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THE CAUCHY FUNCTIONAL EQUATION AND SCALAR PRODUCT IN VECTOR SPACES

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1. In this paper $R = \{t, s, \ldots\}$ denotes the set of all real numbers and $X = \{x, y, \ldots\}$ a real vector space.

A functional $n : X \to R$ is termed a *quadratic functional* if

$$n(x+y) + n(x-y) = 2\,n(x) + 2\,n(y) \tag{1}$$

holds for all $x, y \in X$ [3]. A quadratic functional n is *continuous along rays* if the function $t \to n(tx)$ is continuous in t, for any x.

Improving some results of M. Fréchet [1], P. Jordan and J. v. Neumann [2] have proved the following well-known theorem:

Let X be a complex vector space with distance defined in terms of a norm $|x|$, so that

$$|x+y| \leq |x| + |y|, \; |ix| = |x| \text{ and } \lim_{t \to 0} |tx| = 0.$$

Then the identity

$$|x+y|^2 + |x-y|^2 = 2\,|x|^2 + 2\,|y|^2$$

is characteristic for the existence of an inner product (x, y) connected with the norm by the relations

$$(x, y) = \frac{1}{4}\left[|x+y|^2 - |x-y|^2\right] + \frac{i}{4}\left[|x+iy|^2 - |x-iy|^2\right],$$

$$|x|^2 = (x, x).$$

In Section **2.** of this paper we extend this result to an arbitrary real vector space on which a quadratic functional $n(x)$ is defined, which is bounded on every segment of X. By a segment $\Delta = [x, y]$ of X we understand the set of all vectors $z \in X$ of the form $z = tx + (1-t)\,y$ with $0 \leq t \leq 1$.

In Section **3.** we replace the condition that $n(x)$ is bounded on every segment by the condition $n(tx) = t^2 n(x)$ $(t \in R, x \in X)$ and we obtain some related results which will be used in Section **5.**

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In Section **5.** we find the general form (with respect to an algebraic basic set of X) of a quadratic functional $n(x)$ on X which has the property that $n(tx) = t^2 n(x)$ $(t \in R, x \in X)$. It turns out that $n(x)$ is expressed by use ot d· ivates on R.

The principal results of this paper are summerised in the Main Theorem. It turns out, therefore, that the problem of introducing a scalar product in X by use of a quadratic functional is closely related to some problems with the Cauchy functional equation, i. e. with a function $f: R \to R$ such that

$$f(t+s) = f(t) + f(s) \tag{2}$$

holds for all $t, s \in R$. It is well-known that a function f, which satisfies functional equation (2), is not necessarily continuous. But, if f is continuous or bounded (above or below) on an interval, then it is of the form $f(t) = tf$ (1) [4].

In Section **4.** we give a theorem about functional equation (2) which is used in Section **3.**

A derivative on R is a solution of the Cauchy functional equation (2) which has the additional property that

$$f(ts) = tf(s) + sf(t)$$

holds for all t, s R. The existence and abundance of nontrivial derivatives on R follows from [6] (pp. 120—131).

2. Theorem 1. *Let X be a real vector space and $n: X \to R$ a quadratic functional.*

If for any $x \in X$ there are two positive numbers $A_x > 0$ and B_x such that

$$|t| \leq A_x \text{ implies } |n(tx)| \leq B_x \tag{3}$$

then

$$n(tx) = t^2 n(x) \tag{4}$$

holds for any $t \in R$ and $x \in X$.

Proof. If we set

$$f(t; x) = n(tx),$$

then by use of (1) we get

$$f(t+s; x) + f(t-s; x) = 2f(t; x) + 2f(s; x), \tag{5}$$

for all $t, s \in R$ and $x \in X$. For $t = s = 0$, (5) implies $f(0; x) = 0$. Now, $t = 0$ and (5) lead to $f(-s; x) = f(s; x)$. Hence, $f(2t; x) = 4f(t; x)$ and by induction $f(kt; x) = k^2 f(t; x)$, for any natural number k. Using this, we have $f(1; x) = f(k\frac{1}{k}; x) = k^2 f(\frac{1}{k}; x)$,

i. e. $f\left(\frac{1}{k}; x\right) = \frac{1}{k^2} f(1; x)$, which leads to

$$f(r; x) = r^2 f(1, x),$$

for any rational number r. Now, (3) and Theorem 1 of [3] imply $f(t; x) = t^2 f(1, x)$, i. e. $n(tx) = t^2 n(x)$.

