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MAPPINGS OF INVERSE SYSTEMS

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1. Preliminaries

In this paper we are concerned with inverse systems $\{X_\alpha ; \pi_{\alpha\alpha'}\}$ $(\alpha, \alpha' \in (A, \leqq))$ of topological spaces X_α as defined, for instance, in Chapter VIII of [2]. Sometimes the inverse system is denoted merely by $\{X ; \pi\}$. The spaces X_α shall always be compact and Hausdorff and sometimes compact and metric. All the bonding mappings $\pi_{\alpha\alpha'} : X_{\alpha'} \to X_\alpha$, $\alpha \leqq \alpha'$, shall be (continuous) mappings onto. If the directed set $(A, \leqq)$ has the property that each $\alpha \in A$ has only finitely many predecessors, we shall say that A is of *finite type*. If the directed set A is the set of natural numbers $\{1, 2, \ldots\}$, we speak about inverse *sequences* $\{X_i ; \pi_{ii'}\}$ $(i, i' = 1, 2, \ldots)$.

Given two inverse systems $\{X_\alpha ; \pi_{\alpha\alpha'}\}$ $(\alpha, \alpha' \in A)$ and $\{Y_\beta ; \varrho_{\beta\beta'}\}$ $(\beta, \beta' \in B)$, by a mapping of inverse systems $F : \{X ; \pi\} \to (Y ; \varrho\}$ we mean an order-preserving function $\alpha(\beta)$ of B into A and, for each $\beta \in B$, a mapping $f_\beta : X_{\alpha(\beta)} \to Y_\beta$ such that

$$f_\beta \, \pi_{\alpha(\beta)\, \alpha(\beta')} = \varrho_{\beta\beta'} \, f_{\beta'} \, , \tag{1}$$

whenever $\beta \leqq \beta'$. Sometimes we write $F = \{f_\beta\}$.

With every inverse system $\{X ; \pi\}$ is associated its limit space $X = \operatorname{Inv} \lim \{X ; \pi\}$ as well as natural projections $\pi_\alpha : X \to X_\alpha$. If $\alpha \leqq \alpha'$, then

$$\pi_{\alpha\alpha'} \, \pi_{\alpha'} = \pi_\alpha \, . \tag{2}$$

If all X_α are compact, then X is compact too and the mappings $\pi_\alpha : X \to X_\alpha$ are mappings onto. If we have an inverse sequence of metric compacta X_i, then X is a metric compactum.

The mapping $F = \{f_\beta\} : \{X ; \pi\} \to \{Y ; \varrho\}$ induces a mapping $f : X \to Y$ between inverse limit spaces X and Y. By definition,

$$f_\beta \, \pi_{\alpha(\beta)} = \varrho_\beta \, f \, , \tag{3}$$

for each $\beta \in B$. Sometimes we write $f = \operatorname{Inv} \lim \{f_\beta\}$.

Following papers [6] and [7] we also consider a class Π of (compact) polyhedra and say that a Hausdorff compact space X is

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Π-like, provided for every open covering u of X there exists a u-mapping $g_u : X \to P_u$ *onto* some polyhedron $P_u \in \Pi$. The class of all Π-like Hausdorff compacta X is denoted by $[\Pi]$. Metrizable members of the class $[\Pi]$ form a subclass (Π). Clearly, $\Pi \subseteq (\Pi) \subseteq [\Pi]$. A metric compactum X belongs to (Π) if and only if it admits, for each $\varepsilon > 0$, an ε-mapping $g_\varepsilon : X \to P_\varepsilon$ onto some polyhedron $P_\varepsilon \in \Pi$.

Several examples for these notions have been given in [6] and [7]. Let us point out the following two:

Example 1. If Π is the class of all polyhedra (connected polyhedra), then (Π) is the class of all metric compacta (continua) and $[\Pi]$ the class of all Hausdorff compacta (continua).

Example 2. Let $\Pi = \{I\}$ consists of a single polyhedron — the real line segment $I = [0,1]$. Then $[\Pi]$ is the class of all chainable continua and (Π) the class of all metric chainable continua (also known as snake-like continua [1]).

Here we quote, for future application, some results from [6] and [7].

In [7] we find (as Theorem 1*) the following

Theorem A. *Let Π be a class of connected polyhedra. Then the class (Π) of metric Π-like continua coincides with the class of inverse limits of inverse sequences $\{X_i ; \pi_{ii'}\}$, where the mappings $\pi_{ii'}$ are onto and X_i are polyhedra from Π.*

In [6] we find (as Theorem 3; also cf. [5], Proof of Lemma 5 and Remark on p. 287) the following

Theorem B. *Let Π be a class of connected polyhedra. Then the class $[\Pi]$ of all Π-like continua coincides with the class of inverse limits of inverse systems $\{X_a ; \pi_{aa'}\}$ $(a, a' \in A)$, where all X_a are metric Π-like continua, $X_a \in (\Pi)$, and all $\pi_{aa'}$ are mappings onto. Moreover, one can achieve that the directed set A be of finite type and of power $k(A)$ equal to the weight[1] $w(X)$ of X.*

Examples given in [4], [5] and [6] show that it is not always possible to expand a continuum $X \in [\Pi]$ into an inverse system of polyhedra from Π.

