Nonlocal Operators on Domains

Contents

1 Motivation and overview

Let $\Omega \subset \mathbb{R}^d$ be an open set. The guiding local equation is the heat equation in Ω:

$$\partial_t u = \Delta u \qquad \text{in } \Omega.$$

Given an initial datum f, one has to specify the realization of the Laplacian that is considered in the equation and the boundary condition (Dirichlet, Neumann, Robin). For example, with

Dirichlet boundary condition the local problem is

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x), & t > 0,\ x \in \Omega, \\ u(0,x) = f(x), & x \in \Omega \\ u(t,x) = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

Equivalently, if Δ_D denotes the Dirichlet realization of the Laplacian on Ω, then

$$\frac{d}{dt}u(t) = \Delta_D u(t), \qquad u(0) = f.$$

From an analysis point of view, it is often cleaner to work with the nonnegative self-adjoint operator

$$L := -\Delta_D.$$

Then the same equation becomes

$$\frac{d}{dt}u(t) = -Lu(t), \qquad u(0) = f,$$

and the solution is

$$u(t,x) = e^{-tL}f(x) = T_t f(x) = \mathbb{E}_x[f(X_t)].$$

Here $(T_t)_{t\geq 0}$ is the semigroup associated with the operator L, and $X = (X_t : t \geq 0)$ is the Markov process with generator $-L$.

Boundary condition	Analytic realisation	Corresponding process
$u = 0$	Dirichlet Laplacian	killed Brownian motion
$\partial_\nu u = 0$	Neumann Laplacian	reflected Brownian motion
$\partial_\nu u + \beta u = 0$	Robin Laplacian	reflected Brownian motion killed by boundary local time

Our goal is to take a closer look at the nonlocal analogues of these operators on a domain Ω, i.e. operators of the form

$$\phi(-\Delta), \qquad \phi(-\Delta)|_\Omega, \qquad \phi(-\Delta_\Omega), \qquad \phi(-\Delta)_{\mathrm{reg}}.$$

It is important to note that these nonlocal regional operators are genuinely different and should not be viewed as different realizations of the same operator.

The model examples are fractional operators, corresponding to the power function

$$\phi(\lambda) = \lambda^\alpha, \qquad \alpha \in (0,1).$$

Here the operator $\phi(-\Delta)$ is called a subordinate Laplacian: it is obtained by applying the Bernstein function ϕ to the positive operator $-\Delta$ via the spectral (or Phillips) functional calculus. This transformation is closely connected to the notion of Bochner subordination of semigroups, and probabilistically subordination of Feller processes.

Our first goal is to give a short overview of the notions of Bernstein functions and subordinators, and motivate the transformations mentioned above.

2 Bernstein Functions and Subordinators

A comprehensive overview of the theory of Bernstein functions, subordinators and generalised fractional derivatives can be found in [16].

Definition 2.1. *A function $\phi : [0,\infty) \to [0,\infty)$ is a Bernstein function if $\phi \in C^\infty((0,\infty))$ and*

$$(-1)^{n+1}\phi^{(n)}(\lambda) \geq 0, \qquad \lambda > 0, \quad n \in \mathbb{N}.$$

We write $\phi \in \mathrm{BF}$.

Equivalently,

$$\phi \in \mathrm{BF} \iff \phi' \in \mathrm{CM},$$

where CM denotes the completely monotone functions:

$$\mathrm{CM} = \left\{ f \in C^\infty((0,\infty)) : (-1)^n f^{(n)}(\lambda) \geq 0, \; n \in \mathbb{N}_0, \; \lambda > 0 \right\}.$$

The following crucial analytic representation appeared in a 1929 paper [2] by Sergei Bernstein and is essentially a version of the probabilistic Lévy–Khintchine formula.

Theorem 2.2 (Lévy–Khintchine representation)**.** *A function ϕ is a Bernstein function if and only if there exist constants $a, b \geq 0$ and a measure ν on $(0,\infty)$ such that*

$$\int_0^\infty (1 \wedge t)\, \nu(\,dt) < \infty \tag{1}$$

and

$$\phi(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \nu(\,dt), \qquad \lambda \geq 0. \tag{2}$$

The triple (a, b, ν) is called the Lévy triplet of ϕ. The measure ν is called the Lévy measure, a is the killing term, and b is the drift term.

If $\phi \in \mathrm{BF}$, then ϕ is increasing and concave. In particular,

$$(1 \wedge \lambda)\phi(1) \leq \phi(\lambda) \leq (1 \vee \lambda)\phi(1), \qquad \lambda > 0. \tag{3}$$

Remark 2.3. *For $\phi(\lambda) = \lambda^\alpha$, $\alpha \in (0,1)$, the Lévy triplet is of the form*

$$\left(0, 0, \frac{\alpha}{\Gamma(1-\alpha)} t^{-1-\alpha}\; dt\right).$$

2.1 Complete and special Bernstein functions

Definition 2.4. *A Bernstein function ϕ is a complete Bernstein function, written $\phi \in \mathrm{CBF}$, if its Lévy measure has a completely monotone density:*

$$\nu(\,dt) = \nu(t)\; dt, \qquad \nu \in \mathrm{CM}.$$

Note that for $\alpha \in (0,1)$, $\phi(\lambda) = \lambda^\alpha$ is a complete Bernstein function.

Definition 2.5. *A Bernstein function ϕ is special, written $\phi \in \mathrm{SBF}$, if*

$$\phi^*(\lambda) = \frac{\lambda}{\phi(\lambda)}$$

is again a Bernstein function. The function ϕ^* is called the conjugate Bernstein function.

The conjugate triplet (a^*, b^*, ν^*) is completely determined by the triplet (a, b, ν) of ϕ. The complete Bernstein functions form a subclass of the special Bernstein functions:

$$\mathrm{CBF} \subset \mathrm{SBF}.$$

Moreover, if $\phi \in \mathrm{CBF}$, then $\phi^* \in \mathrm{CBF}$.

Remark 2.6 (Some closure properties)**.** *Let $\phi_1, \phi_2 \in \mathrm{CBF}$ and $t_1, t_2 > 0$*

(i) $t_1\phi_1 + t_2\phi_2 \in \mathrm{CBF}$, *thus* CBF *is a convex cone;*

(ii) $\phi_1 \circ \phi_2 \in \mathrm{CBF}$;

(iii) $\lambda \mapsto \phi_1(t_1\lambda) \in \mathrm{CBF}$;

(iv) if $t_1 + t_2 \leq 1$, *then* $\phi_1^{t_1}\phi_2^{t_2} \in \mathrm{CBF}$.

2.2 Subordinators

From the probabilistic point of view, Bernstein functions are exactly the Laplace exponents of subordinators.

