Block Jacobi annihilators and operators

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OUTLINE

- Block Jacobi method
- Block Jacobi annihilators and operators
- Pivot orderings
- Application of block Jacobi annihilators and operators

E. Begović Kovač, V. Hari: Convergence of the Cyclic and Quasi-cyclic Block Jacobi Methods. arXiv:1604.05825 [math.NA]

JACOBI METHOD

- **Jacobi method** for solving the eigenproblem of symmetric matrices is an iterative process of the form

$$\mathbf{A}^{(k+1)} = \mathbf{U}_k^T \mathbf{A}^{(k)} \mathbf{U}_k, \quad k \geq 0,$$

  where $\mathbf{U}_k$ are orthogonal matrices. (Jacobi, 1846)

- Goal of the k-th step:
  make $\mathbf{A}^{(k+1)}$ "more diagonal" than $\mathbf{A}^{(k)}$.

- **Off-norm** of X: $S^2(X) = \dfrac{\|X - \mathrm{diag}(X)\|_F^2}{2}$

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- **Off-norm** of X: $S^2(X) = \dfrac{\|X - \mathrm{diag}(X)\|_F^2}{2}$

- **Convergence:** $\mathbf{A}^{(k)} \to \mathbf{\Lambda}$, $\mathbf{\Lambda}$ diagonal matrix

- $S(\mathbf{A}') \leq \gamma S(\mathbf{A}), \quad 0 < \gamma \leq 1,$
  where $\mathbf{A}'$ is the matrix obtained from $\mathbf{A}$ after one full cycle and constant γ does not depend on $\mathbf{A}$.

BLOCK JACOBI METHOD

- $\pi = (n_1, n_2, \ldots, n_m)$ is a partition of n,

$$n_1 + n_2 + \cdots + n_m = n, \quad n_i \geq 1,\ 1 \leq i \leq m.$$

- $\mathbf{A}^{(0)} = \mathbf{A}$ is a symmetric square **block matrix** of order n,

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1m} \\ A_{21} & A_{22} & & A_{2m} \\ \vdots & & \ddots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mm} \end{bmatrix} \begin{matrix} n_1 \\ n_2 \\ \vdots \\ n_m \end{matrix}.$$

BLOCK JACOBI METHOD

- $\mathbf{U}_k$, $k \geq 0$ are orthogonal **elementary block matrices**,

$$\mathbf{U}_k = \mathbf{U}_{i(k),j(k)} = \begin{bmatrix} I & & & & \\ & U_{ii} & & U_{ij} & \\ & & I & & \\ & U_{ji} & & U_{jj} & \\ & & & & I \end{bmatrix} \begin{matrix} \\ n_i \\ \\ n_j \\ \\ \end{matrix} .$$

- Matrices

$$\hat{\mathbf{U}}_{ij} = \begin{bmatrix} U_{ii} & U_{ij} \\ U_{ji} & U_{jj} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{A}}_{ij} = \begin{bmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{bmatrix}$$

are **pivot submatrices** and $(i,j) = (i(k), j(k))$ is **pivot pair**.

- $\mathbf{U}_{ij} = \mathcal{E}(i, j, \hat{\mathbf{U}}_{ij})$

- k-th step:

$$\begin{bmatrix} \Lambda_{ii}^{(k+1)} & 0 \\ 0 & \Lambda_{jj}^{(k+1)} \end{bmatrix} = \begin{bmatrix} U_{ii}^{(k)} & U_{ij}^{(k)} \\ U_{ji}^{(k)} & U_{jj}^{(k)} \end{bmatrix}^T \begin{bmatrix} A_{ii}^{(k)} & A_{ij}^{(k)} \\ (A_{ij}^{(k)})^T & A_{jj}^{(k)} \end{bmatrix} \begin{bmatrix} U_{ii}^{(k)} & U_{ij}^{(k)} \\ U_{ji}^{(k)} & U_{jj}^{(k)} \end{bmatrix}$$