Finally, we quote the basic factorization theorem (Corollary 4 in [6]) of [6] as

Theorem C. *Let Π be a class of connected polyhedra and let $X, P_1, \ldots, P_n$ be Hausdorff compact spaces, X being Π-like. Furthermore, let $f_i : X \to P_i$, $i = 1, \ldots, n$, be mappings. Then there exists a Π-like continuum Q and mappings $g : X \to Q$, $p_i : Q \to P_i$, $i = 1, \ldots, n$, such that $f_i = p_i g$, $i = 1, \ldots, n$, g is onto and the weight $w(Q) \leq \mathrm{Max}(w(P_1), \ldots, w(P_n))$.*

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[1] The weight $w(X)$ is the minimal cardinal of a basis of open sets of X.

Theorems A and B establish the possibility of expanding continua $X \leqslant (\Pi)$ and $X \leqslant [\Pi]$ into inverse sequences of polyhedra from Π and into inverse systems of Π-like metric continua respectively.

The purpose of this paper is to investigate the possibility of expanding mappings $f : X \to Y$ into inverse systems of mappings, i. e. of expressing mappings as limits of mappings F of inverse systems. More precisely, we ask whether the following two statements are true or false:

Statement A. *Let Π and Σ be two classes of connected polyhedra. Then, for each mapping $f : X \to Y$ of the metric Π-like continuum $X \leqslant (\Pi)$ onto the metric Σ-like continuum $Y \leqslant (\Sigma)$, there exist inverse sequences $\{X_i ; \pi_{ii'}\}$, $\{Y_j ; \varrho_{jj'}\}$ of polyhedra $X_i \leqslant \Pi$, $Y_j \leqslant \Sigma$ with mappings $\pi_{ii'}$ and $\varrho_{jj'}$ onto, and there exist a mapping $F = \{f_j\} : \{X ; \pi\} \to \{Y ; \varrho\}$ and homeomorphisms $h : X \to$ $\to X' = \operatorname{Inv}\lim \{X ; \pi\}$ and $k : Y \to Y' = \operatorname{Inv}\lim \{Y ; \varrho\}$ such that $f' h = k f$, where $f' = \operatorname{Inv}\lim \{f_j\}$.*

Statement B. *Let Π and Σ be two classes of connected polyhedra. Then, for each mapping $f : X \to Y$ of the Π-like continuum $X \leqslant [\Pi]$ onto the Σ-like continuum $Y \leqslant [\Sigma]$, there exist inverse systems $\{X_\alpha ; \pi_{\alpha\alpha'}\}$ $(\alpha, \alpha' \leqslant A)$, $\{Y_\beta ; \varrho_{\beta\beta'}\}$ $(\beta, \beta' \leqslant B)$ of metric continua $X_\alpha \leqslant (\Pi)$, $Y_\beta \leqslant (\Sigma)$ with $\pi_{\alpha\alpha'}$ and $\varrho_{\beta\beta'}$ onto. There also exist a mapping $F = \{f_\beta\} : \{X ; \pi\} \to \{Y ; \varrho\}$ and homeomorphisms $h : X \to$ $\to X' = \operatorname{Inv}\lim \{X ; \pi\}$, $k : Y \to Y' = \operatorname{Inv}\lim \{Y ; \varrho\}$, such that $f' h = k f$, where $f' = \operatorname{Inv}\lim \{f_\beta\}$. Moreover, one can achieve that $(A, \leqq)$ and $(B, \leqq)$ be directed sets of finite type and of power $w(X)$ and $w(Y)$ respectively.*

The main result of this paper asserts that Statement A is false (see Theorem 1), while Statement B is true (see Theorems 3 and 4).

2. A chainable continuum which admits mappings not expandable into mappings of arcs.

In this section we prove that Statement A is false in the case when Σ is the class of all connected polyhedra and $\Pi = \{I\}$, where $I = [0,1]$ is the real line segment. More precisely, we prove

Theorem 1. *There exist a metric chainable continuum X and a mapping $f : X \to I$ onto I such that it is not possible to find an inverse sequence of arcs $\{X_i ; \pi_{ii'}\}$ $(i, i' = 1, 2, \ldots)$, $X_i = I$, an inverse sequence of metric compacta $\{Y_j ; \varrho_{jj'}\}$ $(j, j' = 1, 2, \ldots)$ and a mapping $F = \{f_j\} : \{X ; \pi\} \to \{Y ; \varrho\}$ such that there exist homeomorphisms $h : X \to X' = \operatorname{Inv}\lim \{X ; \pi\}$, $k : Y \to Y' = \operatorname{Inv}\lim \{Y ; \varrho\}$, for which $f' h = k f$, where $f' = \operatorname{Inv}\lim \{f_j\}$ (here we need not require that $\pi_{ii'}$ and $\varrho_{jj'}$ be onto).*