Definition 2.7. *A subordinator is a stochastic process $S = (S_t : t \geq 0)$ with values in $[0, \infty)$ such that*

(i) $S_0 = 0$*;*

(ii) $t \mapsto S_t$ *is càdlàg almost surely;*

(iii) S *has stationary and independent increments.*

One can easily see that this definition implies that S has almost surely non-decreasing paths.
Let $a \geq 0$, and let $e_a \sim \mathrm{Exp}(a)$ be independent of S. For $a = 0$ we interpret $e_a = +\infty$ almost surely. The killed subordinator $\widehat{S}^a$ is defined by

$$\widehat{S}^a_t = \begin{cases} S_t, & t < e_a, \\ +\infty, & t \geq e_a. \end{cases}$$

Let μ_t denote the transition law of $\widehat{S}^a_t$:

$$\mu_t(B) = \mathbb{P}(\widehat{S}^a_t \in B), \qquad B \in \mathcal{B}([0, \infty)).$$

Its Laplace transform is of the form

$$\mathcal{L}\mu_t(\lambda) = \int_0^\infty e^{-\lambda s}\, \mu_t(\,ds) = \mathbb{E}\left[e^{-\lambda \widehat{S}^a_t}\right] = \mathbb{E}\left[e^{-\lambda S_t} \mathbf{1}_{\{t < e_a\}}\right] = \mathbb{E}\left[e^{-\lambda S_t}\right] e^{-at}, \tag{4}$$

where in the last equality we applied the independence of S and e_a. The stationary independent increments of S and (4) imply that

$$\mathcal{L}\mu_{t+s} = \mathcal{L}\mu_t\, \mathcal{L}\mu_s,$$

and therefore

$$\mu_{t+s} = \mu_t * \mu_s.$$

Theorem 2.8 (Convolution semigroups and Bernstein functions)**.** *A family $(\mu_t)_{t \geq 0}$ is a convolution semigroup of sub-probability measures on $[0, \infty)$ if and only if there exists a unique Bernstein function ϕ such that*

$$\mathcal{L}\mu_t(\lambda) = e^{-t\phi(\lambda)}, \qquad t \geq 0, \quad \lambda \geq 0. \tag{5}$$

Proof. The essence of the proof is the Bernstein theorem for completely monotone functions:

$$f \in \mathrm{CM} \iff f \text{ is the Laplace transform of a positive measure on } [0, \infty).$$

For $\lambda > 0$, set

$$F_t(\lambda) := \int_{[0,\infty)} e^{-\lambda s}\, \mu_t(ds).$$

Since $(\mu_t)_{t \geq 0}$ is a convolution semigroup, $\mu_{t+s} = \mu_t * \mu_s$, so Laplace transforms satisfy

$$F_{t+s}(\lambda) = F_t(\lambda) F_s(\lambda).$$

Vague continuity of convolution semigroups and $\mu_0 = \delta_0$ imply

$$F_t(\lambda) \to 1 \qquad \text{as } t \downarrow 0.$$

Hence, for each fixed $\lambda > 0$, the continuous multiplicative semigroup $t \mapsto F_t(\lambda)$ has the form

$$F_t(\lambda) = e^{-t\phi(\lambda)}$$

for some $\phi(\lambda) \geq 0$. Thus

$$\int_{[0,\infty)} e^{-\lambda s}\, \mu_t(ds) = e^{-t\phi(\lambda)}.$$

Since F_t is the Laplace transform of a positive measure, it is completely monotone in λ. Hence $e^{-t\phi}$ is completely monotone for every $t > 0$. By the standard Bochner characterization of Bernstein functions, this implies that ϕ is a Bernstein function.

Conversely, let ϕ be a Bernstein function. Then $e^{-t\phi(\lambda)}$ is completely monotone for every $t > 0$. By Bernstein's theorem, there exists a sub-probability measure μ_t on $[0, \infty)$ such that

$$e^{-t\phi(\lambda)} = \int_{[0,\infty)} e^{-\lambda s}\, \mu_t(ds).$$

Properties of the Laplace transform and $e^{-t\phi(\lambda)} \to 1$ as $t \downarrow 0$ imply that $(\mu_t)_{t\geq 0}$ is a convolution semigroup. □

Consequently, there is a one-to-one correspondence between killed subordinators and Bernstein functions. This is essentially a one-dimensional subordinator case of the Lévy–Khintchine theory. The function ϕ in (5) is called the *Laplace exponent* of $\widehat{S}^a$,

$$\mathbb{E}\left[e^{-\lambda \widehat{S}^a_t}\right] = e^{-t\phi(\lambda)}, \qquad \lambda \geq 0. \tag{6}$$

Note that the intensity parameter a of the exponential distribution of the random variable e_a is exactly the killing term in the Lévy–Khinchine representation of ϕ in (2).

Example 2.9. *(i) Recall that for $\phi(\lambda) = \lambda^\alpha$, $\alpha \in (0, 1)$, $a = b = 0$, and S is an α-stable subordinator. It has the scaling property*

$$\phi(c\lambda) = c^\alpha \phi(\lambda),$$

which is equivalent to self-similarity, *i.e.*

$$S_{ct} \stackrel{d}{=} c^{1/\alpha} S_t, \qquad c > 0.$$

(ii) Let $c > 0$ and $\phi(\lambda) = c(1 - e^{-\lambda})$. Then the triplet is $(0, 0, c\delta_1)$, and S is a Poisson process with intensity c. More generally, let ν be a finite measure on $(0, \infty)$, and put

$$c := \nu(0, \infty) \in (0, \infty).$$

Then ν trivially satisfies the integrability condition (1) and

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda s})\nu(ds)$$

is a Bernstein function. Let $N = (N_t)_{t\geq 0}$ be a Poisson process with rate c, and let $(Y_k)_{k\geq 1}$ be iid random variables with distribution $\frac{1}{c}\nu$, independent of N. Then

$$S_t := \sum_{k=1}^{N_t} Y_k$$

is a subordinator, called a compound Poisson subordinator. Indeed,

$$\begin{aligned}\mathbb{E}e^{-\lambda S_t} &= \mathbb{E}\left[\left(\mathbb{E}e^{-\lambda Y_1}\right)^{N_t}\right] = \exp\left(-ct\left(1 - \tfrac{1}{c}\int_0^\infty e^{-\lambda s}\nu(ds)\right)\right) \\ &= \exp\left(-t\int_0^\infty (1 - e^{-\lambda s})\nu(ds)\right) = e^{-t\phi(\lambda)}.\end{aligned}$$

(iii) *Other classical examples include* $\phi(\lambda) = \lambda^\alpha + \lambda^\beta$ *and* $\phi(\lambda) = \ell(\lambda)\lambda^\alpha$*, where* ℓ *is slowly varying at* ∞*.*

(iv) *For* $a, b > 0$ *and*

$$\phi(\lambda) = a \log\left(1 + \frac{\lambda}{b}\right),$$

the resulting process S *is the so-called gamma subordinator with Lévy measure*

$$\nu(\,dx) = a\frac{e^{-bx}}{x}\,dx.$$

2.3 The generator of a subordinator

Assume for this section that $a = 0$, so there is no killing. Then S is a Lévy process and hence a Feller process on $[0, \infty)$. Its semigroup is

$$P_t f(x) = \mathbb{E}_x[f(S_t)], \qquad f \in C_0([0, \infty)).$$

The generator is

$$Af(x) = \lim_{t\downarrow 0} \frac{P_t f(x) - f(x)}{t}, \tag{7}$$

for

$$f \in D(A) = \{u \in C_0([0, \infty)) : \text{the limit in (7) exists in } C_0([0, \infty))\}.$$

Using the Lévy–Khintchine representation of ϕ, one obtains the integral form of the generator,

$$Af(x) = bf'(x) + \int_0^\infty (f(x + y) - f(x))\,\nu(\,dy) \tag{8}$$

for $f \in C_0^1([0, \infty))$, see [1, Theorem 3.3.3]. Note that the integrability condition (1) is essential for $C_0^1([0, \infty)) \subset D(A)$.