PREPROCESSING

$$A = \left[\begin{array}{ccc|cc|c|cc}
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x \\
\hline
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x \\
\hline
x & x & x & x & x & x & x & x \\
\hline
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x
\end{array}\right] \quad \mapsto \quad A^{(0)} = \left[\begin{array}{ccc|cc|c|cc}
x & 0 & 0 & x & x & x & x & x \\
0 & x & 0 & x & x & x & x & x \\
0 & 0 & x & x & x & x & x & x \\
\hline
x & x & x & x & 0 & x & x & x \\
x & x & x & 0 & x & x & x & x \\
\hline
x & x & x & x & x & x & x & x \\
\hline
x & x & x & x & x & x & x & 0 \\
x & x & x & x & x & x & 0 & x
\end{array}\right]$$

UBC MATRICES

- **Uniformly bounded cosine (UBC) matrices** $\mathbf{U}_k = (U_{rs}^{(k)})$ (Drmač, 2009)

$$\min_{1\le r\le m} \sigma_{min}(U_{rr}^{(k)}) \ge c > 0, \quad k \ge k_0 \ge 0.$$

- This is generalizetion of the Forsythe-Henrici condition:

  The cosine of the rotation angles has to be uniformly bounded by a positive constant.

- Elementary UBC matrices are denoted by UBCE.

PIVOT STRATEGIES

- **Pivot strategy** can be identified with a function

$$I : \mathbb{N}_0 \to \mathbf{P}_m,$$

where $\mathbb{N}_0 = \{0, 1, \ldots\}$ and $\mathbf{P}_m = \{(i, j) \mid 1 \leq i < j \leq m\}$.

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- **Periodic** pivot strategy with period T

- **Cyclic**: if $T = M \equiv \frac{m(m-1)}{2}$ and $\{I(k) \big| 0 \leq k \leq M - 1\} = \mathbf{P}_m$

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- For a given cyclic strategy I, **pivot ordering** is the sequence

$$\mathcal{O}_I = (i(0), j(0)), \ldots, (i(M-1), j(M-1)) \in \mathcal{O}(\mathbf{P}_m).$$

Jacobi annihilators and operators

FUNCTION vec

- Column vector of X

$$\mathrm{col}(X) = [x_{11}, x_{21}, \ldots, x_{p1}, \ldots, x_{1q}, \ldots, x_{pq}]^T$$

- Let A be a symmetric block matrix with partition $\pi = (n_1, \ldots, n_m)$. Then

$$\mathrm{vec}_\pi(A) = \begin{bmatrix} c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}, \quad \text{where } c_j = \begin{bmatrix} \mathrm{col}(A_{1j}) \\ \mathrm{col}(A_{2j}) \\ \vdots \\ \mathrm{col}(A_{j-1,j}) \end{bmatrix}, \ 2 \le j \le m.$$

- $\mathrm{vec}_\pi : \mathbf{S}_n \to \mathbb{R}^K, \qquad K = N - \sum_{i=1}^{m} \frac{n_i(n_i-1)}{2}, \ N = \frac{n(n-1)}{2},$

  where $\mathbf{S}_n$ is the space of symmetric matrices of order n, is a **linear operator**.

FUNCTION vec_0 and OPERATOR $\mathcal{N}_{ij}$

- Let $\mathbf{S}_{0,n}$ be the space of symmetric matrices of order n whose diagonal blocks are zero. Then $\mathrm{vec}_0 = \mathrm{vec}|_{\mathbf{S}_{0,n}}$ is **bijection**.

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- Linear operator $\mathcal{N}_{ij} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ diagonalize the pivot submatrix of the argument matrix,

$$\mathcal{N}_{ij}(\mathbf{X}) = \begin{bmatrix} X_{11} & & \cdots & & X_{1m} \\ & \mathrm{diag}(X_{ii}) & & 0 & \\ \vdots & & & & \vdots \\ & 0 & & \mathrm{diag}(X_{jj}) & \\ X_{m1} & & \cdots & & X_{mm} \end{bmatrix}.$$

BLOCK JACOBI ANNIHILATORS

Definition

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n. Let $\hat{\mathbf{U}} = \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{bmatrix}$, be orthogonal matrix of the order $n_i + n_j$ and let $\mathbf{U} = \mathcal{E}(i, j, \hat{\mathbf{U}})$ be the appropriate elementary block matrix.