The proof follows easily from this

Theorem 2. *There exist a metric chainable continuum X, a mapping $f : X \to I$ onto I and numbers $\varepsilon > 0$, $\delta > 0$, such that it is not possible to find an ε-mapping $\pi : X \to I$ into I, a δ-mapping $\varrho : I \to Q$ into some metric compactum Q and a mapping $\varphi : I \to Q$, for which*

$$\varphi \pi = \varrho f . \tag{1}$$

We first prove that

Theorem 2 implies Theorem 1. Indeed, let $X, f, \varepsilon, \delta$ have properties stated in Theorem 2. Assume that there exist inverse sequences $\{X_i ; \pi_{ii'}\}$, $\{Y_j ; \varrho_{jj'}\}$ ($\pi_{ii'}, \varrho_{jj'}$ not necessarily onto) and a mapping $F = \{f_j\} : \{X ; \pi\} \to \{Y ; \varrho\}$ such that $X_i = I$, Y_j be metric compacta and that there exist homeomorphisms $h : X \to X' = \text{Inv} \lim \{X ; \pi\}$, $k : I \to Y' = \text{Inv} \lim \{Y ; \varrho\}$, for which $f' h = k f$, where $f' = \text{Inv} \lim \{f_j\}$. Clearly, for a sufficiently large j the mapping $\varrho_j k : I \to Y_j$ would be a δ-mapping, while $\pi_{i(j)}\, h : X \to X_{i(j)}$ would be an ε-mapping. Nevertheless, we would have

$$f_j (\pi_{i(j)}\, h) = \varrho_j f' h = (\varrho_j k) f , \tag{2}$$

contrary to the assertion of Theorem 2.

Proof of Theorem 2. Let C denote the Cantor triadic set. C is obtained from $I = [0,1]$ by successively deleting first the open middle-third U of I, second the open middle-thirds U_0 and U_1 of each of the two segments remaining, third the open middle-thirds U_{00}, U_{01}, U_{10}, U_{11} of each of the four segments remaining, etc. Notice that diam $(U_{i_1} \dots {}_{i_p}) = 1/3^{p+1} \to 0$, for $p \to \infty$.

Let X (see *Fig.* 1) be the subset of the square $I \times I$ given by

$$\begin{aligned} X = (C \times I) \cup (U \times 0) \cup (U_0 \cup U_1) \times 1 \cup \\ \cup (U_{00} \cup U_{01} \cup U_{10} \cup U_{11}) \times 0 \cup \\ \cup (U_{000} \cup U_{001} \cup \dots \cup U_{111}) \times 1 \cup \dots \subseteq I \times I. \end{aligned} \tag{3}$$

X is a metric chainable continuum (i. e. it admits ε-mappings onto arcs, for each $\varepsilon > 0$) considered already by B. Knaster for other purposes.

Let $g : I \to I$ be the Cantor step-function, which shrinks closures of the intervals $U_{i_1} \dots {}_{i_p}$ into points. The mapping g is order-preserving and $g(C) = I$. We define now the mapping $f : X \to I$ by putting

$$f(s, t) = g(s) , \ (s, t) \in X \subseteq I \times I . \tag{4}$$

Clearly, f is a continuous mapping of X onto I.

Observe that

$$x = (s, t) \in X , \ x' = (s', t') \in X , \ s < s', \tag{5}$$

implies

$$f(x) < f(x'),\tag{6}$$

except in the case when both s and s' belong to the closure of the same interval $U_{i_1 \dots i_p}$.

Finally, let $\varepsilon = \frac{1}{2}$ and $\delta = 1$.

We claim that, for X, f, ε and δ chosen in this way, holds the assertion of Theorem 2.

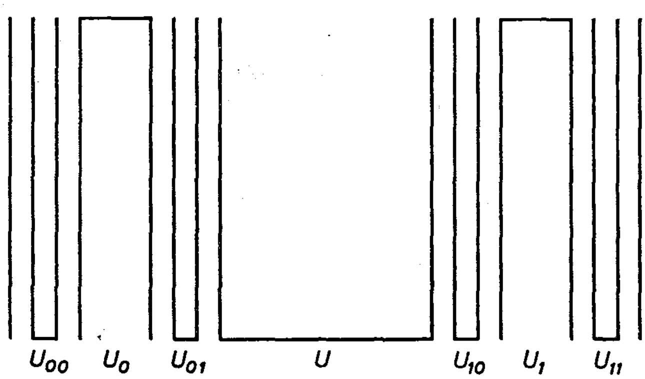

Fig. 1. The chainable continuum X.

Assume on the contrary, that there exist a metric compactum Q, an ε-mapping $\pi: X \to I$ into I, a δ-mapping $\varrho: I \to Q$ into Q and a mapping $\varphi: I \to Q$, such that (1) holds. We shall bring this assumption to a contradiction.

Denote by $M \subseteq I$ the set

$$M = \{y \mid y \in I,\ \varrho(y) = \varrho(0)\},\tag{7}$$

and let

$$y_0 = \text{l. u. b.}(M).\tag{8}$$

M being closed, y_0 belongs to M and thus

$$\varrho(y_0) = \varrho(0).\tag{9}$$

Clearly,

$$y_0 < 1,\tag{10}$$

for $y_0 = 1$ and (9) would imply $\varrho(1) = \varrho(0)$, contrary to the assumption that ϱ is a δ-mapping, for $\delta = 1$.

