On the Global Convergence of the Cyclic Jacobi Methods for the Symmetric Matrix of Order 4

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Abstract

We prove the global convergence of the general cyclic Jacobi method for symmetric matrices of order four. In particular, we study the method under the strategies that enable full parallelization of the method. These strategies, unlike the serial ones, can force the method to be very slow/fast within one cycle, depending on the underlying matrix. This implies that for the global convergence proof one has to consider several adjacent cycles.

Jacobi method

- **Jacobi method** is an iterative method of the form

$$A^{(k+1)} = R_k^T A^{(k)} R_k, \qquad k \geq 0, \quad A^{(0)} = A,$$

where R_k are plane rotations and A is a symmetric matrix.

- **Off-norm** of A is defined by

$$S(A) = \frac{\sqrt{2}}{2}\|A - \mathrm{diag}(A)\|_F = \sqrt{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_{ij}^2}.$$

- The method is **globally convergent** if, for each starting A, $\lim_{k\to\infty} A^{(k)}$ is a diagonal matrix, which is equivalent to

$$\lim_{k\to\infty} S(A^{(k)}) = 0.$$

- It is sufficient to show that

$$S^2(A^{(K)}) \leq \gamma S^2(A), \qquad 0 \leq \gamma < 1, \ K \in \mathbb{N}.$$

Pivot strategies

- **Pivot strategy** $I : k \mapsto (i(k), j(k)), \qquad k \in \mathbb{N}.$
- **Periodic strategy** with period T: $I(k+T) = I(k)$, for all $k \geq 0$.
- **Cyclic strategy**: $T = N \equiv \frac{n(n-1)}{2}$ and all off-diagonal elements are annihilated exactly once during one period.
- **Pivot ordering**: $\mathcal{O}_I = I(0), \ldots, I(N-1)$.
- A cyclic strategy I can be represented by the matrix $M_I = (m_{ij})$, where

$$m_{ij} = m_{ji} = k, \quad \text{if } I(k) = (i,j), \ i < j.$$

- **Equivalent, shift-equivalent and weakly equivalent strategies**
- On order 4 matrix we can assume $I(0) = (1,2)$. There are $5! = 120$ **possible pivot orderings**.

Parallel strategies (16/120)

Two representatives are:

$$\begin{bmatrix} * & 4 & 0 & 2 \\ 4 & * & 3 & 1 \\ 0 & 3 & * & 5 \\ 2 & 1 & 5 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 4 & 2 & 0 \\ 4 & * & 1 & 3 \\ 2 & 1 & * & 5 \\ 0 & 3 & 5 & * \end{bmatrix}.$$

Classes of strategies with permutations inside columns/rows (70/120)

$$\mathcal{C}_1 = \{(1,2), (\pi_3(1),3), (\pi_3(2),3), (\pi_4(1),4), (\pi_4(2),4), (\pi_4(3),4)\},$$

where $\pi_3 = \begin{pmatrix} 1 & 2 \\ \pi_3(1) & \pi_3(2) \end{pmatrix}$ i $\pi_4 = \begin{pmatrix} 1 & 2 & 3 \\ \pi_4(1) & \pi_4(2) & \pi_4(3) \end{pmatrix}$.

E.g. $\begin{bmatrix} * & 0 & 2 & 5 \\ 0 & * & 1 & 3 \\ 2 & 1 & * & 4 \\ 5 & 3 & 4 & * \end{bmatrix}$.

$$\mathcal{C}_2 = \{(3,4), (2,\tau_2(3)), (2,\tau_2(4)), (1,\tau_1(2)), (1,\tau_1(3)), (1,\tau_1(4))\},$$

where $\tau_2 = \begin{pmatrix} 3 & 4 \\ \tau_2(3) & \tau_2(4) \end{pmatrix}$ i $\tau_1 = \begin{pmatrix} 2 & 3 & 4 \\ \tau_1(2) & \tau_1(3) & \tau_1(4) \end{pmatrix}$.