The transformation $\Im_{ij}(\hat{\mathbf{U}})$ determined by

$$\Im_{ij}(\hat{\mathbf{U}})(vec(\mathbf{A})) = vec(\mathcal{N}_{ij}(\mathbf{U}^T\mathbf{A}\mathbf{U})), \qquad \mathbf{A} \in \mathbf{S}_n$$

is called *ij*-**block Jacobi annihilator**.

For each pair $1 \leq i < j \leq m$,

$$\Im_{ij} = \left\{ \Im_{ij}(\hat{\mathbf{U}}) \mid \hat{\mathbf{U}} \text{ is orthogonal matrix of order } n_i + n_j \right\}$$

is the *ij*-**class of block Jacobi annihilators**.

Computing $\Im_{ij}(\hat{U})a$

$a \in \mathbb{R}^K$ an arbitrary vector
$A = vec_0^{-1}(a)$

// **A**

For $r = 1, \ldots, m$
  $A'_{ri} = A_{ri}U_{ii} + A_{rj}U_{ji}$
  $A'_{rj} = A_{ri}U_{ij} + A_{rj}U_{jj}$
EndFor

// $\mathbf{U}^T\mathbf{AU}$

For $r = 1, \ldots, m$
  $A'_{ir} = U_{ii}^T A_{ir} + U_{ji}^T A_{jr}$
  $A'_{jr} = U_{ij}^T A_{ir} + U_{jj}^T A_{jr}$
EndFor

$A'_{ij} = 0$, $A'_{ji} = 0$
$A'_{ii} = \text{diag}(A'_{ii})$, $A'_{jj} = \text{diag}(A'_{jj})$

// $\mathcal{N}_{ij}(\mathbf{U}^T\mathbf{AU})$

$a' = vec(A')$

BLOCK JACOBI ANNIHILATORS - Example

Let $A \in \mathbb{R}^{8\times 8}$, $\pi = (2,2,2,2)$, $i = 1$, $j = 2$. Then $K = 24$ and

$$
\mathfrak{S}_{12}(\hat{\mathbf{U}}) = \left[\begin{array}{c|c|c|c|c|c}
\begin{smallmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{smallmatrix} & & & & & \\ \hline
 & \begin{smallmatrix} U_{11}^T & \\ & U_{11}^T \end{smallmatrix} & \begin{smallmatrix} U_{21}^T & \\ & U_{21}^T \end{smallmatrix} & & & \\ \hline
 & \begin{smallmatrix} U_{12}^T & \\ & U_{12}^T \end{smallmatrix} & \begin{smallmatrix} U_{22}^T & \\ & U_{22}^T \end{smallmatrix} & & & \\ \hline
 & & & \begin{smallmatrix} U_{11}^T & \\ & U_{11}^T \end{smallmatrix} & \begin{smallmatrix} U_{21}^T & \\ & U_{21}^T \end{smallmatrix} & \\ \hline
 & & & \begin{smallmatrix} U_{12}^T & \\ & U_{12}^T \end{smallmatrix} & \begin{smallmatrix} U_{22}^T & \\ & U_{22}^T \end{smallmatrix} & \\ \hline
 & & & & & \begin{smallmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{smallmatrix}
\end{array}\right],
$$

where U_{11}, U_{12}, U_{21}, U_{22} are the blocks of order 2 of $\hat{\mathbf{U}} \in \mathbb{R}^{4\times 4}$ and $\hat{\mathbf{U}}$ is orthogonal.

BLOCK JACOBI ANNIHILATORS - Properties

- $\|\mathfrak{J}_{ij}(\hat{\mathbf{U}})\|_2 = 1$,
  except for $m = 2$ when $\mathfrak{J}_{12}(\hat{\mathbf{U}}) = 0$.

- $\mathfrak{J}$ differs from the identity matrix I_K in exactly $m - 1$ principal submatrices.

