AN INTRODUCTION TO KRYLOV'S L_p-THEORY FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this lecture, we introduce Krylov's L_p-theory for stochastic partial differential equations. More precisely, we prove that there exists a unique solution u to the following stochastic heat equation

$$du(t,x) = \left(a^{ij}(t)u_{x^ix^j}(t,x) + f(t,x)\right)dt + g(t,x)dw_t, \quad (t,x) \in [0,T] \times \mathbf{R}^d$$
$$u(0,x) = u_0(x), \tag{0.1}$$

where w_t is Brownian motion (Wiener process), $a^{ij}(t) = a^{ij}(\omega,t)$ is predictable, symmetric, and satisfy

$$\kappa^{-1}|\xi|^2 \geq a^{ij}(t)\xi^i\xi^j \geq \kappa|\xi|^2 \qquad \forall(\omega,t,\xi) \in \Omega \times [0,T] \times \mathbf{R}^d \tag{0.2}$$

with $\kappa > 0$,

$$f, g \in L_p\left(\Omega \times [0,T], \mathcal{P}, dP \times dt\right),$$

$T \in (0,\infty)$, $p \in [2,\infty)$, $\mathcal{P}$ is the predictable σ-algebra, and Einstein's summation convention is used.

1. Cauchy's problem

The problem solving equations with initial conditions or boundary conditions such as (0.1) is usually called Cauchy's problem, which is named after Augustin Louis Cauchy. Solvability of equation (0.1) heavily depends on conditions of free terms such as u_0, f and g. In other words, u varies depending on u_0, f, and g and it is even possible that we cannot solve (0.1) if they are too bad. Therefore the fundamental questions related to (0.1) are the following :

- "Can we solve (0.1)? What conditions should be given on u_0, f, and g to solve (0.1)?"
- "If we can solve it, is a solution unique?"
- "What is the meaning of solutions? What is an appropriate function space to handle solutions ?".

Answering these questions is called "well-posed problem", which stems from a definition given by Jacques Hadamard. Especially, we focus on answering these questions when f and g are contained in appropriate L_p-classes.

2. A Deterministic L_2-theory

Assume $g = 0$ and consider the deterministic equation first. In other words, we investigate the solvability of the equation

$$u_t(t,x) = a^{ij}(t)u_{x^ix^j}(t,x) + f(t,x) \quad (t,x) \in [0,T] \times \mathbf{R}^d$$
$$u(0,x) = u_0(x). \tag{2.1}$$

For a while, assume that $f \in C_c^\infty\left((0,T)\times \mathbf{R}^d\right)$, $u_0 \in C_c^\infty(\mathbf{R}^d)$, and a very good solution u exists. Recall the definition of the Fourier transform and the inverse Fourier transform. For a very nice function f, we define its Fourier transform and inverse Fourier transform as

$$\mathcal{F}[f](\xi) = \int_{\mathbf{R}^d} e^{-i\xi\cdot x} f(x)dx$$

and

$$\mathcal{F}^{-1}[f](x) = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} e^{ix\cdot\xi} f(\xi)d\xi,$$

respectively. If f has a decay $|x|\to\infty$, then for all $j,k \in \{1,\ldots,d\}$,

$$\begin{aligned}\mathcal{F}[f_{x^jx^k}](\xi) &= \int_{\mathbf{R}^d} e^{-i\xi\cdot x} f_{x^jx^k}(x)dx \\ &= (i\xi^j)(i\xi^k)\int_{\mathbf{R}^d} e^{-i\xi\cdot x} f(x)dx \\ &= -\xi^j\xi^k\mathcal{F}[f](\xi)\end{aligned}$$

due to the integration by parts. It is well-known that if f is a very nice function, then

$$\mathcal{F}^{-1}\left[\mathcal{F}[f]\right](x) = f(x).$$

Taking the Fourier transform to both sides of (2.1) with respect to x, we have

$$\left(\mathcal{F}[u(t,\cdot)](\xi)\right)_t = -\xi^i\xi^j a^{ij}(t)\mathcal{F}[u(t,\cdot)](\xi) + \mathcal{F}[f(t,\cdot)](\xi).$$

Solving the above ordinary differential equations, we have

$$\mathcal{F}[u(t,\cdot)](\xi) = e^{-\int_0^t a^{ij}(r)dr\xi^i\xi^j}\mathcal{F}[u_0](\xi) + \int_0^t e^{-\int_s^t a^{ij}(r)dr\xi^i\xi^j}\mathcal{F}[f(s,\cdot)](\xi)ds.$$

Therefore by the inverse Fourier transform,

$$\begin{aligned}u(t,x) &= \mathcal{F}^{-1}\left[e^{-\int_0^t a^{ij}(r)dr\xi^i\xi^j}\mathcal{F}[u_0](\xi)\right](x) \\ &\quad + \mathcal{F}^{-1}\left[\int_0^t e^{-\int_s^t a^{ij}(r)dr\xi^i\xi^j}\mathcal{F}[f(s,\cdot)](\xi)ds\right](x) \\ &= p(t,\cdot)*u_0(x) + \int_0^t p(s,t,\cdot)*f(s,\cdot)(x)ds, \end{aligned} \tag{2.2}$$

where

$$p(s,t,x) := \mathcal{F}^{-1}\left[e^{-\int_s^t a^{ij}(r)dr\xi^i\xi^j}\right](x)$$

and

$$p(t,x) := \begin{cases} p(0,t,x) & \text{if } t>0 \\ \delta_0(x), & \end{cases}$$

where $\delta_0(x)$ is the Dirac delta function concentrated at 0. Moreover, one can easily check that u defined by (2.2) is indeed a solution to (2.1). Let $\sigma^{ij}(t)$ be a matrix-valued function on $[0,T]$ such that

$$\frac{1}{2}\sigma^{ik}(t)\sigma^{kj}(t) = a^{ij}(t)$$

For a d-dimensional Wiener process W_t and $t\in[0,\infty)$, we define

$$X_t^i := \int_0^t \sigma^{ik}(s)dW_t^k.$$

and

$$X_t = (X^i_t) \qquad (i = 1, \dots, d).$$

Then $p(t,x)$ is the probability density function of X_t and $p(s,t,x)$ is the probability density function of $X_t - X_s$. Thus one can easily check that a solution u is given by

$$u(t,x) = \mathbb{E}[u_0(x - X_t)] + \int_0^t \mathbb{E}[f(s, x + X_t - X_s]ds.$$

