)$ and $(\tilde{\Gamma}^*, \tilde{\Delta}^*, \tilde{P})$ have the same $M$-matrix, then $\Delta = U\tilde{\Delta}$, $\Gamma = U\tilde{\Gamma}$ and $P = U\tilde{P}U^*$ where $U = \Gamma\tilde{\Gamma}^* + \Delta\tilde{\Delta}^*$ is a unitary matrix.

Corollary

If the problems $(\Gamma^*, \Delta^*, P)$ and $(\Gamma^*, \Delta^*, \tilde{P})$ have the same $M$-matrix and $M$ commutes with $\Gamma, \Delta$ then $P = \tilde{P}$. 
Note that given a set of eigenvalues $\lambda_n$ and the terminal value,

$$\Delta^* F_{n,j}(1) + \Gamma^* F'_{n,j}(1) = c_{n,j},$$

$M_n$ is uniquely determined and the above corollaries apply.

The above note is actually a more appealing result since it means that from the eigenvalues and the data at the nodes of the given graph, i.e. the terminal conditions, we can recover the boundary conditions and the potential. It does not rely on the superficial nodes inserted into each edge only on the original given nodes.
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