Peaks of the parabolic Anderson model with white noise potential

Jaeyun Yi

June E Huh Center for Mathematical Challenges (HCMC)
at Korea Institute for Advanced Study (KIAS)

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Outline

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Diffusion in disorderd systems

- **Anderson Hamiltonian:** In "Absence of Diffusion in Certain Random Lattices"(1959), P.W. Anderson suggested that

$$\mathscr{H} = \Delta + \xi$$

describes the spin diffusion or conduction in "impurity band" where ξ is a **random potential** (*random Schrödinger operator*).

Diffusion in disorderd systems

- **Anderson Hamiltonian:** In "Absence of Diffusion in Certain Random Lattices"(1959), P.W. Anderson suggested that

$$\mathscr{H} = \Delta + \xi$$

  describes the spin diffusion or conduction in "impurity band" where ξ is a **random potential** (*random Schrödinger operator*).
- Anderson observed in the nuclear magnetic resonance that electrons becomes trapped, classical diffusion is suppressed, and some localisation occurs (**Anderson localisation**).

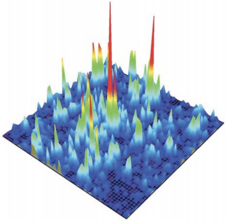

Figure 1: The energy density of 2.4-MHz elastic wave localised in the network of aluminum beads (generated by John Page)

White noise

- **White noise potential** ξ: A distribution-valued Gaussian process with mean zero and covariance

$$\mathbf{E}[\xi(x)\xi(y)] = \delta_0(x-y)\,, \quad x, y \in \mathbf{R}^d\,.$$

- Note that white noise is distribution-valued and defined for any compactly supported smooth functions $\varphi, \psi : \mathbf{R}^d \to \mathbf{R}$,

$$\mathbf{E}[\xi(\varphi)\xi(\psi)] = \int_{\mathbf{R}^d} \varphi(x)\psi(x)\,\mathrm{d}x\,.$$

- Why is it called "white"? : All Fourier modes are uniform, i.e.

$$\mathbf{E}[\xi(e_n)^2] = \int_{\mathbf{R}^d} e_n(x)^2\,\mathrm{d}x = 1\,, \quad \text{for all } n \in \mathbf{N}$$

where $(e_n)_{n\in\mathbf{N}}$ is an L^2-orthonormal basis.

1D Anderson Hamiltonian

- Indeed, white noise is the derivative of a Brownian motion $(B(x))_{x\in\mathbf{R}}$, i.e. the AH is written as

$$\mathscr{H} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}B}{\mathrm{d}x}.$$

- In 1D, [Fukushima-Nakao'77] showed that the bilinear form

$$\mathscr{E}(f,g) = \int fg + \int (f'g + fg')B$$

is closed in H_0^1, which leads to construction of a self-adjoint operator $\mathscr{H}$. Therefore, it admits a pure point spectrum.

- In higher dimensions...?

Spectrum of Anderson Hamiltonian on $Q_L = [0, L]$

- [McKean'94] proved that as the interval length $L \to \infty$, the (shifted) **first eigenvalue with Dirichlet boundary** condition

$$4\sqrt{a_L}(\lambda_L^{(1)} - a_L)$$

converges to a **Gumbel distribution** where $a_L \approx (3/8 \log L)^{2/3}$.

- Let $(\varphi_L^{(k)})_{k\in\mathbf{N}}$ be associated eigenfunctions of $(\lambda_L^{(k)})_{k\in\mathbf{N}}$. Then [Dumaz-Labbé'20] proved that as $L \to \infty$

$$\text{Rescaled } (\lambda_L^{(k)}, \varphi_L^{(k)})_{k\in\mathbf{N}} \to (\lambda^{(k)}, \delta_{U_k})_{k\in\mathbf{N}} \,,$$

where $(\lambda^{(k)}, U_k)_{k\in\mathbf{N}}$ is a Poisson point process, and U_k is the limit of the location of maximum of $|\varphi_L^{(k)}|$.

- In other words, **eigenfunctions are localised** as $L \to \infty$.

