On the Global Convergence of the Block Jacobi Method

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hrzz Hrvatska zaklada za znanost

OUTLINE

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$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix} \begin{matrix} n_1 \\ n_2 \\ \\ n_m \end{matrix}, \qquad A_{ij} = A_{ji}^*$$

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$$\mathbf{U}_{ij} = \begin{bmatrix} I & & & & \\ & U_{ii} & & U_{ij} & \\ & & I & & \\ & U_{ji} & & U_{jj} & \\ & & & & I \end{bmatrix} \begin{matrix} \\ n_i \\ \\ n_j \\ \\ \end{matrix}, \qquad i < j, \qquad \textit{unitary}$$

BLOCK JACOBI METHOD

Block Jacobi method for A is iterative process of the form

$$A^{(k+1)} = U_k^* A^{(k)} U_k, \quad k \geq 0; \qquad A^{(0)} = A,$$

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At step k, the pivot submatrix of $A^{(k)}$ is diagonalized:

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$n_1 = n_2 = \cdots = n_m = 1 \;\longrightarrow\;$ standard (element-wise) Jacobi method

Pivot Strategy

$$I : \mathbb{N}_0 \to P_m \quad \text{pivot strategy}$$

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- the period is $M \equiv \frac{m(m-1)}{2}$
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- $A^{(k)} \longrightarrow A^{(k+1)}$ is the kth step of the method
- $A^{((r-1)M)} \longrightarrow A^{(rM-1)}$ is the rth cycle or sweep of the method.

The Serial Pivot Strategies

Typical cyclic strategies are the column and row-cyclic strategies.

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\begin{aligned}
\mathcal{O}_c &= (1,2),(1,3),(2,3),\ \ldots\ ,(1,m),(2,m),\ldots,(m-1,m) \\
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$$\mathcal{O}_c \longleftrightarrow \begin{bmatrix} * & 0 & 1 & 3 & 6 \\ 0 & * & 2 & 4 & 7 \\ 1 & 2 & * & 5 & 8 \\ 3 & 4 & 5 & * & 9 \\ 6 & 7 & 8 & 9 & * \end{bmatrix}, \qquad \mathcal{O}_r \longleftrightarrow \begin{bmatrix} * & 0 & 1 & 2 & 3 \\ 0 & * & 4 & 5 & 6 \\ 1 & 4 & * & 7 & 8 \\ 2 & 5 & 7 & * & 9 \\ 3 & 6 & 8 & 9 & * \end{bmatrix}.$$

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$\mathcal{O}_c, \mathcal{O}_r \in \mathcal{O}(P_m)$ — the set of all orderings of the set P_m.

Off-norm and Convergence

$$S^2(A) = \sum_{r \neq s} \|A_{rs}\|_F^2, \qquad A = A^*$$

$S(A) \geq 0$ is departure from the block diagonal form (block off-norm) of A.

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Definition

A block Jacobi method is convergent on A if the obtained sequence of matrices $(A^{(k)})$ converges to some diagonal matrix Λ. The method is globally convergent if it is convergent on every hermitian matrix A.

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is a necessary condition for the global convergence.

Is it also a sufficient condition for the global convergence?

A Sufficient Condition for Convergence

Theorem

Let A be a hermitian matrix and $A^{(k)}$, $k \geq 0$ be the sequence obtained by applying the block Jacobi method to A. Let the pivot strategy be cyclic and let $\lim_{k\to\infty} S(A^{(k)}) = 0$.

(i) *If the algorithm which diagonalizes the pivot submatrix always delivers $\mathrm{diag}(\Lambda_{ii}^{(k+1)}, \Lambda_{jj}^{(k+1)})$ with non-increasingly (non-decreasingly) ordered diagonal elements, then $\lim_{k\to\infty} A^{(k)} = \Lambda$, where Λ is diagonal with diagonal elements of Λ non-increasingly (non-decreasingly) ordered.*

(ii) *If the algorithm which diagonalizes the pivot submatrix is any standard (i.e. element-wise) globally convergent Jacobi method, then $\lim_{k\to\infty} A^{(k)} = \Lambda$ is diagonal.*

Thus, we can focus on proving $S(A^{(k)}) \to 0$ as $k \to \infty$.

