Cosine-Sine Decompositions

Some Open Problems and Some Applications

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This work has been fully supported by Croatian Science Foundation
under the project IP–2014–09–3670.

PART 1

How to efficiently diagonalize a symmetric matrix of small dimension ?

(draw attention to the topic)

PART 2

CS decomposition

(why and where such diagonalizations are important ?)

PART 1

How to efficiently diagonalize a symmetric matrix of small dimension ?

Spectral theorem (spectral decomposition).
Let A be a real symmetric matrix of order n.
Then, there exists an orthogonal matrix Q such that

$$Q^T AQ = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

For the purpose of improvement and acceleration of diagonalization methods, it is important to have an efficient algorithm for diagonalizing the matrices of small dimensions ($n = 2,3,4$).

$n = 2$

$$Q = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \quad \text{or} \quad Q = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$$

rotation reflection

$$Q = S_1 \begin{pmatrix} |\cos\phi| & -|\sin\phi| \\ |\sin\phi| & |\cos\phi| \end{pmatrix} S_2$$

where S_1, S_2 are diagonal matrices of signs.

$n = 2$

If

$$A = \begin{pmatrix} a & f \\ f & b \end{pmatrix}$$

then

$$Q^T A Q = \Lambda \quad \Rightarrow \quad \tan 2\phi = \frac{2f}{a-b}, \quad \phi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right],$$

where Q is Jacobi rotation and ϕ is rotation angle.