Choose a point $x_0 \in X$ such that

$$f(x_0) = y_0\tag{11}$$

and that x_0 be of the form

$$x_0 = (c_0,\ 1/2),\ c_0 \in C.\tag{12}$$

Moreover, if c_0 is an end-point of some interval $U_{i_1} \dots {}_{i_p}$, let it be the right end-point. Because of (10), we have

$$c_0 < 1 . \tag{13}$$

Therefore, we can find, arbitrarily close to c_0, points $c_1 \leqslant C$, $c_0 < c_1$, for which

$$y_0 = f(x_0) = g(c_0) < g(c_1) = f(x_1), \tag{14}$$

where

$$x_1 = (c_1, t_1), t_1 \leqslant I . \tag{15}$$

Now consider the points $x_0' = (c_0, 0) \leqslant X$ and $x_0'' = (c_0, 1) \leqslant X$. Clearly, the points $\pi(x_0)$, $\pi(x_0')$ and $\pi(x_0'')$ are three distinct points, their mutual distances being at least $1/2 = \varepsilon$.

We claim that $\pi(x_0)$ lies between the points $\pi(x_0')$ and $\pi(x_0'')$ on the segment I. If this were not so, we would have, say, $\pi(x_0')$ lying between the two remaining points $\pi(x_0)$ and $\pi(x_0'')$. Then we could consider the segment $c_0 \times [1/2, 1]$ whose end-points are x_0 and x_0''. The image of this segment under the mapping π would be a connected set on I joining $\pi(x_0)$ and $\pi(x_0'')$ and therefore necessarily containing the intermediate point $\pi(x_0')$. In other words, we would have a point $\xi \leqslant c_0 \times [1/2, 1]$ with $\pi(\xi) = \pi(x_0')$. However, this is impossible, the distance between ξ and x_0' being at least $1/2 = \varepsilon$.

Now consider two disjoint connected neighbourhoods U' and U'' about the points $\pi(x_0')$ and $\pi(x_0'')$ respectively, and let U' and U'' be so small that they do not contain $\pi(x_0)$. Choose neighbourhoods V' and V'' about x_0' and x_0'' such that

$$\pi(V') \subseteq U', \ \pi(V'') \subseteq U''. \tag{16}$$

Then it is possible to find a point $c_1 \leqslant C$, $c_0 < c_1$, such that

$$(c_1, 0) \leqslant V', (c_1, 1) \leqslant V'', \tag{17}$$

and that (14) hold, for all $x_1 = (c_1, t_1), t_1 \leqslant I$.

The end-points of the segment $c_1 \times I$ map under π into U' and U'' respectively. Therefore, $\pi(c_1 \times I)$ must contain the intermediate point $\pi(x_0)$. Consequently, there exists a point $x_1 = (c_1, t_1) \leqslant c_1 \times I$, such that

$$\pi(x_1) = \pi(x_0) . \tag{18}$$

From the commutativity relation (1) and from (18), (11) and (9), we obtain

$$\varrho(f(x_1)) = \varphi\, \pi(x_1) = \varphi\, \pi(x_0) = \varrho\, f(x_0) = \varrho(y_0) = \varrho(0),$$

which proves that

$$f(x_1) \leqslant M . \tag{19}$$

This is, however, impossible because (14) implies

$$f(x_1) > y_0 = \text{l. u. b.}(M). \tag{20}$$

This completes the proof of Theorems 2 and 1.

Remark 1. Let X and f be those described in the proof of Theorem 2. Then there exist an inverse sequence $\{X_i; \pi_{ii'}\}$ of connected 2-dimensional polyhedra X_i with mappings $\pi_{ii'}$ onto, a sequence $\{Y_j; \varrho_{jj'}\}$ of arcs $Y_j = I$ with mappings $\varrho_{jj'}$ onto, a mapping $F = \{f_j\} : \{X; \pi\} \to \{Y; \varrho\}$ and homeomorphisms $h: X \to X' = \text{Inv lim}\{X; \pi\}$, $k: I \to Y' = \text{Inv lim}\{Y; \varrho\}$ such that $f'h = kf$, for $f' = \text{Inv lim}\{f_j\}$.

Remark 2. It would be interesting to know whether Statement A is true or false in the case when $\Pi = \Sigma$ is the class of all connected polyhedra.

Remark 3. Theorem 1 gives a partial answer to a problem raised recently by J. Mioduszewski ([8], Remark on p. 40; also cf. Problem **P 389**, Colloq. Math. **10** (1963), p. 185).