Theorem 2. *Let X be a real vector space and $n : X \to R$ a real functional. If*

a) $n(x+y) + n(x-y) = 2n(x) + 2n(y) \quad (x, y \leqslant X)$, *and*

b) $\sup_{x \in \Delta} |n(x)| < +\infty$ *holds for every segment Δ of X,*

then

$$m(x, y) = \frac{1}{4}\left[n(x+y) - n(x-y)\right]$$

is a bilinear functional on X. Furthermore $m(x, x) = n(x)$.

Corollary 1. *Let X be a real vector space and $n : X \to R$ a real functional. If*

a) $n(x+y) + n(x-y) = 2n(x) + 2n(y) \quad (x, y \leqslant X)$,

b) $n(tx) = t^2 n(x) \quad (t \leqslant R,\ x \leqslant X)$,

c) $\inf_{x \in X} n(x) > -\infty \quad (x \leqslant X)$ *and*

d) $n(x) = 0 \Longleftrightarrow x = 0$,

then X is a normed vector space with $|x| = [n(x)]^{1/2}$ as a norm of x. Moreover X is a unitary space with a scalar product

$$(x, y) = \frac{1}{4}\left[n(x+y) - n(x-y)\right], \quad (x, x) = n(x).$$

For the proof of Theorem 2 and for later use we need the following lemma.

Lemma 1. *Let X be a real vector space and $n : X \to R$ a quadratic functional.*

Then the functional

$$m(x, y) = \frac{1}{4}\left[n(x+y) - n(x-y)\right] \tag{6}$$

is symmetric and additive, i. e.

$$m(x, y) = m(y, x) \text{ and } \tag{7}$$

$$m(x+y, z) = m(x, z) + m(y, z) \quad (x, y, z \leqslant X) \text{ (cf. [5])}. \tag{8}$$

Proof. Using (1), we have $n(-x) = n(x)$, so that $m(x, y) = m(y, x)$ holds. Furthermore we have

$$\begin{aligned}
4m(x+y, z) &= n(x+y+z) - n(x+y-z) = \\
&= 2n(x+z) + 2n(y) - n[x+(z-y)] - n[x-(z-y)] = \\
&= 2n(x+z) + 2n(y) - 2n(y-z) - 2n(x) = \\
&= n(x+z) + 2n(x) + 2n(z) - n(x+z) + 2n(y) - 2n(y-z) - 2n(x) = \\
&= n(x+z) - n(x-z) + n(y+z) - n(y-z) = \\
&= 4m(x, z) + 4m(y, z).
\end{aligned}$$

Thus, (8) is also proved.

Proof of Theorem 2. Using Theorem 1, and assumptions $a)$ and $b)$ of Theorem 2, for $\Delta = [-x, x]$, we have $n(tx) = t^2 n(x)$. Hence, by $a)$ and $b)$,

$$\frac{n(tx+y) + n(tx-y)}{2} = n(tx) + n(y) = t^2 n(x) + n(y),$$

which implies

$$n(tx+y) = n(y) + t^2 n(x) + 2m(t; x, y), \tag{9}$$

where

$$m(t; x, y) = m(tx, y). \tag{10}$$

If in (8) we set tx instead of x, sy instead of y and y instead of z, we get

$$m(t+s; x, y) = m(t; x, y) + m(s; x, y) \ (t, s \in R; x \in X). \tag{11}$$

Now, the assumption $b)$ of Theorem 2 together with (9) implies

$$\sup_{0 \leq t \leq 1} |m(t; x, y)| < +\infty.$$

From here it follows that the function $t \to m(t; x, y)$ is bounded on some interval. Since it satisfies the Cauchy functional equation (11) it is continuous and, therefore,

$$m(t; x, y) = t\, m(1; x, y) \tag{12}$$

holds, for all $t \in R$ and $x, y \in X$. By use of (12), (9) becomes

$$n(tx+y) = n(y) + 2t\, m(x, y) + t^2 n(x). \tag{13}$$

Replacing y by $-y$ in (13), we get

$$n(tx-y) = n(y) - 2t\, m(x, y) + t^2 n(x),$$

which together with (13) leads to

$$\frac{1}{4}\left[n(tx+y) - n(tx-y) \right] = t\, m(x, y), \text{ i. e.}$$

$$m(tx, y) = t\, m(x, y) \quad (t \in R; x, y \in X). \tag{14}$$

Using (14) and the symmetry of the functional m we obtain Theorem 2.