Eigenfunctions of Anderson Hamiltonian

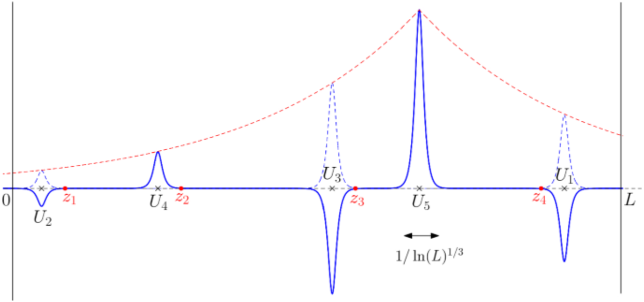

Figure 2: A schematic plot of eigenfunctions. The main peak lies at U_k. The height of the peak near U_k decays exponentially [Dumaz-Labbé'20].

Beyond 1D Anderson Hamiltonian

- The work in [Dumaz-Labbé'20] successfully established the **Anderson localisation** in terms of localised eigenfunctions.
- However, their proof relies on the **Riccati transform**, which only works in 1D (a SDE for $X_\lambda := \varphi'/\varphi$ with an eigenpair (λ, φ)).

Conjecture

The rescaled point process of $(\lambda_L^{(k)}, \varphi_L^{(k)})_{k\in\mathbf{N}}$ *converges to a Poisson point process as* $L \to \infty$ *for* $d \geqslant 2$.

Anderson Hamiltonian in higher dimensions

There have been some fruitful results in $d = 2, 3$ recently.

- [Chouk-Zuijlen'22, Hsu-Labbé'23]: **Large deviation** and **asymptotics** of eigenvalues $(\lambda_L^{(k)})_{k\in\mathbf{N}}$ on $Q_L := (-L, L)^d$.
- [Hsu-Labbé'26]: Construction of $\mathscr{H}$ on $\mathbf{R}^2$ and $\mathbf{R}^3$. The spectrum is $\mathbf{R}$ almost surely.
- [Hsu'26]: Construction of $\mathscr{H}_A := (i\nabla + A)^2 + \xi$ and asympotics of eigenvalues (AH subjected to a magnetic field A).

Parabolic Anderson model

Equipped with the construction of $\mathscr{H}$, we are interested in the **dynamical model**: Cauchy problem $\partial_t u = \mathscr{H} u$, i.e.

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \xi(x) u(t,x)\,, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d\,, \\ u(0,x) \equiv 1 & x \in \mathbf{R}^d\,. \end{cases}$$

We call the equation the **Parabolic Anderson Model** (PAM) with white noise potential.

The PAM is a canonical example of **stochastic partial differential equations** (SPDEs) with multiplicative noise.

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Parabolic Anderson model

The PAM $\partial_t u = \Delta u + \xi u$ has many motivation.

- Random walk in random potential.
- Branching particles following diffusion.
- Random Schrödigner equation.

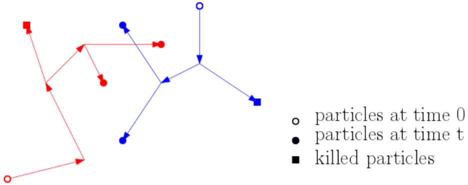

Figure 3: Two particles end up with four particles after branching and dying [Corwin-Shen'20]

Feynman-Kac solution

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \xi(x) u(t,x)\,, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d\,, \\ u(0,x) \equiv 1 & x \in \mathbf{R}^d\,. \end{cases}$$

If ξ is smooth and bounded, we have the **Feynman-Kac** representation of the solution u:

$$u(t,x) = \mathbf{E}_x\left[e^{\int_0^t \xi(B_s)\,\mathrm{d}s} \right]$$

where $(B_t)_{t\geq 0}$ is a d dimensional Brownian motion with $B_0 = x \in \mathbf{R}^d$.

Proof idea: Apply Itô's formula to $M_s := \exp\left(\int_0^s \xi(B_r)\,\mathrm{d}r\right) u(t-s, B_s)$ to show $(M_t)_{t\geqslant 0}$ is a martingale. Then $\mathbf{E}_x[M_t] = \mathbf{E}_x[M_0]$ completes the proof.

Feynman-Kac solution for white noise

Since the white noise ξ is not function-valued, we cannot define the Brownian integral directly.

