Peaks of the parabolic Anderson model with white noise potential

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Main results

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Review

Consider the parabolic Anderson model (PAM) in $d \leqslant 3$:

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \xi(x) u(t,x), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d, \\ u(0,x) \equiv 1 & x \in \mathbf{R}^d, \end{cases}$$

where ξ is white noise potential, i.e. mean-zero Gaussian process with covariance $\mathbf{E}[\xi(x)\xi(y)] = \delta_0(x-y)$.

In particular, we are interested in

- **Spatial asymptotics** as $|x| \to \infty$
- **Macroscopic Hausdorff dimension** of tall peaks

Main results

For $d \leqslant 3$, define the random set of peaks

$$\mathscr{P}_t^d(\alpha) := \left\{ x \in \mathbf{R}^d : u(t,x) \geqslant \exp\left(\alpha\, t (\log|x|)^{2/(4-d)}\right)\right\},$$

for $\alpha, t > 0$.

Theorem (Ghosal-Y.'26 (AOP))

There exist $\mathfrak{c}_d > 0$ and $t_0 = t_0(\alpha, d) > 0$ such that for all $t \geqslant t_0$

$$\mathrm{Dim}_{\mathrm{H}}[\mathscr{P}_t^d(\alpha)] = (d - \alpha^{(4-d)/2}\mathfrak{c}_d) \vee 0, \qquad a.s.$$

where $\mathrm{Dim}_{\mathrm{H}}$ *denotes the macroscopic Hausdorff dimension. Moreover,*

$$\limsup_{|x|\to\infty} \frac{\log u(t,x)}{(\log|x|)^{2/(4-d)}} = \left(\frac{d}{\mathfrak{c}_d}\right)^{\frac{2}{4-d}} t, \qquad a.s.$$

We say that peaks of the PAM are *macroscopically multifractal*.

Remark on main results

- We believe that tall peaks of the PAM are concentrated on small regions (intermittency).
- However, due to the roughness of ξ, it is extremely hard to study geometric characterization of intermittency.
  - Since every moment $\mathbf{E}[u(t,x)^p]$ blows up at finite time, we cannot derive tail probability $\mathbf{P}(u(t,x) > z)$ from moment asymptotics.
  - The PAM in 2D and 3D is singular, so we have to concern many regularity problems and renormlisation, etc.
- Our result presents **precise asymptotics of tall peaks**, and show that they have **infinitely many different length scales (i.e. multifractal)** for the PAM in 2D and 3D.

Related work

- [König-Perkowski-Zuijlen'22] studied the PAM with 2D white noise potential started from the Dirac delta initial data.
- They proved that the total mass $U(t) := \int_{\mathbf{R}^d} u^{\delta_0}(t,x)\,\mathrm{d}x$ satisfies

$$\lim_{t\to\infty} \frac{\log U(t)}{t\log t} = \frac{2}{\mathfrak{c}_2},$$

which is equivalent to that for all $x \in \mathbf{R}^2$

$$\lim_{t\to\infty} \frac{\log u(t,x)}{t\log t} = \frac{2}{\mathfrak{c}_2}.$$

- We also extended the above result to the 3D case.

Remark on intermittency

- [König-Perkowski-Zuijlen'22] also proved that for $L = L_t = t^a$ with $a \in (0,1)$ as $t \to \infty$

$$\log\left(\min_{x \in Q_L} u^{\delta_0}(t,x)\right) \asymp \log\left(\sup_{x \in \mathbf{R}^2} u^{\delta_0}(t,x)\right) \asymp 2\mathfrak{c}_2^{-1}\, t \log t\,.$$

- This shows that we cannot observe intermittency at least in the t^a-window.
- Therefore, it seems reasonable to investigate tall peaks at exponential scales.

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Macroscopic Hausdorff dimension

- We are interested tall peaks at large scales.
- However, the classical Hausdorff dimension is not suitable for our purpose. For instance, $\dim_{\mathrm{H}}(\mathbf{Z}) = 0$.
- Therefore, we use the macroscopic Hausdorff dimension $\mathrm{Dim}_{\mathrm{H}}$ introduced by [Barlow-Taylor'89,92].