For comparison, a general Feller generator on $C_c^\infty(\mathbb{R}^d)$ has the Courrege-type form

$$\begin{aligned} Af(x) = &-c(x)f(x) + \sum_i b_i(x)\partial_i f(x) + \sum_{i,j} a_{ij}(x)\partial_{ij} f(x) \\ &+ \int_{\mathbb{R}^d} \Big(f(x + y) - f(x) - \nabla f(x) \cdot y\,\mathbf{1}_{\{|y|\le 1\}}\Big)\,\nu(x,\,dy), \end{aligned}$$

where

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2)\nu(x,\,dy) < \infty, \qquad x \in \mathbb{R}^d.$$

2.4 Trajectories of subordinators

The material on the pathwise behaviour of subordinators presented in this subsection is based on the lecture notes [3]. The Lévy triplet (a, b, ν) determines the pathwise behavior of $\widehat{S}^a$.

If the killing term $a = \phi(0) > 0$, then the killed subordinator dies after an independent exponential random time e_a. Throughout most of the discussion we assume $a = 0$, hence $e_a = +\infty$ and $S = \widehat{S}^a$.

Note that by the Lévy-Khinchine representation, the subordinator with triplet $(0, b, \nu)$ can be represented as

$$S_t = bt + T_t, \tag{9}$$

where T is a subordinator with triplet $(0, 0, \nu)$, therefore b is the *drift* parameter. Moreover,

$$b = \lim_{\lambda\to\infty} \frac{\phi(\lambda)}{\lambda} = (\mathbb{P}) \lim_{t\downarrow 0} \frac{S_t}{t}.$$

Note that for $b > 0$, the trajectories of S are strictly increasing.

The jump activity of a subordinator is governed by the total mass of its Lévy measure. We distinguish two cases:

- **Finite Lévy measure – the compound Poisson case:** If $\nu(0,\infty) < \infty$, then
$$\phi(\lambda) = b\lambda + \nu(0,\infty) \int_0^\infty (1 - e^{-\lambda t}) \frac{\nu(\,dt)}{\nu(0,\infty)}, \tag{10}$$
which corresponds to a compound Poisson process with drift:
$$S_t = bt + \sum_{i=1}^{N_t} Y_i. \tag{11}$$
Here $N = (N_t)_{t\geq 0}$ is a Poisson process with intensity $\nu(0,\infty)$, and $(Y_i)_{i\in\mathbb{N}}$ are iid with law $\frac{\nu(\,dt)}{\nu(0,\infty)}$. Thus S is a random walk time-changed by an independent Poisson process. This is a finite-activity process and, in many ways, behaves like a discrete-time process. If additionally $b = 0$, then S is a step process, and the holding times are exponential with parameter $\nu(0,\infty)$. In this case ϕ is bounded.

- **Infinite Lévy measure – infinite activity:** If $\nu(0,\infty) = \infty$, we say that S has infinite activity. Formally, the Lévy–Itô decomposition of S implies
$$S_t = bt + \sum_{s\leq t} \Delta S_s = bt + \int_{(0,t]\times(0,\infty)} y\, N(\,ds,\,dy) \qquad \text{a.s.}, \tag{12}$$
where $\Delta S_s = S_s - S_{s-}$ and the jumps are described by a Poisson random measure N with characteristic measure ν (intensity measure $dt \times \nu$). More precisely, for $A \in \mathcal{B}(0,\infty)$, $\overline{A} \subset (0,\infty)$ with $\nu(A) < \infty$,
$$N((s,t] \times A) \sim \text{Poisson}((t-s)\nu(A)).$$
Even if $b = 0$, the paths are strictly increasing. Indeed, for $t_1 < t_2$,
$$\begin{aligned}\mathbb{P}(S_{t_1} = S_{t_2}) &= \mathbb{P}\left(\sum_{t_1<s\leq t_2} \Delta S_s = 0\right) \\ &= \mathbb{P}\left(\bigcap_{n\geq 1} \{N((t_1,t_2] \times (1/n,\infty)) = 0\}\right) \\ &= \lim_{n\to\infty} \exp\left(-(t_2 - t_1)\nu((1/n,\infty))\right) = 0,\end{aligned}$$
provided $\nu((1/n,\infty)) \to \nu((0,\infty)) = \infty$. Hence $S_{t_1} < S_{t_2}$ a.s.

Since each path is increasing and càdlàg, it has at most countably many discontinuity points (jump times) at any time interval. Further, almost all times are continuity points, i.e. for each fixed $t > 0$,
$$\mathbb{P}(\Delta S_t > 0) = 0,$$
so every deterministic time is almost surely a continuity point.

2.5 Bochner subordination

Bochner subordination is a method for obtaining a new semigroup (and in the Markov case, a new process), from a strongly continuous semigroup by random time change (via a convolution semigroup on $[0,\infty)$).

Let $(B, \|\cdot\|)$ be a Banach space, and let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on B.[1] Let S be a subordinator with transition laws $(\mu_t)_{t\geq 0}$ and Laplace exponent ϕ,
$$\int_0^\infty e^{-\lambda s}\, \mu_t(\,ds) = e^{-t\phi(\lambda)}.$$

[1] For a Feller process $X = (X_t : t \geq 0)$, take $T_t f(x) = \mathbb{E}_x[f(X_t)]$.

For $t \geq 0$, define

$$T_t^\phi f := \int_0^\infty T_s f \, \mu_t(\, ds). \tag{13}$$

We say that $(T_t^\phi)_{t\geq 0}$ is subordinated to $(T_t)_{t\geq 0}$ in the sense of Bochner. Note that this is an operator-valued integral, and that the pointwise evaluation needs additional justification.

