Peaks of the parabolic Anderson model with white noise potential

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4th Korean Croatian Summer Probability Camp
Dubrovnik, July 1, 2026

Outline

Main results

Proof outline

Table of Contents

Review

Consider the parabolic Anderson model (PAM) in $d \leqslant 3$:

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \xi(x) u(t,x)\,, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d\,, \\ u(0,x) \equiv 1 & x \in \mathbf{R}^d\,, \end{cases}$$

where ξ is white noise potential, i.e. mean-zero Gaussian process with covariance $\mathbf{E}[\xi(x)\xi(y)] = \delta_0(x-y)$.

In particular, we are interested in

- **Spatial asymptotics** as $|x| \to \infty$
- **Macroscopic Hausdorff dimension** of tall peaks

Main results

For $d \leqslant 3$, define the random set of peaks

$$\mathscr{P}_t^d(\alpha) := \left\{x \in \mathbf{R}^d : u(t,x) \geqslant \exp\left(\alpha\, t(\log|x|)^{2/(4-d)}\right)\right\},$$

for $\alpha, t > 0$.

Theorem (Ghosal-Y.'26 (AOP))

There exist $\mathfrak{c}_d > 0$ and $t_0 = t_0(\alpha, d) > 0$ such that for all $t \geqslant t_0$

$$\mathrm{Dim}_{\mathrm{H}}[\mathscr{P}_t^d(\alpha)] = (d - \alpha^{(4-d)/2}\mathfrak{c}_d) \vee 0, \qquad a.s.$$

where $\mathrm{Dim}_{\mathrm{H}}$ denotes the macroscopic Hausdorff dimension. Moreover,

$$\limsup_{|x|\to\infty} \frac{\log u(t,x)}{(\log|x|)^{2/(4-d)}} = \left(\frac{d}{\mathfrak{c}_d}\right)^{\frac{2}{4-d}} t, \qquad a.s.$$

We say that peaks of the PAM are *macroscopically multifractal*.

Table of Contents

Proof outline for the lower bound

- We show that $\mathscr{P}_t^d(\alpha)$ contains γ-thick set for all $\gamma > \mathfrak{c}_d \alpha^{(4-d)/2}/d$, which proves

$$\mathrm{Dim}_{\mathrm{H}}[\mathscr{P}_t^d(\alpha)] \geq d(1-\gamma) \to d - \alpha^{(4-d)/2}\mathfrak{c}_d\,.$$

- Indeed, for each n, we can find points $\{x_i\}_{i=1}^{m(n)}$ satistying
  - $x_i \in B(x, e^{n\gamma})$ for $x \in (e^n, e^{n+1}]^d \cap e^{n\gamma}\mathbf{Z}$ and $i = 1, \ldots, m(n)$,
  - $|x_i - x_j| \geqslant e^{n\theta}$ whenever $1 \leqslant i < j \leqslant m(n)$,
  - $m(n) \approx_d e^{nd(\gamma-\theta)}$.
- Then we show that

$$\sum_{n=0}^{\infty} \mathbf{P}\left(\min_{x \in (e^n, e^{n+1}]^d \cap e^{n\gamma}\mathbf{Z}} \max_{1 \leqslant i \leqslant m(n)} \frac{\log u(t, x_i)}{(\log|x_i|)^{2/(4-d)}} \leqslant \alpha t \right) < \infty.$$

Letting $\gamma \to \mathfrak{c}_d \alpha^{(4-d)/2}/d$ as $\theta \to 0$, the conclusion follows from lower bound of the tail estimate and the Borel-Cantelli lemma.

Proof outline for the upper bound

- We prove that for all $\rho \in (d - \mathfrak{c}_d \alpha^{(4-d)/2}, d)$,

$$\sum_{n=0}^{\infty} \nu_\rho^n(\mathscr{P}_t^d(\alpha)) < \infty \qquad \text{a.s.},$$

  which concludes the upper bound by taking $\rho \to d - \mathfrak{c}_d \alpha^{(4-d)/2}$.
- Recalling the definition of ν_ρ^n, we show that

$$\sum_{n=0}^{\infty} e^{-n\rho} \sum_{\substack{y \in \mathbf{Z}^d \\ B(y,1) \subseteq \mathbb{S}_n}} \mathbf{P}\left(\sup_{x \in B(y,1)} \frac{\log u(t,x)}{(\log|x|)^{2/(4-d)}} \geqslant \alpha t \right) < \infty,$$

  which implies $\mathbf{E}[\sum_{n=0}^{\infty} \nu_\rho^n(\mathscr{P}_t^d(\alpha))] < \infty$.