3. Expanding mappings of general Π-like continua into mappings of metric Π-like continua

In this section we prove

Theorem 3. *Statement* B *is a true theorem.*

In fact, we shall prove a more precise result, which implies Theorem 3. It reads as follows:

Theorem 4. *Let Π be a class of connected polyhedra, X and Y two Hausdorff continua, X being Π-like, $X \in [\Pi]$, and let $f: X \to Y$ be a mapping onto Y. Furthermore, let $\{Y_\beta; \varrho_{\beta\beta'}\}$ $(\beta, \beta' \in B)$ be an inverse system of metric compacta Y_β with mappings $\varrho_{\beta\beta'}$ onto and such that $Y = \text{Inv lim}\{Y_\beta; \varrho_{\beta\beta'}\}$ and that $(B, \leqq)$ be of finite type. Then there exists an inverse system $\{X_\alpha; \pi_{\alpha\alpha'}\}$ $(\alpha, \alpha' \in A)$ of metric Π-like continua $X_\alpha \in (\Pi)$, with mappings $\pi_{\alpha\alpha'}$ onto, such that $(A, \leqq)$ be of finite type. Furthermore, there exist a mapping $F = \{f_\beta\} : \{X; \pi\} \to \{Y; \varrho\}$ and a homeomorphism $h: X \to X' = \text{Inv lim}\{X; \pi\}$ such that $f'h = f$, where $f' = \text{Inv lim}\{f_\beta\}$. If in addition $k(B) \leqq w(Y)$, then one can achieve that $k(A) \leqq w(X)$.*

Remark 4. Theorem 4 is a generalization of almost all earlier results of the author concerning inverse system expansions in the non-metric case. In particular, Theorem 4 readily implies Theorems B and C. However, the proof of Theorem 4 given below depends on Theorem C (in the case when $P_1, \ldots, P_n$ are of countable weight, i. e. are metric).

Theorem 4 implies Theorem B. Let Π be a class of connected polyhedra and let $X \in [\Pi]$. Put $Y = X$ and let $i: X \to Y$ be the identity. $Y = X$ can be embedded in the direct product ΠI_λ, $\lambda \in \Lambda$, where $I_\lambda = I = [0,1]$ and Λ is a set of power $k(\Lambda) = w(X)$. Clearly, ΠI_λ is the inverse limit of the inverse system of all the finite products $I_{\lambda_1} \times \ldots \times I_{\lambda_r}$ with natural projections as bonding mappings, the set of indexes B being the set of all finite subsets $\beta = \{\lambda_1, \ldots, \lambda_r\}$, ordered by inclusion. B is of finite type and $k(B) = k(\Lambda) = w(X)$ (for infinite $w(X)$). By appropriate restrictions we obtain an inverse system $\{Y_\beta; \varrho_{\beta\beta'}\}$ $(\beta, \beta' \in B)$ of metric continua $Y_\beta = \varrho_\beta(X)$ with mappings $\varrho_{\beta\beta'}$ onto and such that Inv lim $\{Y; \varrho\} = X$. An application of Theorem 4 to $i: X \to X =$ Inv lim $\{Y; \varrho\}$ immediately yields the assertion of Theorem B.

Theorem 4 implies Theorem C. It suffices to prove Theorem C in the simplest case when $n = 1$. The general case then follows easily by considering the mapping $f = f_1 \times \ldots \times f_n : X \to P_1 \times \ldots \times P_n = P$ and applying Theorem C (case $n = 1$) to this situation.

So, let Π be a class of connected polyhedra, let X and P be Hausdorff compact spaces, $X \in [\Pi]$, and $f: X \to P$ a mapping. Take an expansion of P into a system of metric compacta $\{Y_\beta; \varrho_{\beta\beta'}\}$ $(\beta, \beta' \in B)$ (use the argument given above). We can assume in addition that $k(B) = w(P)$ and that B is of finite type.

The application of Theorem 4 to this situation yields an inverse system $\{X_\alpha; \pi_{\alpha\alpha'}\}$ $(\alpha, \alpha' \in A)$ of metric Π-like continua $X_\alpha \in (\Pi)$, a mapping $F = \{f_\beta\}: \{X; \pi\} \to \{Y; \varrho\}$ and a homeomorphism $h: X \to X' =$ Inv lim $\{X; \pi\}$ such that $f' h = f$, $f' =$ Inv lim $\{f_\beta\}$.

Let $\alpha(\beta)$ be the order-preserving function from B into A, which occurs in the definition of F, and let $A_0 = \alpha(B) \subseteq A$. Clearly, A_0 is also a directed set of power $k(A_0) \leqq k(B) = w(P)$. Consider now the inverse subsystem $\{X_\alpha; \pi_{\alpha\alpha'}\}$ $(\alpha, \alpha' \in A_0)$ and let X_0 be its inverse limit. Clearly, $F = \{f_\beta\}$ maps this subsystem into $\{Y; \varrho\}$ and thus induces a limit mapping $f_0: X_0 \to Y$, defined by

$$\varrho_\beta f_0 = f_\beta \pi_{\alpha(\beta)0}, \tag{1}$$

where $\pi_{\alpha(\beta)0}: X_0 \to X_{\alpha(\beta)}$ is the natural projection.

Moreover, there is a natural mapping $p: X' \to X_0$, defined by

$$\pi_{\alpha 0} p = \pi_\alpha, \quad \alpha \in A_0. \tag{2}$$

Clearly,

$$f_0 p = f', \tag{3}$$

because of

$$\varrho_\beta f_0 p = f_\beta \pi_{\alpha(\beta)0} p = f_\beta \pi_{\alpha(\beta)} = \varrho_\beta f',$$

for all $\beta \in B$.