Let $\mathbb{V}_n := [e^{-n}, e^n)^d$, $\mathbb{S}_0 := \mathbb{V}_0$, and $\mathbb{S}_{n+1} := \mathbb{V}_{n+1} \setminus \mathbb{V}_n$ for $n \geqslant 1$. Let $\mathscr{B}(x, r) := \prod_{i=1}^d [x_i, x_i + r)$ for $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$ and $r \geqslant 1$. For any subset $E \subseteq \mathbf{R}^d$, $\rho > 0$, and $n \geqslant 1$, define

$$\nu_\rho^n(E) := \inf\left\{ \sum_{i=1}^m \left(\frac{s(B_i)}{e^n} \right)^\rho : B_i \subseteq \mathbb{S}_n, E \cap \mathbb{S}_n \subseteq \cup_{i=1}^m B_i \right\},$$

where B_i has the form of $B(x, r)$ and $s(B_i) = r$. Then,

$$\mathrm{Dim}_{\mathrm{H}}(E) := \inf\left\{ \rho > 0 : \sum_{n=1}^\infty \nu_\rho^n(E) < \infty \right\}.$$

Macroscopic Hausdorff dimension

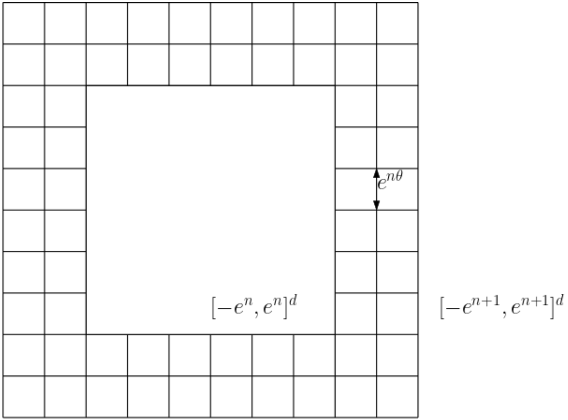

Figure 1: $\mathrm{Dim}_{\mathrm{H}}(E) = d(1-\theta)$ if E is θ-thick set: E contains a point in each cell of side length $e^{n\theta}$ in the outer shell of $[-e^n, e^n)^d$ for all large n.

In other words, we want to measure how densely tall peaks occur across exponential scales.

Examples

- $\mathrm{Dim_H}(\mathbf{Z}^d) = \mathrm{Dim_H}(\mathbf{R}^d) = d$.
- Let $B(t)$ denote a one-dimensional Browian motion and $U(t) := e^{-t/2} B(e^t)$ be an Ornstein-Uhlenbeck process. Then, for all $\gamma > 0$, almost surely

  $$\mathrm{Dim_H}\left\{ t \geqslant 0 : \frac{B(t)}{(2t \log\log t)^{1/2}} \geqslant \gamma \right\} = \begin{cases} 1 & \text{if } \gamma \leqslant 1, \\ 0 & \text{if } \gamma > 1. \end{cases}$$

  and
  $$\mathrm{Dim_H}\left\{ t \geqslant 0 : \frac{U(t)}{(2 \log t)^{1/2}} \geqslant \gamma \right\} = (1 - \gamma^2) \vee 0 .$$
- Therefore, we say that the Brownian motion is (macroscopically) monofractal whereas the OU process is multifractal [Khoshnevisan-Kim-Xiao’17,18].

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Proof outline

- We will only present the proof outline for the macroscopic Hausdorff dimension since it is harder.
- The main input for the proof is a sharp upper and lower bound for the tail estimate

$$\mathbf{P}\left(\sup_{x\in\mathbb{S}^n}\frac{\log u(t,x)}{(\log|x|)^{2/(4-d)}}\geqslant\alpha t\right),$$

where $\mathbb{S}^n$ is an exponential shell $[-e^n, e^n)^d \setminus [-e^{n-1}, e^{n-1})^d$.