Before going to next step, we introduce Banach space-valued function spaces. For a Banach space F with the norm $\|\cdot\|_F$, $C([0,T];F)$ denotes the space of all functions u such that

$$\|u\|_{C([0,T];F)} := \sup_t \|u(t,\cdot)\|_F < \infty$$

and $\|u(t,\cdot)\|_F$ is continuous with respect to t on $[0,T]$.

For $p \in [1,\infty)$, a Banach space F, and a measure space $(X, \mathcal{M}, \mu)$, by $L_p(X, \mathcal{M}, \mu; F)$, we denote the space of all F-valued $\mathcal{M}^\mu$-measurable functions u so that

$$\|u\|_{L_p(X,\mathcal{M},\mu;F)} := \left(\int_X \|u(x)\|_F^p \, \mu(dx) \right)^{1/p} < \infty,$$

where $\mathcal{M}^\mu$ denotes the completion of $\mathcal{M}$ with respect to the measure μ. If there is no confusion for the given measure and σ-algebra, we usually omit the measure and the σ-algebra. In particular, we set $L_2(\mathbf{R}^d) := L_2(\mathbf{R}^d, \mathcal{L}, \ell; \mathbf{R})$ and $L_2\left((0,T); L_2(\mathbf{R}^d)\right) := L_2\left([0,T], \mathcal{L}, \ell; L_2(\mathbf{R}^d)\right)$, where $\mathcal{L}$ and ℓ denote the Lebesgue measurable sets and Lebesgue measure, respectively.

Lemma 2.1. *Let $p \in [1,\infty)$, $K(s,t,x)$ be integrable function on $[0,T] \times [0,T] \times \mathbf{R}^d$, $f \in L_p\left((0,T); L_p(\mathbf{R}^d)\right)$, and*

$$v(t,x) := \int_0^t K(s,t,\cdot) * f(s,\cdot)(x) ds := \int_0^t \int_{\mathbf{R}^d} K(s,t,y) f(s.x - y) dy ds.$$

Then

$$\int_0^T \|v(t,\cdot)\|^p_{L_p(\mathbf{R}^d)} dt \leq \left(\int_0^T \sup_{t \leq T} \|K(t-s,t,\cdot)\|_{L_1(\mathbf{R}^d)} ds \right)^p \int_0^T \|f(t,\cdot)\|^p_{L_p(\mathbf{R}^d)} dt. \tag{2.3}$$

In particular, if f is a constant with respect to t, i.e. $f(t,x) = f(x)$ for all $t \in [0,T]$, then

$$\int_0^T \|v(t,\cdot)\|^p_{L_p(\mathbf{R}^d)} dt \leq T \left(\int_0^T \sup_{t \in [0,T]} \|K(t-s,t,\cdot)\|_{L_1(\mathbf{R}^d)} ds \right)^p \|f(\cdot)\|^p_{L_p(\mathbf{R}^d)} dt.$$

Proof.

$$\begin{aligned} \left\| \int_{\mathbf{R}^d} K(s,t,y) f(s.\cdot - y) dy \right\|_{L_p(\mathbf{R}^d)} &\leq \int_{\mathbf{R}^d} |K(s,t,y)| \, \|f(s.\cdot - y)\|_{L_p(\mathbf{R}^d)} \, dy \\ &= \|f(s,\cdot)\|_{L_p(\mathbf{R}^d)} \int_{\mathbf{R}^d} |K(s,t,y)| dy \\ &= \|K(s,t,\cdot)\|_{L_1(\mathbf{R}^d)} \, \|f(s,\cdot)\|_{L_p(\mathbf{R}^d)} . \end{aligned}$$

Thus by applying generalized Minkowski's inequality,

$$\|v(t,\cdot)\|_{L_p} \leq \int_0^t \|K(t-s,t,\cdot)\|_{L_1(\mathbf{R}^d)} \, \|f(t-s,\cdot)\|_{L_p(\mathbf{R}^d)} \, ds.$$

Finally, by generalized Minkowski's inequality and Hölder's inequality ,

$$\begin{aligned}
&\int_0^T \|v(t,\cdot)\|_{L_p}^p dt \\
&\leq \left(\int_0^T \left(\int_0^T 1_{0<t-s} \|K(t-s,t,\cdot)\|_{L_1(\mathbf{R}^d)} \, \|f(t-s,\cdot)\|_{L_p(\mathbf{R}^d)} \, ds \right)^p dt \right)^{p/p} \\
&\leq \left(\int_0^T \sup_{t\in[0,T]} \|K(t-s,t,\cdot)\|_{L_1(\mathbf{R}^d)} \left(\int_0^T \|1_{0<t-s} f(t-s,\cdot)\|_{L_p(\mathbf{R}^d)}^p \, dt \right)^{1/p} ds \right)^p \\
&\leq \left(\int_0^T \sup_{t\in[0,T]} \|K(t-s,t,\cdot)\|_{L_1(\mathbf{R}^d)} ds \right)^p \int_0^T \|f(t,\cdot)\|_{L_p(\mathbf{R}^d)}^p \, dt.
\end{aligned}$$