Proof outline

We first focus on the tail estimate $\mathbf{P}(u(t,x) > z)$.

Step 1. Approximate $u(t,x)$ by $u_L(t,x)$ where $L = L(t) = t^a$ using Feynman-Kac representation.

Step 2. Approximate $u_L(t,x)$ by $\exp(\lambda_L t)$ where λ_L is the principal eigenvalue of the Anderson hamiltonian on Q_L. Thus, we have

$$\mathbf{P}(u(t,x) > z) \approx \mathbf{P}(u_L(t,x) > z) \approx \mathbf{P}\left(\lambda_L > \frac{\log z}{t}\right).$$

Step 3. Apply upper and lower bounds for tail probability $\mathbf{P}(\lambda_L > \log z / t)$.

Tail probability of the eigenvalue

[Chouk-Zuijlen'21], [Hsu-Labbé'22]: Tail probability for the principal eigenvalue.

Theorem

Fix $\varepsilon \in (0,1)$. *There exist* $c_2 > c_1 > 0$ *and* s_0 *such that for all* $L \geqslant 1$ *and* $s \geqslant s_0$

$$\mathbf{P}(\lambda_L \leqslant s) \leqslant \exp\left(-c_2 s^{d/2} e^{d \log L - (1+\varepsilon)\mathfrak{c}_d s^{(4-d)/2}}\right)$$

$$\mathbf{P}(\lambda_L \geqslant s) \leqslant c_1 s^{d/2} \exp\left(d \log L - (1-\varepsilon)\mathfrak{c}_d s^{(4-d)/2}\right).$$

Proof idea: Large deviation for the eigenvalue of the Anderson Hamiltonian. The constant $\mathfrak{c}_d$ and the exponent $(4-d)/2$ comes from the corresponding variational problem.

Monotonicity of eigenvalues

For each $x \in Q_L$, we know that $\mathbf{P}(u(t,x) > z) \approx \mathbf{P}(\lambda_L > \log z/t)$. Then how do we derive $\mathbf{P}(\sup_{x\in B(y,1)} u(t,x) > z)$?

Let $\lambda_L(x)$ be the principal eigenvalue on $Q_L^x = [x - L/2, x + L/2]$. For $L \geqslant r \geqslant 1$ and $x, y \in \mathbf{R}^d$ such that $Q_r^y \subseteq Q_L^x$

$$\lambda_L(y) \leqslant \lambda_L(x)\,.$$

Therefore, for $L \geqslant 1$

$$\mathbf{P}\left(\sup_{x\in B(y,1)} u(t,x) > z\right) \leqslant \mathbf{P}\left(\lambda_L(y) \geqslant \frac{\log z}{t}\right)\,.$$

Independence of eigenvalues

For the lower bound, we need to bound

$$\mathbf{P}\left(\max_{1\leqslant i\leqslant m} u(t,x_i)\leqslant z\right)\approx \mathbf{P}\left(\max_{1\leqslant i\leqslant m}\lambda_L(x_i)\leqslant \frac{\log z}{t}\right),$$

for $|x_i - x_j| \geqslant e^{n\theta}$ with $\theta > 0$.

Note that if $|x_i - x_j| \geqslant 2L$, $\lambda_L(x_i)$ and $\lambda_L(x_j)$ are independent since the noise ξ restricted on $Q_L^{x_i}$ and $Q_L^{x_j}$ are independent. Therfore,

$$\mathbf{P}\left(\max_{1\leqslant i\leqslant m}\lambda_L(x_i)\leqslant \frac{\log z}{t}\right)=\prod_{i=1}^{m}\mathbf{P}\left(\lambda_L(x_i)\leqslant \frac{\log z}{t}\right).$$

Step 1

By the Feynman-Kac formula, we have

$$u(t,x) - u_L(t,x) = \mathbf{E}_x\left[e^{\int_0^t (\xi_\varepsilon(B_s) - C_\varepsilon)\,\mathrm{d}s} \mathbb{1}_{B[0,t] \not\subseteq Q_L} \right].$$

We can easily get

$$\mathbf{P}(B[0,t] \not\subseteq Q_L) \approx \mathbf{P}\left(\sup_{s\in[0,t]} |B_t - x| > L \right) \lesssim e^{-cL^2/t}.$$

However, $\xi_\varepsilon - C_\varepsilon$ is not bounded in L^∞ as $\varepsilon \to 0$.