The mappings $\pi_{\alpha\alpha'}$ being onto, it is easy to show that $p: X' \to X_0$ is also a mapping onto. Indeed, if $x_0 \in X_0$, then the sets

$\pi_\alpha^{-1}(\pi_{\alpha 0}(x_0)) \subseteq X$, $\alpha \in A_0$, form a centered system of non-empty closed sets. Therefore, the intersection of all these sets is not empty, and obviously maps under p into x_0. Furthermore, $X_0 \in [\Pi]$, $[\Pi]$ being closed with respect to inverse limits (see [6], Theorem 1). Finally, $k(A_0) \leqq w(P)$ and the fact that $w(X_\alpha) \leqq \aleph_0$ imply that $w(X_0) \leqq w(P)$. Putting $Q = X_0$, $g = p$, $p = f_0$, we thus obtain Theorem C.

Theorem 4 implies Theorem 3. This is immediate. It suffices to apply Theorem B and expand Y into an inverse system of Σ-like metric continua with an indexing set B of finite type, and then apply Theorem 4.

Proof of Theorem 4. Let Π be a class of connected polyhedra, X a Π-like continuum, $\{Y_\beta\,;\varrho_{\beta\beta'}\}$ $(\beta, \beta' \in B)$ an inverse system of metric compacta Y_β with mappings $\varrho_{\beta\beta'}$ onto and such that B be a directed set of finite type. Let $Y = \operatorname{Inv}\lim\{Y\,;\varrho\}$ and let $f: X \to Y$ be a mapping onto Y. We can assume that the weights $w(X)$ and $w(Y)$ are infinite cardinals. Clearly, $w(Y) \leqq w(X)$.

It is easy to see that there exists an inverse system $\{X_\alpha\,;\pi_{\alpha\alpha'}\}$ $(\alpha, \alpha' \in A)$ of metric continua with mappings $\pi_{\alpha\alpha'}$ onto, and with the indexing set $(A, \leqq)$ of finite type and cardinality $k(A) \leqq w(X)$, and such that $X = \operatorname{Inv}\lim\{X\,;\pi\}$ (apply the argument described above or Theorem B taking for Π the class of all connected polyhedra).

Consider now the set $(C, \leqq) = (A, \leqq) \times (B, \leqq)$ provided with the product ordering, i. e. let

$$\gamma = (\alpha, \beta) \leqq (\alpha', \beta') = \gamma' \tag{4}$$

if and only if

$$\alpha \leqq \alpha' \text{ and } \beta \leqq \beta'. \tag{5}$$

C is readily seen to be directed and of finite type. Moreover, if $k(B) \leqq w(Y)$, then

$$k(C) = k(A)\,k(B) \leqq w(X)\,w(Y) = w(X). \tag{6}$$

For each $\gamma = (\alpha, \beta)$, let $\alpha(\gamma) = \alpha$ and $\beta(\gamma) = \beta$. Now we wish to define, for each $\gamma \in C$, a metric Π-like continuum $Z_\gamma \in (\Pi)$, a mapping $\chi_\gamma : X \to Z_\gamma$ onto Z_γ and mappings $\varphi_\gamma : Z_\gamma \to X_{\alpha(\gamma)}$, $\psi_\gamma : Z_\gamma \to$ $\to Y_{\beta(\gamma)}$ such that

$$\varphi_\gamma \chi_\gamma = \pi_{\alpha(\gamma)}, \tag{7}$$

$$\psi_\gamma \chi_\gamma = \varrho_{\beta(\gamma)} f. \tag{8}$$

Moreover, we wish to define mappings $\sigma_{\gamma\gamma'} : Z_{\gamma'} \to Z_\gamma$, for each pair $\gamma \leqq \gamma'$, $\gamma, \gamma' \in C$, in such a way that

$$\sigma_{\gamma\gamma'}\chi_{\gamma'} = \chi_\gamma, \tag{9}$$

for all $\gamma \leqq \gamma'$. In other words, we require the commutativity of the diagram

$$\begin{tikzcd}
X_{\alpha(\gamma)} & & X \arrow[ll, "\pi_{\alpha(\gamma)}"'] \arrow[dll, "\chi_\gamma"'] \arrow[dl, "\chi_{\gamma'}"] \arrow[dd, "f"] \\
Z_\gamma \arrow[u, "\varphi_\gamma"] \arrow[d, "\psi_\gamma"'] & Z_{\gamma'} \arrow[l, "\sigma_{\gamma\gamma'}"] & \\
Y_{\beta(\gamma)} & & Y \arrow[ll, "\varrho_{\beta(\gamma)}"]
\end{tikzcd}$$

The objects Z_γ, χ_γ, φ_γ, ψ_γ, $\sigma_{\gamma\gamma'}$ are defined by induction. First observe that C (being of finite type) has a set of first elements γ_0 (elements with no predecessors). For every such γ_0, consider $\pi_{\alpha(\gamma_0)}: X \to X_{\alpha(\gamma_0)}$ and $\varrho_{\beta(\gamma_0)} f: X \to Y_{\beta(\gamma_0)}$, and apply Theorem C. We obtain a metric Π-like continuum Z_{γ_0}, a mapping $\chi_{\gamma_0}: X \to Z_{\gamma_0}$ onto Z_{γ_0} and mappings $\varphi_{\gamma_0}: Z_{\gamma_0} \to X_{\alpha(\gamma_0)}$, $\psi_{\gamma_0}: Z_{\gamma_0} \to Y_{\beta(\gamma_0)}$, such that (7) and (8) hold, for $\gamma = \gamma_0$.

