Diagonalization Methods for Solving Definite Generalized Eigenvalue Problem

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OUTLINE

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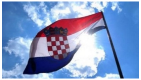
- GEP (DGEP, PGEP)

This work has been fully supported by Croatian Science Foundation under the project IP-09-2014-3670.



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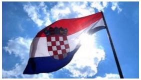

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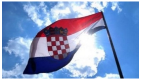

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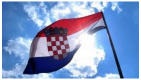

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- Properties (convergence, global and asymptotic),
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- Block algorithms
- Global convergence of block algorithms

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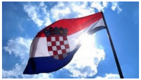

Hrvatska zaklada za znanost

Why Element-wise, Two-sided Jacobi Methods?

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- Two sided methods can smoothly, timely and cost effectively stop the process.

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For a definite pair (A, B) there exists a nonsingular matrix F such that

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Hence, probably the best choice for the kernel algorithm is some element-wise diagonalization method.

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The methods are connected: the FL method can be viewed as the HZ or CJ method with "fast scaled" transformations.

We have also derived their "equally promising" complex counterparts.

The Real and Complex FL Method

Starting with a definite pair (A, B) of Hermitian matrices, FL generates a sequence of "congruent" matrix pairs

$$(A, B) = (A^{(0)}, B^{(0)}), (A^{(1)}, B^{(1)}), \ldots$$

by the rule

$$A^{(k+1)} = F_k^* A^{(k)} F_k \ , \qquad B^{(k+1)} = F_k^* B^{(k)} F_k \ , \quad k \geq 0.$$

Here F_k is an elementary plane matrix defined by the pivot pair $(i(k), j(k))$

$$F_k = \begin{bmatrix} I & & & & \\ & 1 & & \alpha_k & \\ & & I & & \\ & \beta_k & & 1 & \\ & & & & I \end{bmatrix} \begin{matrix} \\ i(k) \\ \\ j(k) \\ \\ \end{matrix} , \qquad \alpha_k, \beta_k \in \mathbf{C},$$

Derivation of the Complex FL Method

The goal is to compute complex numbers α_k, β_k such that the pivot elements $a_{ij}^{(k)}$, $b_{ij}^{(k)}$ of $A^{(k)}$, $B^{(k)}$ are annihilated.

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We simplify notation: $A = A^{(k)}$, $A' = A^{(k+1)}$, $F = F_k$, $(i,j) = (i(k), j(k))$.

Pivot submatrices $\hat{A}$, $\hat{B}$, $\hat{F}$ of A, B, F are 2×2 principal submatrices obtained on the intersection of pivot rows and columns i and j.

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We have

$$A' = F^* A F, \quad B' = F^* B F \qquad \left(\hat{A}' = \hat{F}^* \hat{A} \hat{F}, \quad \hat{B}' = \hat{F}^* \hat{B} \hat{F} \right)$$

and F is chosen to obtain $a'_{ij} = 0$ and $b'_{ij} = 0$.

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$$\begin{bmatrix} 1 & \bar{\beta} \\ \bar{\alpha} & 1 \end{bmatrix} \begin{bmatrix} a_{ii} & a_{ij} \\ \bar{a}_{ij} & a_{jj} \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ \beta & 1 \end{bmatrix} = \begin{bmatrix} a'_{ii} & 0 \\ 0 & a'_{jj} \end{bmatrix}$$

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This leads us to solving a system of two nonlinear equations

$$e_1 = a_{ii}\alpha + a_{jj}\bar{\beta} + \bar{a}_{ij}\alpha\bar{\beta} + a_{ij} = 0$$
$$e_2 = b_{ii}\alpha + b_{jj}\bar{\beta} + \bar{b}_{ij}\alpha\bar{\beta} + b_{ij} = 0.$$

Solution if Matrices are Real and Symmetric

$$
\begin{aligned}
\Im_{ii} &= a_{ii}b_{ij} - a_{ij}b_{ii} = \begin{vmatrix} a_{ii} & b_{ii} \\ a_{ij} & b_{ij} \end{vmatrix} \\
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\Im &= \Im_{ij}^2 + 4\Im_{ii}\Im_{jj} \\
\nu &= (\Im_{ij} + \mathrm{sgn}(\Im_{ij})\sqrt{\Im})/2 \\
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If $\begin{bmatrix} a_{ii} & a_{ij} \\ \bar{a}_{ij} & a_{jj} \end{bmatrix}$ and $\begin{bmatrix} b_{ii} & b_{ij} \\ \bar{b}_{ij} & b_{jj} \end{bmatrix}$ are proportional, all $\Im_{ii}$, $\Im_{jj}$, $\Im_{ij}$, $\Im$ and ν are zero and a special algorithm is required. This is the real FL algorithm.