Proof of Corollary 1. It follows from c) that $\inf_{t\in R} n(tx) > -\infty$, which together with $n(tx) = t^2 n(x)$ implies $n(x) \geq 0$, i. e. the functional n is positive on X. But this and (9) lead to

$$2m(t; x, y) \geq -n(y) - t^2 n(x).$$

From here it follows that the function $t \to m(t; x, y)$ is bounded from below on some interval. Since it satisfies the Cauchy functional equation it is continuous and therefore (12) is satisfied, which by use of d) implies the assertion of Corollary 1.

Remark 1. *If X is a complex vector space and n a complex valued functional defined on X such that*

a) $n: X \to R$ *is a quadratic functional,*

b') $n(tx) = t^2 n(x) \quad (t \in R; x \in X)$,

b'') $n(ix) = n(x) \quad (x \in X)$,

c) $n(x) \geq 0 \quad (x \in X)$ *and*

d) $n(x) = 0 \Longleftrightarrow x = 0$,

then X is a unitary space with a scalar product

$$(x, y) = \frac{1}{4}\Big[n(x+y) - n(x-y)\Big] + \frac{i}{4}\Big[n(x+iy) - n(x-iy)\Big],$$

$$(x, x) = n(x).$$

In order to prove this we note that, in the same way as in the proof of Theorem 2, we find that $m(tx, y) = tm(x, y)$ holds for all $t \in R$ and $x, y \in X$, where the functional m is defined by (6). Now we set

$$(x, y) = m(x, y) + i\, m(x, iy) \tag{15}$$

and we find by use of b'') that

$$(ix, y) = i(x, y) \text{ and } (x, y) = \overline{(y, x)}.$$

Hence for a complex number $c = t + is$ $(t, s \in R)$ we have

$$(cx, y) = (tx + isx, y) = (tx, y) + (isx, y) = t(x, y) + \\ + is(x, y) = c(x, y),$$

which leads to the assertion. Remark 1 shows that in questions of introducing scalar product by use of the quadratic functional which posseses properties b') and b'') for all real numbers t, it is sufficient to treat the case of a real vector space.

If $b')$ $c)$ and $d)$ are replaced by $\sup_{x \in \Delta} |n(x)| < +\infty$, for every segment Δ, than (x, y) is a sesquilinear functional on X (i. e. $x \to (x, y)$, $\overline{(y, x)}$ are linear functionals, for every $y \in X$). The case, when $b')$, $b'')$, $c)$ and $d)$ are replaced by $n(\lambda x) = |\lambda|^2 n(x)$, for all complex λ, will be treated in the forthcomming paper »Quadratic and sesquilinear functionals«.

3. Theorem 3. *Let X be a real vector space and $n : X \to R$ a functional such that*

$a)$ $n(x + y) + n(x - y) = 2n(x) + 2n(y)$ $(x, y \in X)$ *and*

$b)$ $n(tx) = t^2 n(x)$ $(t \in R, x \in X)$

hold true.

Then the functional

$$a(t; x, y) = \frac{m(tx, y) - m(x, ty)}{2}$$

is a skew symmetric bilinear functional in x and y, for every $t \in R$. Functionals $a(t; x, y)$ and $n(x)$ are connected by the equation:

$$n(tx + y) = n(y) + \frac{t}{2}\Big[n(x + y) - n(x - y)\Big] + \\ + t^2 n(x) + a(t; x, y). \tag{16}$$

Furthermore, as a function of t, the functional $a(t; x, y)$ satisfies the following functional equations:

$$\begin{aligned} a(t+s; x, y) &= a(t; x, y) + a(s; x, y), \\ a(t \cdot s; x, y) &= t\, a(s; x, y) + s\, a(t; x, y), \end{aligned} \tag{17}$$

for all $t, s \in R$ and $x, y \in X$.