UBC Orthogonal Transformations

What kind of transformation matrices to use?
If $\pi = (1, 1, \ldots, 1)$ the transformations are plane rotations and satisfy

$$\cos\phi_k \geq \frac{\sqrt{2}}{2}, \ k \geq 0 \qquad \text{(recall the Forsythe-Henrici condition!)}$$

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If $\mathbf{U}_{ij}$ is UBCE, then its diagonal blocks satisfy

$$\sigma_{min}(U_{ii}) = \sigma_{min}(U_{jj}) \geq \gamma_{ij} > \tilde{\gamma}_{n_i+n_j} \geq \tilde{\gamma}_n,$$

where

$$\gamma_{ij} = \frac{3}{\sqrt{(4^{n_i} + 6n_j - 1)(n_j + 1)}}, \quad \tilde{\gamma}_r = \frac{3\sqrt{2}}{\sqrt{4^r + 26}}.$$

What is known?

Drmač: SIAM J. Mat. Anal. Appl. 31, 2009

Convergence proof for the serial block Jacobi methods:

$$S(A') \leq c_n S(A), \qquad 0 \leq c_n < 1,$$

$A' \longleftarrow A$ after one sweep, c_n is a constant depending just on n.
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Hari: Numer. Math. 129, 2015

More general setting, global convergence results for general block Jacobi-type methods under the serial and weak-wavefront strategies.

What Pivot Strategies?

So far we know the following:

- it is sufficient to consider the problem $S(A^{(k)}) \to 0$ as $k \to \infty$
- The transformation matrices must be UBCE

The remaining problems include:

- to find some larger class of usable pivot strategies for which the proof can be made
- to try to obtain the result in the form:

$$S(A') \leq c_n S(A), \qquad 0 \leq c_n < 1,$$

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Cyclic Pivot Strategies

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$$I \text{ cyclic iff } I = I_{\mathcal{O}} \text{ for some } \mathcal{O} \in \mathcal{O}(P_m)$$

$\mathcal{O}$ is also called pivot ordering. It generally has the form:

$$\mathcal{O} = (i_0, j_0), (i_1, j_1), \ \ldots \ , (i_{M-1}, j_{M-1}), \qquad M = m(m-1)/2.$$

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To visually depict $\mathcal{O}$, we use the symmetric matrix $\mathrm{M}_{\mathcal{O}} = (\mathrm{m}_{rt})$, defined by

$$\mathrm{m}_{i(k)j(k)} = \mathrm{m}_{j(k)i(k)} = k, \quad k = 0, 1, \ldots \ , M-1.$$

We set $\mathrm{m}_{ss} = -1$, $1 \leq s \leq m$. Since $(s, s) \notin \mathcal{O}$ we depict them by $*$.

Inverse Ordering, Inverse Cyclic Strategy

With

$$\mathcal{O} = (i_0, j_0), (i_1, j_1), \ \ldots \ , (i_{M-1}, j_{M-1})$$

is associated its inverse or reverse ordering

$$\mathcal{O}^{\leftarrow} = (i_{M-1}, j_{M-1}), \ \ldots \ , (i_1, j_1), (i_0, j_0)$$

$I_{\mathcal{O}^{\leftarrow}}$ is the inverse (reverse) strategy of $I_{\mathcal{O}}$. Note: $\mathcal{O}^{\leftarrow\leftarrow} = \mathcal{O}$.

$$\mathrm{M}_{\mathcal{O}_c^{\leftarrow}} = \begin{bmatrix} * & 9 & 8 & 6 & 3 \\ 9 & * & 7 & 5 & 2 \\ 8 & 7 & * & 4 & 1 \\ 6 & 5 & 4 & * & 0 \\ 3 & 2 & 1 & 0 & * \end{bmatrix}, \qquad \mathrm{M}_{\mathcal{O}_r^{\leftarrow}} = \begin{bmatrix} * & 9 & 8 & 7 & 6 \\ 9 & * & 5 & 4 & 3 \\ 8 & 5 & * & 2 & 1 \\ 7 & 4 & 2 & * & 0 \\ 6 & 3 & 1 & 0 & * \end{bmatrix}.$$

Permutation Equivalent Strategies

Two pivot orderings $\mathcal{O}, \mathcal{O}' \in \mathcal{O}(P_m)$ are permutation equivalent if

$$\mathrm{M}_{\mathcal{O}'} = \mathrm{P}^T \mathrm{M}_{\mathcal{O}} \mathrm{P}$$

holds for some permutation matrix P. Then we write $\mathcal{O}' \overset{\mathrm{p}}{\sim} \mathcal{O}$, $I_{\mathcal{O}'} \overset{\mathrm{p}}{\sim} I_{\mathcal{O}}$.