The Complex FL Algorithm

$$
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\Im'_{ij} &= a_{ii}b_{jj} - a_{jj}b_{ii} = \begin{vmatrix} a_{ii} & b_{ii} \\ a_{jj} & b_{jj} \end{vmatrix} \\
\imath\Im''_{ij} &= a_{ij}\bar{b}_{ij} - \bar{a}_{ij}b_{ij} = \begin{vmatrix} a_{ij} & b_{ij} \\ \bar{a}_{ij} & \bar{b}_{ij} \end{vmatrix} = -2\imath \begin{vmatrix} \mathrm{Re}(a_{ij}) & \mathrm{Re}(b_{ij}) \\ \mathrm{Im}(a_{ij}) & \mathrm{Im}(b_{ij}) \end{vmatrix} \\
\Im_{ij} &= \Im'_{ij} + \imath\Im''_{ij} \\
\Im &= \Im_{ij}^2 + 4\bar{\Im}_{ii}\Im_{jj} = (\Im'_{ij})^2 - (\Im''_{ij})^2 + 2\imath\Im'_{ij}\Im''_{ij} + 4\bar{\Im}_{ii}\Im_{jj} \\
\nu &= (\Im_{ij} + \mathrm{sgn}(\Im'_{ij})\sqrt{\Im})/2, \\
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- Difficult and challenging for making a good numerical code (to many freedoms, all we have $\alpha A + \beta B \succ O$, when to stop iterations?)
- Theoretical results are lacking (all we have is quadratic asymptotic convergence result)

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$$Z_k = \begin{bmatrix} I & & & & \\ & * & & * & \\ & & I & & \\ & * & & * & \\ & & & & I \end{bmatrix} \begin{matrix} \\ i(k) \\ \\ j(k) \\ \\ \end{matrix}$$

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The selection of pivot pairs $(i(k), j(k))$ defines pivot strategy.

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At step k we denote: $\quad A^{(k)} \mapsto A, \quad A^{(k+1)} \mapsto A', \quad Z_k \mapsto Z,$

$$\hat{A} = \begin{bmatrix} a_{ii} & a_{ij} \\ \bar{a}_{ij} & a_{jj} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 & b_{ij} \\ \bar{b}_{ij} & 1 \end{bmatrix}, \quad \hat{Z} = \begin{bmatrix} c & -s \\ \tilde{s} & \tilde{c} \end{bmatrix}.$$

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Then $A' = Z^* A Z, \quad B' = Z^* B Z$ implies $\hat{A}' = \hat{Z}^* \hat{A} \hat{Z}, \quad \hat{B}' = \hat{Z}^* \hat{B} \hat{Z}$.

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$$\hat{A} = \begin{bmatrix} a_{ii} & a_{ij} \\ \bar{a}_{ij} & a_{jj} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 & b_{ij} \\ \bar{b}_{ij} & 1 \end{bmatrix}, \quad \hat{Z} = \begin{bmatrix} c & -s \\ \tilde{s} & \tilde{c} \end{bmatrix}.$$

$\hat{A}$, $\hat{B}$, $\hat{Z}$ are pivot submatrices of A, B, Z.

Then $A' = Z^* A Z$, $B' = Z^* B Z$ implies $\hat{A}' = \hat{Z}^* \hat{A} \hat{Z}$, $\hat{B}' = \hat{Z}^* \hat{B} \hat{Z}$.

$\hat{Z}$ is chosen to diagonalize $\hat{A}'$ and to make $\hat{B}'$ identity matrix I_2.

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$\hat{Z}$ is chosen to diagonalize $\hat{A}'$ and to make $\hat{B}'$ identity matrix I_2.