Proof. In the proof of Theorem 3 we will make use of Theorem 4 about functional equations, which will be proved later on. Using (6), (10) and the assumption $b)$: $n(tx) = t^2 n(x)$, we get

$$m\left(\frac{1}{t}; x, y\right) = \frac{1}{4}\left[n\left(\frac{1}{t} x + y\right) - n\left(\frac{1}{t} x - y\right)\right] = \\ = \frac{1}{t^2}\frac{1}{4}\Big[n(ty + x) - n(ty - x)\Big], \quad \text{i. e.} \tag{18}$$

$$m(t; x, y) = t^2 m\left(\frac{1}{t}; y, x\right),$$

for all $t \in R$, $t \neq 0$ and all $x, y \in X$. If we set

$$a(t; x, y) = \frac{m(t; x, y) - m(t; y, x)}{2}, \tag{19}$$

and

$$b(t; x, y) = \frac{m(t; x, y) + m(t; y, x)}{2}, \tag{20}$$

we find, by use of (20), (19), (18) and (8),

$$\begin{aligned} a(t+s; x, y) &= a(t; x, y) + a(s; x, y), \\ a(t; x, y) &= -t^2 a\left(\frac{1}{t}; x, y\right) \end{aligned} \tag{21}$$

and

$$\begin{aligned} b(t+s; x, y) &= b(t; x, y) + b(s; x, y), \\ b(t; x, y) &= t^2 b\left(\frac{1}{t}; x, y\right). \end{aligned} \tag{22}$$

In Theorem 4 we will prove that (21) leads to (17) and that (22) implies $b(t; x, y) = t\, b(1; x, y)$. Hence,

$$m(t; x, y) = b(t; x, y) + a(t; x, y) = t\, m(x, y) + a(t; x, y),$$

which together with (9) implies (16).

From the definition of a it follows that

$$a(t; x, y) = -a(t; y, x) \text{ and } a(t; x, -y) = -a(t; x, y). \tag{23}$$

Now, (16) implies

$$\begin{aligned} n(ts \cdot x + y) = n(y) + \frac{ts}{2}\Big[n(x+y) - n(x-y)\Big] + \\ + (ts)^2 n(x) + a(ts; x, y), \end{aligned}$$

$$\begin{aligned} n(t \cdot sx + y) = n(y) + \frac{t}{2}\Big[n(sx+y) - n(sx-y)\Big] + \\ + t^2 n(sx) + a(t; sx, y). \end{aligned}$$

From here we get

$$\frac{2}{t}\Big[a(ts; x, y) - a(t; sx, y)\Big] =$$

$$= n(sx+y) - n(sx-y) - s\Big[n(x+y) - n(x-y)\Big].$$

Using once again (16), for $n(sx+y)$ and $n(sx-y)$, we get

$$a(ts; x, y) = t\, a(s; x, y) + a(t; sx, y),$$

which together with (17) leads to

$$a(t; sx, y) = s\, a(t; x, y). \tag{24}$$

Obviously (24), (23) and (8) imply that $a(t; x, y)$ is a bilinear functional in x and y, for every $t \in R$.

4. In this section we study functions $f: R \to R$, which satisfy the Cauchy functional equation and some other subsidiary conditions.

Theorem 4. *Let f and $g \neq 0$ be two solutions of the Cauchy functional equation.*

If

$$g(t) = P(t)\, f(1/t)$$

holds for all $t \neq 0$, where $P(t)$ is a continuous function such that $P(1) = 1$, then $P(t) = t^2$ and

$$f(t) + g(t) = 2t\, g(1).$$

Furthermore, the function

$$F(t) = f(t) - t\, f(1)$$

satisfies the following functional equations

$$F(t+s) = F(t) + F(s),$$
$$F(ts) = t\,F(s) + s\,F(t),$$

for all $t, s \in R$.