Remark 2.10. *The operators T_t^ϕ are contractions:*

$$\|T_t^\phi u\| \leq \int_0^\infty \|T_s u\| \, \mu_t(\, ds) \leq \|u\| \int_0^\infty \mu_t(\, ds) \leq \|u\|.$$

The semigroup property follows from the convolution property of (μ_t):

$$\begin{aligned} T_t^\phi T_s^\phi u &= \int_0^\infty \int_0^\infty T_v T_w u \, \mu_s(\, dw) \mu_t(\, dv) \\ &= \int_0^\infty \int_0^\infty T_{v+w} u \, \mu_s(\, dw) \mu_t(\, dv) \\ &= \int_0^\infty T_r u \, (\mu_s * \mu_t)(\, dr) \\ &= \int_0^\infty T_r u \, \mu_{s+t}(\, dr) = T_{s+t}^\phi u. \end{aligned}$$

Strong continuity is obtained from

$$\begin{aligned} \|T_t^\phi u - u\| &= \left\| \int_0^\infty T_s u \, \mu_t(\, ds) - u \right\| \\ &\leq \int_0^\infty \|T_s u - u\| \, \mu_t(\, ds) + \|u\|(1 - \mu_t([0,\infty))), \end{aligned}$$

using $\mu_t \to \delta_0$ vaguely as $t \downarrow 0$. In the conservative case $\mu_t([0,\infty)) = 1$.

Remark 2.11 (The generator of the subordinate semigroup)**.** *Let A be the generator of $(T_t)_{t\geq 0}$. Formally, the generator of $(T_t^\phi)_{t\geq 0}$ is denoted by*

$$A^\phi = -\phi(-A). \tag{14}$$

This notation is justified by the following informal calculation. If A is a self-adjoint dissipative operator on a Hilbert space with spectral resolution $E(\, d\lambda)$, then

$$A = \int_{-\infty}^0 \lambda \, E(\, d\lambda), \qquad T_t = \int_{-\infty}^0 e^{\lambda t} \, E(\, d\lambda).$$

Informally,

$$T_t^\phi u = \int_0^\infty \int_{-\infty}^0 e^{\lambda s} E(\, d\lambda) u \, \mu_t(\, ds) = \int_{-\infty}^0 e^{-t\phi(-\lambda)} E(\, d\lambda) u = e^{-t\phi(-A)} u.$$

Thus $A^\phi u = -\phi(-A)u$. Phillips' formula gives

$$A^\phi u = -\phi(-A)u = -au + bAu + \int_0^\infty (T_t u - u) \, \nu(\, dt), \tag{15}$$

for suitable $u \in D(A)$.

Remark 2.12. *The subordinate semigroup inherits many important properties from $(T_t)_{t\geq 0}$:*

(i) $(T_t^\phi)_{t\geq 0}$ is a strongly continuous contraction semigroup;

(ii) if (T_t) is Feller, then (T_t^ϕ) is Feller;

(iii) if (T_t) is conservative and $a = 0$, then (T_t^ϕ) is conservative.

Probabilistically, let $X = (X_t)_{t\geq 0}$ be the Markov process associated with $(T_t)_{t\geq 0}$, and let $S = (S_t)_{t\geq 0}$ be a subordinator with Laplace exponent ϕ, independent of X. Then

$$X_t^\phi(\omega) = X_{S_t(\omega)}(\omega), \qquad t \geq 0,$$

defines the Markov process corresponding to $(T_t^\phi)_{t\geq 0}$.

3 Frequent assumptions

3.1 Classes of domains

First we note several conditions on the domain Ω imposed throughout these notes. Throughout we assume that $\Omega \subseteq \mathbb{R}^d$ is an open set, $d \geq 2$. The regularity assumptions imposed on Ω vary according to the role played by the boundary in the statement under consideration. Some constructions only require Ω to be open. For instance, killed Brownian motion and the Dirichlet form with domain $H_0^1(\Omega)$ are well-defined on arbitrary open sets. However, once one wants to speak about traces, normal derivatives, Robin or Neumann boundary conditions, reflected Brownian motion, or sharp boundary estimates, more structure of $\partial\Omega$ is needed.

In the local setting, Lipschitz regularity is typically enough to define Sobolev traces and boundary measures, C^1 or C^2 regularity makes the outward normal derivative and reflected Brownian motion easier to formulate, and $C^{1,1}$ or κ-fat regularity is often imposed when one needs precise heat kernel estimates, boundary Harnack principles, etc. We recall definitions of several different regularity classes for open domains:

Definition 3.1. *The set Ω is a d-set if there are constants $c_1, c_2 > 0$ such that, for all r small enough and $x_0 \in \partial\Omega$,*

$$c_1 r^d \leq |\Omega \cap B(x_0, r)| \leq c_2 r^d.$$

Definition 3.2. *An open set Ω is κ-fat if there exist $\kappa \in (0, 1/2]$ and $R > 0$ such that for every $Q \in \partial\Omega$ and every $r \leq R$, there is a point $A_r(Q) \in \Omega \cap B(Q, r)$ such that*

$$B(A_r(Q), \kappa r) \subseteq \Omega \cap B(Q, r).$$

Definition 3.3. *A domain Ω is $C^{1,1}$ if near each boundary point $Q \in \partial\Omega$, after a change of coordinates, $\partial\Omega$ is represented as the graph of a $C^{1,1}$ function. A C^2 domain is open and connected, and the boundary is locally C^2.*

3.2 Assumptions on subordinators

Throughout these notes we let $S = (S_t)_{t\geq 0}$ be a subordinator with Laplace exponent ϕ, so that

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \qquad \lambda \geq 0.$$

By the Lévy–Khintchine representation,

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\nu(dt),$$

where $b \geq 0$ and ν is a measure on $(0, \infty)$ satisfying the integrability condition (2).

We are interested in purely nonlocal, pure-jump, regional nonlocal operators with singular kernels. A usual assumption in this setting:

$$b = 0 \quad \text{and} \quad \nu(0, \infty) = \infty.$$

Note that the condition $b = 0$ removes the deterministic drift part (otherwise the subordinated Brownian motion would also contain a local Brownian component). Finite Lévy measure would lead to a finite jump activity (finitely many jumps in any finite time interval) and a bounded operator A^ϕ.

Additional assumptions on ϕ are imposed for example when one wants fine analytic and potential estimates, such as heat kernel estimates, Green function estimates, or boundary Harnack principles. One usual assumption is that $\phi \in \mathrm{CBF}$ and that it satisfies a suitable weak scaling. Before introducing this condition, we recall the basic upper and lower bounds for Bernstein functions in (3).

Definition 3.4. *We say that $\phi \in \mathrm{BF}$ satisfies a weak scaling condition, written $\mathrm{WSC}(\alpha, \beta, c, R)$, if there exist constants $\alpha, \beta \in (0, 1)$, $c > 1$ and $R \geq 0$ such that*

$$c^{-1}\,\lambda^\alpha \phi(r) \leq \phi(\lambda r) \leq c\,\lambda^\beta \phi(r), \qquad \lambda \geq 1,\ r > R.$$

The optimal exponents α and β are called the Matuszewska indices for ϕ.