$\hat{Z}$ is sought in the form of a product of two Jacobi rotations and one or two diagonal matrices.

Real Algorithm: $\hat{Z}$ is sought in the form:

(a) $$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1+b_{ij}}} & 0 \\ 0 & \frac{1}{\sqrt{1-b_{ij}}} \end{bmatrix} \begin{bmatrix} \cos(\theta-\frac{\pi}{4}) & -\sin(\theta-\frac{\pi}{4}) \\ \sin(\theta-\frac{\pi}{4}) & \cos(\theta-\frac{\pi}{4}) \end{bmatrix}$$

(b) $$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1-b_{ij}}} & 0 \\ 0 & \frac{1}{\sqrt{1+b_{ij}}} \end{bmatrix} \begin{bmatrix} \cos(\theta+\frac{\pi}{4}) & -\sin(\theta+\frac{\pi}{4}) \\ \sin(\theta+\frac{\pi}{4}) & \cos(\theta+\frac{\pi}{4}) \end{bmatrix}$$

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Both approaches yield the same algorithm.

Essential Part of the Real Algorithm

$$\xi = \frac{b_{ij}}{\sqrt{1+b_{ij}} + \sqrt{1-b_{ij}}}, \quad \rho = \frac{1}{2}(\sqrt{1+b_{ij}} + \sqrt{1-b_{ij}}), \quad \xi^2 + \rho^2 = 1,$$

$$\tan(2\theta) = \frac{2a_{ij} - (a_{ii} + a_{jj})\, b_{ij}}{\sqrt{1-(b_{ij})^2}\,(a_{ii} - a_{jj})}, \qquad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4},$$

$$\begin{aligned}
\cos\phi &= \rho\cos\theta - \xi\sin\theta \\
\sin\phi &= \rho\sin\theta + \xi\cos\theta \\
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$$\boxed{\hat{Z} = \frac{1}{\sqrt{1-b_{ij}^2}} \begin{bmatrix} \cos\phi & -\sin\phi \\ \cos\psi & \sin\psi \end{bmatrix}.}$$

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$$\begin{aligned}
a'_{ii} &= a_{ii} + \frac{1}{1-b_{ij}^2}\left[(b_{ij}^2-\sin^2\phi)\,a_{ii} + 2\cos\phi\sin\psi\,a_{ij} + \sin^2\psi\,a_{jj}\right] \\
a'_{jj} &= a_{jj} - \frac{1}{1-b_{ij}^2}\left[(\sin^2\psi-b_{ij}^2)\,a_{jj} + 2\cos\psi\sin\phi\,a_{ij} + \sin^2\phi\,a_{ii}\right]
\end{aligned}$$

Complex Algorithm: $\hat{Z}$ is sought in the form:

$$\hat{B} \to \text{diag} \qquad\qquad \hat{B} \to I_2$$

$$\uparrow \qquad\qquad \uparrow$$

$$\hat{Z} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} e^{\imath \arg(b_{ij})} \\ \frac{\sqrt{2}}{2} e^{-\imath \arg(b_{ij})} & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{1+|b_{ij}|}} & 0 \\ 0 & \frac{1}{\sqrt{1-|b_{ij}|}} \end{bmatrix}$$

$$\cdot \begin{bmatrix} \cos(\theta - \frac{\pi}{4}) & -e^{\imath\alpha} \sin(\theta - \frac{\pi}{4}) \\ e^{-\imath\alpha} \sin(\theta - \frac{\pi}{4}) & \cos(\theta - \frac{\pi}{4}) \end{bmatrix} \cdot \begin{bmatrix} e^{\imath\omega_i} & 0 \\ 0 & e^{\imath\omega_j} \end{bmatrix}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\hat{A} \to \text{diag} \qquad\qquad \text{diag}(\hat{Z}) \succ O$$

Essential Part of the Complex Algorithm

Let

$$b = |b_{ij}|, \quad t = \sqrt{1 - b^2}, \quad e = a_{jj} - a_{ii}, \qquad \epsilon = \begin{cases} 1, & e \geq 0 \\ -1, & e < 0 \end{cases},$$