We need a change of measure or *Girsanov theorem.*

Step 1

Consider a semilinear elliptic equation for Y with a large $\eta > 0$

$$(\eta - \frac{1}{2}\Delta)\, Y = \frac{1}{2}|\nabla X|^2 - C + \nabla Y \cdot \nabla X + \frac{1}{2}|\nabla Y|^2 \, .$$

Then since $X - \frac{1}{2}\Delta X = \xi$ (recall $X = (1 - \frac{1}{2}\Delta)^{-1}\xi$), we have

$$\xi - C = X + \eta Y - \frac{1}{2}\Delta(X + Y) - \frac{1}{2}|\nabla(X + Y)|^2 \, ,$$

or equivalently,

$$\frac{1}{2}\Delta(X + Y) = -(\xi - C) + X + \eta Y - \frac{1}{2}|\nabla(X + Y)|^2 \, .$$

On the other hand, by Itô's formula, we obtain

$$(X + Y)(B_t) = (X + Y)(B_0) + \frac{1}{2}\int_0^t \Delta(X + Y)(B_s)\, \mathrm{d}s + \int_0^t \nabla(X + Y)(B_s)\, \mathrm{d}B_s.$$

Step 1

Therefore, we have

$$\exp\left(\int_0^t (\xi(B_s) - C)\,\mathrm{d}s\right) = \mathcal{N}_t \cdot \mathscr{D}_{0,t}\,,$$

where

$$\mathcal{N}_t = \exp\left(\int_0^t \nabla(X+Y)(B_s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t |\nabla(X+Y)(B_s)|^2\,\mathrm{d}s\right)$$

and

$$\mathscr{D}_{s,t} = \exp\left(\int_s^t (X+\eta Y)(B_r)\,\mathrm{d}r + (X+Y)(B_s) - (X+Y)(B_t)\right)\,.$$

Note that $X \in \mathscr{C}^{2-d/2-\kappa}$ and $Y \in \mathscr{C}^{3-d/2-\kappa}$. In particualr, they are bounded and continuous if $d \leqslant 3 \implies \|\mathscr{D}_{0,t}\|_{L^\infty} < \infty$ for all $t > 0$.

Step 1

By the Girsanov theorem,

$$u(t,x) - u_L(t,x) = \mathbf{E}_x\left[e^{\int_0^t (\xi(B_s)-C)\,\mathrm{d}s}\mathbb{1}_{B[0,t]\not\subseteq Q_L}\right] = \bar{\mathbf{E}}_x[\mathscr{D}_{0,t}\mathbb{1}_{Z[0,t]\not\subseteq Q_L}]$$

where $\bar{\mathbf{E}}_x$ is taken w.r.t $\bar{\mathbf{P}}_x$ under which the coordinate process Z_t satisfies

$$Z_t = x + \int_0^t \nabla(X+Y)(Z_s)\,\mathrm{d}s + B'_s\,,$$

for a Brownian motion B' under $\bar{\mathbf{P}}_x$.

To estimate $\bar{\mathbf{P}}_x(Z[0,t]\not\subseteq Q_L)$, we will obtain the transition density Γ_t of Z_t. Γ is the heat kernel of the Cauchy problem

$$\partial_t w = \frac{1}{2}\Delta w + \mu\cdot\nabla w\,, \quad w(T,\cdot) = \delta_y\,,$$

where $\mu = \nabla(X+Y) \in \mathscr{C}^{1-d/2-\kappa}$ is distribution valued if $d \geqslant 2$.

Step 1

Note that for $d = 3$, $\mu = \nabla(X + Y) \in \mathscr{C}^{-1/2-\kappa}$, then the solution w to

$$\partial_t w = \frac{1}{2}\Delta w + \mu \cdot \nabla w, \quad w(T, \cdot) = \delta_y$$

belong to at most $w \in \mathscr{C}^{3/2-\kappa}$. Thus, $\nabla w \in \mathscr{C}^{1/2-\kappa}$, which implies that $\mu \cdot \nabla w$ is not well-defined since $(-1/2-\kappa) + (1/2-\kappa) < 0$.

Therefore, the above equation is also singular. We introduce a pair (w, w') (*paracontrolled distribution*) such that

$$w^\sharp(t) := w(t) - \nabla w(t) \prec \mathscr{I}(\mu)(t) \in \mathscr{C}^{2-2\kappa}.$$

Here, $\prec$ is a praproduct and $\mathscr{I}(\mu)(t) := \int_t^T P_{s-t}\mu(s)\, \mathrm{d}s$.