Now assume that we have defined already $Z_\gamma, \chi_\gamma, \varphi_\gamma, \psi_\gamma, \sigma_{\gamma''\gamma'}$, for all $\gamma \leqq \gamma', \gamma'' \leqq \gamma$, in such a way that (7), (8) and (9) hold, that $Z_\gamma \in (\Pi)$ and $\chi_\gamma(X) = Z_\gamma$. We extend the definitions to γ' as follows.

Let $\gamma_1, \ldots, \gamma_r$ be all the predecessors of γ'. Consider the mappings $\pi_{\alpha(\gamma')}: X \to X_{\alpha(\gamma')}$, $\varrho_{\beta(\gamma')} f: X \to Y_{\beta(\gamma')}$, $\chi_{\gamma_1}: X \to Z_{\gamma_1}, \ldots, \chi_{\gamma_r}: X \to$ $\to Z_{\gamma_r}$ and apply Theorem C. We obtain a Π-like metric continuum $Z_{\gamma'} \in (\Pi)$, a mapping $\chi_{\gamma'}: X \to Z_{\gamma'}$ onto $Z_{\gamma'}$, mappings $\varphi_{\gamma'}: Z_{\gamma'} \to$ $\to X_{\alpha(\gamma')}$, $\psi_{\gamma'}: Z_{\gamma'} \to X_{\beta(\gamma')}$ and mappings $\sigma_{\gamma\gamma'}: Z_{\gamma'} \to Z_\gamma$, for all $\gamma \leqq \gamma'$, i. e. for $\gamma = \gamma_1, \ldots, \gamma_r$. These mappings satisfy relations (7) and (8), for $\gamma = \gamma'$, and satisfy (9).

It follows that, by induction, we can define $Z_\gamma, \chi_\gamma, \varphi_\gamma, \psi_\gamma, \sigma_{\gamma\gamma'}$, for all $\gamma, \gamma' \in C$, $\gamma \leqq \gamma'$, in such a manner that $Z_\gamma \in (\Pi)$, χ_γ is onto and (7), (8) and (9) hold good.

Indeed, C being of finite type, it is easy to see that C satisfies the following principle of induction:

Let $\Gamma \subseteq C$ be a set such that:

(*i*) Γ contains all the first elements γ_0 of C,

(*ii*) if Γ contains all the predecessors of $\gamma \in C$, then $\gamma \in \Gamma$.

Then Γ coincides with C.

Now, it is easy to see that $\{Z_\gamma; \sigma_{\gamma\gamma'}\}$ $(\gamma, \gamma' \in C)$ is an inverse system with mappings $\sigma_{\gamma\gamma'}$ onto. Indeed, (9) and the fact that χ_γ is a mapping onto imply that $\sigma_{\gamma\gamma'}$ also is a mapping onto. Furthermore, if $\gamma \leqq \gamma' \leqq \gamma''$, then, by (9),

$$\sigma_{\gamma\gamma'}\, \sigma_{\gamma'\gamma''}\, \chi_{\gamma''} = \sigma_{\gamma\gamma'}\, \chi_{\gamma'} = \chi_\gamma = \sigma_{\gamma\gamma''}\, \chi_{\gamma''},$$

and since $\chi_{\gamma''}$ is onto, it follows

$$\sigma_{\gamma\gamma'}\, \sigma_{\gamma'\gamma''} = \sigma_{\gamma\gamma''}. \tag{10}$$

Let X' denote the inverse limit of $\{Z_\gamma ; \sigma_{\gamma\gamma'}\}$ and $\sigma_\gamma : X' \to Z_\gamma$ the corresponding natural projections. The mappings $\chi_\gamma : X \to Z_\gamma$ induce a mapping $h : X \to X'$, which is onto, and is defined by

$$\sigma_\gamma\, h = \chi_\gamma . \tag{11}$$

$h : X \to X'$ is in fact a homeomorphism, because it is one-to-one. Indeed, if $x \neq x'$, $x, x' \in X$, then

$$\pi_\alpha(x) \neq \pi_\alpha(x'), \tag{12}$$

for some $\alpha \in A$. Take any $\beta \in B$ and let $\gamma = (\alpha, \beta) \in C$. Then, $\alpha(\gamma) = \alpha$ and $\beta(\gamma) = \beta$. Clearly,

$$\chi_\gamma(x) \neq \chi_\gamma(x'), \tag{13}$$

because $\chi_\gamma(x) = \chi_\gamma(x')$ would imply, by (7),

$$\pi_\alpha(x) = \varphi_\gamma\, \chi_\gamma(x) = \varphi_\gamma\, \chi_\gamma(x') = \pi_\alpha(x'),$$

