

Jacobi method for small matrices

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- Why small matrices?
- Jacobi method and pivot strategies
- Parallel strategies on 4×4 matrices
- Conclusion

PIVOT STRATEGIES

Pivot strategy is the function $k \mapsto (i(k), j(k))$.

- Cyclic strategies
- Serial strategies (row and column)
- Strategies with permutations inside columns / rows
- Examples:

$$\begin{bmatrix} * & 0 & 2 & 4 & 6 \\ & * & 1 & 5 & 8 \\ & & * & 3 & 7 \\ & & & * & 9 \\ & & & & * \end{bmatrix} \in \mathcal{C}_1, \quad \begin{bmatrix} * & 6 & 9 & 7 & 8 \\ & * & 3 & 5 & 4 \\ & & * & 2 & 1 \\ & & & * & 0 \\ & & & & * \end{bmatrix} \in \mathcal{C}_2,$$
$$\begin{bmatrix} * & 9 & 8 & 4 & 2 \\ & * & 7 & 6 & 1 \\ & & * & 5 & 0 \\ & & & * & 3 \\ & & & & * \end{bmatrix} \in \mathcal{C}_3, \quad \begin{bmatrix} * & 3 & 1 & 0 & 2 \\ & * & 6 & 5 & 4 \\ & & * & 8 & 7 \\ & & & * & 9 \\ & & & & * \end{bmatrix} \in \mathcal{C}_4.$$

Jacobi method:

- J. Demmel, K. Veselić: *Jacobi's method is more accurate than QR*. SIAM J. Matrix Anal. Appl. 13 (4) (1992) 1204–1245.
- Z. Drmač: *A global convergence proof of cyclic Jacobi methods with block rotations*. SIAM J. Matrix Anal. Appl. 31 (3) (2009) 1329–1350.
- V. Hari: *Convergence to Diagonal Form of Block Jacobi-type Methods*. Numer. Math. 129 (3) (2015) 449–481
- C. G. J. Jacobi: *Über ein Leichtes Verfahren die in der Theorie der Seculr Störungen Vorkommenden Gleichungen Numerisch Aufzulösen*. Crelle's J. 30 (1846) 51–96.

Pivot strategies:

- E. R. Hansen: *On Cyclic Jacobi Methods*. SIAM J. Appl. Math. 11 (2) (1963) 448–459.
- W. F. Mascarenhas: *On the Convergence of the Jacobi Methods for Arbitrary Orderings*. SIAM J. Matrix Anal. Appl. 16 (4) (1995) 1197–1209.
- G. Shroff, R. Schreiber: *On the Convergence of the Cyclic Jacobi Method for Parallel Block Orderings*. SIAM J. Matrix Anal. Appl. 10 (3) (1989) 326–346.

PARALLEL STRATEGIES ON 4×4 MATRICES

We restrict our attention to those cyclic pivot strategies (16 strategies) that enable full parallelization of the method.

It will be sufficient to study pivot strategy

$$\begin{bmatrix} * & 4 & 0 & 2 \\ 4 & * & 3 & 1 \\ 0 & 3 & * & 5 \\ 2 & 1 & 5 & * \end{bmatrix},$$

which corresponds to parallel strategy

$$\begin{bmatrix} * & 2 & 0 & 1 \\ 2 & * & 1 & 0 \\ 0 & 1 & * & 2 \\ 1 & 0 & 2 & * \end{bmatrix}.$$

- Off-norm of a symmetric matrix

$$S(A) = \frac{\sqrt{2}}{2} \text{off}(A) = \frac{\sqrt{2}}{2} \|A - \text{diag}(A)\|_F = \sqrt{\sum_{i=1}^{n-1} \sum_{j=i+1}^n a_{ij}^2}, \quad A = A^T.$$

We find the constant γ , $0 \leq \gamma < 1$, such that

$$S^2(A^{[3]}) \leq \gamma S^2(A),$$

where $A^{[3]}$ is obtained from A after three full cycles.

