Indefinitna CS dekompozicija

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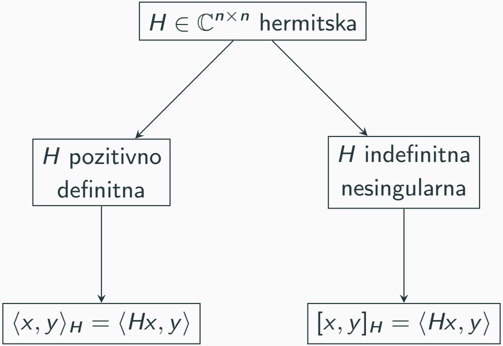
Indefinitni skalarni produkti



Indefinitni skalarni produkti

1. linearnost u prvom argumentu

$$[\alpha x + \beta y, z]_H = \alpha [x, z]_H + \beta [y, z]_H, \quad \forall \alpha, \beta \in \mathbb{C}, \quad \forall x, y, z \in \mathbb{C}^n$$

2. hermitičnost

$$[x, y]_H = \overline{[y, x]}_H, \quad \forall x, y \in \mathbb{C}^n$$

3. nedegeneriranost

$$[x, y]_H = 0, \quad \forall y \in \mathbb{C}^n \Rightarrow x = 0$$

$[\cdot, \cdot] : \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \to \mathbb{C}$ sa svojstvima (1)-(3) je indefinitan skalarni produkt

Indefinitni skalarni produkti

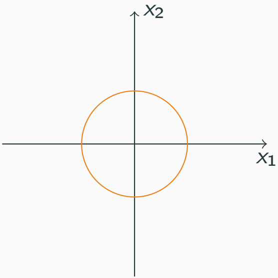
$$H = U\Lambda U^* = U|\Lambda|^{\frac{1}{2}} J \underbrace{|\Lambda|^{\frac{1}{2}} U^*}_{G} = G^* J G$$

$$J = \operatorname{diag}(\pm 1, \ldots, \pm 1) \quad (= \operatorname{diag}(I_p, -I_{n-p}))$$

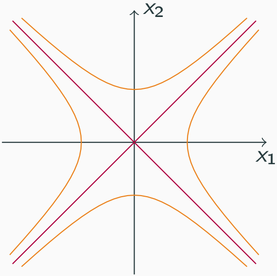
$$\Downarrow$$

$$[x, y]_H = [Gx, Gy]_J$$

Indefinitni skalarni produkti



$x^* I x = 1$



$x^* J x = \pm 1$

Indefinitni skalarni produkti

$x \in \mathbb{C}^n$ je neutralan ako je $[x, x] = 0$.

$$[x, x] < 0, \quad [y, y] > 0, \quad t \in \mathbb{R}$$

$$\Downarrow$$

$$0 = [x + ty, x + ty] = [x, x] + 2\,\mathrm{Re}(x^* J y)t + t^2[y, y]$$

Problem: $[x, y] = 0, \forall y \in \mathrm{span}\{x\}$

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Degenerirani potprostori

Potprostor $\mathcal{M} \subset \mathbb{C}^n$ je degeneriran ako postoji $0 \neq x \in \mathcal{M}$ takav da je

$$[x, y] = 0, \forall y \in \mathcal{M}.$$

U suprotnom je $\mathcal{M}$ nedegeneriran.

Nedegenerirani potprostori

J-ortogonalni komplement potprostora $\mathcal{M} \subset \mathbb{C}^n$ je potprostor

$$\mathcal{M}^{[\perp]} = \{x \in \mathbb{C}^n \mid [x, y] = 0, \quad \forall y \in \mathcal{M}\}$$

$$\mathcal{M} \text{ degeneriran} \quad \Leftrightarrow \quad \mathcal{M} \cap \mathcal{M}^{[\perp]} \neq \emptyset$$

Nedegenerirani potprostori

Za potprostor $\mathcal{M} \subset \mathbb{C}^n$ su sljedeće tvrdnje ekvivalentne:

1. $\mathcal{M}$ je nedegeneriran
2. A^*JA je nesingularna, gdje stupci matrice A tvore bazu za $\mathcal{M}$
3. $\mathcal{M} \oplus \mathcal{M}^{[\perp]} = \mathbb{C}^n$
4. $\mathcal{M}$ ima J-ortonormalnu bazu.