Proof outline for the lower bound

- We show that $\mathscr{P}_t^d(\alpha)$ contains γ-thick set for all $\gamma > \mathfrak{c}_d \alpha^{(4-d)/2}/d$, which proves

$$\mathrm{Dim}_{\mathrm{H}}[\mathscr{P}_t^d(\alpha)] \geq d(1-\gamma) \to d - \alpha^{(4-d)/2}\mathfrak{c}_d\,.$$

- Indeed, for each n, we can find points $\{x_i\}_{i=1}^{m(n)}$ satistying
  - $x_i \in B(x, e^{n\gamma})$ for $x \in (e^n, e^{n+1}]^d \cap e^{n\gamma}\mathbf{Z}$ and $i = 1, \dots, m(n)$,
  - $|x_i - x_j| \geqslant e^{n\theta}$ whenever $1 \leqslant i < j \leqslant m(n)$,
  - $m(n) \approx_d e^{nd(\gamma-\theta)}$.
- Then we show that

$$\sum_{n=0}^{\infty} \mathbf{P}\left(\min_{x \in (e^n, e^{n+1}]^d \cap e^{n\gamma}\mathbf{Z}} \max_{1 \leqslant i \leqslant m(n)} \frac{\log u(t, x_i)}{(\log|x_i|)^{2/(4-d)}} \leqslant \alpha t \right) < \infty.$$

Letting $\gamma \to \mathfrak{c}_d \alpha^{(4-d)/2}/d$ as $\theta \to 0$, the conclusion follows from lower bound of the tail estimate and the Borel-Cantelli lemma.

Proof outline for the upper bound

- We prove that for all $\rho \in (d - \mathfrak{c}_d \alpha^{(4-d)/2}, d)$,

$$\sum_{n=0}^{\infty} \nu_\rho^n(\mathscr{P}_t^d(\alpha)) < \infty \qquad \text{a.s.}\,,$$

  which concludes the upper bound by taking $\rho \to d - \mathfrak{c}_d \alpha^{(4-d)/2}$.
- Recalling the definition of ν_ρ^n, we show that

$$\sum_{n=0}^{\infty} e^{-n\rho} \sum_{\substack{y \in \mathbf{Z}^d \\ B(y,1) \subseteq \mathbb{S}_n}} \mathbf{P}\left(\sup_{x \in B(y,1)} \frac{\log u(t,x)}{(\log|x|)^{2/(4-d)}} \geqslant \alpha t \right) < \infty\,,$$

  which implies $\mathbf{E}[\sum_{n=0}^{\infty} \nu_\rho^n(\mathscr{P}_t^d(\alpha))] < \infty$.

Proof outline

We first focus on the tail estimate $\mathbf{P}(u(t,x) > z)$.

Step 1. Approximate $u(t,x)$ by $u_L(t,x)$ where $L = L(t) = t^a$ using Feynman-Kac representation.

Step 2. Approximate $u_L(t,x)$ by $\exp(\lambda_L t)$ where λ_L is the principal eigenvalue of the Anderson hamiltonian on Q_L. Thus, we have

$$\mathbf{P}(u(t,x) > z) \approx \mathbf{P}(u_L(t,x) > z) \approx \mathbf{P}\left(\lambda_L > \frac{\log z}{t}\right).$$

Step 3. Apply upper and lower bounds for tail probability $\mathbf{P}(\lambda_L > \log z/t)$.

Tail probability of the eigenvalue

[Chouk-Zuijlen'21], [Hsu-Labbé'22]: Tail probability for the principal eigenvalue.

Theorem

Fix $\varepsilon \in (0,1)$. There exist $c_2 > c_1 > 0$ and s_0 such that for all $L \geqslant 1$ and $s \geqslant s_0$

$$\mathbf{P}(\lambda_L \leqslant s) \leqslant \exp\left(-c_2 s^{d/2} e^{d \log L - (1+\varepsilon)\mathfrak{c}_d s^{(4-d)/2}}\right)$$

$$\mathbf{P}(\lambda_L \geqslant s) \leqslant c_1 s^{d/2} \exp\left(d \log L - (1-\varepsilon)\mathfrak{c}_d s^{(4-d)/2}\right).$$

Proof idea: Large deviation for the eigenvalue of the Anderson Hamiltonian. The constant $\mathfrak{c}_d$ and the exponent $(4-d)/2$ comes from the corresponding variational problem.