SLOW CONVERGENCE - EXAMPLE

$$A = A^{(0)} = \begin{bmatrix} d + p_1 + p_2 & 0 & \epsilon + p_1 & -a + p_1 \\ 0 & d + p_2 & a & -\epsilon \\ \epsilon + p_1 & a & d + p_1 & 0 \\ -a + p_1 & \epsilon & 0 & d \end{bmatrix},$$

where $a = 1$, $\epsilon = 10^{-k-10}$, $d = 1 + \epsilon$, $p_2 = 10^{-1.5k}$, $p_3 = 10^{-1.5k-10}$ and

$$S(A) = 1.41421356237309504880168872420969807856967187537694.$$

SLOW CONVERGENCE - EXAMPLE

As long as the rotation angle is “close enough” to $\pm\frac{\pi}{4}$, the off-norm will not change in the next two steps.

k	(i, j)	$\sin \phi(i, j)$
1	(1, 3)	-0.70710678118654752440084434442717950962116550533689
2	(2, 4)	0.70710678118654752440084434442717950962114782766736
3	(1, 4)	0.70710678118654752440084436210484903221376812582298
4	(2, 3)	-0.70710678118654752440084436210484903221376812582299
5	(1, 2)	0.70710678118654752440084436210484903928483593768847
6	(3, 4)	0.70710678118654752440084436210484903928481826001894
7	(1, 3)	0.2499999999999998750000000000000390625000000000390e-15
8	(2, 4)	-0.2499999999999998750000000000000390625000000000390e-15

CHANGING THE APPROACH

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Pivot strategy $\mathcal{O} = (1, 3), (2, 4), (1, 4), (2, 3), (1, 2), (3, 4)$ on matrix

$$A = \begin{bmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & 0 \\ a_{14} & a_{24} & 0 & a_{44} \end{bmatrix}.$$

Let

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad Q^T X Q = \begin{bmatrix} x_{11} & x_{13} & x_{14} & -x_{12} \\ x_{31} & x_{33} & x_{34} & -x_{32} \\ x_{41} & x_{43} & x_{44} & -x_{42} \\ -x_{21} & -x_{23} & -x_{24} & x_{22} \end{bmatrix}.$$

Definition

For $A \in \mathcal{S}_4$ we define

$$\mathcal{T} = (R(1, 3, \phi)R(2, 4, \psi)Q)^T A (R(1, 3, \phi)R(2, 4, \psi)Q),$$

where $R(1, 3, \phi)$ and $R(2, 4, \psi)$ are Jacobi rotations which annihilate the elements a_{13} and a_{24} of A , respectively, and $\phi, \psi \in [-\frac{\pi}{4}, \frac{\pi}{4}]$.

Properties of \mathcal{T} :

- $\mathcal{T}^k(A) = (Q^k)^T A^{(2k)} Q^k, \quad k \geq 0,$
- $S(\mathcal{T}^k(A)) = S(A^{(2k)}), \quad k \geq 0.$

MAIN THEOREM

Proposition

Let $A \in \mathcal{S}_4$ be such that $a_{12} = 0$, $a_{34} = 0$ and $\epsilon = 10^{-5}$. Then

$$S(\mathcal{T}^6(A)) \leq (1 - \epsilon)S(A).$$

Theorem

Let $A \in \mathcal{S}_4$ be such that $a_{12} = 0$, $a_{34} = 0$ and let $A^{(12)}$ be obtained by applying 12 steps of the Jacobi method under the strategy $\mathcal{O} = (1, 3), (2, 4), (1, 4), (2, 3), (1, 2), (3, 4)$ to A . Then

$$S(A^{(12)}) \leq (1 - 10^{-5})S(A).$$

- E. Begović: *Convergence of Block Jacobi Methods*. Ph.D. thesis, University of Zagreb, 2014.
- E. Begović, V. Hari: *On the Global Convergence of the Jacobi Method for Symmetric Matrices of order 4 under Parallel Strategies*. in preparation

Theorem

Let $A \in \mathcal{S}_4$ and let $A^{[3]}$ be obtained by applying 3 full cycles of the Jacobi method under **any** cyclic strategy on A . Then there is a constant γ such that

$$S^2(A^{[3]}) \leq \gamma S^2(A), \quad 0 \leq \gamma < 1.$$

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Thank you!