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Let p be a permutation of $\{1, 2, \ldots, m\}$ such that

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$\overset{\mathrm{p}}{\sim}$ is equivalence relation on $\mathcal{O}(P_m)$.

Equivalent Strategies

Let $\mathcal{O} \in \mathcal{O}(P_m)$. The number of pairs in $\mathcal{O}$, $|\mathcal{O}|$, is the length of $\mathcal{O}$.

An admissible transposition on $\mathcal{O}$ is any transposition of two adjacent terms $(i_r, j_r), (i_{r+1}, j_{r+1}) \to (i_{r+1}, j_{r+1}), (i_r, j_r)$, provided that $\{i_r, j_r\}$ and $\{i_{r+1}, j_{r+1}\}$ are disjoint (we say that such pairs commute).

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Cyclic pivot strategy $I_{\mathcal{O}'}$ is equivalent to $I_{\mathcal{O}}$ (we write $I_{\mathcal{O}'} \sim I_{\mathcal{O}}$) if $\mathcal{O}' \sim \mathcal{O}$.

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An admissible transposition on $\mathcal{O}$ is any transposition of two adjacent terms $(i_r, j_r), (i_{r+1}, j_{r+1}) \to (i_{r+1}, j_{r+1}), (i_r, j_r)$, provided that $\{i_r, j_r\}$ and $\{i_{r+1}, j_{r+1}\}$ are disjoint (we say that such pairs commute).

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Cyclic pivot strategy $I_{\mathcal{O}'}$ is equivalent to $I_{\mathcal{O}}$ (we write $I_{\mathcal{O}'} \sim I_{\mathcal{O}}$) if $\mathcal{O}' \sim \mathcal{O}$.

$\sim$ is equivalence relation on $\mathcal{O}(P_m)$.

Weakly Equivalent Strategies

Two pivot orderings $\mathcal{O}, \mathcal{O}' \in \mathcal{O}(P_m)$, are

(ii) shift-equivalent ($\mathcal{O} \overset{s}{\sim} \mathcal{O}'$) if $\mathcal{O} = [\mathcal{O}_1, \mathcal{O}_2]$ and $\mathcal{O}' = [\mathcal{O}_2, \mathcal{O}_1]$, where [,] stands for concatenation.

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(iii) weak equivalent ($\mathcal{O} \overset{w}{\sim} \mathcal{O}'$) if there are orderings $\mathcal{O}_i \in \mathcal{O}(P_m)$, such that in the sequence $\mathcal{O} = \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_r = \mathcal{O}'$ each two neighboring terms are equivalent or shift-equivalent.

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$\overset{s}{\sim}$ and $\overset{w}{\sim}$ are equivalence relations on $\mathcal{O}(P_m)$.

$$I_{\mathcal{O}'} \overset{s}{\sim} I_{\mathcal{O}} \quad \text{iff} \quad \mathcal{O}' \overset{s}{\sim} \mathcal{O}, \qquad\qquad I_{\mathcal{O}'} \overset{w}{\sim} I_{\mathcal{O}} \quad \text{iff} \quad \mathcal{O}' \overset{w}{\sim} \mathcal{O}.$$

The Main Theorem

Let $A = A^*$ of order n, $\pi = (n_1, \ldots, n_m)$ and $\mathcal{O} \in \mathcal{O}(P_m)$. Apply to A the cyclic block Jacobi method, defined by π, $I_{\mathcal{O}}$, and let the transformation matrices be UBCE. Let A' be obtained from A after one cycle. Suppose that

$$S(A') \leq c_n S(A), \qquad 0 \leq c_n < 1. \tag{1}$$

Let $\tilde{A} = \tilde{A}^*$ be of order n, $\tilde{\pi} = (\tilde{n}_1, \ldots, \tilde{n}_m)$ partition of n and $\tilde{\mathcal{O}} \in \mathcal{O}(P_m)$. Apply to $\tilde{A}$ the cyclic block Jacobi method, defined by $\tilde{\pi}$, $I_{\tilde{\mathcal{O}}}$ and let the transformation matrices be UBCE. Let $\tilde{A}'$ be obtained from $\tilde{A}$ after one sweep.