Proof. If we take $t \neq 0$ and a rational number $r \neq 0$, then $g(rt) = P(rt)\, f\left(\frac{1}{rt}\right)$ together with $g(rt) = r\,g(t)$ and

$$f\left(\frac{1}{rt}\right) = \frac{1}{r} f\left(\frac{1}{t}\right)$$

imply

$$g(t) = \frac{P(rt)}{r^2} f\left(\frac{1}{t}\right).$$

Hence,

$$\left[\frac{P(rt)}{r^2} - P(t)\right] f\left(\frac{1}{t}\right) = 0, \tag{25}$$

for all $r \neq 0$. If $s \neq 0$ is any real number and $r_n \neq 0$ a sequence of rational numbers which tends to t, then (25) implies:

$$\left[\frac{P(st)}{s^2} - P(t)\right] f\left(\frac{1}{t}\right) = 0.$$

Since $f \neq 0$, we get

$$P(st) = s^2 P(t),$$

for all $s \neq 0$ and for at least one $t \neq 0$. If we take $s = 1/t$ we find $P(t) = t^2 P(1) = t^2$. Hence $P(s, t) = (s\,t)^2$. If in this relation we replace s by s/t we find $P(s) = s^2$. Thus,

$$g(t) = t^2 f\left(\frac{1}{t}\right). \tag{26}$$

From (26) it follows $g(1) = f(1)$. Now, we set

$$F(t) = f(t) - t f(1) \quad \text{and} \quad G(t) = g(t) - t g(1).$$

Using (26) we find

$$G(t) = t^2 F(1/t) \tag{27}$$

and we conclude that F and G are solutions of the Cauchy functional equation. Furthermore $F(r) = G(r) = 0$, for any rational number r. We have, therefore,

$$\begin{aligned} G(t) &= G(1+t) = (1+t)^2 F\left(\frac{1}{1+t}\right) = (1+t)^2 F\left(1 - \frac{t}{1+t}\right) = \\ &= -(1+t)^2 F\left(\frac{t}{1+t}\right) = -(1+t)^2 \left(\frac{t}{1+t}\right)^2 G\left(\frac{1+t}{t}\right) = \\ &= -t^2 G\left(\frac{1}{t}\right) = -F(t). \end{aligned}$$

Thus,

$$F(t) = -G(t), \tag{28}$$

which together with (27) leads to

$$F(t) = -t^2 F\left(\frac{1}{t}\right) \tag{29}$$

and

$$f(t) - t f(1) = -(g(t) - t g(1)), \text{ i. e. } f(t) + g(t) = 2 t f(1).$$

It remains to prove that (29) and $F(t+s) = F(t) + F(s)$ imply $F(t s) = t F(s) + s F(t)$. By using (29) we have

$$\begin{aligned} F(t) + \frac{1}{t^2} F(t) &= F\left(t - \frac{1}{t}\right) = F\left(\frac{t^2 - 1}{t}\right) = -\left(\frac{t^2-1}{t}\right)^2 \cdot \\ \cdot F\left(\frac{t}{t^2-1}\right) &= -\left(\frac{t^2-1}{t}\right)^2 F\left(\frac{1}{t-1} - \frac{1}{t^2-1}\right) = -\left(\frac{t^2-1}{t}\right)^2 \cdot \\ &\cdot \frac{-1}{(t-1)^2} F(t-1) + \left(\frac{t^2-1}{t}\right)^2 \frac{-1}{(t^2-1)^2} F(t^2 - 1) = \\ &= \left(\frac{t+1}{t}\right)^2 F(t) - \frac{1}{t^2} F(t^2). \end{aligned}$$

From here,

$$F(t^2) = 2t\,F(t). \tag{30}$$

Replacing in (30) t by $t+s$ and using (30) we get

$$F(ts) = t\,F(s) + s\,F(t).$$

Corollary 2. *If a function $f: R \to R$ satisfies the Cauchy functional equation and*

$$f(t) = t^2 f(1/t)$$

holds, for all $t \neq 0$, then $f(t) = t f(1)$[1].

Remark 2. From Theorem 3 one can see that derivatives on reals, i. e. functions $F: R \to R$ such that

$$\begin{aligned} F(t+s) &= F(t) + F(s) \\ F(ts) &= tF(s) + sF(t),\ F \neq 0 \end{aligned} \tag{31}$$

holds for all $t, s \in R$, will play an important role in studying a quadratic functional.

