On the Complex Falk-Langemeyer Method

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89th GAMM Annual Meeting
March 19-23, 2018
Munich, Germany

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- This work has been fully supported by Croatian Science Foundation under the project IP-09-2014-3670.

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- For a definite pair (A, B) there is a nonsingular matrix F such that

$$F^*AF = \Lambda_A \,, \qquad F^*BF = \Lambda_B ,$$

$\Lambda_A = \text{diag}(\alpha_1, \ldots, \alpha_n), \quad \Lambda_B = \text{diag}(\beta_1, \ldots, \beta_n)$ are real matrices

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- The eigenpairs are: $(\alpha_i/\beta_i, Fe_i)$, $1 \leq i \leq n$; $I_n = [e_1, \ldots, e_n]$.

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- If neither A nor B is definite,one can try to maximize the minimum eigenvalue of B_φ by rotating the pair

$$(A, B) \mapsto (A_\varphi, B_\varphi) = (A\cos\varphi + B\sin\varphi, -A\sin\varphi + B\cos\varphi),$$

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We follow the last choice!

Jacobi Methods for PGEP

We have at disposal several diagonalization methods for PGEP with real matrices:

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All three methods have excellent numerical properties, in particular they are indicated as high relative accurate on well-behaved positive definite matrices.

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The same can be said for the CJ and FL method.

Few Facts about Real FL method

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- High relative accuracy (HRA) of the FL method not yet proved

  (numerical tests indicate HRA of the method)

Derivation of the CFL Method

Starting with a definite pair (A, B) of complex Hermitian matrices, CFL generates a sequence of "congruent" matrix pairs

$$(A, B) = (A^{(0)}, B^{(0)}),\ (A^{(1)}, B^{(1)}),\ (A^{(2)}, B^{(2)}) \dots$$

by the rule

$$A^{(k+1)} = F_k^* A^{(k)} F_k \ , \qquad B^{(k+1)} = F_k^* B^{(k)} F_k \ , \quad k \geq 0.$$

Here F_k is an elementary plane matrix defined by the *pivot pair* $(i(k), j(k))$

$$F_k = \begin{bmatrix} I & & & & \\ & 1 & & \alpha_k & \\ & & I & & \\ & \beta_k & & 1 & \\ & & & & I \end{bmatrix} \begin{matrix} \\ i(k) \\ \\ j(k) \\ \\ \end{matrix} , \qquad \alpha_k, \beta_k \in \mathbf{C},$$

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We simplify notation: $A \leftarrow A^{(k)}$, $A' \leftarrow A^{(k+1)}$, $F \leftarrow F_k$, $(i,j) \leftarrow (i(k),j(k))$.

Pivot submatrices $\hat{A}$, $\hat{B}$, $\hat{F}$ of A, B, F are 2×2 principal submatrices obtained on the intersection of pivot rows and columns i and j.

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We have

$$A' = F^* A F, \quad B' = F^* B F \qquad \left(\hat{A}' = \hat{F}^* \hat{A} \hat{F}, \quad \hat{B}' = \hat{F}^* \hat{B} \hat{F} \right)$$

and F is chosen to obtain $a'_{ij} = 0$, $b'_{ij} = 0$.

Derivation of the CFL Method ($n = 2$)

Further simplification: $(1,2) \leftarrow (i,j)$, $a_1 \leftarrow a_{ii}$, $a_2 \leftarrow a_{ij}$, $a_3 \leftarrow a_{jj}$, $a_1' \leftarrow a_{ii}'$, ...