J-adjungirana matrica

$A^{[*]} \in \mathbb{C}^{n \times n}$ je J-adjungirana matrica matrice $A \in \mathbb{C}^{n \times n}$ ako je

$$[Ax, y] = [x, A^{[*]}y], \forall x, y \in \mathbb{C}^n.$$

$$A^{[*]} = JA^*J = \begin{pmatrix} A_{11}^* & -A_{21}^* \\ -A_{12}^* & A_{22}^* \end{pmatrix}$$

J-hermitske matrice

$A \in \mathbb{C}^{n \times n}$ je J-hermitska ako je

$$[Ax, y] = [x, Ay], \forall x, y \in \mathbb{C}^n.$$

$$A = JA^*J = \begin{pmatrix} A_{11} & -A_{21}^* \\ A_{21} & A_{22} \end{pmatrix}, \quad A_{11} = A_{11}^*, A_{22} = A_{22}^*$$

J-unitarne matrice

Matrica $Q \in \mathbb{C}^{n \times n}$ je J-unitarna ako je

$$[Qx, Qy] = [x, y], \forall x, y \in \mathbb{C}.$$

$$Q^* J Q = J$$

$$Q^{[*]} = Q^{-1}$$

Napomene:

1. J-unitarne matrice čine multiplikativnu grupu zatvorenu na adjungiranje i J-adjungiranje
2. $Q \in \mathbb{C}^{n \times n}$ nesingularna s J-ortonormiranim stupcima $\Rightarrow$ Q nije nužno J-unitarna

Primjena: prigušeni sustav

$$M\ddot{x} + C\dot{x} + Kx = f$$

$$M > 0, \quad C \geq 0, \quad K > 0$$

$$\Downarrow$$

$$M = L_2 L_2^T \quad K = L_1 L_1^T$$

$$\Downarrow$$

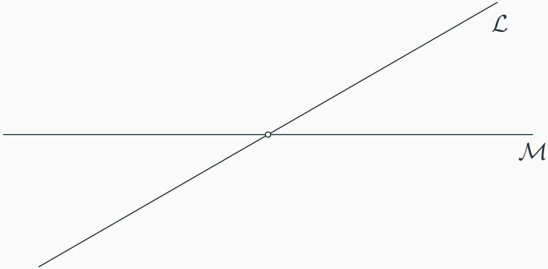
$$y_1 = L_1^T x$$

$$y_2 = L_2^T \dot{x}$$

$$\Downarrow$$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & L_1^T L_2^{-T} \\ -L_2^{-1} L_1 & -L_2^{-1} C L_2^{-T} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ L_2^{-1} f \end{bmatrix}$$

Kutevi među potprostorima



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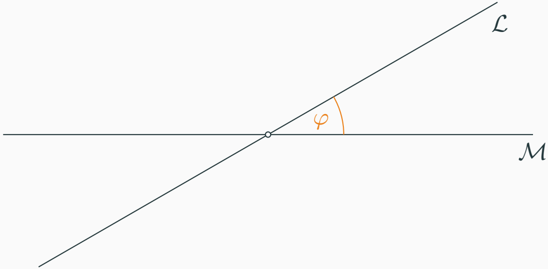

$$0 \leq \varphi \leq \frac{\pi}{2}$$

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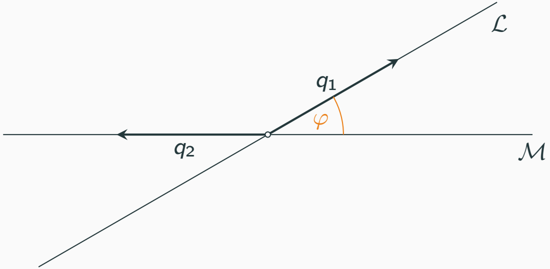

$$0 \le \varphi \le \frac{\pi}{2}$$
$$\|q_1\|_2 = \|q_2\|_2 = 1$$

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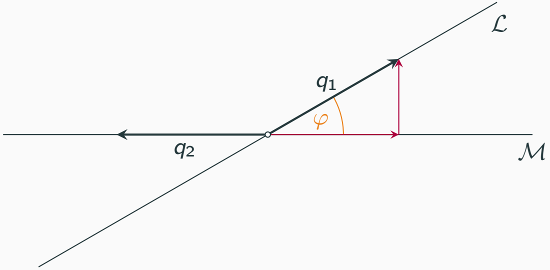