Let us prove that $F(t) = 0$ for any algebraic number t. Indeed, if t is an algebraic number and

$$t^n + a_1 t^{n-1} + \ldots + a_{n-1} t + a_n = 0 \tag{32}$$

is the »least equation« for t with integral coefficients $a_1, \ldots, a_n$, then applying F on (32) we get:

$$[n\,t^{n-1} + (n-1)\,a_1 t^{n-2} + \ldots + a_{n-1}]\,F(t) = 0,$$

which implies $F(t) = 0$. However, from here it is easy to conclude that the set $\{t \mid F(t) = 0,\ t \in R\}$ is an algebraically closed field, but it is in general different from R.

5. Using results obtained so far in this section we find the explicit expression for a quadratic functional $n(x)$ $(n(tx) = t^2 n(x)$, $t \in R$, $x \in X)$ in an algebraic basic set. We start with

Remark 3. If F is a nontrivial solution of (31) and if e_1 and e_2 is a basic set in a real two dimensional vector space X, then the functional

$$n(te_1 + se_2) = \begin{vmatrix} F(t) & F(s) \\ t & s \end{vmatrix} \qquad (t, s \in R)$$

satisfies all conditions of Theorem 3, but the function $t \to n(te_1 + se_2)$ is not continuous, for all s.

More generally, we have the following Theorem.

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[1] This result was obtained independently by Prof. W. B. Jurkat at the end of 1963. and will appear in Proc. Amer. Math. Soc. The present author had this result at the end of October in 1963. and he presented it in the Seminar on Algebra and Analysis in Zagreb on October 30, 1963.

Theorem 5. *Let X be a real k-dimensional vector space $(k > 1)$ and $n: X \to R$ a quadratic functional on X such that $n(tx) = t^2 n(x)$ holds for all $t \in R$ and $x \in X$. If $e_1, \dots, e_k$ is a basic set in X, then*

$$n\left(\sum_{i=1}^{k} t_i e_i\right) = \sum_{i,j=1}^{k} b_{ij} t_i t_j + \sum_{1 \leqslant i < j \leqslant k} \begin{vmatrix} a_{ij}(t_i) & a_{ij}(t_j) \\ t_i & t_j \end{vmatrix} \tag{33}$$

holds for all $t_1, \dots, t_k \in R$, where $b_{ij} = b_{ji} \in R$ are constants and each function $t \to a_{ij}(t)$ satisfies both functional equations (31).

Proof. Applying (16) for $x = e_1$, $y = \sum_{i=2}^{k} t_i e_i$ and $t = t_1$, we have

$$\begin{aligned} n\left(\sum_{i=1}^{k} t_i e_i\right) &= t_1^2 n(e_1) + n\left(\sum_{i=2}^{k} t_i e_i\right) + 2t_1 m\left(e_1, \sum_{i=2}^{k} t_i e_i\right) + \\ &\quad + a\left(t_1; e_1, \sum_{i=2}^{k} t_i e_i\right) = \\ &= t_1^2 n(e_1) + n\left(\sum_{i=2}^{k} t_i e_i\right) + 2t_1 \sum_{i=2}^{k} m(e_1, t_i e_i) + \\ &\quad + \sum_{i=2}^{k} t_i a(t_1; e_1, e_i). \end{aligned} \tag{34}$$

Applying once again (16), we find

$$m(t_i; e_i, e_1) = t_i m(e_i, e_1) - \frac{1}{2} a(t_i; e_1, e_i). \tag{35}$$

Now (35) and (34) lead to

$$\begin{aligned} n\left(\sum_{i=1}^{k} t_i e_i\right) &= n\left(\sum_{i=2}^{k} t_i e_i\right) + t_1^2 n(e_1) + 2 \sum_{i=2}^{k} t_1 t_i m(e_1, e_i) + \\ &\quad + \sum_{i=2}^{k} \left[t_i a(t_1; e_1, e_i) - t_1 a(t_i; e_1, e_i)\right]. \end{aligned}$$