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The goal is to compute α and β which satisfy the matrix equations

$$\begin{bmatrix} 1 & \bar{\beta} \\ \bar{\alpha} & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ \bar{a}_2 & a_3 \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ \beta & 1 \end{bmatrix} = \begin{bmatrix} a'_1 & 0 \\ 0 & a'_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \bar{\beta} \\ \bar{\alpha} & 1 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ \bar{b}_2 & b_3 \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ \beta & 1 \end{bmatrix} = \begin{bmatrix} b'_1 & 0 \\ 0 & b'_3 \end{bmatrix}.$$

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This leads us to solving a system of two nonlinear equations:

$$e_1 = a_1\alpha + a_3\bar{\beta} + \bar{a}_2\alpha\bar{\beta} + a_2 = 0, \quad (1)$$

$$e_2 = b_1\alpha + b_3\bar{\beta} + \bar{b}_2\alpha\bar{\beta} + b_2 = 0. \quad (2)$$

Derivation of the CFL Method

To solve the obtained system of equation, we use the following quantities:

$$
\begin{aligned}
\Im_1 &= a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\
\Im_3 &= a_3 b_2 - a_2 b_3 = \begin{vmatrix} a_3 & b_3 \\ a_2 & b_2 \end{vmatrix} \\
\Im_2 &= \Im_2' + i\Im_2'', \qquad\qquad \Im_2',\ \Im_2' \ \text{real} \\
\Im_2' &= a_1 b_3 - a_3 b_1 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \\
i\Im_2'' &= a_2 \bar{b}_2 - \bar{a}_2 b_2 = \begin{vmatrix} a_2 & b_2 \\ \bar{a}_2 & \bar{b}_2 \end{vmatrix} = i\left(-2\begin{vmatrix} \mathrm{Re}(a_2) & \mathrm{Re}(b_2) \\ \mathrm{Im}(a_2) & \mathrm{Im}(b_2) \end{vmatrix}\right) \\
\Im &= \Im_2^2 + 4\bar{\Im}_1\Im_3.
\end{aligned}
$$

The First Result

Recall, $\Im = \Im_2^2 + 4\bar{\Im}_1\Im_3$.

Lemma

Suppose the pair $(\hat{A}, \hat{B})$ *is definite. Then*

(i) $\Im \geq 0$

(ii) The following statements are equivalent

(a) $\Im = 0$

(b) $\Im_1 = \Im_2 = \Im_3 = 0$

(c) $\sigma\hat{A} + \omega\hat{B} = 0$ *for some real* σ, ω, $|\sigma| + |\omega| > 0$.

The Second Result

Lemma

Let $(\hat{A}, \hat{B})$ *be definite and* $\Im > 0$. *Then*

$$\begin{aligned} &(i) && \alpha = 0 \quad \textit{iff} \quad \Im_3 = 0 \\ &(ii) && \beta = 0 \quad \textit{iff} \quad \Im_1 = 0 \\ &(iii) && \alpha = \beta = 0 \quad \textit{iff} \quad \Im_1 = \Im_3 = 0. \end{aligned}$$

The Third Result

Lemma

Suppose $(\hat{A}, \hat{B})$ *is definite and* $\Im > 0$. *Then the solution* (α, β) *of the system* $e_1 - e_2$ *is given by*

$$\alpha = \frac{\Im_3}{\nu}, \quad \beta = -\frac{\bar{\Im}_1}{\nu}, \tag{3}$$

where ν *is any nonzero solution of the equation*

$$\nu^2 - \Im_2 \nu - \bar{\Im}_1 \Im_3 = 0. \tag{4}$$

The General Solution

Theorem

Let the pair $(\hat{A}, \hat{B})$ be definite.

(i) *If* $\Im > 0$ *then* $\alpha = \frac{\Im_3}{\nu}, \quad \beta = -\frac{\bar{\Im}_1}{\nu},$
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(ii) *If* $\Im = 0$ *then the equations in the system* e_1–e_2 *are proportional and there is infinite number of solutions.*

(a) *Let* $\hat{A} \neq 0$. *If* $|a_1| + |a_2| > 0$ *then*

$$\alpha = -\frac{\bar{\gamma} a_3 + a_2}{a_1 + \bar{\gamma} \bar{a}_2}, \quad \beta = \gamma, \quad \gamma \in \{z \in \mathbf{C}; a_1 + \bar{z} a_2 \neq 0\}.$$