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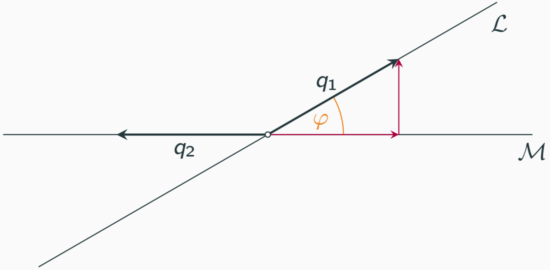
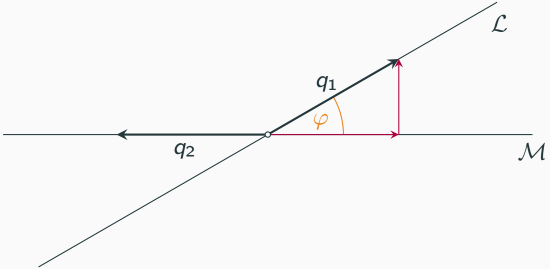


$$0 \leq \varphi \leq \frac{\pi}{2}$$
$$\|q_1\|_2 = \|q_2\|_2 = 1$$
$$\cos\varphi = \|q_2 q_2^* q_1\|_2 = \|P_2 P_1\|_2$$
$$\sin\varphi = \|(I - q_2 q_2^*) q_1\|_2 = \|(I - P_2) P_1\|_2$$

Kutevi među potprostorima



$$0 \le \varphi \le \frac{\pi}{2}$$
$$\|q_1\|_2 = \|q_2\|_2 = 1$$
$$\cos\varphi = \|q_2 q_2^* q_1\|_2 = \|P_2 P_1\|_2$$
$$\sin\varphi = \|(I - q_2 q_2^*) q_1\|_2 = \|(I - P_2) P_1\|_2$$
$$\varphi = 0 \Leftrightarrow \mathcal{M} = \mathcal{L}$$
$$\varphi = \frac{\pi}{2} \Leftrightarrow \mathcal{M} \perp \mathcal{L}$$

Kutevi među potprostorima

$$\dim \mathcal{L} = l \qquad \dim \mathcal{M} = m$$
$$\mathcal{R}(Q_1) = \mathcal{L} \qquad \mathcal{R}(Q_2) = \mathcal{M}$$
$$Q_1^* Q_1 = I_l \qquad Q_1^* Q_2 = I_m$$

$$|q_2^* q_1| \quad \longrightarrow \quad Q_2^* Q_1$$

tražimo kosinuse:

1. $0 \leq c_i \leq 1$
2. unitarno invarijantni

Kutevi među potprostorima

$$\begin{pmatrix} Q_2^* \\ \tilde{Q}_2^* \end{pmatrix} \begin{pmatrix} Q_1 & \tilde{Q}_1 \end{pmatrix} = \begin{pmatrix} Q_2^* Q_1 & Q_2^* \tilde{Q}_1 \\ \tilde{Q}_2^* Q_1 & \tilde{Q}_2^* \tilde{Q}_1 \end{pmatrix} = \begin{pmatrix} U_1 & \\ & U_2 \end{pmatrix} \begin{pmatrix} \Gamma & W_{12} \\ W_{21} & \tilde{\Gamma} \end{pmatrix} \begin{pmatrix} V_1^* & \\ & V_2^* \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} I & & \\ & C & \\ & & 0 \end{pmatrix} \qquad \tilde{\Gamma} = \begin{pmatrix} I & & \\ & \tilde{C} & \\ & & 0 \end{pmatrix}$$

$$1 > c_1 \geq \cdots \geq c_r > 0$$

$$1 > \tilde{c}_1 \geq \cdots \geq \tilde{c}_s > 0$$

Kutevi među potprostorima

$$\begin{pmatrix} I & 0 & 0 & W_{14} & W_{15} & W_{16} \\ 0 & C & 0 & W_{24} & W_{25} & W_{26} \\ 0 & 0 & 0 & W_{34} & W_{35} & W_{36} \\ W_{41} & W_{42} & W_{43} & I & 0 & 0 \\ W_{51} & W_{52} & W_{53} & 0 & \tilde{C} & 0 \\ W_{61} & W_{62} & W_{63} & 0 & 0 & 0 \end{pmatrix}$$