□

Since $p(s,t,x)$ is the probability density function of $X_t - X_s$,

$$\sup_{t\in[0,T]} \|p(t-s,t,\cdot)\|_{L_1(\mathbf{R}^d)} = 1,$$

by Lemma 2.1, we have

$$\int_0^T \|u(t,\cdot)\|_{L_p(\mathbf{R}^d)}^p dt \leq T^{1+p} \|u\|_{L_p(\mathbf{R}^d)}^p + T \int_0^T \|f(t,\cdot)\|_{L_p(\mathbf{R}^d)}^p dt.$$

Our next questions are that can we have

$$\begin{aligned}
&\int_0^T \int_{\mathbf{R}^d} |u_{x^i}(s,x)|^p \, dxds + \int_0^T \int_{\mathbf{R}^d} |u_{x^i x^j}(s,x)|^p \, dxds \\
&\leq N(p,T) \left(\int_0^T \int_{\mathbf{R}^d} |u_0(x)|^p \, dxds + \int_0^T \int_{\mathbf{R}^d} |f(s,\cdot)(x)|^p \, dxds \right) ?
\end{aligned}$$

Here $i,j \in (1,\ldots,d)$. Similarly to estimating u, we can easily check that

$$\begin{aligned}
&\int_0^T \|u_{x^i}(t,\cdot)\|_{L_p(\mathbf{R}^d)}^p dt \\
&\leq \left(\int_0^T \sup_{t\in[0,T]} \|p_{x^i}(t-s,t,\cdot)\|_{L_1(\mathbf{R}^d)} dt \right)^p \left(T\|u_0\|_{L_p(\mathbf{R}^d)} + \int_0^T \|f(t,\cdot)\|_{L_p(\mathbf{R}^d)}^p dt \right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \|u_{x^i x^j}(t,\cdot)\|_{L_p(\mathbf{R}^d)}^p dt \\
&\leq \left(\int_0^T \sup_{t\in[0,T]} \|p_{x^i x^j}(s,t,\cdot)\|_{L_1(\mathbf{R}^d)} dt \right)^p \left(T\|u_0\|_{L_p(\mathbf{R}^d)} + \int_0^T \|f(t,\cdot)\|_{L_p(\mathbf{R}^d)}^p dt \right).
\end{aligned}$$

Exercise 2.2. *Show that there exists a positive constant $c > 0$ such that for all $0 < s < t$*

$$c \le (t-s)^{-1/2} \le \int_{\mathbf{R}^d} |p_{x^i}(s,t,x)| dx \le c^{-1}(t-s)^{-1/2}$$

and

$$c \le (t-s)^{-1} \int_{\mathbf{R}^d} |p_{x^i x^j}(t,x)| dt \le c^{-1} t^{-1}.$$

Thus we can control the L_p-norm of u_{x^i}. But we do know that the L_p-norm of $u_{x^i x^j}$ also can be controlled by L_p-norms of u_0 and f yet. Since we want handle non-smooth u_0 and f, it is needed to introduce the concept of weak-derivatives.

Definition 2.3 (Weak-derivative)**.** *Let f be a locally integrable function on $\mathbf{R}^d$. We say that a locally integrable function f_{x^i} is the i-th weak-derivative of f ($i = 1, \ldots, d$) iff for all $\phi \in C_c^\infty(\mathbf{R}^d)$ the following equality holds:*

$$\int_{\mathbf{R}^d} f_{x^i}(x)\phi(x) ds = (-1) \int_{\mathbf{R}^d} f(x)\phi_{x^i}(x) dx.$$

Moreover generally, for any multi-index α, a locally integrable function $D^\alpha f$ is called the α-th weak-derivative of f if for all $\phi \in C_c^\infty(\mathbf{R}^d)$

$$\int_{\mathbf{R}^d} D^\alpha f(x)\phi(x) dx = (-1)^{|\alpha|} \int_{\mathbf{R}^d} f(x) D^\alpha \phi(x) dx.$$

Definition 2.4 (Sobolev space)**.** *Let $p \in [1,\infty)$ and $n \in \mathbb{N}$. We define the Sobolev space with the exponent p and order k as*

$$H_p^n(\mathbf{R}^d) := \{f \in L_p(\mathbf{R}^d) : \|D^\alpha f\|_{L_p(\mathbf{R}^d)} < \infty \quad \forall |\alpha| \le n\},$$

where $D^\alpha f$ is the α-th weak-derivative of f.

Remark 2.5. It is well-known that $H_p^n(\mathbf{R}^d)$ is complete with the norm

$$\|f\|_{H_p^n(\mathbf{R}^d)} := \sum_{|\alpha| \le n} \|D^\alpha f\|_{L_p(\mathbf{R}^d)}.$$

Definition 2.6 (Weak solution)**.** *We say that a function $u \in L_p\left([0,T]; L_p(\mathbf{R}^d)\right)$ is a solution to (2.1) iff for all $\phi \in C_c^\infty$ and $t \in [0,T]$*

$$\begin{aligned} &(u(t,\cdot),\phi)_{L_2(\mathbf{R}^d)} \\ &= (u_0,\phi)_{L_2(\mathbf{R}^d)} + \int_0^t \left(u(s,\cdot), a^{ij}(s)\phi_{x^i x^j}\right)_{L_2(\mathbf{R}^d)} ds + \int_0^t (f(s,\cdot),\phi)_{L_2(\mathbf{R}^d)} ds. \end{aligned} \tag{2.4}$$

Theorem 2.7 (L_2-theory)**.** *Let $T \in (0,\infty)$. Assume that $a^{ij}(t)$ is measurable and satisfy the ellipticity condition that*

$$\kappa|\xi|^2 \le a^{ij}(t)\xi^i\xi^j \le \kappa^{-1}|\xi|^2 \qquad \forall (t,\xi) \in [0,T] \times \mathbf{R}^d. \tag{2.5}$$

Then for all $f \in L_2((0,T); L_2(\mathbf{R}^d))$ and $u_0 \in H_2^1(\mathbf{R}^d)$, there exists a unique solution $u \in C\left([0,T]; L_2(\mathbf{R}^d)\right) \cap L_2\left([0,T]; H_2^2(\mathbf{R}^d)\right)$ to equation (0.1) such that

$$\begin{aligned} &\sup_{t\in[0,T]} \|u(t,\cdot)\|^2_{H_2^1(\mathbf{R}^d)} + \int_0^T \|u_{xx}(t,\cdot)\|^2_{L_2(\mathbf{R}^d)} dt \\ &\le N \left(\int_0^T \|f(t,\cdot)\|^2_{L_2(\mathbf{R}^d)} dt + \|u_0\|^2_{H_2^1(\mathbf{R}^d)} \right), \end{aligned} \tag{2.6}$$

where N depends only on p and T.