If $|a_2| + |a_3| > 0$ *then*

$$\alpha = \gamma, \quad \beta = -\frac{\bar{\gamma} a_1 + \bar{a}_2}{\bar{\gamma} a_2 + a_3}, \quad \gamma \in \{c \in \mathbf{C}; a_3 + \bar{z} a_2 \neq 0\}.$$

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If $|a_2| + |a_3| > 0$ *then*

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(b) *Let* $\hat{B} \neq 0$. *Then the solutions are as in the case (a) provided that* a_1, a_2, a_3 *are replaced by* b_1, b_2, b_3, *resp.*

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The second criterion ensures the smallest flop count per step of the method.

The third criterion ensures that (α, β) is determined by a more reliable set of input data.

The Case $\Im > 0$: The Standard Solution

The theorem gives the solution:

$$\alpha = \frac{\Im_3}{\nu}, \qquad \beta = -\frac{\bar{\Im}_1}{\nu}$$

where ν is any nonzero solution of the equation

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Respecting the first criterion we choose larger (by absolute value) ν:

$$\nu = \frac{\Im_2' + \imath \Im_2'' + \mathrm{sgn}(\Im_2')\sqrt{\Im}}{2}.$$

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This is referred to as the standard solution.

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The theorem and the three criteria imply the following solution:

if $|a_1| + |b_1| \geq |a_3| + |b_3|$ **then** $\beta = 0, \quad \alpha = -\dfrac{a_2}{a_1} \quad \left(= -\dfrac{b_2}{b_1}\right),$

else $\alpha = 0, \quad \beta = -\dfrac{\bar{a}_2}{a_3} \quad \left(= -\dfrac{\bar{b}_2}{b_3}\right)$

end

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$$\textbf{else} \quad \alpha = 0, \qquad \beta = -\frac{\bar{a}_2}{a_3} \quad \left(= -\frac{\bar{b}_2}{b_3} \right)$$

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The probability for $\Im = 0$ is zero. We have to consider the case $\Im \approx 0$.

The Case $\Im \approx 0$

Let $\Im_1 = \Im_1' + \imath \Im_1''$, $\Im_3 = \Im_3' + \imath \Im_3''$, $a_2 = a_2' + \imath a_2''$, $b_2 = b_2' + \imath b_2''$.

$$\begin{aligned}
|\Im| &= |(\Im_2' - \Im_2'')(\Im_2' + \Im_2'') + 4\mathrm{Re}(\bar{\Im}_1 \Im_3)| \\
&\leq \max\{(\Im_2')^2\,,\,(\Im_2'')^2\} + 4|\Im_1'\Im_3' + \Im_1''\Im_3''| \\
&\leq \max\{(|a_1 b_3| + |b_1 a_3|)^2, 4(|a_2' b_2''| + |a_2'' b_2'|)^2\} + \\
&\quad 4[|a_1 a_3||b_2|^2 + |b_1 b_3||a_2|^2 + (|a_1 b_3| + |b_1 a_3|)(|a_2' b_2'| + |a_2'' b_2''|)|] \\
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- If $\mathrm{fl}(\Im) < -\varrho\epsilon$ we consider (A, B) not definite and abort comput.
- If $\varrho\epsilon^2 \leq \mathrm{fl}(\Im)$, we employ the standard solution for α, β.

The Case $\Im \approx 0, \quad \mathrm{fl}(\Im) \in (-\varrho\epsilon^2, \varrho\epsilon^2)$

If $\mathrm{fl}(\Im) \in (0, \varrho\epsilon^2)$, then severe cancelations take place and the computed ν, α and β will have large relative errors.

