# Structure-preserving low multilinear rank approximation of antisymmetric tensors

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Hausdorff School: Low-rank Tensor Techniques

Joint work with Daniel Kressner (EPF Lausanne)

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Hrvatska zaklada za znanost

# ANTISYMMETRIC TENSORS

- **Antisymmetric tensor**

$$\mathcal{A}(i_1, i_2, \ldots, i_d) = (-1)^{|\sigma|}\mathcal{A}(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(d)})$$

**Löwdin rules:**

$$\begin{aligned} (i)\ \mathcal{A}(i,j,k) &= 0, \quad \text{if } i = j \text{ or } i = k \text{ or } j = k, \\ (ii)\ \mathcal{A}(i,j,k) &= \mathcal{A}(j,k,i) = \mathcal{A}(k,i,j) \\ &= -\mathcal{A}(j,i,k) = -\mathcal{A}(k,j,i) = -\mathcal{A}(i,k,j), \text{ otherwise.} \end{aligned}$$

- **Antisymmetrizer** anti($\mathcal{X}$) - projection of the tensor $\mathcal{X}$ on the space of antisymmetric tensors

# MULTILINEAR RANK

- **Multilinear rank** of tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ is $d$-tuple

$$(r_1, r_2, \ldots, r_d), \quad \text{where } r_k = \text{rank}(\mathbf{X}_{(k)}),\ 1 \leq k \leq d.$$

- If $\mathcal{A}$ is antisymmetric, then $r_1 = r_2 = \cdots = r_d = r$.
- We say that $\mathcal{A}$ has multilinear rank $r$ and denote it $\mathcal{A} \in \mathcal{M}_r$.

- **Minimization problem:** For given antisymmetric $\mathcal{A}$ find antisymmetric $\hat{\mathcal{A}} \in \mathcal{M}_r$ such that

$$\|\mathcal{A} - \hat{\mathcal{A}}\|^2 \to \min.$$

- Dual **maximization problem** (De Lathauwer, 2000):

$$\|\mathcal{A} \times_1 \mathbf{U}_1^T \times_2 \mathbf{U}_2^T \times_3 \cdots \times_d \mathbf{U}_d^T\|^2 \to \max.$$

# MULTILINEAR RANK OF ANTISYMMETRIC TENSOR

**Theorem (B., Kressner)**

Let $\mathcal{A} \in \mathbb{R}^{n \times n \times \cdots \times n}$ be an antisymmetric tensor of order $d \geq 3$. Then the multilinear rank $r$ of $\mathcal{A}$ satisfies

(i) $r = 0$ for $n < d$;

(ii) $r \leq d$ for $n = d$ or $n = d + 1$;

(iii) $r \leq n$ for $n \geq d + 2$.

There exist tensors $\mathcal{A}$ for which equality is attained in $(i)$–$(iii)$.

**Corrolary (B., Kressner)**

Let $\mathcal{A}$ be an antisymmetric tensor of order $d$. Then for the multilinear rank $r$ of $\mathcal{A}$ can attain the values from the set

$$\{0, d, d + 2, \ldots, n\}.$$

# JACOBI METHOD

**Idea:** Apply Jacobi rotations on $\mathcal{A}$ in order to increase the norm of $(r, r, r)$-subtensor in "upper left front" corner.
(Ishteva et al., 2013.)

Jacobi rotations are $n \times n$ matrices

$$R(i,j,\phi) = \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & \cos\phi & & & & -\sin\phi & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & \sin\phi & & & & \cos\phi & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{matrix},$$

where $(i,j) = (i(k), j(k))$ is called $k$-th pivot pair.

# JACOBI METHOD

## Algorithm

Take initial $Q$. (Using HOSVD with $Q = U$ or $Q = I_n$.)
$\mathcal{A}_1 = \mathcal{A} \times_1 Q^T \times_2 Q^T \times_3 Q^T$
REPEAT
  Choose $(i(k), j(k))$.
  Find $\phi$.
  $R_k = R(i(k), j(k), \phi(k))$
  $Q_{k+1} = Q_k R_k$
  $\mathcal{A}_{k+1} = \mathcal{A}_k \times_1 R_k^T \times_2 R_k^T \times_3 R_k^T$
UNTIL convergence
$U = Q(1:n, 1:r)$
$\hat{\mathcal{A}} = (\mathcal{A} \times_1 U^T \times_2 U^T \times_3 U^T) \times_1 U \times_2 U \times_3 U$