E.g. $\begin{bmatrix} * & 3 & 5 & 4 \\ 3 & * & 2 & 1 \\ 5 & 2 & * & 0 \\ 4 & 1 & 0 & * \end{bmatrix}$.

$$\mathcal{C}_3 = \{(\pi_4(1),4), (\pi_4(2),4), (\pi_4(3),4), (\pi_3(1),3), (\pi_3(2),3), (1,2)\}$$

E.g. $\begin{bmatrix} * & 5 & 3 & 0 \\ 5 & * & 4 & 2 \\ 3 & 4 & * & 1 \\ 0 & 2 & 1 & * \end{bmatrix}$.

$$\mathcal{C}_4 = \{(1,\tau_1(2)), (1,\tau_1(3)), (1,\tau_1(4)), (2,\tau_2(3)), (2,\tau_2(4)), (3,4)\}.$$

E.g. $\begin{bmatrix} * & 2 & 0 & 1 \\ 2 & * & 3 & 4 \\ 0 & 3 & * & 5 \\ 1 & 4 & 5 & * \end{bmatrix}$.

Nazareth strategies (11/120)

E.g. $\begin{bmatrix} * & 0 & 2 & 1 \\ 0 & * & 3 & 5 \\ 2 & 3 & * & 4 \\ 1 & 5 & 4 & * \end{bmatrix}$.

Permutationally equivalent (23/120)

Two strategies I_1 and I_2 are permutationally equivalent if there is a permutation matrix P such that

$$M_{I_2} = P^T M_{I_1} P.$$

Main theorem

Let $A \in \mathbb{C}^{4\times 4}$ be a symmetric matrix. Let I be a cyclic strategy and by $A^{[2]}$ denote a matrix obtained from A after two full cycles of the Jacobi method defined by the strategy I, with rotation angles from the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$. There is a constant γ that does not depend on I such that

$$S^2(A^{[2]}) \leq \gamma S^2(A), \quad 0 \leq \gamma < 1.$$

Numerical example

$$A = A^{(0)} = \begin{bmatrix} d+p_1+p_2 & 0 & \epsilon+p_1 & -a+p_1 \\ 0 & d+p_2 & a & -\epsilon \\ \epsilon+p_1 & a & d+p_1 & 0 \\ -a+p_1 & \epsilon & 0 & d \end{bmatrix},$$

with $a = 1$, $d = 1$, $\epsilon = 10^{-52}$, $p_1 = \epsilon$, $p_2 = \epsilon^{1.5}$.

$$S(A) = 1.41421356237309504880168872420969807856967187537694$$

k	$(i(k), j(k))$	$S(A^{(k)})$
1	$(1,3)$	1.41421356237309504880168872420969807856967187537694
2	$(2,4)$	1.41421356237309504880168872420969807856967187537694
3	$(1,4)$	1.41421356237309504880168872420969807856967187537694
4	$(2,3)$	1.41421356237309504880168872420969807856967187537694
5	$(1,2)$	1.41421356237309504880168872420969807856967187537694
6	$(3,4)$	1.41421356237309504880168872420969807856967187537694
7	$(1,3)$	0.99999999999999999999999999999999999999999999999999
8	$(2,4)$	$0.17677669529663688110021108266947024663734760219051e{-26}$

Theorem - Arbitrary slow convergence within one cycle

For every $\epsilon > 0$ and $\mathbf{n \geq 4}$, there exists a symmetric matrix A_ϵ of order n, depending on ϵ, and a cyclic strategy I, such that

$$S(A^{[1]}) > (1-\epsilon) S(A_\epsilon),$$

where $A^{[1]}$ is obtained from A_ϵ by applying a full cycle of the Jacobi method defined by the strategy I.

Global convergence

Jacobi method for the symmetric matrix of order four, with rotation angles from the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$, is **globally convergent under any cyclic pivot strategy.**

References

- E. Begović, V. Hari: *On the global convergence of the Jacobi method for symmetric matrices of order 4 under parallel strategies*, submitted.
- V. Hari, E. Begović: *All cyclic Jacobi methods for symmetric matrix of order four are globally convergent*, in preparation.

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