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We can assume $\alpha\beta = 0$. Let $\beta = 0$. Then the equations

$$\begin{aligned} e_1 &= a_1\alpha + a_3\bar{\beta} + \bar{a}_2\alpha\bar{\beta} + a_2 = 0 \\ e_2 &= b_1\alpha + b_3\bar{\beta} + \bar{b}_2\alpha\bar{\beta} + b_2 = 0 \end{aligned}$$

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$$\begin{aligned} e_1 &= a_1\alpha + a_2 = 0 \\ e_2 &= b_1\alpha + b_2 = 0 \end{aligned}$$

and we can look for the least square (LS) solution.

The Case $\Im \approx 0, \quad \beta = 0$

Let $\tilde{a}_1 = \sqrt{a_1^2 + b_1^2}$, $c_1 = a_1/\tilde{a}_1$, $s_1 = b_1/\tilde{a}_1$. We obtain

$$\begin{aligned}
\| \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \alpha + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \|_2^2 &= \| \begin{bmatrix} \tilde{a}_1 \\ 0 \end{bmatrix} \alpha + \begin{bmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \|_2^2 \\
&= \left| \tilde{a}_1 \alpha + \frac{a_1 a_2 + b_1 b_2}{\tilde{a}_1} \right|^2 + \frac{|\Im_1|^2}{a_1^2 + b_1^2},
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where $\| \cdot \|_2$ stands for the Euclidean vector norm. The solution is

$$\alpha = -\frac{a_1 a_2 + b_1 b_2}{a_1^2 + b_1^2} \qquad \text{with the residual error} \qquad \frac{|\Im_1|}{\sqrt{a_1^2 + b_1^2}}.$$

The Case $\Im \approx 0$, the LS solution

The case $\alpha = 0$ is treated in the similar way. We obtain

$$\beta = -\frac{a_3 \bar{a}_2 + b_3 \bar{b}_2}{a_3^2 + b_3^2} \qquad \text{with the residual error} \qquad \frac{|\Im_3|}{\sqrt{a_3^2 + b_3^2}},$$

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This leads us to the following algorithm:

$$\textbf{if} \quad \frac{|\Im_1|}{\sqrt{a_1^2 + b_1^2}} \leq \frac{|\Im_3|}{\sqrt{a_3^2 + b_3^2}} \quad \textbf{then} \quad \alpha = -\frac{a_1 a_2 + b_1 b_2}{a_1^2 + b_1^2}, \quad \beta = 0$$

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endif

Since $(\hat{A}, \hat{B})$ is definite, we should have $a_1^2 + b_1^2 > 0$ and $a_3^2 + b_3^2 > 0$.

Toward the Complex Falk-Langemeyer Algorithm

Moving from 2×2 to $n \times n$ GEP. We are dealing with an iterative process.

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$(1, 2) \longrightarrow (i, j) = (i(k), j(k))$ pivot pair in step k

$\left(\hat{A}, \hat{B}\right) \longrightarrow \left(\hat{A}_{ij}^{(k)}, \hat{B}_{ij}^{(k)}\right)$

$a_1,\ a_2,\ a_3 \longrightarrow a_{ii}^{(k)},\ a_{ij}^{(k)},\ a_{jj}^{(k)}, \qquad b_1,\ b_2,\ b_3 \longrightarrow b_{ii}^{(k)},\ b_{ij}^{(k)},\ b_{jj}^{(k)}$

$\Im_1,\ \Im_3 \longrightarrow \Im_i^{(k)},\ \Im_j^{(k)},$

$\Im_2 = \Im_2' + \imath \Im_2'' \longrightarrow \Im_{ij}^{(k)} = \mathrm{Re}(\Im_{ij}^{(k)}) + \imath \mathrm{Im}(\Im_{ij}^{(k)})$

Pivot strategy:

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Pivot strategy: assume the serial one, say, the row-cyclic one

The Complex Falk-Langemeyer Method

Input data: $A = A^*$, $B = B^*$ of order n and the logical variable *eivec*

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1. Set $k = 0$, $A^{(k)} = A$, $B^{(k)} = B$. If *eivec* then set $F^{(k)} = I_n$

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1. Set $k = 0$, $A^{(k)} = A$, $B^{(k)} = B$. If *eivec* then set $F^{(k)} = I_n$
2. Repeat
   - (a) Choose the pivot pair $(i, j) = (i(k), j(k))$
   - (b) Compute the parameters (α_k, β_k) of F_k
   - (c) Compute $A^{(k+1)} = F_k^* A^{(k)} F_k$, $B^{(k+1)} = F_k^* B^{(k)} F_k$

     if *eivec* then compute $F^{(k+1)} = F^{(k)} F_k$.