Kutevi među potprostorima

$$\begin{pmatrix} I & 0 & 0 & W_{14} & W_{15} & W_{16} \\ 0 & C & 0 & W_{24} & W_{25} & W_{26} \\ 0 & 0 & 0 & W_{34} & W_{35} & W_{36} \\ W_{41} & W_{42} & W_{43} & I & 0 & 0 \\ W_{51} & W_{52} & W_{53} & 0 & \tilde{C} & 0 \\ W_{61} & W_{62} & W_{63} & 0 & 0 & 0 \end{pmatrix}$$

Kutevi među potprostorima

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & W_{25} & W_{26} \\ 0 & 0 & 0 & 0 & W_{35} & W_{36} \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & W_{52} & W_{53} & 0 & \tilde{C} & 0 \\ 0 & W_{62} & W_{63} & 0 & 0 & 0 \end{pmatrix}$$

Kutevi među potprostorima

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & W_{25} & W_{26} \\ 0 & 0 & 0 & 0 & W_{35} & W_{36} \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & W_{52} & W_{53} & 0 & \tilde{C} & 0 \\ 0 & W_{62} & W_{63} & 0 & 0 & 0 \end{pmatrix}$$

Kutevi među potprostorima

$$
\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & C & 0 & 0 & W_{25} & 0 \\
0 & 0 & 0 & 0 & W_{35} & W_{36} \\
0 & 0 & 0 & I & 0 & 0 \\
0 & W_{52} & W_{53} & 0 & \tilde{C} & 0 \\
0 & 0 & W_{63} & 0 & 0 & 0
\end{pmatrix}
$$

Kutevi među potprostorima

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & W_{25} & 0 \\ 0 & 0 & 0 & 0 & W_{35} & W_{36} \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & W_{52} & W_{53} & 0 & \tilde{C} & 0 \\ 0 & 0 & W_{63} & 0 & 0 & 0 \end{pmatrix}$$

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$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & W_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & W_{36} \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & W_{52} & 0 & 0 & \tilde{C} & 0 \\ 0 & 0 & W_{63} & 0 & 0 & 0 \end{pmatrix}$$

Kutevi među potprostorima

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & W_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & W_{52} & 0 & 0 & \tilde{C} & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{pmatrix}$$

Kutevi među potprostorima

$$\begin{pmatrix} C & W_{25} \\ W_{52} & \tilde{C} \end{pmatrix}$$

$$\left.\begin{aligned} C^2 + W_{52}^* W_{52} = I \\ \tilde{C}^2 + W_{52} W_{52}^* = I \end{aligned}\right\} \quad \Rightarrow \quad W_{52} \text{ kvadratna i nesingularna}$$

$$\left.\begin{aligned} CW_{25} + W_{52}^* \tilde{C} = 0 \\ CW_{52}^* + W_{25} \tilde{C} = 0 \end{aligned}\right\} \quad \Rightarrow \quad W_{52} C^2 = \tilde{C}^2 W_{52} \quad \Rightarrow \quad \tilde{C}$$

$$W_{52}^* W_{52} = I - C^2 \quad \Rightarrow \quad W_{52} = \tilde{W}_{52} S, \quad S = (I - C^2)^{\frac{1}{2}},$$

$$\Rightarrow \quad \tilde{W}_{52} \text{ unitarna, blok-konformalna s } C$$

$$\Rightarrow \quad W_{25} = -S \tilde{W}_{52}^*$$

Kutevi među potprostorima

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & -S\tilde{W}_{52}^* & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & S\tilde{W}_{52} & 0 & 0 & \tilde{C} & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{pmatrix}$$