**Convergence:** to the stationary point of minimization problem.

# Numerical examples - Approximation error

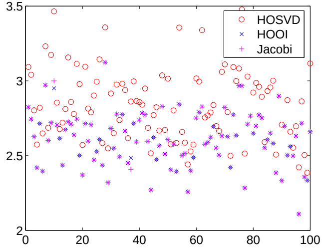


Figure: Multilinear rank 3 approximation of 100 random antisymmetric $10 \times 10 \times 10$ tensors.

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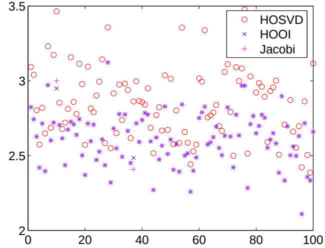


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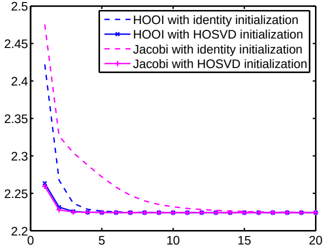


Figure: Multilinear rank 6 approximation of a random $10 \times 10 \times 10$ tensor.

# RANK $d$ APPROXIMATION

- For a given antisymmetric tensor $\mathcal{A}$ find its best rank 1 approximation $\mathcal{B}$.
- $\mathcal{B}$ of rank 1 $\dashrightarrow$ $\hat{\mathcal{A}}$ of multilinear rank $d$

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**Theorem (B., Kressner)**

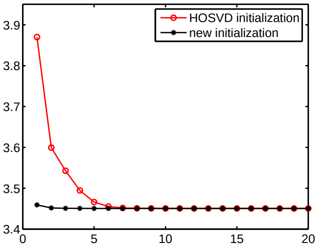
Let $\mathcal{A} \in \mathbb{R}^{n\times\cdots\times n}$ be an antisymmetric tensor of order $d$. It holds

$$\begin{aligned}
&\max\left\{\|\mathcal{A}\times_1 U^T\cdots\times_d U^T\| :\ U\in\mathbb{R}^{n\times d}, U^TU = I_d\right\}\\
=&d!\max\left\{|\mathcal{A}\times_1 u_1^T\cdots\times_d u_d^T| :\ [u_1,\ldots,u_d]^T[u_1,\ldots,u_d] = I_d\right\}\\
=&d!\max\left\{|\mathcal{A}\times_1 v_1^T\cdots\times_d v_d^T| :\ \|v_1\|_2 = \cdots = \|v_d\|_2 = 1\right\}.
\end{aligned}$$

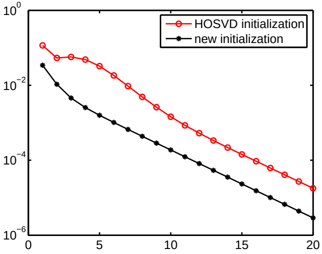
- If $\mathcal{B} = \alpha u_1 \circ u_2 \circ \cdots \circ u_d$, $u_k$ orthonormal, $1 \leq k \leq d$, then

$$\hat{\mathcal{A}} = d!\ \mathrm{anti}(\mathcal{B}).$$

# Case $d = 4$: New initialization of HOPM for rank 1 approximation - Convergence




(a) Approximation error




(b) Norm of gradient

Figure: Multilinear rank 4 approximation of a random $10 \times 10 \times 10 \times 10$ tensor.

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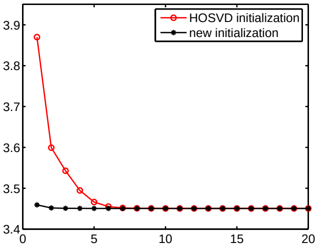


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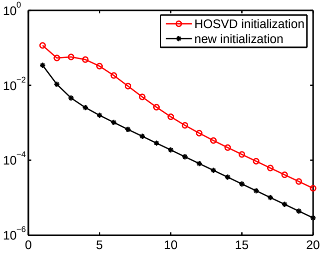


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**THANK YOU!**