   Until convergence

One Step of the CFL Method: **2(b)**-part

The superscipt (k) is omitted, $\quad \mathbf{u}$ is the unit round-off

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(i) Renormalize $\hat{A}$, $\hat{B}$ and compute:

$$\Im'_{ij} = a_{ii} b_{jj} - a_{jj}\, b_{ii}; \quad \Im''_{ij} = -2\,(a'_{ij}\, b''_{ij} - b'_{ij}\, a''_{ij}); \quad \Im_{ij} = \Im'_{ij} + \imath\, \Im''_{ij};$$

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$$4\left[\, |a_{ii} a_{jj}| |b_{ij}|^2 + |b_{ii} b_{jj}| |a_{ij}|^2 + (|a_{ii} b_{jj}| + |b_{ii} a_{jj}|)(|a'_{ij} b'_{ij}| + |a''_{ij} b''_{ij}|)\,\right];$$

One Step of the CFL Method: **(b)**-part

(ii) Set $job = 0$;

If $\Im > \varrho \mathbf{u}^2$ then $\nu = \frac{1}{2}(\Im_{ij} + \mathrm{sgn}(\Im'_{ij})\sqrt{\Im})$, $\alpha = \frac{\Im_j}{\nu}$, $\beta = -\frac{\bar{\Im}_i}{\nu}$

elseif $\Im < -\varrho \mathbf{u}$ then $job = -1$

else if $|\Im_i| \sqrt{a_{jj}^2 + b_{jj}^2} \leq |\Im_j| \sqrt{a_{ii}^2 + b_{ii}^2}$

then $\alpha = -\frac{a_{ii}\, a_{ij} + b_{ii}\, b_{ij}}{a_{ii}^2 + b_{ii}^2}$, $\beta = 0$

else $\alpha = 0$, $\beta = -\frac{a_{jj}\, \bar{a}_{ij} + b_{jj}\, \bar{b}_{ij}}{a_{jj}^2 + b_{jj}^2}$

endif

endif

Properties of the CFL Method

Theorem

Let (A, B) be a definite pair of Hermitian matrices and let $(A^{(k)}, B^{(k)}), k \geq 0$ be the sequence of pairs generated by applying the CFL algorithm to (A, B). Then for each k the following assertions hold:

(i) F_k *is nonsingular*

(ii) $|\alpha_k \beta_k| \leq 1$

(iii) $|\alpha_k \beta_k| = 1$ *iff* $Re(\Im_{ij}^{(k)}) = 0$ *and* $|a_{ij}^{(k)}| + |b_{ij}^{(k)}| > 0.$

We also have $\alpha_k \beta_k = -1$ *iff* $\Im_{ij}^{(k)} = 0.$

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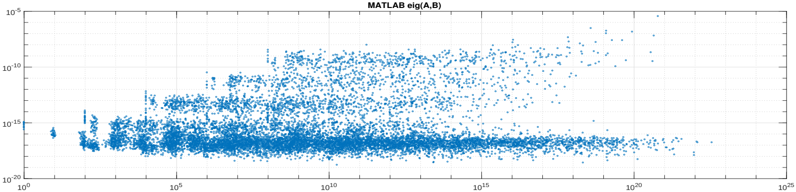
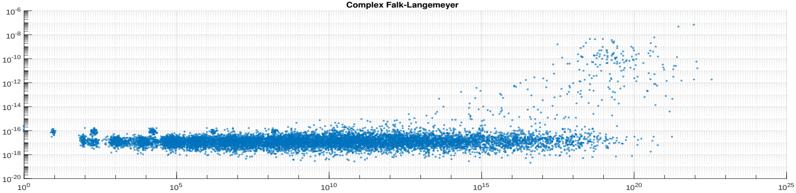
We also have $\alpha_k \beta_k = -1 \quad \textit{iff} \quad \Im_{ij}^{(k)} = 0.$