Proof. **Part I.** (A priori estimate)

First we show that any solution $u \in C\left([0,T];H_2^1(\mathbf{R}^d)\right) \cap L_2\left([0,T];H_2^2(\mathbf{R}^d)\right)$ to equation (2.1) satisfies (2.6). We use Sobolev mollifiers. Fix a nonnegative $\varphi \in C_c^\infty$ with the unit integral and for $\varepsilon > 0$, denote $\varphi^\varepsilon(x) = e^{-d}\varphi(x/\varepsilon)$, $u_0^\varepsilon(x) = u_0 * \varphi^\varepsilon(x)$, $f^\varepsilon(s,x) = f(s,\cdot) * \varphi^\varepsilon(x)$, and $u^\varepsilon(t,x) = u(t,\cdot) * \varphi^\varepsilon(\cdot)(x)$. Putting $\varphi^\varepsilon(x-\cdot)$ in (2.4), for all $(t,x) \in (0,T) \times \mathbf{R}^d$, we have

$$u^\varepsilon(t,x) = u_0^\varepsilon(t,x) + \int_0^t a^{ij}(s) u^\varepsilon_{x^ix^j}(s,x)ds + \int_0^t f^\varepsilon(s,x)ds.$$

By the chain rule, Fubini's theorem, and the integration by parts,

$$\begin{aligned}\int_{\mathbf{R}^d} |u^\varepsilon(t,x)|^2 dx = \int_{\mathbf{R}^d} |u_0^\varepsilon(x)|^2 dx - \int_0^t \int_{\mathbf{R}^d} 2u^\varepsilon_{x^j}(s,x) a^{ij}(s) u^\varepsilon_{x^i}(s,x) dsdx \\ + 2\int_0^t \int_{\mathbf{R}^d} u^\varepsilon(s,x) f^\varepsilon(s,x) dxds.\end{aligned}$$

Therefore, by the ellipticity condition (0.2), the Cauchy-Bunyakovsky-Schwarz inequality and arithmetic-geometric mean inequality, for all $t \in [0,T]$ and $\delta \in (0,\infty)$,

$$\begin{aligned}&\int_{\mathbf{R}^d} |u^\varepsilon(t,x)|^2 dx + 2\kappa \int_0^t \int_{\mathbf{R}^d} \left|u^\varepsilon_x(s,x)\right|^2 dsdx \\ &\leq \int_{\mathbf{R}^d} |u^\varepsilon(t,x)|^2 dx + 2\int_0^t \int_{\mathbf{R}^d} a^{ij}(s) u^\varepsilon_{x^i}(s,x) u^\varepsilon_{x^j} dsdx \\ &\leq \int_{\mathbf{R}^d} |u_0^\varepsilon(x)|^2 dx + 2\int_0^T \int_{\mathbf{R}^d} u^\varepsilon(s,x) f^\varepsilon(s,x) dxds \\ &\leq \int_{\mathbf{R}^d} |u_0^\varepsilon(x)|^2 dx + \delta \int_0^T \int_{\mathbf{R}^d} |u^\varepsilon(s,x)|^2 dxds + \delta^{-1} \int_0^T \int_{\mathbf{R}^d} |f^\varepsilon(s,x)|^2 dxds. \quad (2.7)\end{aligned}$$

Taking $\delta = 1/(2T)$, we have

$$\begin{aligned}&\sup_{t\in[0,T]} \int_{\mathbf{R}^d} |u^\varepsilon(t,x)|^2 dx + \int_0^T \int_{\mathbf{R}^d} \left|u^\varepsilon_{x^i}(s,x)\right|^2 dsdx \\ &\leq N(T,\kappa) \left(\int_{\mathbf{R}^d} |u_0^\varepsilon(x)|^2 dx + \int_0^T \int_{\mathbf{R}^d} |f^\varepsilon(s,x)|^2 dxds \right). \quad (2.8)\end{aligned}$$

Set $v^k(t,x) = u_{x^k}$ for $(k = 1,\dots,d)$. Then v^k satisfies

$$(v^k)^\varepsilon(t,x) = (u_0)^\varepsilon_{x^k}(t,x) + \int_0^t a^{ij}(s)(v^k)^\varepsilon_{x^ix^j}(s,x)ds + \int_0^t f^\varepsilon_{x^k}(s,x)ds.$$

By the integration by parts and the ellipticity condition, we have

$$\begin{aligned}&\int_{\mathbf{R}^d} |(v^k)^\varepsilon(t,x)|^2 dx + \kappa \int_0^t \int_{\mathbf{R}^d} \left|(v^k_x)^\varepsilon(s,x)\right|^2 dsdx \\ &\leq \int_{\mathbf{R}^d} |(u_0)^\varepsilon_{x^k}(x)|^2 dx + 2\int_0^t \int_{\mathbf{R}^d} \left(v^k_{x^k}\right)^\varepsilon (s,x) f^\varepsilon(s,x) dxds.\end{aligned}$$

