On the Global Convergence of the Jacobi Method

Vjeran Hari[1,*] and **Erna Begović Kovač**[2]

[1] Department of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia
[2] Faculty of Chemical Engineering and Technology, University of Zagreb, Marulićev trg 19, 10000 Zagreb, Croatia

The paper is concerned with the global convergence of the Jacobi method for symmetric matrices under a special class of cyclic pivot strategies. That class generalizes the well-known class of serial strategies.

Copyright line will be provided by the publisher

1 Introduction

Jacobi method for solving the eigenproblem of symmetric matrices is an iterative process of the form

$$A^{(k+1)} = R_k^T A^{(k)} R_k, \quad k \geq 0;\ A^{(0)} = A. \tag{1}$$

Here A is a real symmetric matrix of order n and $R_k = R(i(k), j(k), \varphi_k)$ are plane rotations, where $R(p, q, \phi)$ stands for the rotation in the (p, q)-plane by the angle ϕ. The goal of the k-th step of the method is to minimize the off-norm of $A^{(k)}$, which is the Frobenius norm of the off-diagonal part of $A^{(k)}$. Actually, we shall use the measure $S(X) = \|X - \text{diag}(X)\|_F/\sqrt{2}$. In order to achieve that, the rotation angle φ_k of R_k is chosen to annihilate $a^{(k)}_{i(k),j(k)}$, $i(k) < j(k)$, the *pivot element* of $A^{(k)}$. This choice reduces $S^2(A^{(k)})$ by $|a^{(k)}_{i(k),j(k)}|^2$ and makes a known formula for $\tan 2\varphi_k$. We assume that $|\varphi_k| \leq \pi/4$ for all $k \geq 0$.

Selection of the pivot elements is determined by *pivot strategy*, which we identify with a function $I : \mathbf{N}_0 \to \mathbf{P}_n$, where $\mathbf{N}_0 = \{0, 1, \ldots\}$ and $\mathbf{P}_n = \{(i, j) \mid 1 \leq i < j \leq n\}$. If there is $T \in \mathbf{N}$ such that $I(k + T) = I(k)$, $k \geq 0$, I is *periodic* with period T. In addition, if $T = N \equiv \frac{n(n-1)}{2}$ and $\{I(k) \mid 0 \leq k \leq N - 1\} = \mathbf{P}_n$, the pivot strategy is *cyclic*. Cyclic (pivot) strategies are important because they ensure the quadratic asymptotic convergence of the method under certain additional conditions (see [3]).

The Jacobi method is known for its high relative accuracy on well behaved matrices and efficiency on parallel computers. The method is globally convergent if, for any initial $A^{(0)}$, the sequence $A^{(k)}$ generated by (1) converges to some diagonal matrix. This condition is equivalent to the convergence of $S(A^{(k)})$ to zero. The global convergence of the Jacobi method has been studied in many papers, of which we mention [2, 4, 7–9]. For the cyclic pivot strategies considered here, the global convergence result shall be given in the stronger form

$$S^2(A') \leq \gamma_n S^2(A), \quad 0 \leq \gamma_n < 1, \quad \gamma_n \text{ is a constant which may depend on } n \text{ and not on } A,$$

where A' is the matrix obtained from A after one cycle of the Jacobi method. Such a result combined with the theory of Jacobi annihilators and operators can be used for proving the global convergence of other Jacobi-type methods [4, 6].

The choice of the pivot strategy plays an essential role in the global convergence consideration of the symmetric Jacobi method. Here we present four classes of cyclic pivot strategies which emerge from the well known column- and row-cyclic strategies and also their generalization. They have been used in [1, 5] in the context of the block Jacobi methods. In this short communication, we describe them for the element-wise method and formulate an appropriate convergence result.

The paper is divided into three sections. In Section 2 we define the serial pivot strategies with permutations and formulate the appropriate global convergence result. In Section 3 we briefly show how that class of "convergent" pivot orderings can be enlarged.

2 Serial pivot strategies with permutations

Let $\mathcal{O}(\mathbf{P}_n)$ denote the set of all orderings of $\mathbf{P}_n$. An ordering of $\mathbf{P}_n$ is a sequence of pairs of length N, containing all elements of $\mathbf{P}_n$. For a given cyclic strategy I, *pivot ordering* is the sequence $\mathcal{O}_I = (i(0), j(0)), (i(1), j(1)), \ldots, (i(N-1), j(N-1)) \in \mathcal{O}(\mathbf{P}_n)$. Conversely, for a given ordering $\mathcal{O} = (i_0, j_0), (i_1, j_1), \ldots, (i_{N-1}, j_{N-1}) \in \mathcal{O}(\mathbf{P}_n)$, the cyclic strategy $I_\mathcal{O}$ defined by $\mathcal{O}$ is defined by $I_\mathcal{O}(k) = (i_{\tau(k)}, j_{\tau(k)})$, where $0 \leq \tau(k) \leq N - 1$ and $k \equiv \tau(k) (\mathrm{mod}\ N)$, $k \geq 0$.