Monotonicity of eigenvalues

For each $x \in Q_L$, we know that $\mathbf{P}(u(t,x) > z) \approx \mathbf{P}(\lambda_L > \log z/t)$. Then how do we derive $\mathbf{P}(\sup_{x\in B(y,1)} u(t,x) > z)$?

Let $\lambda_L(x)$ be the principal eigenvalue on $Q_L^x = [x - L/2, x + L/2]$. For $L \geqslant r \geqslant 1$ and $x, y \in \mathbf{R}^d$ such that $Q_r^y \subseteq Q_L^x$

$$\lambda_L(y) \leqslant \lambda_L(x) \,.$$

Therefore, for $L \geqslant 1$

$$\mathbf{P}\left(\sup_{x\in B(y,1)} u(t,x) > z\right) \leqslant \mathbf{P}\left(\lambda_L(y) \geqslant \frac{\log z}{t}\right) .$$

Independence of eigenvalues

For the lower bound, we need to bound

$$\mathbf{P}\left(\max_{1\leqslant i\leqslant m} u(t,x_i)\leqslant z\right)\approx \mathbf{P}\left(\max_{1\leqslant i\leqslant m}\lambda_L(x_i)\leqslant \frac{\log z}{t}\right),$$

for $|x_i-x_j|\geqslant e^{n\theta}$ with $\theta>0$.

Note that if $|x_i-x_j|\geqslant 2L$, $\lambda_L(x_i)$ and $\lambda_L(x_j)$ are independent since the noise ξ restricted on $Q_L^{x_i}$ and $Q_L^{x_j}$ are independent. Therfore,

$$\mathbf{P}\left(\max_{1\leqslant i\leqslant m}\lambda_L(x_i)\leqslant \frac{\log z}{t}\right)=\prod_{i=1}^{m}\mathbf{P}\left(\lambda_L(x_i)\leqslant \frac{\log z}{t}\right).$$

Step 1

By the Feynman-Kac formula, we have

$$u(t,x) - u_L(t,x) = \mathbf{E}_x \left[e^{\int_0^t (\xi_\varepsilon(B_s) - C_\varepsilon)\, \mathrm{d}s} \mathbb{1}_{B[0,t] \not\subseteq Q_L} \right].$$

We can easily get

$$\mathbf{P}(B[0,t] \not\subseteq Q_L) \approx \mathbf{P}\left(\sup_{s\in[0,t]} |B_t - x| > L \right) \lesssim e^{-cL^2/t}.$$

However, $\xi_\varepsilon - C_\varepsilon$ is not bounded in L^∞ as $\varepsilon \to 0$.

We need a change of measure or *Girsanov theorem.*

Step 1

Consider a semilinear elliptic equation for Y with a large $\eta > 0$

$$(\eta - \frac{1}{2}\Delta)\,Y = \frac{1}{2}|\nabla X|^2 - C + \nabla Y \cdot \nabla X + \frac{1}{2}|\nabla Y|^2 .$$

Then since $X - \frac{1}{2}\Delta X = \xi$ (recall $X = (1 - \frac{1}{2}\Delta)^{-1}\xi$), we have

$$\xi - C = X + \eta Y - \frac{1}{2}\Delta(X+Y) - \frac{1}{2}|\nabla(X+Y)|^2 ,$$

or equivalently,

$$\frac{1}{2}\Delta(X+Y) = -(\xi - C) + X + \eta Y - \frac{1}{2}|\nabla(X+Y)|^2 .$$

On the other hand, by Itô's formula, we obtain

$$(X+Y)(B_t) = (X+Y)(B_0) + \frac{1}{2}\int_0^t \Delta(X+Y)(B_s)\,\mathrm{d}s + \int_0^t \nabla(X+Y)(B_s)\,\mathrm{d}B_s.$$