Next we consider high relative accuracy (HRA) of the method!

Relative errors: *CFL* vs. MATLAB eig(A,B)





Theorem

Let $A = A^T \succ O$, $B = B^T \succ O$ *and* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\lambda_i \in \sigma(A, B)$.
Let $A_S = D_A^{-1/2} A D_A^{-1/2}$, $B_S = D_B^{-1/2} B D_B^{-1/2}$, $D_A = diag(A)$, $D_B = diag(B)$

Let δA, δB *be symmetric perturbations and* $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_n$ *the eigenvalues of* $(A + \delta A, B + \delta B)$.

Let

$$\varepsilon_{A_S} = \|(\delta A)_S\|_2 / \|A_S\|_2, \quad \varepsilon_{B_S} = \|(\delta B)_S\|_2 / \|B_S\|_2$$

where

$$(\delta A)_S = D_A^{-1/2} \delta A D_A^{-1/2}, \quad (\delta B)_S = D_B^{-1/2} \delta B D_B^{-1/2} \ .$$

If

$$\varepsilon_{A_S} \kappa_2(A_S) < 1 \quad \textit{and} \quad \varepsilon_{B_S} \kappa_2(B_S) < 1,$$

then

$$\max_{1 \leq i \leq n} \frac{|\tilde{\lambda}_i - \lambda_i|}{\lambda_i} \leq \frac{\varepsilon_{A_S} \kappa_2(A_S) + \varepsilon_{B_S} \kappa_2(B_S)}{1 - \varepsilon_{B_S} \kappa_2(B_S)}.$$

Theoretical Background

- From the theorem we see that one class of "well-behaved matrix pairs" is made ofpairs of Hermitian positive definite matrices that can be well-scaled, i.e. for which $\kappa_2(A_S)$ and $\kappa_2(B_S)$ are small.
- For a well-behaved pair, the perturbations also have to be special, i.e. the numbers ε_{A_S} and ε_{B_S} have to be small. Then we shall have tiny relative errors.
- For those well-behaved pairs we have to find out what methods generate at every step only tiny relative errors $\varepsilon_{A_S^{(k)}}$, $\varepsilon_{B_S^{(k)}}$ and in the same time matrices with small or modest $\kappa_2(A_S^{(k)})$ and $\kappa_2(B^{(k)})$.

Nonetheless, this is a demanding task, so we shall go for a shortcut.

How to detect high relative accuracy of a method?

Recall the assertion of the theorem

$$\max_{1\le i\le n} \frac{|\tilde{\lambda}_i - \lambda_i|}{\lambda_i} \le \frac{\varepsilon_{A_S}\kappa_2(A_S) + \varepsilon_{B_S}\kappa_2(B_S)}{1 - \varepsilon_{B_S}\kappa_2(B_S)}, \qquad \text{it implies}$$

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holds for a larger sample Υ of well-behaved pairs (A, B)! Here

- $\tilde{\lambda}_i$ are the computed eigenvalues of (A, B)
- $f(n)$ is a slowly growing function of n and $\mathbf{u}$ is the round off unit
- Rel. (5) should not depend on $\kappa_2(A^{(0)})$ and $\kappa_2(B^{(0)})$.