The weak scaling assumptions are a suitable substitute for the exact scaling present in the fractional case, $\phi(\lambda) = \lambda^\alpha$, $\alpha \in (0, 1)$. Note that in this case one has

$$\phi(\rho\lambda) = \rho^\alpha \phi(\lambda), \qquad \lambda,\ \rho > 0.$$

Note that under WSC, ϕ grows between the two powers λ^α and λ^β. This property essentially allows one to extend many arguments from the stable case to more general subordinate Brownian motions. In this setting, it is often convenient to introduce the scale function

$$\Phi(r) := \frac{1}{\phi(r^{-2})},$$

representing the spatial scale of the subordinate Brownian motion. Note that Φ satisfies a corresponding weak scaling condition near zero. Thus the spatial scale has a behaviour between the powers $r^{2\beta}$ and $r^{2\alpha}$ so the corresponding jump process behaves, at small scales, like a stable process whose order lies between 2α and 2β.

4 Subordinate Laplacian on $\mathbb{R}^d$

The basic model in $\mathbb{R}^d$ is the nonlocal operator $L = (-\Delta)^{\alpha/2}$, $\alpha \in (0, 2)$, and more generally $L = \phi(-\Delta)$ is the subordinate Laplacian from Remark 2.11 for $\phi \in \mathrm{BF}$. Although many of the statements hold more generally, we concentrate on the case when ϕ has the Lévy triplet $(0, 0, \nu(t)\,dt)$, where the Lévy measure ν is infinite. First we recall several equivalent ways of defining this operator, other than via Bochner subordination of the corresponding semigroup.[2] A comprehensive overview of equivalent ways of defining the fractional Laplacian can be found in [13].

The Fourier transform definition. Observe that the Laplace operator Δ takes diagonal form in the Fourier variable: it is a Fourier multiplier with symbol $-|\xi|^2$, namely

$$\mathcal{F}(\Delta f)(\xi) = -|\xi|^2 \mathcal{F} f(\xi)$$

for Schwartz functions $f \in \mathcal{S}(\mathbb{R}^d)$. Following this approach one can define

$$\mathcal{F}\big[\phi(-\Delta) f\big](\xi) = \phi(|\xi|^2) \mathcal{F} f(\xi).$$

[2] Equivalent in the sense that the definitions of Lf agree on a rich class of functions f, e.g. $C_b^2(\mathbb{R}^d)$.

Since the characteristic function of the subordinate Brownian motion is of the form

$$\mathbb{E}\left[e^{i\xi\cdot B_{S_t}}\right] = \mathbb{E}\left[e^{-S_t|\xi|^2}\right] = e^{-t\phi(|\xi|^2)},$$

i.e. $\phi(|\xi|^2)$ is the symbol of the positive operator $\phi(-\Delta)$, we note that this definition is equivalent to the one in Remark 2.11.

The singular integral form. Recall that the heat kernel of the Brownian semigroup, i.e. Brownian motion $B = (B_t,\ t \geq 0)$,

$$T_t f(x) = \mathbb{E}_x[f(B_t)] = e^{t\Delta} f(x) = \int_{\mathbb{R}^d} p_t^B(x-y) f(y)\, dy, \tag{16}$$

is of the form

$$p_t^B(z) = (4\pi t)^{-d/2} \exp\left(-\frac{|z|^2}{4t}\right).$$

Now, evaluating the subordinate Laplacian of f at x in (15), we obtain

$$\begin{aligned}
-\phi(-\Delta)f(x) &= \int_0^\infty \left(\int_{\mathbb{R}^d} p_s^B(x-y) f(y)\, dy - f(x)\right) \nu(ds) \\
&= \int_0^\infty \int_{\mathbb{R}^d} (f(y) - f(x))^B p_s(x-y)\, dy\, \nu(ds) \\
&= \int_0^\infty \int_{\mathbb{R}^d} (f(x+z) - f(x)) p_s^B(z)\, dz\, \nu(ds).
\end{aligned}$$

Interchanging the order of integration suggests the definition

$$j(|z|) := \int_0^\infty p_s^B(z)\, \nu(ds) = \int_0^\infty (4\pi s)^{-d/2} \exp\left(-\frac{|z|^2}{4s}\right) \nu(ds). \tag{17}$$

Note that the kernel j is radially symmetric and singular near 0. Thus one can define the operator $-\phi(-\Delta)$ in a pointwise sense as

$$\begin{aligned}
\phi(-\Delta)f(x) &= \mathrm{P.V.} \int_{\mathbb{R}^d} (f(x) - f(y)) j(|x-y|)\, dy \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d\setminus B(x,\varepsilon)} (f(x) - f(y)) j(|x-y|)\, dy \\
&= \int_{\mathbb{R}^d} (f(x) - f(y) - \nabla f(x)\cdot(y-x)) j(|x-y|)\, dy.
\end{aligned}$$

For the stable case $\phi(\lambda) = \lambda^\alpha$, $\alpha \in (0,1)$,

$$j(r) = \frac{4^\alpha \alpha \Gamma\left(\frac{d}{2}+\alpha\right)}{\pi^{d/2}\Gamma(1-\alpha)} r^{-d-2\alpha},$$

and the formula reduces to the usual singular integral representation of the fractional Laplacian.

Heat kernel of the subordinate Laplacian. Using the heat kernel representation of the Brownian semigroup (16) and Tonelli's theorem, we get that the subordinated semigroup T_t^ϕ from (13) satisfies

$$\begin{aligned}
T_t^\phi f(x) &= \int_0^\infty \int_{\mathbb{R}^d} f(y) p_s^B(y-x)\, dy\, \mathbb{P}(S_t \in\, ds) \\
&= \int_{\mathbb{R}^d} f(y) \left(\int_0^\infty p_s^B(y-x) \mathbb{P}(S_t \in\, ds)\right) dy.
\end{aligned}$$

Hence, the transition density of the subordinate Brownian motion is of the form

$$p(t,x,y) = \int_0^\infty p_s^B(y-x)\,\mathbb{P}(S_t \in\, ds).$$

Since the Brownian heat kernel is translation invariant and rotationally symmetric, the same is true of the subordinated kernel, and therefore we often use the notation

$$p(t,x,y) = p(t,y-x) = p(t,|x-y|).$$

In general, this transition density does not have a closed form. In the stable setting $\phi(r) = r^\alpha$, for $\alpha = \frac{1}{2}$, $p(t,z)$ is the Poisson kernel of the half-space in $\mathbb{R}^{d+1}$,

$$p(t,z) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \frac{t}{\left(t^2+|z|^2\right)^{(d+1)/2}}.$$

For $\alpha \in (0,1)$ one can obtain the two-sided estimate for the transition density of an isotropic symmetric 2α-stable Lévy process:

$$p(t,x,y) \asymp t^{-d/2\alpha} \wedge \frac{t}{|x-y|^{d+2\alpha}}.$$

This result can be generalised for ϕ satisfying (WSC), for small time, see e.g. [6, Theorem 1.2]

$$p(t,x,y) \asymp \left(\Phi^{-1}(t)\right)^{-d} \wedge \frac{t}{|x-y|^d \Phi(|x-y|)},\ t \le 1.$$