Let $\mathcal{O} = (i_0, j_0), (i_1, j_1), \ldots, (i_{N-1}, j_{N-1}) \in \mathcal{O}(\mathbf{P}_n)$. Its *reverse ordering* is defined as $\mathcal{O}^{\leftarrow} = (i_{N-1}, j_{N-1}), \ldots, (i_1, j_1), (i_0, j_0) \in \mathcal{O}(\mathbf{P}_n)$. We also say that pivot strategy $I_\mathcal{O}^{\leftarrow} = I_{\mathcal{O}^{\leftarrow}}$ is reverse to $I_\mathcal{O}$.

* Corresponding author: e-mail hari@math.hr,

Copyright line will be provided by the publisher

Starting from the column-wise ordering of $\mathbf{P}_n$, we modify it in such a way that within each column the pivot elements can be taken in an arbitrary order. This means that such orderings start with the position $(1,2)$, continue with the pairs which address positions of the third column, then of the forth column, and so on, until the n-th column. Let $\Pi^{(l_1,l_2)}$ denote the set of all permutations of the set $\{l_1, l_1+1, \ldots, l_2\}$. We define the class of *column-wise orderings with permutations* as

$$\mathcal{C}_c^{(n)} = \{\mathcal{O} \in \mathcal{O}(\mathbf{P}_n) \mid \mathcal{O} = (1,2),(\pi_3(1),3),(\pi_3(2),3),\ldots,(\pi_n(1),n),\ldots,(\pi_n(n-1),n),\ \pi_j \in \Pi^{(1,j-1)},\ 3 \le j \le n\}.$$

In a similar way, we modify the set of row-wise orderings of $\mathbf{P}_n$. We obtain the class of *row-wise orderings with permutations*, which is defined by

$$\mathcal{C}_r^{(n)} = \{\mathcal{O} \in \mathcal{O}(\mathbf{P}_n) \mid \mathcal{O} = (n-1,n),(n-2,\tau_{n-2}(n-1)),(n-2,\tau_{n-2}(n)),\ldots,(1,\tau_1(n)),\ \tau_i \in \Pi^{(i+1,n)},\ 1 \le i \le n-2\}.$$

The other two classes of orderings with permutations are defined using reverse orderings,

$$\overleftarrow{\mathcal{C}}_c^{(n)} = \left\{\mathcal{O} \in \mathcal{O}(\mathbf{P}_n) \mid \mathcal{O}^{\leftarrow} \in \mathcal{C}_c^{(n)}\right\}, \qquad \overleftarrow{\mathcal{C}}_r^{(n)} = \left\{\mathcal{O} \in \mathcal{O}(\mathbf{P}_n) \mid \mathcal{O}^{\leftarrow} \in \mathcal{C}_r^{(n)}\right\}.$$

In other words, the orderings from $\overleftarrow{\mathcal{C}}_c^{(n)}$ ($\overleftarrow{\mathcal{C}}_r^{(n)}$) take the pivot elements column-by-column (row-by-row) from right to left (top to bottom), with permutations inside columns (rows). By

$$\mathcal{C}_{sp}^{(n)} = \mathcal{C}_c^{(n)} \cup \overleftarrow{\mathcal{C}}_c^{(n)} \cup \mathcal{C}_r^{(n)} \cup \overleftarrow{\mathcal{C}}_r^{(n)}$$

we denote the set of serial orderings with permutations. The following global convergence result (and its generalization to the block Jacobi method) is proved in [1,5].

Theorem 2.1 *Let A be a symmetric matrix of order n, $\mathcal{O} \in \mathcal{C}_{sp}^{(n)}$, and let A' be obtained from A after applying one sweep of the cyclic Jacobi method defined by the strategy $I_{\mathcal{O}}$, with rotation angles from the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$. Then there is a constant γ_n depending only on n, such that*

$$S^2(A') \le \gamma_n S^2(A), \quad 0 \le \gamma_n < 1.$$

3 Generalized serial orderings

The class of "convergent" orderings $\mathcal{C}_{sp}^{(n)}$ can be considerably extended using equivalence relations on the set $\mathcal{O}(\mathbf{P}_n)$.