If we set

$a_{1i}(t) = a(t; e_1, e_i)$ $(i = 2, \dots, k)$ and $b_{1i} = b_{i1} = m(e_1, e_i)$, $b_{11} = n(e_1)$, we get

$$n\left(\sum_{i=1}^{k} t_i e_i\right) = b_{11} t_1^2 + 2 \sum_{i=2}^{k} b_{1i} t_1 t_i +$$

$$+ \sum_{j=2}^{k} \begin{vmatrix} a_{1j}(t_1) & a_{1j}(t_j) \\ t_1 & t_j \end{vmatrix} + n\left(\sum_{i=2}^{k} t_i e_i\right),$$

which by induction implies (33).

Now using Theorem 3 and Theorem 5 we can sum up the main results of this paper in the following theorem:

The Main Theorem. *Let X be a real vector space and $n : X \to R$ a real valued functional such that:*

a) $n(x+y) + n(x-y) = 2n(x) + 2n(y) \quad (x, y \in X)$

and

b) $n(tx) = t^2 n(x) \quad (t \in R, x \in X)$.

If $\{e_\alpha \mid 1 \leq \alpha < \Omega\}$ is an algebraic basic set in X then

$$n\left(\sum_{\alpha} t_\alpha e_\alpha\right) = \sum_{1 \leqslant \alpha, \beta < \Omega} b_{\alpha\beta} t_\alpha t_\beta + \sum_{1 \leqslant \alpha < \beta < \Omega} \begin{vmatrix} a_{\alpha\beta}(t_\alpha) & a_{\alpha\beta}(t_\beta) \\ t_\alpha & t_\beta \end{vmatrix} \tag{36}$$

holds true for all $t_\alpha \in R$, where in the sum only a finite number of terms may be different from zero; $b_{\alpha\beta} = b_{\beta\alpha}$ are real constants and $t \to a_{\alpha\beta}(t)$ is a derivative on the set of all reals, i. e.

$$a_{\alpha\beta}(t+s) = a_{\alpha\beta}(t) + a_{\alpha\beta}(s) \text{ and } a_{\alpha\beta}(ts) = t\, a_{\alpha\beta}(s) + s\, a_{\alpha\beta}(t) \tag{37}$$

holds for all $t, s \in R$ and $1 \leq \alpha < \beta < \Omega$.

If in addition $\sup_{x \in \Delta} |n(x)| < +\infty$ holds, for every segment Δ of X, then in (36) the second sum equals to zero and

$$n\left(\sum_{\alpha} t_\alpha e_\alpha\right) = \sum_{\alpha, \beta} b_{\alpha\beta} t_\alpha t_\beta .$$

Conversely, if $\{e_\alpha \mid 1 \leq \alpha < \Omega\}$ is an algebraic basic set in X, $b_{\alpha\beta} = b_{\beta\alpha}$ real constants and $t \to a_{\alpha\beta}(t)$ satisfies (37), for all α, β ($1 \leq \alpha < \beta < \Omega$), then by (36), a quadratic functional $X \ni x \to n(x) \in R$ is defined such that $n(tx) = t^2 n(x)$.

Remark 4. From the proofs of theorems derived in this paper and from $n(x+y) + n(x-y) = 2n(x) + 2n(y)$ it is obvious how one can extend results of this paper to the case when n takes values among matrices or vectors of some rather general vector spaces. Since such generalisations are straightforward, we discussed the case of the real — valued quadratic functional only.

A c k n o w l e d g e m e n t. The motivation for these investigations were the following questions communicated to us by Prof. J. A c z é l and raised by Prof. I. R. H a l p e r i n while lecturing in Paris in 1963 on Hilbert spaces.

1. Suppose that the function $f: R \to R$ satisfies the Cauchy functional equation and that $f(t) = t^2 f(1/t)$, for all $t \neq 0$. Does this imply the continuity of f ?

Corollary 2 gives an affirmative answer to this question.

2. Suppose that X is a real, complex or quaternionic vector space and that n is a functional such that

a) $n(x+y) + n(x-y) = 2n(x) + 2n(y) \quad (x, y \in X)$ and

b) $n(\lambda x) = |\lambda|^2 n(x)$

holds respectively, for all real, complex or quaternionic λ. Do a) and b) imply the continuity of the function $t \to n(tx+y) - n(tx-y)$, for t real and for all $x, y \in X$.