Therefore following (2.7) and (2.8), we have

$$\sup_{t\in[0,T]}\int_{\mathbf{R}^d}|u^\varepsilon_{x^k}(t,x)|^2\,dx+\int_0^T\int_{\mathbf{R}^d}|u^\varepsilon_{x^ix^k}(s,x)|^2\,dsdx$$
$$\leq N\left(\int_{\mathbf{R}^d}|(u_0)^\varepsilon_{x^k}(x)|^2dx+\int_0^T\int_{\mathbf{R}^d}|f^\varepsilon(s,x)|^2dxds\right), \tag{2.9}$$

Applying the same idea to $u^{\varepsilon_1}-u^{\varepsilon_2}$ with $\varepsilon_1,\varepsilon_2>0$, we have

$$\sup_{t\in[0,T]}\left(\int_{\mathbf{R}^d}|(u^{\varepsilon_1}-u^{\varepsilon_2})(t,x)|^2\,dx+\int_0^T\int_{\mathbf{R}^d}|(u^{\varepsilon_1}_x-u^{\varepsilon_2}_x)(s,x)|^2\,dxds\right)$$
$$+\int_{\mathbf{R}^d}\left|\left(u^{\varepsilon_1}_{x^ix^j}-u^{\varepsilon_2}_{x^ix^j}\right)(t,x)\right|^2dx$$
$$\leq N(d,T)\bigg(\int_{\mathbf{R}^d}|(u^{\varepsilon_1}_0-u^{\varepsilon_2}_0)(x)|^2dx+\int_{\mathbf{R}^d}|((u_0)^{\varepsilon_1}_x-(u_0)^{\varepsilon_2}_x)(x)|^2\,dx$$
$$+\int_0^T\int_{\mathbf{R}^d}|(f^{\varepsilon_1}-f^{\varepsilon_2})(s,x)|^2dxds\bigg),$$

which implies that u^ε converges to

$$v\in C\left([0,T];H^1_2(\mathbf{R}^d)\right)\cap L_2\left([0,T];H^2_2(\mathbf{R}^d)\right)$$

as $\varepsilon\downarrow 0$. Since for each $t>0$, $u^\varepsilon(t,x)\to u(t,x)$ for almost every x, we conclude $u=v$ as an element of

$$C\left([0,T];L_2(\mathbf{R}^d)\right)\cap L_2\left([0,T];H^2_2(\mathbf{R}^d)\right).$$

Observing

$$\left(\int_{\mathbf{R}^d}|u^\varepsilon_0(x)|^pdx+\int_0^T\int_{\mathbf{R}^d}|f^\varepsilon(s,x)|^pdxds\right)$$
$$\leq\left(\int_{\mathbf{R}^d}|u_0(x)|^pdx+\int_0^T\int_{\mathbf{R}^d}|f(s,x)|^pdxds\right)$$

and taking $\varepsilon\downarrow 0$ in (2.8) and (2.9), we finally get (2.6).

Part II. (Existence)

We already show that $u(t,x)$ is a solution to equation (0.1) if $u_0\in C^\infty_c(\mathbf{R}^d)$ and $f\in C^\infty_c\left([0,T]\times\mathbf{R}^d\right)$. Thus it only remains to weaken the conditions on u_0 and f.

Choose sequences $u^n_0\in C^\infty_c(\mathbf{R}^d)$ and $f^n\in C^\infty_c((0,T)\times\mathbf{R}^d)$ so that

$$u^n_0\to u_0\quad\text{in}\quad L_2\qquad\text{and}\qquad f^n\to f\qquad\text{in}\quad L_2\left((0,T);L_2(\mathbf{R}^d)\right)$$

as $n\to\infty$. Then

$$u^n(t,x):=\mathbb{E}[u^n_0(x-X_t)]+\int_0^t\mathbb{E}[f^n(s,x+X_t-X_s]ds \tag{2.10}$$

satisfies

$$u^n_t(t,x)=a^{ij}(t)u^n_{x^ix^j}(t,x)+f^n(t,x)\qquad(t,x)\in(0,T)\times\mathbf{R}^d$$
$$u^n(0,x)=u^n_0(x).$$

Due to (2.6), u^n becomes a Cauchy sequence in

$$C\left([0,T];L_2(\mathbf{R}^d)\right)\cap L_2\left([0,T];H_2^2(\mathbf{R}^d)\right)$$

and thus there exists u such that $u_n\to u$ in

$$C\left([0,T];L_2(\mathbf{R}^d)\right)\cap L_2\left([0,T];H_2^2(\mathbf{R}^d)\right).$$

Since

$$u_0^n\to u_0 \quad \text{in} \quad L_2 \qquad \text{and} \qquad f^n\to f \qquad \text{in} \quad L_2\left([0,T];L_2(\mathbf{R}^d)\right),$$

one can easily check that u is a solution to equation (0.1).

Part III. (Uniqueness) We already showed that any solution

$$u\in C\left([0,T];L_2(\mathbf{R}^d)\right)\cap L_2\left([0,T];H_2^2(\mathbf{R}^d)\right).$$

to equation (0.1) satisfies (2.6). Therefore, the uniqueness is obvious. □

Remark 2.8. By taking the limit to both sides in (2.10), we have

$$u(t,x)=\mathbb{E}[u_0(x-X_t)]+\int_0^t\mathbb{E}[f(s,x+X_t-X_s]ds \tag{2.11}$$

for almost every $(t,x)\in[0,T]\times\mathbf{R}^d$ even though u_0 and f are merely contained in L_2 and $L_2\left([0,T];L_2(\mathbf{R}^d)\right)$, respectively, without smoothness.

Remark 2.9. Theorem 2.7 holds even if $p\neq 2$ and $p\in(1,\infty)$. Classically, this theorem can be proved on basis of singular integral theory. But these days, lots of kernel free estimates are developed. Since L_p-theories are beyond the scope of this lecture, I am not going to give details.