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$$\begin{aligned}
u + \imath\, v &= e^{-\imath \arg(b_{ij})}\, a_{ij}, \qquad \tan\gamma = 2\tfrac{v}{e}, \quad -\tfrac{\pi}{2} < \gamma \leq \tfrac{\pi}{2} \\
\tan 2\theta &= \epsilon \tfrac{2u-(a_{ii}+a_{jj})b}{t\sqrt{e^2+4v^2}}, \quad -\tfrac{\pi}{4} < \theta \leq \tfrac{\pi}{4} \\
2\cos^2\phi &= 1 + b\sin 2\theta + t\cos 2\theta \cos\gamma, \qquad 0 \leq \phi \leq \tfrac{\pi}{2} \\
2\cos^2\psi &= 1 - b\sin 2\theta + t\cos 2\theta \cos\gamma, \qquad 0 \leq \psi \leq \tfrac{\pi}{2} \\
e^{\imath\alpha}\sin\phi &= \tfrac{e^{\imath \arg(b_{ij})}}{2\cos\psi}\left[\sin 2\theta - b - \imath t\cos 2\theta \sin\gamma\right] \\
e^{-\imath\beta}\sin\psi &= \tfrac{e^{-\imath \arg(b_{ij})}}{2\cos\phi}\left[\sin 2\theta + b + \imath t\cos 2\theta \sin\gamma\right].
\end{aligned}$$

Then

$$\hat{Z} = \frac{1}{\sqrt{1-b^2}} \begin{bmatrix} \cos\phi & e^{\imath\alpha}\sin\phi \\ -e^{-\imath\beta}\sin\psi & \cos\psi \end{bmatrix}$$

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- We derive the complex CJ algorithm
- The real CJ algorithm is obtained by simplifying the complex one

Derivation of Complex LL^*J Algorithm, $B \succ O$

Consider the Cholesky foctorization of $\hat{B}$: $\hat{B} = \hat{L}\hat{L}^*$,

$$\begin{bmatrix} 1 & b_{ij} \\ \bar{b}_{ij} & 1 \end{bmatrix} = \hat{B} = \hat{L}\hat{L}^* = \begin{bmatrix} 1 & 0 \\ \bar{a} & \bar{c} \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & a \\ \bar{a} & |a|^2 + |c|^2 \end{bmatrix}$$

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$$\hat{L} = \begin{bmatrix} 1 & 0 \\ \bar{b}_{ij} & \tau \end{bmatrix}, \qquad \hat{L}^{-1} = \frac{1}{\tau} \begin{bmatrix} \tau & 0 \\ -\bar{b}_{ij} & 1 \end{bmatrix}, \qquad \hat{L}^{-*} = \frac{1}{\tau} \begin{bmatrix} \tau & -b_{ij} \\ 0 & 1 \end{bmatrix}.$$

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Let $\hat{F}_1 = \hat{L}^{-*}$. Then $\hat{F}_1^* \hat{B} \hat{F}_1 = I_2$ and

$$\hat{F}_1^* \hat{A} \hat{F}_1 = \begin{bmatrix} a_{ii} & (a_{ij} - b_{ij} a_{ii})/\tau \\ (\bar{a}_{ij} - \bar{b}_{ij} a_{ii})/\tau & a_{jj} - \frac{a_{ij}\bar{b}_{ij} + \bar{a}_{ij} b_{ij} - (a_{ii} + a_{jj})|b_{ij}|^2}{1 - |b_{ij}|^2} \end{bmatrix}.$$

Derivation of Complex LL^*J Algorithm, $B \succ O$

The final $\hat{F}$ is obtained as product $\hat{F} = \hat{F}_1 \hat{R}_1$ where

$\hat{R}_1$ is the complex Jacobi rotation which diagonalizes $\hat{F}_1^* \hat{A} \hat{F}_1$.