4.1 The corresponding Dirichlet form

Start with the bilinear form associated with $-\phi(-\Delta)$: for $u, v \in C_c^\infty(\mathbb{R}^d)$,

$$\mathcal{E}(u,v) = (\phi(-\Delta)u, v)_{L^2(\mathbb{R}^d)}.$$

Using the symmetry of j,

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(x)-u(y))(v(x)-v(y))j(|x-y|)\,dx\,dy.$$

The domain is

$$D(\mathcal{E}) = \left\{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u,u) < \infty\right\}.$$

With the norm

$$\|u\|_{\mathcal{E}_1} = \left(\mathcal{E}(u,u) + \|u\|^2_{L^2(\mathbb{R}^d)}\right)^{1/2},$$

the space $(D(\mathcal{E}), \|\cdot\|_{\mathcal{E}_1})$ is a Hilbert space. The pair $(\mathcal{E}, D(\mathcal{E}))$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$:

(i) $\mathcal{E}$ is symmetric and bilinear on $D(\mathcal{E})$;

(ii) $D(\mathcal{E})$ is dense in $L^2(\mathbb{R}^d)$ and closed under $\|\cdot\|_{\mathcal{E}_1}$;

(iii) it satisfies the unit-contraction property;

(iv) it has a core, e.g. $C_c(\mathbb{R}^d) \cap D(\mathcal{E})$, dense both in $(C_c(\mathbb{R}^d), \|\cdot\|_\infty)$ and $(D(\mathcal{E}), \|\cdot\|_{\mathcal{E}_1})$.

By the Fukushima correspondence, [10, Theorem 7.2.1], a regular Dirichlet form gives a strong Markov (Hunt) process whose generator $(A, D(A))$ satisfies

$$\mathcal{E}(u,v) = -(Au,v)_{L^2}, \qquad u \in D(A),\ v \in D(\mathcal{E}).$$

5 Dirichlet subordinate Laplacian

Let $f \in C_c^2(\Omega)$ and extend it by zero to a function $\widetilde{f}$ on $\mathbb{R}^d$. Then the Dirichlet operator on Ω is obtained by applying the whole-space operator to the zero extension and then restricting back to Ω:

$$\phi(-\Delta)\big|_\Omega f(x) := \phi(-\Delta)\widetilde{f}(x), \qquad x \in \Omega.$$

Using the singular integral representation,

$$\begin{aligned}\phi(-\Delta)\big|_\Omega f(x) &= \mathrm{P.V.}\int_\Omega (f(x)-f(y))j(|x-y|)\,dy + f(x)\int_{\Omega^c} j(|x-y|)\,dy \\ &= \mathrm{P.V.}\int_\Omega (f(x)-f(y))j(|x-y|)\,dy + \kappa_\Omega(x)f(x),\end{aligned}$$

where

$$\kappa_\Omega(x) = \int_{\Omega^c} j(|x-y|)\,dy.$$

is called the killing density or killing potential.

Let us look at the corresponding stochastic process. Let $X_t = B_{S_t}$ be the subordinate Brownian motion, and define the first exit time of X from Ω,

$$\tau_\Omega = \inf\{t \geq 0 : X_t \notin \Omega\}.$$

The killed process X^Ω is the stochastic process defined by

$$X_t^\Omega = \begin{cases} X_t, & t < \tau_\Omega, \\ \partial, & t \geq \tau_\Omega, \end{cases}$$

where ∂ is a cemetery state.[3] The killed semigroup $(T_t^\Omega)_t$ is then given by

$$T_t^\Omega f(x) = \mathbb{E}_x[f(X_t^\Omega)] = \mathbb{E}_x[f(X_t); t < \tau_\Omega].$$

It is a strongly continuous contraction semigroup on $C_0(\Omega)$, but in general it is only sub-Markovian, because of the killing.

5.1 The Dirichlet form of the killed process

Recall that the whole-space Dirichlet form associated with $-\phi(-\Delta)$ is given by the bilinear form

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(x)-u(y))(v(x)-v(y))j(|x-y|)\,dx\,dy.$$

with domain

$$D(\mathcal{E}) = \left\{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u,u) < \infty\right\}.$$

The killed form on Ω is obtained by zero extension:

$$D(\mathcal{E})_\Omega = \{u \in L^2(\Omega) : \widetilde{u} \in D(\mathcal{E})\},$$

where $\widetilde{u} = u$ on Ω and $\widetilde{u} = 0$ on Ω^c.[4] Then

$$\begin{aligned}\mathcal{E}^\Omega(u,u) = \frac{1}{2}&\int_\Omega\int_\Omega (u(x)-u(y))^2 j(|x-y|)\,dx\,dy \\ &+ \int_\Omega u(x)^2\kappa_\Omega(x)\,dx.\end{aligned}$$

This form can be considered as a Dirichlet form on $L^2(\Omega)$, and it is regular with core $C_c^\infty(\Omega)$.

[3] This is an absorbing state (outside of $\mathbb{R}^d$) representing the process after the killing. Using the one-point compactification $\Omega_\partial = \Omega \cup \{\partial\}$, a function $f : \Omega \to \mathbb{R}$ may be extended by $\widetilde{f}(\partial) = 0$. This identifies $C_0(\Omega) = \{\widetilde{u} \in C(\Omega_\partial) : \widetilde{u}(\partial) = 0\}$.

[4] These equalities need to hold only *quasi everywhere*, for further reading we refer to [10].

5.2 Dirichlet heat kernel estimates

The killed process has a transition density $p_\Omega(t,x,y)$, given by Hunt's formula

$$p_\Omega(t,x,y) = p(t,x,y) - \mathbb{E}_x\left[p(t-\tau_\Omega, X_{\tau_\Omega}, y); \tau_\Omega < t\right].$$

For more precise information on $p_\Omega(t,x,y)$, one needs to consider how the process X leaves Ω. This requires additional regularity assumptions on Ω and on ϕ. Under (WSC) on ϕ and assuming Ω is κ-fat, the Dirichlet heat kernel satisfies the following two-sided estimates obtained in [9],

$$p_\Omega(t,x,y) \asymp \mathbb{P}_x(\tau_\Omega > t)\mathbb{P}_y(\tau_\Omega > t)p(t,x,y), \;\; t \le 1.$$

Moreover, for $C^{1,1}$ domains, the tail distribution of the exit time can be estimated in terms of the distance to the boundary, $\delta_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$, that is

$$\mathbb{P}_x(\tau_\Omega > t) \asymp 1 \wedge \frac{\Phi(\delta_\Omega(x))^{1/2}}{\sqrt{t}}.$$

The quantity $\Phi(\delta_\Omega(x))^{1/2}$ is connected to the exact boundary decay rate of regular harmonic functions in $C^{1,1}$ sets (e. g. balls). Note that in the stable case it reduces to $\delta_\Omega(x)^{\alpha/2}$.

For bounded domains, large-time estimates involve the principal eigenvalue λ_1 of the killed generator and exponential decay terms such as $e^{-\lambda_1 t}$.