$n = 3$

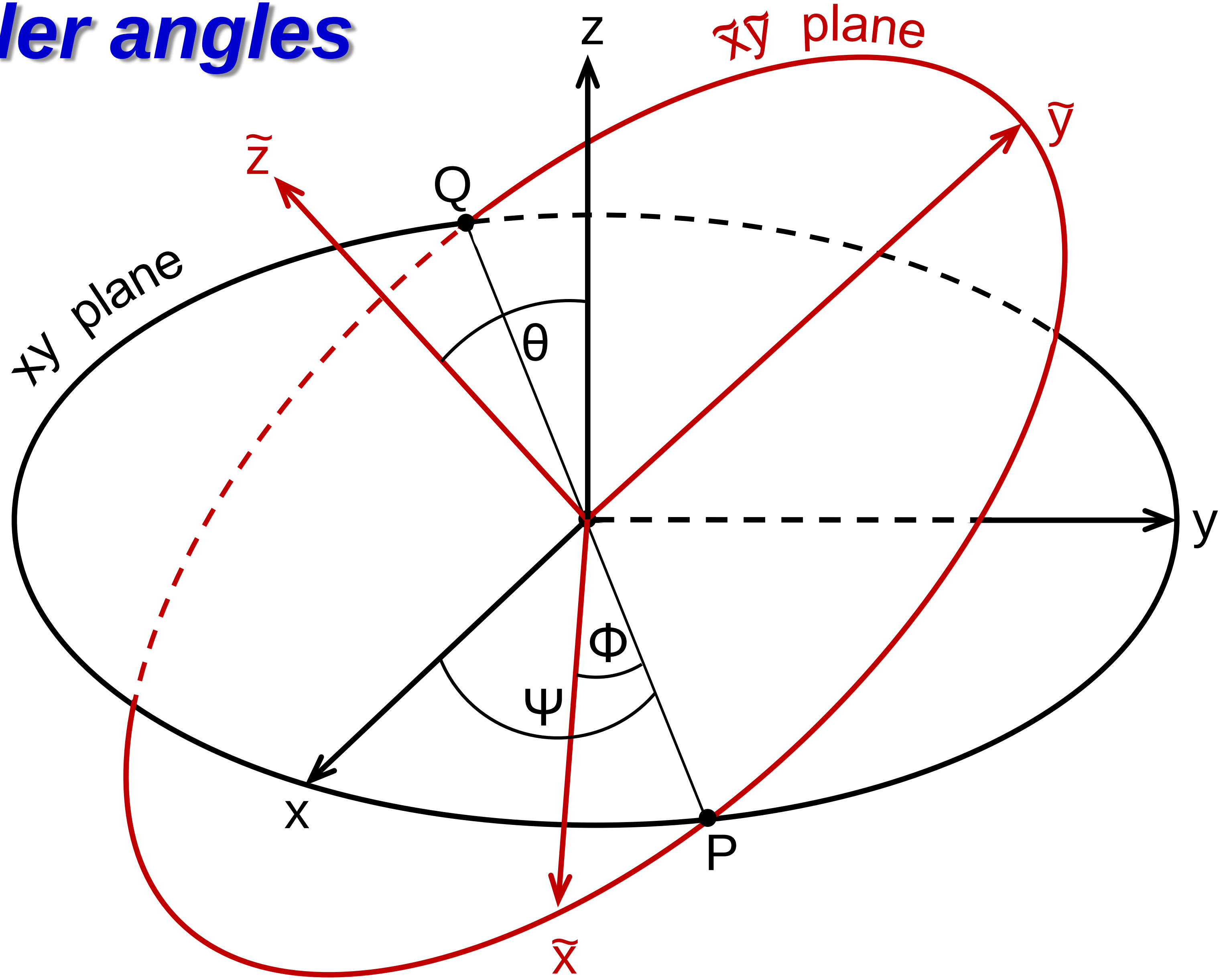
$$Q^T A Q = \Lambda$$

$$Q = S_1 R_{12}(\phi) R_{23}(\theta) R_{12}(\psi) S_2 \ ,$$

where R_{ij} is rotation in the (i, j) plane.

Here ϕ, θ and ψ are the rotation, nutation and precession angles (Euler angles).

Euler angles



$n = 3$

$R_{12}(\phi)$ $R_{12}(\psi)$

Let

$$A_k = \begin{pmatrix} a_k & f_k & g_k \\ f_k & b_k & h_k \\ g_k & h_k & d_k \end{pmatrix}$$

$R_{23}(\theta)$

be the matrix after k rotations (k = 0,1,2,3) on A have been applied. Thus, $A_0 = A$, $A_3 = \Lambda$.

The rotation $R_{12}(\phi)$ prepares the ground to enable $R_{23}(\theta)$ to annihilate g_1 and h_1 simultaneously and then $R_{12}(\psi)$ annihilates f_2.

$n = 3$

$R_{23}(\theta)$ is

Jacobi rotation and Givens rotation

$$\tan 2\theta = \frac{2h_1}{b_1 - d_1} \qquad \tan\theta = \frac{g_1}{f_1}$$

$$h_1\left(f_1^2 - g_1^2\right) = f_1 g_1 \left(b_1 - d_1\right),$$

or equivalently $\frac{f_1}{g_1} - \frac{g_1}{f_1} = \frac{b_1 - d_1}{h_1}$ for $f_1 g_1 h_1 \neq 0$.

$$\Downarrow$$

cubic equation for $\cot\phi$.

A.W. Bojanczyk, A. Lutoborski, Computation of the Euler angles of a symmetric 3x3 matrix, SIAM J. Matrix Anal. Appl. 12, N.1, 41-48 (1991).

$n = 3$

The total cost of computing the Euler angles:

EXPLICITLY (by using trigonometric form of Cardano formulas):

≈ 90 flops + 2 roots + 6 trigonometric evaluations.

APPROXIMATELY

Newton method: ≈ 6 iterations for cubic equation + 30 operations for nutation and precession angle,

or alternatively one can use eigenvalue methods:

QR method (standard shifts, starting with tridiagonal form): ≈ 3 iterations,

Jacobi method (row-cyclic strategy): ≈ 3 sweeps,

with the error $\dfrac{\left\| A - \tilde{Q}\tilde{\Lambda}\tilde{Q}^T \right\|_F}{\left\| A \right\|_F}$ of the same order of magnitude.

$n = 4$

Six rotations,

1\. 5. 3.

$$A = \begin{pmatrix} a & f & g & h \\ f & b & p & q \\ g & p & d & r \\ h & q & r & e \end{pmatrix}$$

4\.
6\.
2\.

1\. and 2. rotation prepare the ground

for 3. and 4. ones which annihilate the entire block.

$\Downarrow$

system of two cubic equations for 1. and 2. rotation angle

How to obtain an efficient eigensolver for *n* = 3 and *n* = 4 ?

- using one or more additional rotations as a preprocessing step (to annihilate or to make some suitable relations among certain matrix elements),
- using the theory of arrowhead matrices,

$$\begin{pmatrix} * & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix}, \quad \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ * & * & * & * \end{pmatrix}, \quad \ldots$$

Any idea or suggestion is welcome.
Invitation for collaboration is open !!

Why these diagonalizations are important ?

PART 2. **CS decomposition**

to be continued …