Let us assume that the $(1,2)$-element of $\hat{R}_1$ is $-e^{\imath \epsilon_1} \sin \vartheta_1$. Then the angles ϑ_1 and ϵ_1 are determined by the formulas

$$\begin{aligned} \epsilon_1 &= \arg(a_{ij} - b_{ij} a_{ii}), \\ \tan(2\vartheta_1) &= \frac{2|a_{ij} - a_{ii} b_{ij}| \sqrt{1 - |b_{ij}|^2}}{a_{ii} - a_{jj} + a_{ij} \bar{b}_{ij} + \bar{a}_{ij} b_{ij} - 2a_{ii}|b_{ij}|^2}, \quad -\frac{\pi}{4} \leq \vartheta_1 \leq \frac{\pi}{4}. \end{aligned}$$

Derivation of Complex LL^*J Algorithm, $B \succ O$

The transformation formulas for the diagonal elements of A read

$$
\begin{aligned}
a'_{ii} &= a_{ii} + \tan\vartheta_1 \cdot \frac{|a_{ij} - a_{ii} b_{ij}|}{\sqrt{1 - b_{ij}^2}} \\
a'_{jj} &= a_{jj} - \frac{a_{ij}\bar{b}_{ij} + \bar{a}_{ij} b_{ij} - (a_{ii} + a_{jj})|b_{ij}|^2}{1 - |b_{ij}|^2} - \tan\vartheta_1 \cdot \frac{|a_{ij} - a_{ii} b_{ij}|}{\sqrt{1 - b_{ij}^2}}.
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In the case $a_{ii} = a_{jj}$, $a_{ij} = a_{ii} b_{ij}$, $\tan(2\vartheta_1)$ has the form $0/0$.

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Then we choose $\vartheta_1 = 0$, so that $a'_{ii} = a_{ii}$ and $a'_{jj} = a_{jj}$.

Let $c_{\vartheta_1} = \cos\vartheta_1$, $s^{\pm}_{\vartheta_1} = e^{\pm\imath\epsilon_1}\sin\vartheta_1$. Then

$$\hat{F} = \frac{1}{\sqrt{1-|b_{ij}|^2}} \begin{bmatrix} \sqrt{1-|b_{ij}|^2} & -b_{ij} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_{\vartheta_1} & -s^{+}_{\vartheta_1} \\ s^{-}_{\vartheta_1} & c_{\vartheta_1} \end{bmatrix}$$

$$= \frac{1}{\sqrt{1-|b_{ij}|^2}} \begin{bmatrix} c_{\tilde{\vartheta}_1} & -s_{\tilde{\vartheta}_1} \\ s^{-}_{\vartheta_1} & c_{\vartheta_1} \end{bmatrix} \qquad \begin{aligned} c_{\tilde{\vartheta}_1} &= c_{\vartheta_1}\sqrt{1-|b_{ij}|^2} - s^{-}_{\vartheta_1} b_{ij} \\ s_{\tilde{\vartheta}_1} &= c_{\vartheta_1} b_{ij} + s^{+}_{\vartheta_1}\sqrt{1-|b_{ij}|^2} \end{aligned}$$

$$= \begin{bmatrix} c1 & -s1 \\ s2 & c2 \end{bmatrix},$$

$$c1 = c_{\vartheta_1} - s^{-}_{\vartheta_1} b_{ij}/\sqrt{1-|b_{ij}|^2}, \qquad c2 = c_{\vartheta_1}/\sqrt{1-|b_{ij}|^2},$$

$$s1 = c_{\vartheta_1} b_{ij}/\sqrt{1-|b_{ij}|^2} + s^{+}_{\vartheta_1}, \qquad s2 = s^{-}_{\vartheta_1}/\sqrt{1-|b_{ij}|^2}.$$

Derivation of Complex RR^*J Algorithm, $B \succ O$

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Assuming positive c, one obtains $a = b_{ij}$, $c = \sqrt{1 - |b_{ij}|^2} = \tau$. Hence

$$\hat{R} = \begin{bmatrix} \tau & b_{ij} \\ 0 & 1 \end{bmatrix}, \qquad \hat{R}^{-1} = \frac{1}{\tau} \begin{bmatrix} 1 & -b_{ij} \\ 0 & \tau \end{bmatrix}, \qquad \hat{R}^{-*} = \frac{1}{\tau} \begin{bmatrix} 1 & 0 \\ -\bar{b}_{ij} & \tau \end{bmatrix}.$$

If we write $\hat{F}_2 = \hat{R}^{-*}$, then $\hat{F}_2^* \hat{B} \hat{F}_2 = \hat{R}^{-1} \hat{B} \hat{R}^{-*} = I_2$ and we have