- If $\tilde{\mathcal{O}} \sim \mathcal{O}$ then $S(\tilde{A}') \leq c_n S(\tilde{A})$
- If $\tilde{\mathcal{O}} \overset{\text{p}}{\sim} \mathcal{O}$ then $S(\tilde{A}') \leq c_n S(\tilde{A})$
- If $\tilde{\mathcal{O}} = \mathcal{O}^{\leftarrow}$ then $S(\tilde{A}') \leq c_n S(\tilde{A})$
- If $\tilde{\mathcal{O}} \overset{w}{\sim} \mathcal{O}$ then the block Jacobi method defined by $I_{\tilde{\mathcal{O}}}$ is globally convergent.

Serial strategies with permutations

$\Pi^{(k_1,k_2)}$ the group of permutations of the set $\{k_1, k_1+1, \ldots, k_2\}$.

Column-wise orderings with permutations of the set $\mathbf{P}_m$

$$\mathcal{B}_c^{(m)} = \Big\{\mathcal{O} \in \mathcal{O}(\mathbf{P}_m) \;\Big|\; \mathcal{O} = (1,2), (\tau_3(1),3), (\tau_3(2),3), \ldots, (\tau_m(1),m), \ldots, (\tau_m(m-1),m),\ \tau_j \in \Pi^{(1,j-1)},\ 3 \le j \le m\Big\}.$$

$\mathcal{O} \in \mathcal{B}_c^{(m)}$ $\qquad$ $\tilde{\mathcal{O}} \overset{w}{\sim} \mathcal{O}$

$$\mathrm{M}_{\mathcal{O}} = \begin{bmatrix} * & 0 & 2 & 4 & 9 & 12 \\ 0 & * & 1 & 5 & 8 & 10 \\ 2 & 1 & * & 3 & 7 & 13 \\ 4 & 5 & 3 & * & 6 & 11 \\ 9 & 8 & 7 & 6 & * & 14 \\ 12 & 10 & 13 & 11 & 14 & * \end{bmatrix}, \qquad \mathrm{M}_{\tilde{\mathcal{O}}} = \begin{bmatrix} * & 7 & 9 & 0 & 2 & 5 \\ 7 & * & 10 & 13 & 14 & 6 \\ 9 & 10 & * & 11 & 12 & 8 \\ 0 & 13 & 11 & * & 1 & 4 \\ 2 & 14 & 12 & 1 & * & 3 \\ 5 & 6 & 8 & 4 & 3 & * \end{bmatrix}.$$

Serial strategies with permutations

The same theorem holds for the class of orderings

$$\mathcal{B}_r^{(m)} = \{\mathcal{O} \in \mathcal{O}(\mathbf{P}_m) \mid \mathcal{O} = (m-1, m), \ \ldots \ , (1, \tau_1(2)), \ldots, (1, \tau_1(m)), \\ \tau_i \in \Pi^{(i+1,m)}, \ 1 \le i \le m-2\}.$$

$\mathcal{O} \in \mathcal{B}_r^{(m)}$	$\tilde{\mathcal{O}} \overset{w}{\sim} \mathcal{O}$

$$M_{\mathcal{O}} = \begin{bmatrix} * & 11 & 13 & 12 & 10 & 14 \\ 10 & * & 9 & 7 & 6 & 8 \\ 11 & 9 & * & 5 & 3 & 4 \\ 12 & 6 & 5 & * & 1 & 2 \\ 13 & 7 & 3 & 1 & * & 0 \\ 14 & 8 & 4 & 2 & 0 & * \end{bmatrix}, \qquad M_{\tilde{\mathcal{O}}} = \begin{bmatrix} * & 14 & 1 & 0 & 11 & 2 \\ 14 & * & 13 & 10 & 7 & 12 \\ 1 & 13 & * & 9 & 6 & 8 \\ 0 & 10 & 9 & * & 4 & 5 \\ 11 & 7 & 6 & 4 & * & 3 \\ 2 & 12 & 8 & 5 & 3 & * \end{bmatrix}$$