We then solve a fixed point problem for $(w, \nabla w)$.

Step 1

By the relation between transition semigroup and the generator, for any $\phi \in C_c(\mathbf{R}^d)$ we have

$$\bar{\mathbf{E}}_x[\phi(Z_t)] = \int_{\mathbf{R}^d} \phi(y)\Gamma_t(x,y)\,\mathrm{d}y\,,$$

where the transition density is defined by

$$\Gamma_t(x,y) = w(t,x)\,.$$

The heat kernel estimate for Γ follows from the classical theory by Strook.

Step 1: Strook's theory

Let $\mu = \nabla U$ where $U \in L^\infty(\mathbf{R}^d)$ ($U = X + Y$ in our case). Then, there exist $C_1, C_2, C_3, C_4 > 0$ depending on d such that

$$\Gamma_t(x,y) \leqslant \frac{1}{t^{d/2}} \exp\left(C_1 \| U\|_{L^\infty} - C_2 \frac{|y-x|^2}{t}\right),$$

$$\Gamma_t(x,y) \geqslant \frac{1}{t^{d/2}} \exp\left(-C_3 \| U\|_{L^\infty} - C_3 \| U\|_{L^\infty} \frac{|x-y|^2}{t}\right).$$

Step 1

Let us summarise what we have estimated: Recall $X = (1 - 2^{-1}\Delta)^{-1}\xi$.

$$\|X\|_{\mathscr{C}^{1/2-\kappa}} + \|Y\|_{\mathscr{C}^{1-2\kappa}} \lesssim \mathfrak{a}(\log L)^2 ,$$

where $\mathfrak{a}$ is a random finite constant, which implies that

$$\exp(-\mathfrak{a}^{N_1}(t-s)(\log L)^{N_2})\mathbb{1}_{Z[s,t]\subseteq Q_L} \leqslant \mathscr{D}_{s,t}\mathbb{1}_{Z[s,t]\subseteq Q_L} \leqslant \exp(\mathfrak{a}^{N_1}(t-s)(\log L)^{N_2}) .$$

By Strook's theory, we have

$$t^{-d/2}\exp\left(-\mathfrak{a}(\log L)^2(1+t^{-1}|x-y|^2)\right) \leqslant \Gamma_t(x,y) \leqslant p_t(x,y)\exp(\mathfrak{a}(\log L)^2) ,$$

where p_t is transition density of the Brownian motion. As a corollary, for $r \leqslant L$ and $|x| < r/4$

$$\bar{\mathbb{P}}_x\left(Z[0,t] \not\subseteq Q_r\right) \lesssim \exp\left(C\mathfrak{a}(\log L)^2 t - \frac{r^2}{Ct}\right) .$$

Step 1

Recall that

$$u(t,x) - u_L(t,x) = \mathbf{E}_x\left[e^{\int_0^t (\xi(B_s)-C)\,\mathrm{d}s} \mathbb{1}_{B[0,t]\not\subseteq Q_L}\right] = \bar{\mathbf{E}}_x[\mathscr{D}_{0,t}\mathbb{1}_{Z[0,t]\not\subseteq Q_L}]\,.$$

Taking $L = r = t^2$, for sufficiently large t, we can make

$$u(t,x) - u_L(t,x) \lesssim \exp\left(\mathfrak{a}^{N_1}(2\log t)^{N_2} t + \mathfrak{a}(2\log t)^2 t - \frac{t^4}{Ct}\right),$$

arbitrarily small. In other words, we have approximated

$$u(t,x) \approx u_L(t,x)\,.$$

Step 2: Upper bound

Recall the spectral representation with an initial profile $\psi \in L^2(Q_L)$

$$u_L^{\psi}(t,x) = \sum_{k\in\mathbf{Z}_+} \exp(\lambda_L^{(k)} t)\langle \varphi_L^{(k)}, \psi\rangle \varphi_L^{(k)}(x)\,.$$

Therefore,

$$\int_{Q_L} u_L^{\psi}(t,y)\,\mathrm{d}y = \sum_{k\in\mathbf{Z}_+} \exp(\lambda_L^{(k)} t)\langle \varphi_L^{(k)}, \psi\rangle \langle \varphi_L^{(k)}, \mathbb{1}_{Q_L}\rangle \leqslant \exp(\lambda_L^{(1)} t)\|\psi\|_{L^2}\|\mathbb{1}_{Q_L}\|_{L^2}$$