The Algorithm RR^*J

$$\hat{F}_2^* \hat{A} \hat{F}_2 = \begin{bmatrix} a_{ii} - \frac{a_{ij}\bar{b}_{ij} + \bar{a}_{ij} b_{ij} - (a_{ii}+a_{jj})|b_{ij}|^2}{\tau^2} & (a_{ij} - a_{jj} b_{ij})/\tau \\ (\bar{a}_{ij} - a_{jj}\bar{b}_{ij})/\tau & a_{jj} \end{bmatrix}.$$

- The final transformation is $\hat{F} = \hat{F}_2 \hat{R}_2$,
- $\hat{R}_2$ is the Jacobi rotation which annihilates $(1,2)$-element of $\hat{F}_2^* \hat{A} \hat{F}_2$
- Let $(1,2)$-element of $\hat{R}_2$ be $-e^{\imath\epsilon_2}\sin\vartheta_2$

Then the parameters ϵ_2 and ϑ_2 are determined by the formulas

$$\begin{aligned} \epsilon_2 &= \arg(a_{ij} - b_{ij} a_{jj}), \\ \tan(2\vartheta_2) &= \frac{2|a_{ij} - a_{jj} b_{ij}|\sqrt{1 - |b_{ij}|^2}}{a_{ii} - a_{jj} - (a_{ij}\bar{b}_{ij} + \bar{a}_{ij} b_{ij}) + 2a_{jj}|b_{ij}|^2}, \quad -\frac{\pi}{4} \leq \vartheta_2 \leq \frac{\pi}{4}. \end{aligned}$$

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a'_{ii} &= a_{ii} - \frac{a_{ij}\bar{b}_{ij} + \bar{a}_{ij}b_{ij} - (a_{ii} + a_{jj})|b_{ij}|^2}{1 - |b_{ij}|^2} + \tan\vartheta_2 \cdot \frac{|a_{ij} - a_{jj}b_{ij}|}{\sqrt{1 - b_{ij}^2}}, \\
a'_{jj} &= a_{jj} - \tan\vartheta_2 \cdot \frac{|a_{ij} - a_{jj}b_{ij}|}{\sqrt{1 - b_{ij}^2}}.
\end{aligned}
$$

If $a_{ii} = a_{jj}$, $a_{ij} = a_{jj}b_{ij}$, ϑ_2 is not well defined and we choose $\vartheta_2 = 0$.

In that case a'_{ii} and a'_{jj} reduce to a_{ii} and a_{jj}, respectively.

The Algorithm RR^*J

Let $c_{\vartheta_2} = \cos\vartheta_2$, $s_{\vartheta_2}^{\pm} = e^{\pm\imath\epsilon_2}\sin\vartheta_2$. Then

$$
\begin{aligned}
\hat{F} &= \frac{1}{\sqrt{1-|b_{ij}|^2}} \begin{bmatrix} 1 & 0 \\ -\bar{b}_{ij} & \sqrt{1-|b_{ij}|^2} \end{bmatrix} \begin{bmatrix} c_{\vartheta_2} & -s_{\vartheta_2}^{+} \\ s_{\vartheta_2}^{-} & c_{\vartheta_2} \end{bmatrix} \\
&= \frac{1}{\sqrt{1-|b_{ij}|^2}} \begin{bmatrix} c_{\vartheta_2} & -s_{\vartheta_2}^{+} \\ s_{\tilde{\vartheta}_2} & c_{\tilde{\vartheta}_2} \end{bmatrix}, \qquad \begin{aligned} c_{\tilde{\vartheta}_2} &= c_{\vartheta_2}\sqrt{1-|b_{ij}|^2} + s_{\vartheta_2}^{+}\bar{b}_{ij} \\ s_{\tilde{\vartheta}_2} &= s_{\vartheta_2}^{-}\sqrt{1-b_{ij}^2} - c_{\vartheta_2}\bar{b}_{ij} \end{aligned} \\
&= \begin{bmatrix} c1 & -s1 \\ s2 & c2 \end{bmatrix}, \quad \text{It is easy to check that } c_{\tilde{\vartheta}}^2 + s_{\tilde{\vartheta}}^2 = 1.
\end{aligned}
$$

$$
c1 = c_{\vartheta_2}/\sqrt{1-b_{ij}^2}, \qquad c2 = c_{\vartheta_2} + s_{\vartheta_2}^{+}\bar{b}_{ij}/\sqrt{1-b_{ij}^2},
$$

$$
s1 = s_{\vartheta_2}^{+}/\sqrt{1-b_{ij}^2}^{\,+}, \qquad s2 = s_{\vartheta_2}^{-} - c_{\vartheta_2}\bar{b}_{ij}/\sqrt{1-b_{ij}^2}.
$$