5.3 The Feller property

In general $(T_t^\Omega)_{t\ge0}$ need not be Feller for arbitrary open sets. A regularity condition for the process is needed: for every $z \in \partial\Omega$,

$$\mathbb{P}_z(\tau_\Omega = 0) = 1.$$

Equivalently, for $z \in \partial\Omega$ and $t > 0$,

$$\lim_{x\to z,\, x\in\Omega} \mathbb{P}_x(\tau_\Omega > t) = 0.$$

This condition ensures, for example, that for $f \in C_0(\Omega)$,

$$T_t^\Omega f(x) \to 0 \qquad \text{as } x \to z \in \partial\Omega.$$

A standard class of domains which are regular for isotropic Lévy processes with unbounded characteristic exponents, and therefore X, are those satisfying the exterior cone condition. An open set $\Omega \subset \mathbb{R}^d$ is said to satisfy the *exterior cone condition* if for every $z \in \partial\Omega$ there exist a finite open cone Γ, a radius $r > 0$, and an orthogonal transformation R such that

$$z + R\Gamma \cap B(z,r) \subset \Omega^c.$$

In particular, Lipschitz (and therefore, $C^{1,1}$) domains are regular for X.

The short argument goes as follows. Since ϕ is unbounded, X is not a compound Poisson process. By isotropy, every nonempty cone is accessible at arbitrarily small times with positive probability, and the exterior cone condition therefore implies that $\mathbb{P}_z(\tau_\Omega = 0) > 0$ for every $z \in \partial\Omega$. Since the event $\{\tau_\Omega = 0\}$ belongs to the germ σ-field, Blumenthal's zero–one law yields $\mathbb{P}_z(\tau_\Omega = 0) = 1$.

6 Spectral subordinate Laplacian

6.1 Subordinating the killed Brownian semigroup

The spectral subordinate Laplacian is obtained in a different order from the restricted operator. First one kills Brownian motion on leaving Ω, obtaining the Dirichlet Laplacian $-\Delta_\Omega$. Then one applies the Bernstein function ϕ to $-\Delta_\Omega$, see Remark 2.11.

Let $(P_t^\Omega)_{t\geq 0}$ be the killed Brownian semigroup on Ω. Define the subordinate semigroup (in the sense of Bochner) as

$$T_t^\phi f = \int_0^\infty P_s^\Omega f \, \mathbb{P}(S_t \in \, ds),$$

and similarly as before, note that its transition density is of the form

$$p_\Omega^\phi(t,x,y) = \int_0^\infty p_\Omega^B(s,x,y) \, \mathbb{P}(S_t \in \, ds),$$

where p_Ω^B is the Dirichlet Brownian heat kernel (transition density of the Brownian motion killed upon exiting Ω).

6.2 The integral representation

The operator satisfies, formally,

$$\phi(-\Delta_\Omega) f = \int_0^\infty (f - P_t^\Omega f) \, \nu(\, dt).$$

This leads to a singular-integral-type representation

$$\phi(-\Delta_\Omega) f(x) = \mathrm{P.\,V.} \int_\Omega (f(x) - f(y)) J_\Omega(x,y) \, dy + \kappa_\Omega^{\mathrm{sp}}(x) f(x),$$

where

$$J_\Omega(x,y) = \int_0^\infty p_\Omega^B(t,x,y) \, \nu(\, dt),$$

and

$$\kappa_\Omega^{\mathrm{sp}}(x) = \int_0^\infty (1 - P_t^\Omega \mathbf{1}(x)) \, \nu(\, dt).$$

Here $\kappa_\Omega^{\mathrm{sp}}$ is the killing density of the spectral subordinate process.

If Ω is regular for Brownian motion, e.g. satisfies the exterior cone property, the subordinate killed Brownian process is Feller; this follows from the Feller property of the killed Brownian semigroup and Phillips' theorem on subordination of Feller semigroups, see also Subsection 5.3.

If Ω is $C^{1,1}$ and ϕ satisfies (WSC), then the jumping kernel of the subordinate killed Brownian motion has estimates of the form

$$J_\Omega(x,y) \asymp \left(1 \wedge \frac{\delta_\Omega(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_\Omega(y)}{|x-y|}\right) j(|x-y|),$$

where away from the boundary the jumping density is comparable to the whole-space density j from (17), see [12].

6.3 Spectral formulation

Assume Ω is bounded, open, and connected, so that the Dirichlet Laplacian has discrete spectrum. Let

$$-\Delta_\Omega e_j = \lambda_j e_j, \qquad j \in \mathbb{N},$$

where $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$. Then

$$\phi(-\Delta_\Omega)e_j = \phi(\lambda_j)e_j.$$

For

$$u = \sum_{j=1}^{\infty} \langle u, e_j\rangle_{L^2(\Omega)} e_j,$$

we have

$$\phi(-\Delta_\Omega)u = \sum_{j=1}^{\infty} \phi(\lambda_j)\langle u, e_j\rangle_{L^2(\Omega)} e_j.$$

The operator domain is

$$D(\phi(-\Delta_\Omega)) = \left\{ u \in L^2(\Omega) : \sum_{j=1}^{\infty} \phi(\lambda_j)^2 |\langle u, e_j\rangle_{L^2(\Omega)}|^2 < \infty \right\}.$$

In smooth domains this can often be identified more explicitly in terms of generalised (fractional) Sobolev spaces, though the exact space depends on ϕ, Ω, and the boundary condition.

7 Regional subordinate Laplacian

7.1 Motivation and definition

The previous constructions were obtained either by killing a process and then subordinating, or by subordinating and then killing. The regional operator is instead obtained by restricting the jumping measure to $\Omega \times \Omega$, which one can show is equivalent to appropriately resurrecting the killed process. Informally, only jumps that remain inside Ω are allowed.

Recall the whole-space singular integral

$$\phi(-\Delta)u(x) = \mathrm{P.\,V.} \int_{\mathbb{R}^d} (u(x) - u(y)) j(|x-y|)\, dy.$$

The regional subordinate Laplacian is formally defined by

$$\phi(-\Delta)_{\mathrm{reg}} u(x) = \mathrm{P.\,V.} \int_{\Omega} (u(x) - u(y)) j(|x-y|)\, dy, \qquad x \in \Omega.$$

Comparing this with the restricted operator for $u \in C_c^2(\Omega)$ gives

$$\phi(-\Delta)\big|_\Omega u(x) = \phi(-\Delta)_{\mathrm{reg}} u(x) + \kappa_\Omega(x) u(x),$$

where

$$\kappa_\Omega(x) = \int_{\Omega^c} j(|x-y|)\, dy.$$

Thus

$$\phi(-\Delta)_{\mathrm{reg}} = \phi(-\Delta)\big|_\Omega - \kappa_\Omega.$$

Therefore, the regional operator is obtained by removing the killing potential, i.e. it is a Schrödinger-type perturbation of the Dirichlet subordinate Laplacian by the potential κ_Ω. In order to view it from this perspective, κ_Ω satisfies standard assumptions, for instance belonging to an appropriate Kato class. The corresponding semigroup is then given in terms of the Feynman–Kac transform via a PCAF

$$A_t = \int_0^t \kappa_\Omega(X_s^\Omega)\, ds,$$

i.e.

$$P_t^\Omega f(x) = \mathbb{E}_x\left[f(X_t^\Omega) e^{A_t} \right],$$

where X^Ω is the killed subordinate Brownian motion. This transform should be interpreted carefully, since the potential has the sign that compensates the killing. A variational approach via Dirichlet forms is often better suited for constructing the corresponding regional process.