- $\mathfrak{J} \in \boldsymbol{\mathfrak{J}}_{ij} \;\Rightarrow\; \mathfrak{J}^T \in \boldsymbol{\mathfrak{J}}_{ij}$
- $\mathfrak{J} \in \boldsymbol{\mathfrak{J}}_{ij}^{\mathrm{UBC}} \;\Rightarrow\; \mathfrak{J}^T \in \boldsymbol{\mathfrak{J}}_{ij}^{\mathrm{UBC}}$

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BLOCK JACOBI ANNIHILATORS - Properties

- $\|\mathfrak{I}_{ij}(\hat{\mathbf{U}})\|_2 = 1$,
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- **Transforming a matrix process into a vector process**

$$\mathbf{A}^{(k+1)} = \mathcal{N}_{ij}(\mathbf{U}_k^T \mathbf{A}^{(k)} \mathbf{U}_k)$$
$$\downarrow$$
$$a^{(k+1)} = \mathfrak{I}_{i(k),j(k)}(\hat{\mathbf{U}}_k) a^{(k)}$$

BLOCK JACOBI OPERATORS

Definition

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n and

$$\mathcal{O} = (i_0, j_0), (i_1, j_1), \ldots, (i_{M-1}, j_{M-1}) \in \mathcal{O}(\mathbf{P}_m).$$

Then

$$\mathcal{J}_{\mathcal{O}} = \{\mathcal{J} \mid \mathcal{J} = \mathfrak{S}_{i_{M-1}j_{M-1}} \mathfrak{S}_{i_{M-2}j_{M-2}} \ldots \mathfrak{S}_{i_0 j_0}, \\ \mathfrak{S}_{i_k j_k} \in \boldsymbol{\mathfrak{S}}_{i_k j_k},\ 0 \leq k \leq M-1\}$$

is the class of block Jacobi operators associated with ordering $\mathcal{O}$. The matrices $\mathcal{J}$ of order K from $\mathcal{J}_{\mathcal{O}}$ are **block Jacobi operators**.

Pivot orderings

PIVOT ORDERINGS

- By $\mathcal{O}(\mathbf{P}_m)$ we denote the set of all finite sequences containing the elements of $\mathbf{P}_m$, assuming that each pair from $\mathbf{P}_m$ appears at least once in each sequence.

- For a given cyclic strategy I, **pivot ordering** is the sequence

$$\mathcal{O}_I = (i(0), j(0)), \ldots, (i(M-1), j(M-1)) \in \mathcal{O}(\mathbf{P}_m).$$

- For a given ordering $\mathcal{O} \in \mathcal{O}(\mathbf{P}_m)$, the cyclic strategy $I_{\mathcal{O}}$ is defined by

$$I_{\mathcal{O}}(k) = (i_{\tau(k)}, j_{\tau(k)}), \qquad 0 \leq \tau(k) \leq M-1,$$

where $k \equiv \tau(k) (\text{mod } M),\ k \geq 0$.

PIVOT ORDERINGS

- Visualization of an ordering $\mathcal{O}$ of $\mathbf{P}_m$:

  symmetric matrix of order m, $\mathrm{M}_{\mathcal{O}} = (\mathrm{m}_{rt})$,

$$\mathrm{m}_{i(k)j(k)} = \mathrm{m}_{j(k)i(k)} = k, \quad k = 0, 1, \ldots, M-1.$$

  We set $\mathrm{m}_{rr} = *$, $1 \leq r \leq m$.

- Example: Serial pivot orderings, column-wise and row-wise

$$\mathrm{M}_{\mathcal{O}_c} = \begin{bmatrix} * & 0 & 1 & 3 & 6 \\ 0 & * & 2 & 4 & 7 \\ 1 & 2 & * & 5 & 8 \\ 3 & 4 & 5 & * & 9 \\ 6 & 7 & 8 & 9 & * \end{bmatrix}, \qquad \mathrm{M}_{\mathcal{O}_r} = \begin{bmatrix} * & 0 & 1 & 2 & 3 \\ 0 & * & 4 & 5 & 6 \\ 1 & 4 & * & 7 & 8 \\ 2 & 5 & 7 & * & 9 \\ 3 & 6 & 8 & 9 & * \end{bmatrix}.$$

EQUIVALENT ORDERINGS

- **Admissible transposition** on $\mathcal{O} \in \mathcal{O}(\mathcal{S})$, $\mathcal{S} \subseteq \mathbf{P}_m$ is a transposition of two adjacent terms

$$(i_r, j_r), (i_{r+1}, j_{r+1}) \to (i_{r+1}, j_{r+1}), (i_r, j_r),$$

provided that $\{i_r, j_r\}$ and $\{i_{r+1}, j_{r+1}\}$ are disjoint.