Kutevi među potprostorima

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & -S & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & S & 0 & 0 & \tilde{C} & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{pmatrix}$$

Kutevi među potprostorima

$$Q_2^* Q_1 \quad \leftrightarrow \quad \begin{pmatrix} Q_2^* Q_1 \\ 0 \end{pmatrix} \quad \leftrightarrow \quad \begin{pmatrix} I \\ 0 \end{pmatrix} Q_2^* Q_1 \quad \leftrightarrow \quad \begin{pmatrix} Q_2^* \\ \tilde{Q}_2^* \end{pmatrix} Q_2 Q_2^* Q_1$$

Indefinitni kutevi

$$J = \begin{pmatrix} I_p & \\ & -I_{n-p} \end{pmatrix}$$

$$\dim \mathcal{L} = p \qquad\qquad \dim \mathcal{M} = p$$

$$\mathcal{R}(Q_1) = \mathcal{L} \qquad\qquad \mathcal{R}(Q_2) = \mathcal{M}$$

$$Q_1^* J Q_1 = I_p \qquad\qquad Q_2^* J Q_2 = I_p$$

$$\Downarrow$$

$$[Q_i x, Q_i x] = x^* Q_i^* J Q_i x = x^* x > 0, \quad x \neq 0$$

$$\begin{pmatrix} Q_i & \tilde{Q}_i \end{pmatrix}^* J \begin{pmatrix} Q_i & \tilde{Q}_i \end{pmatrix} = J$$

Indefinitni kutevi

$$\begin{pmatrix} Q_2^* \\ \tilde{Q}_2^* \end{pmatrix} J \begin{pmatrix} Q_1 & \tilde{Q}_1 \end{pmatrix} = \begin{pmatrix} U_1 & \\ & U_2 \end{pmatrix} \begin{pmatrix} \Gamma & W_{12} \\ W_{21} & \tilde{\Gamma} \end{pmatrix} \begin{pmatrix} V_1^* & \\ & V_2^* \end{pmatrix}$$

$$W_{11}^* W_{11} - W_{21}^* W_{21} = I$$

$$W_{22}^* W_{22} - W_{12}^* W_{12} = I$$

$$\Gamma = \begin{pmatrix} I & \\ & C \end{pmatrix} \qquad \tilde{\Gamma} = \begin{pmatrix} I & \\ & \tilde{C} \end{pmatrix}$$

$$1 < c_1 \leq \cdots \leq c_r$$

$$1 < \tilde{c}_1 \leq \cdots \leq \tilde{c}_s$$

Indefinitni kutevi

$$\begin{pmatrix} I & 0 & W_{13} & W_{14} \\ 0 & C & W_{23} & W_{24} \\ W_{31} & W_{32} & I & 0 \\ W_{41} & W_{42} & 0 & \tilde{C} \end{pmatrix}$$

Indefinitni kutevi

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & C & 0 & W_{24} \\ 0 & 0 & I & 0 \\ 0 & W_{42} & 0 & \tilde{C} \end{pmatrix}$$

Indefinitni kutevi

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & C & 0 & S \\ 0 & 0 & I & 0 \\ 0 & S & 0 & C \end{pmatrix}$$

$$S = (C^2 - I)^{\frac{1}{2}}$$

Q J-unitarna $\Rightarrow$

$$Q = \left(\begin{array}{c|c} U_1 & \\ \hline & U_2 \end{array}\right) \left(\begin{array}{cc|cc} I & & 0 & \\ & C & & S \\ \hline 0 & & I & \\ & S & & C \end{array}\right) \left(\begin{array}{c|c} V_1 & \\ \hline & V_2 \end{array}\right)^*$$

$$\begin{pmatrix} C & S \\ S & C \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}I & -\frac{1}{\sqrt{2}}I \\ \frac{1}{\sqrt{2}}I & \frac{1}{\sqrt{2}}I \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}I & -\frac{1}{\sqrt{2}}I \\ \frac{1}{\sqrt{2}}I & \frac{1}{\sqrt{2}}I \end{pmatrix} \begin{pmatrix} C+S & \\ & C-S \end{pmatrix}$$