Let $\varrho \in \mathscr{C}_c^\infty(\mathbf{R}^d)$ be a symmetric probability density function and $\varrho_\varepsilon(x) = \varepsilon^{-d}\varrho(\varepsilon^{-1}x)$ for $x \in \mathbf{R}^d$ and $\varepsilon > 0$. Set

$$\xi_\varepsilon(x) := \langle \xi, \varrho_\varepsilon(\cdot - x)\rangle\,, \qquad x \in \mathbf{R}^d\,.$$

[Chen'14]: For $d = 1$, under $\mathbf{E} \otimes \mathbf{E}_x$, the L^2-limit

$$\int_0^t \xi(B_s)\,\mathrm{d}s := \lim_{\varepsilon\to 0}\int_0^t \xi_\varepsilon(B_s)\,\mathrm{d}s$$

exists as a mean-zero Gaussian process that is $(1/2-\kappa)$-Hölder continuous for any $\kappa > 0$. Therefore, the Feynman-Kac solution $u(t,x) = \mathbf{E}_x\left[e^{\int_0^t \xi(B_s)\,\mathrm{d}s}\right]$ is well-defined for $d = 1$.

Feynmac-Kac solution for more singular noise

[Chen'14]: Quenched asymptotics (a.s.) of the Feynman-Kac solution for a class of noise ξ in which the covaraince is sufficiently regular (e.g. 1D white noise).

[Chen-Deya-Ouyang-Tindel'21]: Moment estiamtes for the Feynman-Kac solution for a class of noise in which the covariance may blow up.

Let $V_\varepsilon(t) := \int_0^t \xi_\varepsilon(B_s)\, \mathrm{d}s$. Then in *Skorohod* or *Wick* sense

$$u(t,x) := \lim_{\varepsilon \to 0} \mathbf{E}_x \left[\exp\left(V_\varepsilon(t) - \frac{1}{2}\mathbf{E}_x[V_\varepsilon(t)^2] \right) \right]$$

is well-defined although $\mathbf{E}_x[V_\varepsilon(t)^2] \to \infty$.

Interesting topics for the PAMs

- Solution theory for a large class of noise.
- Longtime asymptotics: Almost sure and moment estimates.
- Concentration of total mass on small islands.
- Limit theorems for $\int_{Q_L} u(t,x)\,\mathrm{d}x$ as $L = L_t \to \infty$.
- ⋆ Spatial asymptotics: $\limsup$ behaviour of peaks as $|x| \to \infty$.
- ⋆ Macroscopic Hausdorff fractal dimension of peaks.

Intermittency

In particular, the solution of the PAM is expected to have **tall peaks concetrated on small islands** (cf. Anderson localisation).

This property is called **intermittency** [Carmona-Molchanov'94].

We say that u is intermittent if for $0 < p < q$,

$$\lim_{t\to\infty} \frac{(\mathbf{E}[u(t,0)^p])^{1/p}}{(\mathbf{E}[u(t,0)]^q)^{1/q}} = 0\,.$$

For the 1D PAM, we have $\mathbf{E}[u(t,x)^p] = \exp(p^3 t^3/48)$, hence it has intermittency, which implies that its moment is concentrated on small regions in the probability space.

For $d \geqslant 2$, moments blow up in finite time. If moments are infinite, how do we study features of tall peaks?

Complete concentration in discrete PAMs

Consider the PAM on $\mathbf{Z}^d$:

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \xi(x) u(t,x) , & (t,x) \in \mathbf{R}_+ \times \mathbf{Z}^d , \\ u(0,x) \equiv \delta_0(x) \quad x \in \mathbf{Z}^d , \end{cases}$$

where ξ is i.i.d. with $\mathbf{P}(\xi(x) < z) = 1 - z^{-\alpha}$ for $\alpha > d$.

Theorem (König-Lacoin-Mörters-Sidorova'09)

There exist processes $(Z_t^{(1)})_{t>0}$ and $(Z_t^{(2)})_{t>0}$ such that

$$\lim_{t\to\infty} \frac{u(t,Z_t^{(1)}) + u(t,Z_t^{(2)})}{\sum_{z\in\mathbf{Z}^d} u(t,z)} = 1 \qquad \textit{almost surely.}$$

Moreover,

$$\lim_{t\to\infty} \frac{u(t,Z_t^{(1)})}{\sum_{z\in\mathbf{Z}^d} u(t,z)} = 1 \qquad \textit{in probability.}$$

Open question: Similar results for continuous PAMs?