We can postmultiply $\hat{F}$ by $\mathrm{diag}(1\,,\, \bar{c}_{\tilde{\vartheta}_2}/|c_{\tilde{\vartheta}_2}|)$ provided that $c_{\tilde{\vartheta}_2} \neq 0$. This ensures that (the updated) $\hat{F}$ has nonnegative diagonal elements.

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Our numerical tests show that neither LL^*J nor RR^*J is indicated as a high relative accurate algorithm on pairs of well-behaved positive definite matrices.

The same can be said for the hybrid algorithm that selects the LL^*J and RR^*J algorithms in the opposite way, i.e. selects the RR^*J (LL^*J) algorithm when $a_{ii} \leq a_{jj}$ ($a_{ii} > a_{jj}$).

The Cholesky-Jacobi Algorithm

The CJ is a hybrid algorithm which can be briefly defined as follows:

1. select the pivot pair (i,j)
2. if $a_{ii} \leq a_{jj}$ then employ the LL^*J algorithm

   else employ the RR^*J algorithm

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The same can be said for the hybrid algorithm that selects the LL^*J and RR^*J algorithms in the opposite way, i.e. selects the RR^*J (LL^*J) algorithm when $a_{ii} \leq a_{jj}$ ($a_{ii} > a_{jj}$).

Only the above definition warrants the high relative accuracy of the algorithm and it is in complete agreement with the behavior of the real CJ method.

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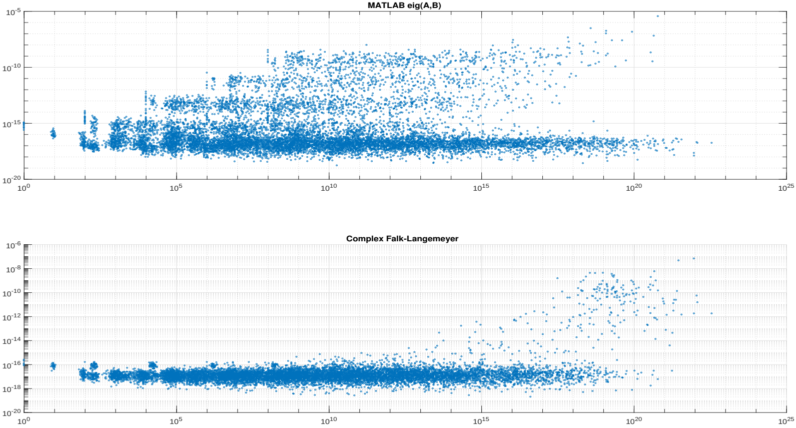
- It requires B to be positive definite (it solves PGEP)

Relative Accuracy, Assume: $A \succ O$, $B \succ O$

$$A_S = [\mathrm{diag}(A)]^{-1/2} A [\mathrm{diag}(A)]^{-1/2}, \quad B_S = [\mathrm{diag}(B)]^{-1/2} B [\mathrm{diag}(B)]^{-1/2}$$

$$\begin{aligned}
\varrho_{(A,B)} &= \max_{1 \le i \le n} \frac{|\tilde{\lambda}_i - \lambda_i|}{\lambda_i} \Big/ \sqrt{\kappa_2^2(A_S) + \kappa_2^2(B_S)} \\
\chi_{(A,B)} &= \sqrt{\kappa_2^2(A^{(0)}) + \kappa_2^2(B^{(0)})} \\
\mathcal{E} &= \{(\chi_{(A,B)}\ ,\ \varrho_{(A,B)}) : (A,B) \in \Upsilon\}.
\end{aligned}$$

Relative errors: *CFL* vs. MATLAB eig(A,B)



Some References and Thank You for Your Patience!

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