Since $\delta_y \notin L^2$, we cannot directly use

$$u_L(t,x) = \int_{Q_L} u_L^{\delta_x}(t,y)\,\mathrm{d}y\,.$$

However, by the Chapman-Kolmogorov equation with $\psi = u_L^{\delta_x}(t_0,\cdot)$,

$$u_L(t,x) \leqslant \exp(\lambda_L^{(1)} t)\|u_L^{\delta_x}(t_0,\cdot)\|_{L^2}\|\mathbb{1}_{Q_L}\|_{L^2} \lesssim \exp(\lambda_L^{(1)} t)\,.$$

Step 2 : Lower bound

Now we estimate the lower bound. Recall that we have

$$u_L(t,x) = \bar{E}_x\left[\mathscr{D}_{0,t}\mathbb{1}_{Z[0,t]\subseteq Q_L}\right], \qquad x \in Q_L .$$

Split into time intervals $[0,\delta]$ and $[\delta,t]$.

$$u_L(t,x) = \bar{E}_x\left[\mathscr{D}_{0,\delta}\mathbb{1}_{Z[0,\delta]\subseteq Q_L}\cdot\mathscr{D}_{\delta,t}\mathbb{1}_{Z[\delta,t]\subseteq Q_L}\right]$$

Using the Markov property at time δ, we have

$$u_L(t,x) \geqslant \exp\left(-C\mathfrak{a}^{N_1}(\log L)^{N_2}\delta\right)\bar{\mathbf{E}}_x\left[\mathbb{1}_{Z[0,\delta]\subseteq Q_L}u_L(t-\delta,Z_\delta)\right].$$

For any $r<L$, we can estimate as

$$\bar{\mathbf{E}}_x\left[\mathbb{1}_{Z[0,\delta]\subseteq Q_L}u_L(t-\delta,Z_\delta)\right] \gtrsim \bar{\mathbf{E}}_x\left[\mathbb{1}_{Z_\delta\in Q_r}u_r(t-\delta,Z_\delta)\right],$$

since $\{Z_\delta\in Q_r\}\subseteq\{Z[0,\delta]\subseteq Q_L\}\cup\{Z[0,\delta]\nsubseteq Q_L\,,\,Z_\delta\in Q_r\}$.

Step 2: Lower bound

Now it remain to bound

$$u_L(t,x) \gtrsim \bar{\mathbf{E}}_x\left[\mathbb{1}_{Z_\delta\in Q_r} u_r(t-\delta, Z_\delta)\right].$$

Using the transition density,

$$\bar{\mathbf{E}}_x\left[\mathbb{1}_{Z_\delta\in Q_r} u_r(t-\delta, Z_\delta)\right] = \int_{Q_r} \Gamma_\delta(x,z)\, u_r(t-\delta,z)\,\mathrm{d}z \gtrsim \int_{Q_r} u_r(t-\delta,z)\,\mathrm{d}z\,.$$

By the Chapman-Kolmogorov equation and the Markov property,

$$u_r^{\delta_z}(t,z) \lesssim u_r(t-\delta,z)\cdot\mathscr{D}_{0,\delta}\mathbb{1}_{Z[0,\delta]\in Q_r} \lesssim u_r(t-\delta,z)\cdot\exp\left(C\mathfrak{a}^{N_1}(\log r)^{N_2}\delta\right).$$

Therefore, we have

$$\int_{Q_r} u_r(t-\delta,z)\,\mathrm{d}z \gtrsim \int_{Q_r} u_r^{\delta_z}(t,z)\,\mathrm{d}z = \sum_{k\in\mathbf{Z}_+} e^{\lambda_r^{(k)}t}\int_{Q_r}(\varphi_r^{(k)}(z))^2\,\mathrm{d}z \geqslant e^{\lambda_r^{(1)}t}.$$

Summary

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \xi(x) u(t,x)\,, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d\,, \\ u(0,x) \equiv 1 & x \in \mathbf{R}^d\,. \end{cases}$$

- We showed spatial asymptotics and macroscopic multifractality of tall peaks of the solution to the PAM for $d \leqslant 3$.
- We employed
  - Paracontrolled distribution for singular SPDE theory.
  - Feynman-Kac representation.
  - Spectral representation.
  - Heat kernel estimates, Girsanov's theorem, etc.

Further questions

Open questions:

- Limit theorems for spatial integrals for the PAM in 2D and 3D.
- Concentration: Total mass of the solution is localised on small regions / several points.
- Solution theory and properties for the 4D PAM. The 4D is the critical dimension since there are infinitely many renormalisations.

Thank you

Thank you!