Limit theorems for the PAM in 1D

Let u be a solution to the PAM and $L(t) = \exp(\alpha^3 t^3/24)$ for $\alpha > 0$.

Theorem (Kim-Kim-Y.'26+)

Let $Q_L = [-L(t)/2, L(t)/2]$. For $\alpha > 2$, the central limit theorem holds for $\int_{Q_L} u(t,x)\,\mathrm{d}x$ as $t \to \infty$. On the other hand, for $\alpha \in (0,2)$, the stable limit theorem holds for $\int_{Q_L} u(t,x)\,\mathrm{d}x$ as $t \to \infty$.

In particular, in the regime $\alpha \in (0,2)$, the above theorem says that the tall peaks of the solution govern the limiting distribution.

Difficulty: Since the noise ξ is distribution-valued and unbounded in $\mathbf{R}^d$, it is difficult to analyse the solution u directly.

Idea: Approximation by solutions localised on $Q_L = [-L/2, L/2]$.

Feynman-Kac representation on a box

Consider the PAM on the box $Q_L = [-L/2, L/2]^d$ with the Dirichlet boundary condition:

$$\begin{cases} \partial_t u_L(t,x) = \Delta u_L(t,x) + \xi_L(x) u_L(t,x)\,, & (t,x) \in \mathbf{R}_+ \times Q_L\,, \\ u_L(0,x) \equiv 1 & x \in Q_L\,. \end{cases}$$

Then we have the Feynman-Kac representation:

$$u_L(t,x) = \mathbf{E}_x \left[e^{\int_0^t \xi_L(B_s)\,\mathrm{d}s} \mathbb{1}_{B[0,t] \subseteq Q_L} \right],$$

where $B[0,t] := \{B_s : s \in [0,t]\}$.

Spectral representation on a box

Consdier the PAM on the box $Q_L = [-L/2, L/2]^d$ with the DC boundary condition:

$$\begin{cases} \partial_t u_L(t,x) = \Delta u_L(t,x) + \xi_L(x) u_L(t,x)\,, & (t,x) \in \mathbf{R}_+ \times Q_L\,, \\ u_L(0,x) \equiv 1 \quad x \in Q_L\,. \end{cases}$$

We have the **spectral representation** of the solution u_L:

$$u_L(t,x) = \sum_{k \in \mathbf{Z}_+} e^{\lambda_L^{(k)} t} \varphi_L^{(k)}(x) \varphi_L^{(k)}(0)\,.$$

Two important tools for estimates:

(i) **Feynman-Kac representation**

(ii) **Spectral representation**

The PAM in higher dimensions: Singular case

The PAM on $\mathbf{R}^d$ and Q_L becomes more *singular* since ξ has lower regularity in higher dimensions.

In $d = 2$, the Wick product successfully makes sense of the Feynman-Kac solution by subtracting a diverging counterterm.

However, in $d = 3$, we need a completely new theory to solve it.

Remark: $d = 4$ case is a critical case. [Gabriel-Rosati'26+] studied the elliptic Anderson model.

The PAM with time-dependent noise

We can also consider the PAM with space-time white noise

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \xi(t,x) u(t,x)\,, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d\,, \\ u(0,x) \equiv 1 & x \in \mathbf{R}^d\,. \end{cases}$$

where the covariance of ξ is given by

$$\mathbf{E}[\xi(t,x)\xi(s,y)] = \delta_0(t-s)\delta_0(x-y)\,.$$

This model has also been studied extensively by Walsh, Krylov, Dalang, Mueller, Khoshnevisan, Kim, Y., etc.

In particular, by the Cole-Hopf transform $h = \log u$, the model is related to the Kardar-Parisi-Zhang equation and many intergrable models.