Step 1

Therefore, we have

$$\exp\left(\int_0^t (\xi(B_s) - C)\,\mathrm{d}s\right) = \mathscr{N}_t \cdot \mathscr{D}_{0,t}\,,$$

where

$$\mathscr{N}_t = \exp\left(\int_0^t \nabla(X+Y)(B_s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t |\nabla(X+Y)(B_s)|^2\,\mathrm{d}s\right)$$

and

$$\mathscr{D}_{s,t} = \exp\left(\int_s^t (X+\eta Y)(B_r)\,\mathrm{d}r + (X+Y)(B_s) - (X+Y)(B_t)\right)\,.$$

Note that $X \in \mathscr{C}^{2-d/2-\kappa}$ and $Y \in \mathscr{C}^{3-d/2-\kappa}$. In particualr, they are bounded and continuous if $d \leqslant 3 \implies \|\mathscr{D}_{0,t}\|_{L^\infty} < \infty$ for all $t > 0$.

Step 1

By the Girsanov theorem,

$$u(t,x) - u_L(t,x) = \mathbf{E}_x\left[e^{\int_0^t (\xi(B_s)-C)\,\mathrm{d}s}\mathbb{1}_{B[0,t]\not\subseteq Q_L}\right] = \bar{\mathbf{E}}_x[\mathscr{D}_{0,t}\mathbb{1}_{Z[0,t]\not\subseteq Q_L}]$$

where $\bar{\mathbf{E}}_x$ is taken w.r.t $\bar{\mathbf{P}}_x$ under which the coordinate process Z_t satisfies

$$Z_t = x + \int_0^t \nabla(X+Y)(Z_s)\,\mathrm{d}s + B'_s\,,$$

for a Brownian motion B' under $\bar{\mathbf{P}}_x$.

To estimate $\bar{\mathbf{P}}_x(Z[0,t] \not\subseteq Q_L)$, we will obtain the transition density Γ_t of Z_t. Γ is the heat kernel of the Cauchy problem

$$\partial_t w = \frac{1}{2}\Delta w + \mu \cdot \nabla w\,, \quad w(T,\cdot) = \delta_y\,,$$

where $\mu = \nabla(X+Y) \in \mathscr{C}^{1-d/2-\kappa}$ is distribution valued if $d \geqslant 2$.

Step 1

Note that for $d = 3$, $\mu = \nabla(X + Y) \in \mathscr{C}^{-1/2-\kappa}$, then the solution w to

$$\partial_t w = \frac{1}{2}\Delta w + \mu \cdot \nabla w, \quad w(T, \cdot) = \delta_y$$

belong to at most $w \in \mathscr{C}^{3/2-\kappa}$. Thus, $\nabla w \in \mathscr{C}^{1/2-\kappa}$, which implies that $\mu \cdot \nabla w$ is not well-defined since $(-1/2 - \kappa) + (1/2 - \kappa) < 0$.

Therefore, the above equation is also singular. We introduce a pair (w, w') (*paracontrolled distribution*) such that

$$w^\sharp(t) := w(t) - \nabla w(t) \prec \mathscr{I}(\mu)(t) \in \mathscr{C}^{2-2\kappa}.$$

Here, $\prec$ is a praproduct and $\mathscr{I}(\mu)(t) := \int_t^T P_{s-t}\mu(s)\,\mathrm{d}s$.

We then solve a fixed point problem for $(w, \nabla w)$.

Step 1

By the relation between transition semigroup and the generator, for any $\phi \in C_c(\mathbf{R}^d)$ we have

$$\bar{\mathbf{E}}_x[\phi(Z_t)] = \int_{\mathbf{R}^d} \phi(y)\Gamma_t(x,y)\,\mathrm{d}y\,,$$

where the transition density is defined by

$$\Gamma_t(x,y) = w(t,x)\,.$$

The heat kernel estimate for Γ follows from the classical theory by Strook.