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provided that $\{i_r, j_r\}$ and $\{i_{r+1}, j_{r+1}\}$ are disjoint.

Two sequences $\mathcal{O}, \mathcal{O}' \in \mathcal{O}(\mathbf{P}_m)$ are:

(i) **equivalent** (we write $\mathcal{O} \sim \mathcal{O}'$) if one can be obtained from the other by a finite set of admissible transpositions,

(ii) **shift-equivalent** ($\mathcal{O} \overset{s}{\sim} \mathcal{O}'$) if $\mathcal{O} = [\mathcal{O}_1, \mathcal{O}_2]$ and $\mathcal{O}' = [\mathcal{O}_2, \mathcal{O}_1]$, where [,] stands for concatenation,

(iii) **weak equivalent** ($\mathcal{O} \overset{w}{\sim} \mathcal{O}'$) if there exist $\mathcal{O}_i \in \mathcal{O}(\mathbf{P}_m)$, $0 \le i \le r$, such that in the sequence $\mathcal{O} = \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_r = \mathcal{O}'$ every two adjacent terms are equivalent or shift-equivalent.

CONVERGENT ORDERINGS

Theorem (Hansen 1963, Shroff and Schreiber 1989)

If a block Jacobi method converges for some cyclic ordering, then it converges for all orderings that are weak equivalent to it.

The block methods defined by equivalent cyclic orderings generate the same matrices after each full cycle and within the same cycle they produce the same sets of orthogonal elementary matrices.

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The block methods defined by equivalent cyclic orderings generate the same matrices after each full cycle and within the same cycle they produce the same sets of orthogonal elementary matrices.

What do we do?
Enlarge the set of “convergent orderings”.

INVERSE AND PERMUTATION EQUIVALENT

- Let $\mathcal{O} \in \mathcal{O}(\mathbf{P}_m)$, $\mathcal{O} = (i_0, j_0), (i_1, j_1), \ldots, (i_{M-1}, j_{M-1})$. Then

$$\mathcal{O}^{\leftarrow} = (i_{M-1}, j_{M-1}), \ldots, (i_1, j_1), (i_0, j_0) \ \in \mathcal{O}(\mathbf{P}_m)$$

  is **inverse ordering** to $\mathcal{O}$.

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- Two pivot orderings $\mathcal{O}, \mathcal{O}' \in \mathcal{O}(\mathbf{P}_m)$ are **permutation equivalent** if

$$\mathrm{M}_{\mathcal{O}'} = \mathrm{P}\mathrm{M}_{\mathcal{O}}\mathrm{P}^T$$

  holds for some permutation matrix P. We write $\mathcal{O}' \overset{\mathrm{p}}{\sim} \mathcal{O}$.

PROPERTIES OF BLOCK JACOBI OPERATORS

If two sequences $\mathcal{O}, \mathcal{O}' \in \boldsymbol{\mathcal{O}}(S)$ are

- **equivalent** $\Rightarrow$ $\mathcal{J}_{\mathcal{O}} = \mathcal{J}_{\mathcal{O}'}$,
- **shift-equivalent** $\Rightarrow$ $\text{spr}(\mathcal{J}_{\mathcal{O}}) = \text{spr}(\mathcal{J}_{\mathcal{O}'})$,
- **weak equivalent** $\Rightarrow$ $\text{spr}(\mathcal{J}_{\mathcal{O}}) = \text{spr}(\mathcal{J}_{\mathcal{O}'})$,
- **permutation equivalent** $\Rightarrow$ $\|\mathcal{J}_{\mathcal{O}}\|_2 = \|\mathcal{J}_{\mathcal{O}'}\|_2$,
- **reverse to each other** $\Rightarrow$ $\|\mathcal{J}_{\mathcal{O}}\|_2 = \|\mathcal{J}_{\mathcal{O}'}\|_2$.