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Hölder-Besov regularity of noise

- Since ξ is distribution-valued, we need a generalised notion of the Hölder regularity.
- Define $\psi \in C_c^\infty(\mathbf{R}^d)$ such that $\mathrm{supp}(\psi) \subseteq B(0,1)$. Let

$$\psi_x^\delta(y) = \delta^{-d}\psi\left(\frac{y-x}{\delta}\right).$$

- To measure the regularity of ξ, we use the Kolmogorov continuity theorem:

$$\mathbf{E}[|\langle \xi, \psi_x^\delta\rangle|^p] \lesssim_p \delta^{-\frac{dp}{2}} \quad \text{for all } \delta > 0, p \geqslant 2, x \in \mathbf{R}^d$$

  implies that $\xi \in \mathscr{C}^{-d/2-\kappa}$ for all $\kappa > 0$.
- This is equivalent to the usual Hölder-Besov space $\mathscr{B}_{\infty,\infty}^{-d/2-\kappa}$.

Why is it called singular?

- A standard PDE argument (Schauder estimate) implies that $u \in \mathscr{C}^{2-d/2-\kappa}$.
- **Problem:** The product $u \cdot \xi$ in the PAM is well-defined only if the sum of the regularity is positive (Young's integral).
- However, for $d \geqslant 2$,

$$\left(2 - \frac{d}{2} - \kappa\right) + \left(-\frac{d}{2} - \kappa\right) = 2 - d - \kappa < 0 \,.$$

- Therefore, the PAM in $d \geqslant 2$ is called singular, and needs a new solution theory.

Heuristic understanding of singular SPDEs (I)

Write the mild formulation (Duhamel's formula):

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta}(u(s)\xi)\,\mathrm{d}s\,.$$

The first order approximation suggests at small scales

$$u(t) \approx \int_0^t e^{(t-s)\Delta}\xi\,\mathrm{d}s = (1 - e^{t\Delta})(-\Delta)^{-1}\xi \approx (-\Delta)^{-1}\xi\,.$$

Therefore, it suffices to analyse the most irregular term of the PAM

$$\xi \cdot u \approx \xi \cdot (-\Delta)^{-1}\xi\,,$$

where the RHS can be understood as an iterated Gaussian integral.

Heuristic understanding of singular SPDEs (II)

- Let $\xi_\varepsilon = \rho_\varepsilon * \xi$ where $(\rho_\varepsilon)_\varepsilon$ is a mollifier. The problem is that

$$\xi_\varepsilon \cdot (1-\Delta)^{-1}\xi_\varepsilon$$

  does not converge as $\varepsilon \to 0$ in any sense.
- **Renormalisation**: Instead, the recentred term

$$\xi_\varepsilon \cdot (1-\Delta)^{-1}\xi_\varepsilon - \mathbf{E}[\xi_\varepsilon \cdot (1-\Delta)^{-1}\xi_\varepsilon]$$

  converges as $\varepsilon \to 0$ (*Wick renormalisation*).
- Here, $C_\varepsilon := \mathbf{E}[\xi_\varepsilon \cdot (1-\Delta)^{-1}\xi_\varepsilon]$ is called the renormalisation constant and diverging to ∞ as $\varepsilon \to 0$. Indeed, for $d = 2$

$$C_\varepsilon \approx \log \frac{1}{\varepsilon}\,.$$

Solving the PAM in 2D (I)

- Define $X_\varepsilon := (1-\Delta)^{-1}\xi_\varepsilon$ and consider the renormalised PAM:

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + (\xi_\varepsilon - C_\varepsilon)\, u_\varepsilon\,.$$

- After the exponential transform $u_\varepsilon = e^{X_\varepsilon} v_\varepsilon$, we have

$$\partial_t v_\varepsilon = \Delta v_\varepsilon + 2\nabla X_\varepsilon \cdot \nabla v_\varepsilon + \left(|\nabla X_\varepsilon|^2 - C_\varepsilon + X_\varepsilon\right) v_\varepsilon\,.$$

- Note that $C_\varepsilon \approx \mathbf{E}[\xi_\varepsilon \cdot (1-\Delta)^{-1}\xi_\varepsilon] \approx \mathbf{E}[|\nabla X_\varepsilon|^2]$ .Therefore, as $\varepsilon \to 0$

$$\partial_t v = \Delta v + 2\nabla X \cdot \nabla v + \left(|\nabla X|^{\diamond 2} + X\right) v$$

has a rigorous meaning where

$$|\nabla X|^{\diamond 2} := \lim_{\varepsilon\to 0} |\nabla X_\varepsilon|^{\diamond 2} := \lim_{\varepsilon\to 0} |\nabla X_\varepsilon|^2 - C_\varepsilon\,.$$