Step 1: Strook's theory

Let $\mu = \nabla U$ where $U \in L^\infty(\mathbf{R}^d)$ $(U = X + Y$ in our case$)$. Then, there exist $C_1, C_2, C_3, C_4 > 0$ depending on d such that

$$\Gamma_t(x,y) \leqslant \frac{1}{t^{d/2}} \exp\left(C_1 \| U \|_{L^\infty} - C_2 \frac{|y-x|^2}{t} \right),$$

$$\Gamma_t(x,y) \geqslant \frac{1}{t^{d/2}} \exp\left(-C_3 \| U \|_{L^\infty} - C_3 \| U \|_{L^\infty} \frac{|x-y|^2}{t} \right).$$

Step 1

Let us summarise what we have estimated: Recall $X = (1 - 2^{-1}\Delta)^{-1}\xi$.

$$\|X\|_{\mathscr{C}^{1/2-\kappa}} + \|Y\|_{\mathscr{C}^{1-2\kappa}} \lesssim \mathfrak{a}(\log L)^2 ,$$

where $\mathfrak{a}$ is a random finite constant, which implies that

$$\exp(-\mathfrak{a}^{N_1}(t-s)(\log L)^{N_2})\mathbb{1}_{Z[s,t]\subseteq Q_L} \leqslant \mathscr{D}_{s,t}\mathbb{1}_{Z[s,t]\subseteq Q_L} \leqslant \exp(\mathfrak{a}^{N_1}(t-s)(\log L)^{N_2}) .$$

By Strook's theory, we have

$$t^{-d/2}\exp\left(-\mathfrak{a}(\log L)^2(1 + t^{-1}|x-y|^2)\right) \leqslant \Gamma_t(x,y) \leqslant p_t(x,y)\exp(\mathfrak{a}(\log L)^2) ,$$

where p_t is transition density of the Brownian motion. As a corollary, for $r \leqslant L$ and $|x| < r/4$

$$\bar{\mathbb{P}}_x\left(Z[0,t] \not\subseteq Q_r\right) \lesssim \exp\left(C\mathfrak{a}(\log L)^2 t - \frac{r^2}{Ct}\right) .$$

Step 1

Recall that

$$u(t,x) - u_L(t,x) = \mathbf{E}_x\left[e^{\int_0^t (\xi(B_s)-C)\,\mathrm{d}s}\mathbb{1}_{B[0,t]\not\subseteq Q_L}\right] = \bar{\mathbf{E}}_x[\mathscr{D}_{0,t}\mathbb{1}_{Z[0,t]\not\subseteq Q_L}]\,.$$

Taking $L = r = t^2$, for sufficiently large t, we can make

$$u(t,x) - u_L(t,x) \lesssim \exp\left(\mathfrak{a}^{N_1}(2\log t)^{N_2}t + \mathfrak{a}(2\log t)^2 t - \frac{t^4}{Ct}\right),$$

arbitrarily small. In other words, we have approximated

$$u(t,x) \approx u_L(t,x)\,.$$

Difficulty in upper and lowe bound by spectral representation

We somehow want a upper and lower bound uniformly in $x \in Q_L$. However, from the spectral representation, we find that the eigenfunctions have some effect. Therefore, we have to ignore them. It is expected to have some fast decay. Therefore, we need to restrict the box size to see some uniform size of peaks. See KPZ'22.

Step 2: Upper bound

Recall the spectral representation with an initial profile $\psi \in L^2(Q_L)$

$$u_L^{\psi}(t,x) = \sum_{k\in\mathbf{Z}_+} \exp(\lambda_L^{(k)} t)\langle \varphi_L^{(k)}, \psi\rangle \varphi_L^{(k)}(x)\,.$$

Therefore,

$$\int_{Q_L} u_L^{\psi}(t,y)\,\mathrm{d}y = \sum_{k\in\mathbf{Z}_+} \exp(\lambda_L^{(k)} t)\langle \varphi_L^{(k)}, \psi\rangle \langle \varphi_L^{(k)}, \mathbb{1}_{Q_L}\rangle \leqslant \exp(\lambda_L^{(1)} t)\|\psi\|_{L^2}\|\mathbb{1}_{Q_L}\|_{L^2}$$

Since $\delta_y \notin L^2$, we cannot directly use

$$u_L(t,x) = \int_{Q_L} u_L^{\delta_x}(t,y)\,\mathrm{d}y\,.$$

However, by the Chapman-Kolmogorov equation with $\psi = u_L^{\delta_x}(t_0,\cdot)$,

$$u_L(t,x) \leqslant \exp(\lambda_L^{(1)} t)\|u_L^{\delta_x}(t_0,\cdot)\|_{L^2}\|\mathbb{1}_{Q_L}\|_{L^2} \lesssim \exp(\lambda_L^{(1)} t)\,.$$