Application

SERIAL STRATEGIES WITH PERMUTATIONS

Examples:

$$\begin{bmatrix} * & 0 & 2 & 4 & 9 & 12 \\ 0 & * & 1 & 5 & 8 & 10 \\ 2 & 1 & * & 3 & 7 & 13 \\ 4 & 5 & 3 & * & 6 & 11 \\ 9 & 8 & 7 & 6 & * & 14 \\ 12 & 10 & 13 & 11 & 14 & * \end{bmatrix} \in \mathcal{B}_c^{(6)}, \qquad \begin{bmatrix} * & 11 & 13 & 12 & 10 & 14 \\ 10 & * & 9 & 7 & 6 & 8 \\ 11 & 9 & * & 5 & 3 & 4 \\ 12 & 6 & 5 & * & 1 & 2 \\ 13 & 7 & 3 & 1 & * & 0 \\ 14 & 8 & 4 & 2 & 0 & * \end{bmatrix} \in \mathcal{B}_r^{(6)}$$

SERIAL STRATEGIES WITH PERMUTATIONS

Formal definition:

$$\mathcal{B}_c^{(m)} = \{\mathcal{O} \in \mathcal{O}(\mathbf{P}_m) \mid \mathcal{O} = (1,2), (\pi_3(1),3), (\pi_3(2),3), \ldots,$$

$$(\pi_m(1), m), \ldots, (\pi_m(m-1), m), \quad \pi_j \in \Pi^{(1,j-1)},\ 3 \le j \le m\},$$

$$\mathcal{B}_r^{(m)} = \{\mathcal{O} \in \mathcal{O}(\mathbf{P}_m) \mid \mathcal{O} = (m-1,m), (m-2, \tau_{m-2}(m)), (m-2, \tau_{m-2}(m-1)), \ldots,$$

$$(1, \tau_1(m)), \ldots, (1, \tau_1(2)), \quad \tau_i \in \Pi^{(i+1,m)},\ 1 \le i \le m-2\},$$

where $\Pi^{(l_1,l_2)}$ stands for the set of all permutations of $\{l_1, l_1+1, \ldots, l_2\}$.

SERIAL STRATEGIES WITH PERMUTATIONS

$$\overleftarrow{\mathcal{B}}_c^{(m)} = \{\mathcal{O} \in \mathcal{O}(\mathbf{P}_m) \mid \mathcal{O}^{\leftarrow} \in \mathcal{B}_1\},$$
$$\overleftarrow{\mathcal{B}}_r^{(m)} = \{\mathcal{O} \in \mathcal{O}(\mathbf{P}_m) \mid \mathcal{O}^{\leftarrow} \in \mathcal{B}_2\}.$$

Examples:

$$\begin{bmatrix} * & 14 & 12 & 10 & 5 & 2 \\ 14 & * & 13 & 9 & 6 & 4 \\ 12 & 13 & * & 11 & 7 & 1 \\ 10 & 9 & 11 & * & 8 & 3 \\ 5 & 6 & 7 & 8 & * & 0 \\ 2 & 4 & 1 & 3 & 0 & * \end{bmatrix} \in \overleftarrow{\mathcal{B}}_c^{(6)}, \qquad \begin{bmatrix} * & 4 & 3 & 2 & 1 & 0 \\ 4 & * & 5 & 8 & 7 & 6 \\ 3 & 5 & * & 9 & 11 & 10 \\ 2 & 8 & 9 & * & 13 & 12 \\ 1 & 7 & 11 & 13 & * & 14 \\ 0 & 6 & 10 & 12 & 14 & * \end{bmatrix} \in \overleftarrow{\mathcal{B}}_r^{(6)}$$

SERIAL STRATEGIES WITH PERMUTATIONS

Set

$$\mathcal{B}_{sp}^{(m)} = \mathcal{B}_{c}^{(m)} \cup \overleftarrow{\mathcal{B}_{c}}^{(m)} \cup \mathcal{B}_{r}^{(m)} \cup \overleftarrow{\mathcal{B}_{r}}^{(m)}.$$

Theorem (BK, Hari)

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n, $\mathcal{O} \in \mathcal{B}_{sp}^{(m)}$ and let $\mathcal{J} \in \mathcal{J}_{\mathcal{O}}^{\mathrm{UBC}}$ be a block Jacobi operator.