An *admissible transposition* on $\mathcal{O} \in \mathcal{O}(\mathbf{P}_n)$ is any transposition of two adjacent terms $(i_r, j_r), (i_{r+1}, j_{r+1})$ of $\mathcal{O}$, provided that $\{i_r, j_r\}$ and $\{i_{r+1}, j_{r+1}\}$ are disjoint. Two orderings $\mathcal{O}, \mathcal{O}' \in \mathcal{O}(\mathbf{P}_n)$ are: *(i) equivalent* (we write $\mathcal{O} \sim \mathcal{O}'$) if one can be obtained from the other using a finite number of admissible transpositions; *(ii) shift-equivalent* ($\mathcal{O} \overset{\mathsf{s}}{\sim} \mathcal{O}'$) if $\mathcal{O} = [\mathcal{O}_1, \mathcal{O}_2]$ and $\mathcal{O}' = [\mathcal{O}_2, \mathcal{O}_1]$, where $[\ ,\ ]$ stands for the concatenation; *(iii) weakly equivalent* ($\mathcal{O} \overset{\mathsf{w}}{\sim} \mathcal{O}'$) if there are $\mathcal{O}_i \in \mathcal{O}(\mathbf{P}_n)$, $0 \le i \le r$, such that in the sequence $\mathcal{O} = \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_r = \mathcal{O}'$ every two adjacent terms are equivalent or shift-equivalent. Furthermore, two orderings $\mathcal{O}, \mathcal{O}' \in \mathcal{O}(\mathbf{P}_n)$ are *permutation equivalent* ($\mathcal{O} \overset{\mathsf{p}}{\sim} \mathcal{O}'$) if there is a permutation p of the set $\{1, 2, \ldots, n\}$ such that $\mathcal{O}' = (\mathsf{p}(i_0), \mathsf{p}(j_0)), (\mathsf{p}(i_1), \mathsf{p}(j_1)), \ldots, (\mathsf{p}(i_{N-1}), \mathsf{p}(j_{N-1}))$ provided that $\mathcal{O} = (i_0, j_0), (i_1, j_1), \ldots, (i_{N-1}, j_{N-1})$. It is easy to check that $\sim$, $\overset{\mathsf{s}}{\sim}$, $\overset{\mathsf{w}}{\sim}$ and $\overset{\mathsf{p}}{\sim}$ are equivalence relations. Two cyclic pivot strategies $I_{\mathcal{O}}$ and $I_{\mathcal{O}'}$ are equivalent, shift-equivalent, weakly equivalent, permutation equivalent if the same holds for their pivot orderings $\mathcal{O}$ and $\mathcal{O}'$, respectively.

Having all these tools, we define the set of *generalized serial orderings* of $\mathcal{O}(\mathbf{P}_n)$ as follows,

$$\mathcal{C}_{sg}^{(n)} = \{\mathcal{O} \in \mathcal{O}(\mathbf{P}_n) \mid \mathcal{O} \overset{\mathsf{p}}{\sim} \mathcal{O}' \overset{\mathsf{w}}{\sim} \mathcal{O}'' \text{ or } \mathcal{O} \overset{\mathsf{w}}{\sim} \mathcal{O}' \overset{\mathsf{p}}{\sim} \mathcal{O}'',\ \mathcal{O}'' \in \mathcal{C}_{sp}^{(n)}\}.$$

It has been proved in [5] that Theorem 2.1 holds in a slightly weaker form if the assumption $\mathcal{O} \in \mathcal{C}_{sp}^{(n)}$ is replaced by $\mathcal{O} \in \mathcal{C}_{sg}^{(n)}$. In addition, such a result holds not only for the element-wise method, but also for the block Jacobi method. This enables us to prove the global convergence of other cyclic (element-wise or block) Jacobi-type methods, such as J-Jacobi, Falk-Langemeyer, Hari-Zimmermann, Paardekooper method etc., for orderings from $\mathcal{C}_{sg}^{(n)}$.

Acknowledgements This work has been fully supported by Croatian Science Foundation under the project IP-2014-09-3670.

References

[1] E. Begović, Ph.D. thesis (University of Zagreb, 2014).
[2] G. E. Forsythe and P. Henrici, Trans. Amer. Math. Soc. **94**, 1–23 (1960).
[3] V. Hari, Numer. Math. **60**, 375–406 (1991).
[4] V. Hari, Numer. Math. **129 (3)** 449–481 (2015).
[5] V. Hari and E. Begović Kovač, arXiv:1604.05825 [math.NA] (2016).
[6] V. Hari and S. Singer and S. Singer, Lin. Alg. and Its Appl. **444**, 1-27 (2014).
[7] P. Henrici and K. Zimmermann, Linear Algebra Appl. **1**, 489–501 (1968).
[8] W. Mascarenhas, SIAM J. Matrix Anal. Appl. **16 (4)** 1197–1206 (1995).
[9] G. Shroff and R. Schreiber, SIAM J. Matrix Anal. Appl. **10 (3)** 326–346 (1989).

Copyright line will be provided by the publisher