Step 2 : Lower bound

Now we estimate the lower bound. Recall that we have

$$u_L(t,x) = \bar{E}_x\left[\mathscr{D}_{0,t}\mathbb{1}_{Z[0,t]\subseteq Q_L}\right], \qquad x \in Q_L .$$

Split into time intervals $[0,\delta]$ and $[\delta,t]$.

$$u_L(t,x) = \bar{E}_x\left[\mathscr{D}_{0,\delta}\mathbb{1}_{Z[0,\delta]\subseteq Q_L}\cdot\mathscr{D}_{\delta,t}\mathbb{1}_{Z[\delta,t]\subseteq Q_L}\right]$$

Using the Markov property at time δ, we have

$$u_L(t,x) \geqslant \exp\left(-C\mathfrak{a}^{N_1}(\log L)^{N_2}\delta\right)\bar{\mathbf{E}}_x\left[\mathbb{1}_{Z[0,\delta]\subseteq Q_L}u_L(t-\delta,Z_\delta)\right].$$

For any $r<L$, we can estimate as

$$\bar{\mathbf{E}}_x\left[\mathbb{1}_{Z[0,\delta]\subseteq Q_L}u_L(t-\delta,Z_\delta)\right] \gtrsim \bar{\mathbf{E}}_x\left[\mathbb{1}_{Z_\delta\in Q_r}u_r(t-\delta,Z_\delta)\right],$$

since $\{Z_\delta\in Q_r\}\subseteq\{Z[0,\delta]\subseteq Q_L\}\cup\{Z[0,\delta]\nsubseteq Q_L\,,\ Z_\delta\in Q_r\}$.

Step 2: Lower bound

Now it remain to bound

$$u_L(t,x) \gtrsim \bar{\mathbf{E}}_x\left[\mathbb{1}_{Z_\delta\in Q_r} u_r(t-\delta, Z_\delta)\right].$$

Using the transition density,

$$\bar{\mathbf{E}}_x\left[\mathbb{1}_{Z_\delta\in Q_r} u_r(t-\delta, Z_\delta)\right] = \int_{Q_r} \Gamma_\delta(x,z)\, u_r(t-\delta,z)\,\mathrm{d}z \gtrsim \int_{Q_r} u_r(t-\delta,z)\,\mathrm{d}z.$$

By the Chapman-Kolmogorov equation and the Markov property,

$$u_r^{\delta_z}(t,z) \lesssim u_r(t-\delta,z)\cdot\mathscr{D}_{0,\delta}\mathbb{1}_{Z[0,\delta]\in Q_r} \lesssim u_r(t-\delta,z)\cdot\exp\left(C\mathfrak{a}^{N_1}(\log r)^{N_2}\delta\right).$$

Therefore, we have

$$\int_{Q_r} u_r(t-\delta,z)\,\mathrm{d}z \gtrsim \int_{Q_r} u_r^{\delta_z}(t,z)\,\mathrm{d}z = \sum_{k\in\mathbf{Z}_+} e^{\lambda_r^{(k)}t}\int_{Q_r}(\varphi_r^{(k)}(z))^2\,\mathrm{d}z \geqslant e^{\lambda_r^{(1)}t}.$$

Summary

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \xi(x) u(t,x)\,, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d\,, \\ u(0,x) \equiv 1 & x \in \mathbf{R}^d\,. \end{cases}$$

- We showed spatial asymptotics and macroscopic multifractality of tall peaks of the solution to the PAM for $d \leqslant 3$.
- We employed
  - Paracontrolled distribution for singular SPDE theory.
  - Feynman-Kac representation.
  - Spectral representation.
  - Heat kernel estimates, Girsanov's theorem, etc.

Further questions

Open questions:

- Limit theorems for spatial integrals for the PAM in 2D and 3D.
- Concentration: Total mass of the solution is localised on small regions / several points.
- Solution theory and properties for the 4D PAM. The 4D is the critical dimension since there are infinitely many renormalisations.

Thank you

Thank you!