Then there are constants γ_π and $\tilde{\gamma}_n$ depending only on π and n, respectively, such that

$$\|\mathcal{J}\|_2 \leq \gamma_\pi, \quad 0 \leq \gamma_\pi < \tilde{\gamma}_n < 1.$$

SERIAL STRATEGIES WITH PERMUTATIONS

Theorem (BK, Hari)

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n, $\mathcal{O} \in \mathcal{B}_{sp}^{(m)}$ and let $\mathbf{A} \in \mathbf{S}_n$ be a block matrix. Let $\mathbf{A}'$ be obtained from $\mathbf{A}$ by applying one sweep of the cyclic block Jacobi method defined by the strategy $I_{\mathcal{O}}$.

If all transformation matrices are from the class *UBCE*, then there are constants γ_π (depending only on π) and $\tilde{\gamma}_n$ (depending only on n) such that

$$S^2(\mathbf{A}') \leq \gamma_\pi S^2(\mathbf{A}), \qquad 0 \leq \gamma_\pi < \tilde{\gamma}_n < 1.$$

GENERALIZED SERIAL STRATEGIES

Set

$$\mathcal{B}_{sg}^{(m)} = \left\{ \mathcal{O} \in \mathcal{O}(\mathbf{P}_m) \mid \mathcal{O} \overset{\mathsf{p}}{\sim} \mathcal{O}' \overset{\mathsf{w}}{\sim} \mathcal{O}'' \text{ or } \mathcal{O} \overset{\mathsf{w}}{\sim} \mathcal{O}' \overset{\mathsf{p}}{\sim} \mathcal{O}'',\ \mathcal{O}'' \in \mathcal{B}_{sp}^{(m)} \right\}.$$

Theorem (BK, Hari)

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n, $\mathcal{O} \in \mathcal{B}_{sg}^{(m)}$ and let $\mathcal{J} \in \boldsymbol{\mathcal{J}}_{\mathcal{O}}^{\mathrm{UBC}}$ be a block Jacobi operator. Suppose the chain connecting $\mathcal{O}$ to $\mathcal{O}'' \in \mathcal{B}_{sp}^{(m)}$ is in canonical form and contains d shift equivalences.

Then there are constants γ_π and $\tilde{\gamma}_n$ depending only on π and n, respectively, such that for any $d+1$ Jacobi operators $\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_{d+1} \in \boldsymbol{\mathcal{J}}_{\mathcal{O}}^{\mathrm{UBC}}$ it holds

$$\|\mathcal{J}_1 \mathcal{J}_2 \cdots \mathcal{J}_{d+1}\|_2 \le \gamma_\pi, \quad 0 \le \gamma_\pi < \tilde{\gamma}_n < 1.$$

GENERALIZED SERIAL STRATEGIES

Theorem (BK, Hari)

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n, $\mathcal{O} \in \mathcal{B}_{sg}^{(m)}$ and let $\mathbf{A} \in \mathbf{S}_n$ be a block matrix. Suppose the chain connecting $\mathcal{O}$ and $\mathcal{O}''$ is in canonical form and contains d shift equivalences. Let $\mathbf{A}'$ be obtained from $\mathbf{A}$ by applying $d+1$ sweeps of the cyclic block Jacobi method defined by the strategy $I_{\mathcal{O}}$.

If all transformation matrices are from the class *UBCE*, then there are constants γ_π and $\tilde{\gamma}_n$ depending only on π and n, respectively, such that

$$S^2(\mathbf{A}') \leq \gamma_\pi S^2(\mathbf{A}), \qquad 0 \leq \gamma_\pi < \tilde{\gamma}_n < 1.$$

SUMMARY

- We defined a class of Jacobi annihilators and operators designed for the block Jacobi method for symmetric matrices.
- This brings a more general view at block Jacobi method which can be used in the convergence considerations.
- Using reverse orderings and permutations, the class of "convergent" pivot strategies is further enlarged.
- The convergence results are given in a stronger form

$$S(\mathbf{A}') \leq \gamma S(\mathbf{A}), \quad 0 < \gamma \leq 1,$$

  where $\mathbf{A}'$ is the matrix obtained from $\mathbf{A}$ after one full cycle and constant γ does not depend on $\mathbf{A}$.
- This approach can be used for more general block Jacobi-type methods.

THANK YOU!