Solving the PAM in 2D (II)

- Therefore, we can give a meaning to

$$\partial_t u = \Delta u + \xi u\,,$$

by defining the continuous solution map

$$\Phi(\xi, |\nabla X|^{\diamond 2}) \mapsto u\,.$$

- In other words,

$$\lim_{\varepsilon\to 0} \Phi(\xi_\varepsilon, |\nabla X_\varepsilon|^{\diamond 2}) = \Phi(\xi, |\nabla X|^{\diamond 2}) = u\,.$$

- This is an analogue of the Itô solution map $\Phi(B, \int B\,\mathrm{d}B) = X$ of

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}X_t\,.$$

Solution theories for singular SPDEs

However, for $d = 3$, the previous argument isn't sufficient due to lower regularity.

Since 2014, there have been several successful achievement on the development of solution theories for solving singular SPDEs.

- [Hairer'14] establishes the regularity structure, a generalised Talyor expansion in terms of noise.
- [Gubinelli-Imkeller-Perkowski'15] developed the paracontrolled distribution, an extension of controlled rough path theory of Gubinelli and Lyon.
- [Kupiainen'15], [Linares-Otto-Tempelmayr-Tsatsoulis'24] [Duch'25].

Paracontrolled distribution for 3D case

The key idea for the 2D case is to interprete the solution as a multiplicative perturbation of $X_\varepsilon = (1-\Delta)^{-1}\xi_\varepsilon$, i.e.

$$u_\varepsilon = e^{X_\varepsilon} v_\varepsilon$$

where v_ε has better regularity.

For 3D case, we need further to cancel irregular terms :

$$X_\varepsilon^{\vee} := (1-\Delta)^{-1}(|\nabla X_\varepsilon|^{\diamond 2}) , \qquad X_\varepsilon^{\vee\!\vee} := (1-\Delta)^{-1}(\nabla X_\varepsilon^{\vee} \cdot \nabla X_\varepsilon) ,$$

$$X_\varepsilon^{\vee\!\vee\!\vee} := (1-\Delta)^{-1}(\nabla X_\varepsilon^{\vee\!\vee} \cdot \nabla X_\varepsilon) , \qquad X_\varepsilon^{\vee\vee\vee} := (1-\Delta)^{-1}(|\nabla X^{\vee}|^{\diamond 2}) .$$

We then make the Ansatz

$$u_\varepsilon = e^{X_\varepsilon + X_\varepsilon^{\vee} + X_\varepsilon^{\vee\!\vee}} \tilde{v}_\varepsilon ,$$

where $\tilde{v}_\varepsilon$ is also decomposed to the *paracontrolled* terms and more regular terms [Gubinelli-Ugurcan-Zachhuber'20].

Regularity structure for 3D case

To give a meaning to the ill-posed product $u \cdot \xi$, [Hairer'14] suggested a generalised Taylor expansion in terms of noise by lifting to some abstract vector space $\mathscr{T}$:

Let U be a function $U : \mathbf{R}^{d+1} \to \mathscr{T}$ such that

$$U(z) = \sum_{\tau \in \mathscr{T}} c_\tau(z)(K * \tau) + R,$$

where c_τ are some continuous functions, K is the heat kernel, τ are stochastic iterated intergrals (elements of the Gaussian multiplicative chaos), and R is a remainder.

U is obtained from iterating via $U = K * \xi + R$. The true solution is obtained by *reconstruction*.

Remark on the 4D case

For the PAM, we call the case of $d \leqslant 3$ the subcritical regime.

The number of renormalisation constants $C_\varepsilon^{(1)}, C_\varepsilon^{(2)}, \ldots$ is finite for $d \leqslant 3$. However, infinitely many renormalisations are required for the critical regime, 4D case.

Indeed, in the series expansion of the mollifed solution

$$U_\varepsilon(z) = \sum_{\tau \in \mathscr{T}} c_{\tau(z),\varepsilon}(K * \tau) + R_\varepsilon \,,$$

the number of τ is infinite. We only know that each term can be convergent as $\varepsilon \to 0$ and the series of the limits is convergent.

However, we don't know if the limit of the series is convergent (*perturbative approach*).

Thank you

Thank you!