Therefore, we are interested in how $\varrho_{(A,B)}$ behaves with respect to $\chi_{(A,B)}$,

$$\chi_{(A,B)} \equiv \kappa_2(A^{(0)}, B^{(0)}) = \sqrt{\kappa_2^2(A^{(0)}) + \kappa_2^2(B^{(0)})}.$$

- For the given sample of well behaved pairs Υ, and for each method, we shall make its graph of relative errors: $\mathcal{E}$,

$$\mathcal{E} = \{(\chi_{(A,B)}\ ,\ \varrho_{(A,B)})\ :\ (A,B) \in \Upsilon\}.$$

- Then we shall depict that graph $\mathcal{E}$ using the M-function
  `scatter(x,y,3)`

- The method will be indicated high relative accurate if the ordinates of the points on the graph are of order $\mathcal{O}(\mathbf{u})$ where $\mathbf{u} \approx 2.2 \cdot 10^{-16}$.

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where D_A and D_B are the diagonal parts of A and B. Then $\kappa_2(A_S^{(0)})$ and $\kappa_2(B^{(0)})$ can be controlled by the diagonal elements of Δ_A, Δ_B, Σ, since

$$\kappa_2(A_S^{(0)}) \leq n\kappa_2^2(\Sigma)\kappa_2(\Delta_A) \quad \text{and} \quad \kappa_2(B^{(0)}) \leq n\kappa_2^2(\Sigma)\kappa_2(\Delta_B),$$

although most often $\kappa_2(A_S^{(0)})$ and $\kappa_2(B^{(0)})$ are much smaller than these bounds.

How to generate matrix pairs?

To simplify the construction we set $\Delta_B = I_n$.

If the method is high relative accurate, then $\varrho_{(A,B)}$ from the relation (5) should not depend on $\kappa_2(\Delta)$.

Note that

$$\kappa_2(A^{(0)}) \leq \kappa_2(A_S^{(0)})\kappa_2^2(\Delta).$$

If we set $\Delta = I_n$ i $(A^{(0)}, B^{(0)}) = (D_B^{-1/2} A D_B^{-1/2}, B_S)$, then we know in advance the eigenvalues of $(A^{(0)}, B^{(0)})$ These are the quotients

$$(\Delta_A)_{jj}/(\Delta_B)_{jj}, \qquad 1 \leq j \leq n.$$

This way can be used when considering behavior of the methods on pairs with multiple eigenvalues.

More Details

- Diagonal matrices are constructed by help of the M-function `diag(d)`
- `d` is a vector, and vectors are constructed by the M-function `logspace(x1,x2,n)`. We use it for the diagonal matrices Σ and Δ_A.
- For the construction of Δ we use our m-function

  `scalvec(k1,k2,k3,n,k)`

  which generates vector of length n, $d = [10^{\mathrm{k1}}, \ldots, 10^{\mathrm{k2}}, \ldots, 10^{\mathrm{k3}}]$ where `k` determines the position of 10^{k2} within the components of d.
- To compute Δ, the function `scalvec` is used within triple loop controlled by the indices `k1`, `k2` and `k3`
- Orthogonal matrices U and V are computed by the command

  `[Q,~]=qr(rand(n))`
- We have generated the sample Υ of 18900 pairs of matrices of order 10. As "exact eigenvalues" we have used the eigenvalues computed by the M-function `eig(A,B)` in variable precision arithmetic (VPA) using 80 decimal digits.

Relative Accuracy

$$\varrho_{(A,B)} = \max_{1\le i\le n} \frac{|\tilde{\lambda}_i - \lambda_i|}{\lambda_i} / \sqrt{\kappa_2^2(A_S) + \kappa_2^2(B_S)} \le \frac{\sqrt{\varepsilon_{A_S}^2 + \varepsilon_{B_S}^2}}{1 - \varepsilon_{B_S}\kappa_2(B_S)}.$$

$$\chi_{(A,B)} = \sqrt{\kappa_2^2(A^{(0)}) + \kappa_2^2(B^{(0)})}\ .$$

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Relative errors: *CFL* vs. MATLAB eig(A,B)