7.2 The regional Dirichlet form

Recall the Dirichlet form of the killed subordinate Brownian motion X^Ω,

$$\mathcal{E}^\Omega(u,u) = \frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))^2 j(|x-y|)\,dx\,dy \\ + \int_\Omega u(x)^2\kappa_\Omega(x)\,dx.$$

Define the regional bilinear form by removing the killing term in $\mathcal{E}^\Omega$,

$$\mathcal{E}^{\mathrm{reg}}_\Omega(u,u) = \mathcal{E}(u,u) - \int_\Omega u(x)^2\kappa_\Omega(x)\,dx.$$

Hence

$$\mathcal{E}^{\mathrm{reg}}_\Omega(u,u) = \frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))^2 j(|x-y|)\,dx\,dy,$$

i.e.,

$$\mathcal{E}^{\mathrm{reg}}_\Omega(u,v) = \frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))(v(x)-v(y)) j(|x-y|)\,dx\,dy.$$

A natural choice of the domain $\mathcal{F}_0$ could be the closure of $C_c^\infty(\Omega)$ functions under the norm generated by $\mathcal{E}^{\mathrm{reg}}$, which would give a regular Dirichlet form on $L^2(\Omega)$. As before, by [10, Theorem 7.2.1], one can associate a symmetric strong Markov process $Y^{\mathrm{cen}} = (Y^{\mathrm{cen}}_t : t \geq 0)$ with this form, called the *censored process* in Ω.

On the other hand, one can choose the domain of the form

$$\mathcal{F} = \left\{ u \in L^2(\Omega) : \int_\Omega\int_\Omega (u(x)-u(y))^2 j(|x-y|)\,dx\,dy < \infty \right\}.$$

If Ω is a d-set, the Dirichlet form $(\mathcal{E}^{\mathrm{reg}}_\Omega, \mathcal{F})$ generates a regular Dirichlet form on $L^2(\Omega)$, with core $C_c^\infty(\overline{\Omega})$.[5] The corresponding process is denoted by $Y^{\mathrm{ref}} = (Y^{\mathrm{ref}}_t : t \geq 0)$, and is called a *reflected process* in Ω. Note that $\mathcal{F}_0 \subset \mathcal{F}$, and when $\mathcal{F}_0 = \mathcal{F}$ the processes Y^{cen} and Y^{ref} are equivalent. Note that the reflected process is conservative.

By construction, both of these processes make only jumps inside Ω. They are strongly Markov and càdlàg, but are not automatically Feller without additional assumptions.

7.3 Pathwise behavior and heat kernel estimates

By the construction above, one can see that the censored process Y^{cen} in an open set Ω is obtained from the underlying pure-jump subordinate Brownian motion X by suppressing all jumps from Ω to Ω^c. Pathwise, it can be constructed ([5], [17]) by piecing together successive copies of the killed process X^Ω in the sense of Ikeda–Nagasawa–Watanabe. Start a copy of the subordinate Brownian motion X in Ω. When X is killed by a jump from Ω to Ω^c (i.e. X^Ω is killed), it is instantaneously resurrected at its left limit $X_{\tau_\Omega -} \in \Omega$, and an independent copy of X^Ω is started from that point. Note that this procedure can be performed countably many times as long as $X_{\tau_\Omega -} \in \Omega$. The lifetime ζ of Y is the accumulation time of these successive resurrection times. The reflected process is obtained as the maximal extension of the censored process beyond its lifetime and coincides with Y up to time ζ.

Consequently, Y^{cen} never jumps onto or across $\partial\Omega$, and on the event $\{\zeta < \infty\}$,

$$\lim_{t\uparrow\zeta} Y_t \in \partial\Omega.$$

When Ω is a d-set the process Y^{cen} can be represented as the reflected process Y^{ref} killed upon hitting the boundary $\partial\Omega$. Therefore the question of Y^{cen} dying out in finite time is equivalent

[5]We sometimes say that $(\mathcal{E}^{\mathrm{reg}}_\Omega, \mathcal{F})$ is a regular Dirichlet form on $L^2(\overline{\Omega})$.

to the question whether Y^{cen} is a true subprocess of Y^{ref}. This question can be anwsered in terms of the *capacity* of the boundary $\partial\Omega$ for X. We will discuss only a special example of this result, for further reading we refer to [5] for the stable case, and [17] for subordinate Brownian motion with ϕ satisfying (WSC).

For symmetric 2α-stable processes X in bounded Lipschitz domains ($\phi(\lambda) = \lambda^\alpha$), $\alpha \in (0,1)$, the behaviour of Y depends on the index α. If $\alpha \le \frac{1}{2}$, then the capacity of the boundary $\partial\Omega$ w.r.t. X is zero, and therefore Y^{cen} is conservative and almost surely never approaches the boundary. If $\alpha > \frac{1}{2}$, then Y has finite lifetime almost surely and converges to a boundary point at its lifetime. For the reflected process on a d-set Ω, [7] shows that the estimates for the corresponding heat kernel $p_{\mathrm{ref}}(t,x,y)$ are of the same form as the estimates for the heat kerne of X, that is

$$p(t,x,y) \asymp t^{-d/2\alpha} \wedge \frac{t}{|x-y|^{d+2\alpha}}.$$

For Ω a $C^{1,1}$-set and $\phi(\lambda) = \lambda^\alpha$, $\alpha \in (\frac{1}{2}, 1)$, the heat kernel estimates for the censored process were obtained in [8],

$$p_{\mathrm{cen}}(t,x,y) \asymp \left(1 \wedge \frac{\delta_\Omega(x)^{2\alpha-1}}{t^{(2\alpha-1)/2\alpha}}\right)\left(1 \wedge \frac{\delta_\Omega(y)^{2\alpha-1}}{t^{(2\alpha-1)/2\alpha}}\right)\left(t^{-d/2\alpha} \wedge \frac{t}{|x-y|^{d+2\alpha}}\right),\ t \le 1.$$

The main obstacle to extending the sharp heat kernel estimates from censored stable processes to general censored subordinate Brownian motions is not the construction of the censored process itself, which is available in considerable generality through the resurrection transform, but rather the lack of a sufficiently precise understanding of its boundary behaviour. In the stable case $\alpha \in (\frac{1}{2}, 1)$, exact self-similarity, the explicit boundary profile $\delta_D^{2\alpha-1}$, the corresponding boundary Harnack principle and the homogeneous Hardy-type killing term combine to give a complete description of how the heat kernel degenerates near the boundary.

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