3. A Stochastic L_2-theory

Let $(\Omega,\mathcal{F},P)$ be a probability space and w_t be a Brownian motion relative to a filtration $\mathcal{F}_t$, i.e. w_t is $\mathcal{F}_t$-measurable and w_t-w_s is independent of $\mathcal{F}_s$ for all $0\leq s<t$. Each $\mathcal{F}_t$ contains all null sets of $\mathcal{F}$. By $\mathcal{P}$, we denote the predictable σ-algebra, i.e. $\mathcal{P}$ is the smallest σ-algebra on $\Omega\times[0,\infty)$ containing all sets of the form of $A\times[s,t)$, where $0\leq s<t$ and $A\in\mathcal{F}_s$. For $T\in(0,\infty)$, a stochastic process $u(t)$ defined on $\Omega\times[0,T]$ is called predictable if $u(t)1_{[0,T]}(t)$ is predictable on $\Omega\times[0,\infty)$. For $p\in[1,\infty)$ and $n\in\mathbb{N}$, define stochastic Banach spaces as follows

$$\mathbb{L}_p(T):=L_p\left(\Omega\times[0,T],\mathcal{P},dP\times dt;L_p(\mathbf{R}^d)\right)$$

and

$$\mathbb{H}_p^n(T):=L_p\left(\Omega\times[0,T],\mathcal{P},dP\times dt;H_p^n(\mathbf{R}^d)\right).$$

$$\mathbb{H}_0^\infty:=\Bigg\{\sum_{i=1}^j 1_{(\tau_{i-1},\tau_i]}(t)g^i(x):\; j\in\mathbb{N},\ g^i\in C_c^\infty(\mathbf{R}^d),\tau_i \text{ are bounded stopping times}\Bigg\}.$$

Then it is well-known that for all $T>0$, $p\in[1,\infty)$, and $n\in\mathbb{N}$, H_0^∞ is dense in $\mathbb{H}_p^n(T)$, i.e. for any $g\in H_p^n(T)$, there exists a sequence of $g_n\in\mathbb{H}_0^\infty$ such that

$$\|g-g_n\|^p_{\mathbb{H}_p^n(T)} := \mathbb{E}\int_0^T \|g(t,\cdot)\|^p_{H_p^n(\mathbf{R}^d)}dt \to 0 \tag{3.1}$$

as $n\to\infty$ (see [1, Theorem 3.11]).

Definition 3.1 ((Stochastic) weak solution). $u\in\mathbb{L}_p(T)$ *is a solution to (0.1) iff for each* $\phi\in C_c^\infty(\mathbf{R}^d)$*, there exists a* $\Omega'\subset\Omega$ *such that* $P(\Omega')=1$ *and*

$$(u(\omega,t,\cdot),\phi) = (u_0(\omega,\cdot),\phi) + \int_0^t \left(a^{ij}(\omega,s)u_{x^ix^j}(\omega,s,\cdot),\phi\right)ds + \int_0^t (f(\omega,s,\cdot),\phi)\,ds + \int_0^t (g(\omega,s,\cdot),\phi)\,dw_s$$

for all $\omega\in\Omega'$ *and* $t\in[0,T]$*. Simply, we say that* u *is a solution to (0.1) iff for each* $\phi\in C_c^\infty(\mathbf{R}^d)$,

$$(u(t,\cdot),\phi) = (u_0,\phi) + \int_0^t \left(u(s,\cdot),a^{ij}(s)\phi_{x^ix^j}\right)ds + \int_0^t (f(s,\cdot),\phi)\,ds + \int_0^t (g(s,\cdot),\phi)\,dw_s \tag{3.2}$$

holds for all $t\in[0,T]$ *with probability one.*

Remark 3.2. (i) Assume that $g=0$ in (0.1) and let u be a weak solution to (0.1). Then due to Definition 3.2, for each fixed $\omega\in\Omega'$, $u(\omega,t,x)$ becomes a weak solution in the sense of Definition 2.6 to the equation

$$u_t(\omega,t,x) = a^{ij}(\omega,t)u_{x^ix^j}(\omega,t,x) + f(\omega,t,x) \qquad (t,x)\in(0,T]\times\mathbf{R}^d$$
$$u(\omega,0,x) = u_0(\omega,x).$$

(ii) For each $p\in[1,\infty)$, there exists a countable subset of $C_c^\infty(\mathbf{R}^d)$ which is dense in $L_p(\mathbf{R}^d)$. Thus we may assume that the equality in (3.2) holds for all $\phi\in C_c^\infty(\mathbf{R}^d)$ and $t\in[0,T]$ with probability one.

From now on, the variable ω is usually omitted for the notational convenience. For instance, we use $u(t,x)$ instead of $u(\omega,t,x)$.

Consider the following simple stochastic equation:

$$du(t,x) = \Delta u(t,x)dt + g(t,x)dw_t \qquad (t,x)\in(0,T]\times\mathbf{R}^d$$
$$u(0,x) = 0. \tag{3.3}$$

We give a naive idea to obtain a solution representation to equation (3.3). Assume that u and g are very nice. Differentiating both sides of (3.3) with respect to t. Then

$$\frac{du}{dt}(t,x) = \Delta u(t,x) + g(t,x)\frac{dw_t}{dt} \qquad (t,x)\in(0,T]\times\mathbf{R}^d.$$

Here $\frac{dw_t}{dt}$ is not a usual RadonNikodym derivative since w_t is not differentiable with respect to t. Assume that there exists a kind of a random measure called "white noisy" satisfying

$$\int_s^t \frac{dw_t}{dt} = w_t - w_s.$$

Thus by (2.2),

$$u(t,x) = \int_0^t \int_{\mathbf{R}^d} p(t-s,x-y)g(s,y)\frac{dw_s}{ds}dyds$$
$$= \int_0^t \int_{\mathbf{R}^d} p(t-s,x-y)g(s,y)dydw_s,$$

where

$$p(t,x) := \begin{cases} (4\pi t)^{-d/2}\exp\left(-|x|^2/(4t)\right) & \text{if } t>0 \\ p(0,x) = \delta_0(x). \end{cases}$$