Serial Strategies with Permutations

Let

$$\begin{aligned} \overleftarrow{\mathcal{B}}_c^{(m)} &= \left\{ \mathcal{O} \in \mathcal{O}(\mathbf{P}_m) \mid \mathcal{O}^{\leftarrow} \in \mathcal{B}_c^{(m)} \right\}, \\ \overleftarrow{\mathcal{B}}_r^{(m)} &= \left\{ \mathcal{O} \in \mathcal{O}(\mathbf{P}_m) \mid \mathcal{O}^{\leftarrow} \in \mathcal{B}_r^{(m)} \right\}. \end{aligned}$$

and

$$\mathcal{B}_s^{(m)} = \mathcal{B}_c^{(m)} \cup \overleftarrow{\mathcal{B}}_c^{(m)} \cup \mathcal{B}_r^{(m)} \cup \overleftarrow{\mathcal{B}}_r^{(m)}$$

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$\{I_{\mathcal{O}} \mid \mathcal{O} \in \mathcal{B}_s^{(m)}\}$ is the set of serial strategies with permutations.

The main theorem holds if $\mathcal{O} \in \mathcal{B}_s^{(m)}$.

Generalized Serial Strategies

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n. Let

$$\begin{aligned}
\mathcal{B}_{sp}^{(m)} &= \left\{ \mathcal{O} \in \mathcal{O}(\mathcal{P}_m) \mid \mathcal{O} \overset{\text{p}}{\sim} \mathcal{O}' \sim \mathcal{O}'' \text{ or } \mathcal{O} \sim \mathcal{O}' \overset{\text{p}}{\sim} \mathcal{O}'',\ \mathcal{O}'' \in \mathcal{B}_s^{(m)} \right\}, \\
\mathcal{B}_{sg}^{(m)} &= \left\{ \mathcal{O} \in \mathcal{O}(\mathcal{P}_m) \mid \mathcal{O} \overset{\text{p}}{\sim} \mathcal{O}' \overset{\text{w}}{\sim} \mathcal{O}'' \text{ or } \mathcal{O} \overset{\text{w}}{\sim} \mathcal{O}' \overset{\text{p}}{\sim} \mathcal{O}'',\ \mathcal{O}'' \in \mathcal{B}_s^{(m)} \right\}.
\end{aligned}$$

$\mathcal{B}_{sg}^{(m)}$ is the class of generalized serial pivot orderings

$\mathcal{B}_{sp}^{(m)}$ is the subclass of $\mathcal{B}_{sg}^{(m)}$ with no shifts.

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$\mathcal{B}_{sg}^{(m)}$ is the class of generalized serial pivot orderings

$\mathcal{B}_{sp}^{(m)}$ is the subclass of $\mathcal{B}_{sg}^{(m)}$ with no shifts.

The main theorem holds for $\left\{ I_{\mathcal{O}} \mid \mathcal{O} \in \mathcal{B}_{sp}^{(m)} \right\}$.

It holds in a weaker form for $\left\{ I_{\mathcal{O}} \mid \mathcal{O} \in \mathcal{B}_{sg}^{(m)} \right\}$, which still ensures the global convergence.

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- As part of these considerations, we are able to show that for given n and $\varepsilon > 0$, there is a symmetric matrix $A(\varepsilon)$ and a cyclic pivot strategy, such that for the standard Jacobi method holds

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- As part of these considerations, we are able to show that for given n and $\varepsilon > 0$, there is a symmetric matrix $A(\varepsilon)$ and a cyclic pivot strategy, such that for the standard Jacobi method holds

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- We have developed tools, such as block Jacobi annihilators and operators for hermitian matrices, which enable us to prove the global convergence of block Jacobi-type methods for other eigen - value problems (e.g. the generalized eigenvalue problem).

THANK YOU FOR YOUR ATTENTION