According to Remark 3 and the Main Theorem the answer to this question is in the negative provided that the space X is real and not one-dimensional. In our forthcomming paper »Quadratic and sesquilinear functionals« we prove that the answer is in the positive if X is a complex or a quaternionic vector space.

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KOŠIJEVA FUNKCIONALNA JEDNADŽBA I SKALARNI PRODUKT U VEKTORSKIM PROSTORIMA

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Sadržaj

Neka je $R = \{t, s, \ldots\}$ skup realnih brojeva i $X = \{x, y, \ldots\}$ vektorski prostor nad R. Funkcinal $n : X \to R$ zove se *kvadratni funkcional,* ako vrijedi (1) za sve x, y iz X.

Teorem 1. *Ako je X realan vektorski prostor i kvadratni funkcional $n : X \to R$ ima svojstvo da za svako x iz X postoje brojevi A_x i B_x takovi da vrijedi* (3) *tada vrijedi i* (4).

Teorem 2. *Neka je X realan vektorski prostor i $n : X \to R$ kvadratni funkcional. Ako je $\sup_{x \in \Delta} |n(x)| < +\infty$ za svaki segment Δ iz X, onda je sa* (6) *zadan bilinearan funkcional $m(x, y)$ na X. Pored toga je $m(x, y) = n(x)$.*

Teorem 3. *Neka je X realan vektorski prostor i $n : X \to R$ kvadratni funkcional. Ako je $n(tx) = t^2 n(x)$, za sve $t \in R$ i $x \in X$, onda vrijedi* (16). *Pri tome je $a(t; x, y)$ bilinearan antisimetričan funkcional u x, y, za svako t, i za svaki par $x, y \in X$ vrijedi* (17).

Teorem 4. *Ako su f i $g \neq 0$ dva rješenja Košijeve funkcionalne jednadžbe* (2) *i ako vrijedi $g(t) = P(t)\, g(1/t)$, za sve $t \neq 0$, pri čemu je P neprekidna funkcija i $P(1) = 1$, onda je $f(t) + g(t) = 2tg(1)$, a funkcija $F(t) = f(t) - tf(1)$ zadovoljava funkcionalne jednadžbe* (31).

Teorem 5. *Neka je X realan k-dimenzionalan vektorski prostor i $n : X \to R$ kvadratni funkcional takav da je $n(tx) = t^2 n(x)$ $(t \in R,\ x \in X)$.*

Ako je $e_1, \ldots, e_n$ baza u X, tada vrijedi (33) *za sve $t_1, \ldots, t_k \in R$, gdje su $b_{ij} = b_{ji}$ realne konstante, a svaka od funkcija $t \to a_{ij}(t)$ zadovoljava obje funkcionalne jednadžbe* (31).

Iz ovih teorema slijedi

Osnovni teorem. *Neka je X realan vektorski prostor i $n : X \to R$ kvadratni funkcional sa svojstvom da je $n(tx) = t^2 n(x)$. Ako je $\{e_\alpha \mid 1 \leq \alpha < \Omega\}$ algebarska baza u X, tada vrijedi* (36), *gdje je $t_\alpha \neq 0$ samo za konačno brojeva α; $b_{\alpha\beta} = b_{\beta\alpha}$ su realne konstante, a funkcije $t \to a_{\alpha\beta}(t)$ zadovoljavaju* (37). *Ako je pritom i $\sup_{x \in \Delta} |n(x)| < +\infty$, za svaki segment Δ iz X, onda drugi član u* (36) *iščezava.*

Obratno, ako je $\{e_\alpha \mid 1 \leq \alpha < \Omega\}$ algebarska baza u X, $b_{\alpha\beta} = b_{\beta\alpha}$ realne konstante i $t \to a_{\alpha\beta}(t)$ funkcije koje zadovoljavaju (37), *onda je sa* (36) *zadan kvadratni funkcional $X \ni x \to n(x) \in R$, koji ima svojstvo da je $n(tx) = t^2 n(x)$, za svako $t \in R$ i $x \in X$.*

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