Our next step is to check that

$$u(t,x) := \int_0^t \int_{\mathbf{R}^d} p(t-s,x-y)g(s,y)dydw_s \tag{3.4}$$

is indeed a solution to equation (3.3), i.e. it suffices to check that u satisfies

$$u(t,x) = \int_0^t \Delta u(s,x)ds + \int_0^t g(s,x)dw_s \qquad (t,x) \in [0,T]\times\mathbf{R}^d \tag{3.5}$$

with probability one. By (stochastic) Fubini's theorem, the property of the kernel p that $p_t(t,x) = \Delta p(t,x)$, and the fundamental theorem of calculus, (formally)

$$\begin{aligned}\int_0^t \Delta u(s,x)ds &= \int_0^t \int_0^s \int_{\mathbf{R}^d} \Delta p(s-r,x-y)g(r,y)dydB_rds \\ &= \int_0^t \int_{\mathbf{R}^d} \int_0^t 1_{r<s}\partial_s\left(p(s-r,x-y)\right)dsg(r,y)dydB_r \\ &= \int_0^t \int_{\mathbf{R}^d} \int_0^t 1_{r<s}\partial_s\left(p(s-r,x-y)\right)dsg(r,y)dydB_r \\ &= \int_0^t \int_{\mathbf{R}^d} \left(p(t-r,x-y)-\delta_0(x-y)\right)g(r,y)dydB_r \\ &= u(t,x) - \int_0^t g(r,x)dB_r. \end{aligned} \tag{3.6}$$

Therefore u defined by (3.5) is a solution to (3.3) at least formally. Indeed, this is true if $g \in \mathbb{H}_0^\infty(T)$. Now we are ready to obtain an L_2-theory for stochastic partial differential equations.

Theorem 3.3 (L_2-theory for a model equation)**.** *Let $T \in (0,\infty)$. Then for all $g \in \mathbb{L}_2(T)$, there exists a unique solution*

$$u \in L_2\left(\Omega,\mathcal{F};C\left([0,T];L_2(\mathbf{R}^d)\right)\right) \cap \mathbb{H}_2^1(T)$$

to equation (3.3) such that

$$\sup_{t\in[0,T]} \mathbb{E}\|u(t,\cdot)\|^2_{L_2(\mathbf{R}^d)} + \mathbb{E}\int_0^T \|u_x(t,\cdot)\|^2_{L_2(\mathbf{R}^d)}dt \leq N\int_0^T \|g(t,\cdot)\|^2_{L_2(\mathbf{R}^d)}dt, \tag{3.7}$$

where N depends only on p and T.

Proof. **Part I** (A priori estimate) Assume that there exists a solution

$$u \in L_2\left(\Omega,\mathcal{F};C\left([0,T];L_2(\mathbf{R}^d)\right)\right) \cap \mathbb{H}_2^1(T)$$

to equation (3.3). Fix $\phi \in C_c^\infty(\mathbf{R}^d)$ which is nonnegative and has a unit integral. For each $\varepsilon > 0$ and $x \in \mathbf{R}^d$, denote $\phi^\varepsilon(x) = \varepsilon^{-d}\phi(\varepsilon^{-1}x)$, $g^\varepsilon(s,x) = g(s,\cdot) * \varphi^\varepsilon(x)$ and $u^\varepsilon(t,x) = u(t,\cdot) * \phi^\varepsilon(x)$. Then from (3.2),

$$u^\varepsilon(t,x) = u_0^\varepsilon(x) + \int_0^t \Delta u^\varepsilon(s,x)ds + \int_0^t g^\varepsilon(s,x)dw_s$$

for all $(t,x) \in [0,T] \times \mathbf{R}^d$ with probability one. By Ito's formula,

$$\begin{aligned}|u^\varepsilon(t,x)|^2 &= |u_0^\varepsilon(x)|^2 + 2\int_0^t u^\varepsilon(s,x)\Delta u^\varepsilon(s,x)ds \\ &\quad + 2\int_0^t u^\varepsilon(s,x)g^\varepsilon(s,x)dw_s + 2\int_0^t |g^\varepsilon(s,x)|^2 ds\end{aligned}$$

for all $(t,x) \in [0,T] \times \mathbf{R}^d$ with probability one. Taking the integration with respect to x and applying Fubini's theorem and the integration by parts, we have

$$\begin{aligned}&\int_{\mathbf{R}^d} |u^\varepsilon(t,x)|^2 dx + 2\int_0^t \int_{\mathbf{R}^d} |u_x^\varepsilon(s,x)|^2 dxds \\ &= \int_{\mathbf{R}^d} |u_0^\varepsilon(x)|^2 dx + 2\int_0^t \int_{\mathbf{R}^d} u^\varepsilon(s,x)g^\varepsilon(s,x)dxdw_s + 2\int_0^t \int_{\mathbf{R}^d} |g^\varepsilon(s,x)|^2 dxds\end{aligned}$$

Taking the $\sup_{t\leq T}$, and the expectation and applying Burkholder-Davis-Gundy inequality , we have

$$\begin{aligned}&\mathbb{E}\sup_{t\leq T}\int_{\mathbf{R}^d} |u^\varepsilon(t,x)|^2 dx + 2\mathbb{E}\int_0^T \int_{\mathbf{R}^d} |u_x^\varepsilon(s,x)|^2 dsdx \\ &\leq \int_{\mathbf{R}^d} |u_0^\varepsilon(x)|^2 dx + N(p)\mathbb{E}\left(\int_0^T \left(\int_{\mathbf{R}^d} |u^\varepsilon(s,x)g^\varepsilon(s,x)|dx\right)^2 ds\right)^{1/2} \\ &\quad + 2\mathbb{E}\int_0^T \int_{\mathbf{R}^d} |g^\varepsilon(s,x)|dxds.\end{aligned}$$

Note that for all $\delta > 0$, applying Cauchy-Bunyakovsky-Schwarz inequality and arithmetic-geometric mean inequality,

$$\begin{aligned}&\mathbb{E}\left(\int_0^T \left(\int_{\mathbf{R}^d} |u^\varepsilon(s,x)g^\varepsilon(s,x)|dx\right)^2 ds\right)^{1/2} \\ &\leq \mathbb{E}\left(\int_0^T \|u^\varepsilon(s,\cdot)\|^2_{L_2(\mathbf{R}^d)}\|g^\varepsilon(s,\cdot)\|^2_{L_2(\mathbf{R}^d)}ds\right)^{1/2} \\ &\leq \mathbb{E}\left(\sup_{t\leq T}\|u^\varepsilon(t,\cdot)\|^2_{L_2(\mathbf{R}^d)}\int_0^T \|g^\varepsilon(s,\cdot)\|^2_{L_2(\mathbf{R}^d)}ds\right)^{1/2} \\ &\leq \frac{\delta}{2}\mathbb{E}\sup_{t\leq T}\|u^\varepsilon(t,\cdot)\|^2_{L_2(\mathbf{R}^d)} + \frac{\delta^{-1}}{2}\mathbb{E}\int_0^T \|g^\varepsilon(s,\cdot)\|^2_{L_2(\mathbf{R}^d)}ds\end{aligned}$$

Thus taking $\delta > 0$ small enough, we have

$$\begin{aligned}&\mathbb{E}\sup_{t\leq T}\int_{\mathbf{R}^d}|u^\varepsilon(t,x)|^2dx+2\mathbb{E}\int_0^T\int_{\mathbf{R}^d}|u^\varepsilon(s,x)|^2dsdx\\&\leq N\left(\mathbb{E}\int_{\mathbf{R}^d}|u_0^\varepsilon(x)|^2dx+2\mathbb{E}\int_0^T\int_{\mathbf{R}^d}|g^\varepsilon(s,x)|dxds\right).\end{aligned}$$

Due to the linearity, for all $n, m \in \mathbb{N}$,

$$\begin{aligned}&\mathbb{E}\sup_{t\leq T}\int_{\mathbf{R}^d}|u^{1/n}(t,x)-u^{1/m}(t,x)|^2dx+2\mathbb{E}\int_0^T\int_{\mathbf{R}^d}|u^{1/n}(s,x)-u^{1/m}(s,x)|^2dsdx\\&\leq N\left(\mathbb{E}\int_{\mathbf{R}^d}|u_0^{1/n}-u_0^{1/m}(x)|^2dx+2\mathbb{E}\int_0^T\int_{\mathbf{R}^d}|g^{1/m}-g^{1/m}(s,x)|dxds\right).\end{aligned}$$

Thus the sequence $u^{1/n}$ is a Cauchy sequence in

$$L_2\left(\Omega,\mathcal{F};C\left([0,T];L_2(\mathbf{R}^d)\right)\right)\cap\mathbb{H}_2^2\left(T\right)$$

and taking $n \to \infty$, we have (3.7) if a solution u exists. Uniqueness easily obtained from the above estimate or it also can be obtained from Theorem 2.7.

If $g \in \mathbb{H}_0^\infty(T)$, then u defined by (3.4) is indeed a solution. In other words, one can easily check that (3.6) holds if $g \in \mathbb{H}_0^\infty(T)$. For general $g \in \mathbb{L}_p(T)$, we use an approximation sequence $g_n \in \mathbb{H}_0^\infty(T)$ so that (3.1) holds. □

Theorem 3.4 (L_2-theory for general equations)**.** *Let $T \in (0,\infty)$. Assume that $a(t) = a(\omega,t)$ is predictable and satisfies the ellipticity condition*

$$\kappa|\xi|^2\leq a^{ij}(t)\xi^i\xi^j\leq\kappa^{-1}|\xi|^2\qquad\forall(\omega,t,\xi)\in\Omega\times[0,T]\times\mathbf{R}^d$$

with a positive constant $\kappa > 0$. Then for all $u_0 \in L_2\left(\Omega,\mathcal{F}_0;H_2^1(\mathbf{R}^d)\right)$, $f \in \mathbb{L}_p(T)$, $g \in \mathbb{H}_2^1(T)$, there exists a unique solution

$$u\in L_2\left(\Omega,\mathcal{F};C\left([0,T];L_2(\mathbf{R}^d)\right)\right)\cap\mathbb{H}_2^2\left(T\right)$$

to equation (0.1) such that

$$\begin{aligned}&\sup_{t\in[0,T]}\mathbb{E}\|u(t,\cdot)\|^2_{H_2^1(\mathbf{R}^d)}+\mathbb{E}\int_0^T\|u_{xx}(t,\cdot)\|^2_{L_2(\mathbf{R}^d)}dt\\&\leq N\left(\mathbb{E}\|u_0\|^2_{H_2^1(\mathbf{R}^d)}+\mathbb{E}\int_0^T\|f(t,\cdot)\|^2_{L_2(\mathbf{R}^d)}dt+\mathbb{E}\int_0^T\|g(t,\cdot)\|^2_{H_2^1(\mathbf{R}^d)}dt\right),\end{aligned}$$

where N depends only on p, κ, and T.

Proof. Since equation (0.1) is linear, we can obtain this theorem from Theorem 2.7 and Theorem 3.3. We left this to reader as an exercise. □

References

[1] Krylov, N. V. "An analytic approach to SPDEs." Stochastic partial differential equations: six perspectives 64 (1999): 185-242.

[2] Krylov, N. V. "Introduction to the theory of random processes." Vol. 43. American Mathematical Soc., 2002.

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