which is in contradiction with (12). However, (13) and (11) imply

$$h(x) \neq h(x'). \tag{14}$$

Furthermore, the mappings $\psi_\gamma : Z_\gamma \to Y_{\beta(\gamma)}$ enable us to define a mapping $F = \{f_\beta\} : \{Z_\gamma ; \sigma_{\gamma\gamma'}\} \to \{Y_\beta ; \varrho_{\beta\beta'}\}$. For this purpose choose a fixed $\alpha_0 \in A$ and assign to each $\beta \in B$ the element $\gamma(\beta) = (\alpha_0, \beta) \in C$. Clearly, $\beta < \beta'$ implies $\gamma(\beta) < \gamma(\beta')$. Then define, for each $\beta \in B$, a mapping $f_\beta : Z_{\gamma(\beta)} \to Y_\beta$ by

$$f_\beta = \psi_{\gamma(\beta)} . \tag{15}$$

For $\beta \leqq \beta'$, we have

$$f_\beta\, \sigma_{\gamma(\beta)\,\gamma(\beta')} = \varrho_{\beta\beta'}\, f_{\beta'}, \tag{16}$$

which means that $F = \{f_\beta\}$ is a mapping of inverse systems. (16) is obtained by applying subsequently (10), (15), (8), (8), (15), as follows:

$$\begin{aligned} f_\beta\, \sigma_{\gamma(\beta)\,\gamma(\beta')}\, \chi_{\gamma(\beta')} &= f_\beta\, \chi_{\gamma(\beta)} = \psi_{\gamma(\beta)}\, \chi_{\gamma(\beta)} = \\ &= \varrho_\beta\, f = \varrho_{\beta\beta'}\, \varrho_{\beta'}\, f = \varrho_{\beta\beta'}\, \psi_{\gamma(\beta')}\, \chi_{\gamma(\beta')} = \\ &= \varrho_{\beta\beta'}\, f_{\beta'}\, \chi_{\gamma(\beta')}, \end{aligned}$$

and by taking into account that $\chi_{\gamma(\beta')}$ is a mapping onto $Z_{\gamma(\beta')}$.

F induces a mapping $f' : X' \to Y$ defined by

$$\varrho_\beta f' = f_\beta \, \sigma_{\gamma(\beta)} \, . \tag{17}$$

By (17), (15), (11) and (8), we obtain

$$\varrho_\beta f' h = f_\beta \, \sigma_{\gamma(\beta)} \, h = \psi_{\gamma(\beta)} \, \sigma_{\gamma(\beta)} \, h = \psi_{\gamma(\beta)} \, \chi_{\gamma(\beta)} = \varrho_\beta f \, ,$$

for all $\beta \leqslant B$, which proves that

$$f' h = f \, . \tag{18}$$

This completes the proof of Theorem 4.

4. A remark concerning fixed points of inverse limits

Concluding this paper we wish to point aut how Theorem A (demonstrated in [7]) can be used to answer a question raised recently by J. Mioduszewski and M. Rochowski (see [9] and [10]). The question is the following:

Let $\{X_i ; \pi_{ii'}\}$ be an inverse sequence of polyhedra X_i, let all $\pi_{ii'}$ be *onto* and let $X = \text{Inv lim } \{X_i ; \pi_{ii'}\}$. Furthermore, let all X_i have the fixed point property, i. e. the property that every mapping of X_i into itself has at least one fixed point. Does it follow that X also has the fixed point property?

The answer is negative. Indeed, let X be the contractible 2-dimensional continuum, described by S. Kinoshita in [3], which fails to have the fixed point property. Let P be the 2-dimensional connected polyhedron, contained in the Euclidean 3-space $E^3 = E^2 \times \times E^1$, and defined by

$$P = (D \times 0) \cup (S \times I) \cup (T \times I) \, , \tag{1}$$

where

$$D = \{(x, y) \, | \, x^2 + y^2 \leqq 1\} \subseteq E^2 \, , \tag{2}$$
$$S = \{(x, y) \, | \, x^2 + y^2 = 1\} \subseteq E^2 \, , \tag{3}$$
$$T = \{(x, y) \, | \, y = 0 \, , \; 0 \leqq x \leqq 1\} \subseteq E^2 \, , \tag{4}$$
$$I = \{z \, | \, 0 \leqq z \leqq 1\} \subseteq E^1 \, . \tag{5}$$

Obviously, P is contractible and, therefore, acyclic. Hence, by the Lefschetz theorem, it has the fixed point property.

It is easy to see that, for each $\varepsilon > 0$, X admits an ε-mapping onto P. Consequently, Theorem A yields an inverse sequence $\{X_i ; \pi_{ii'}\}$ with mappings $\pi_{ii'}$ onto and such that $X_i = P$, for all $i = 1, 2, \ldots,$ and that $X = \text{Inv lim } \{X_i \, ; \, \pi_{ii'}\}$. We have thus an example answering the above question in the negative.*

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* Note added in proof. The same answer was found also by S. I. Iliadis using properties of a 3-dimensional continuum constructed earlier by I. Ya. Verčenko, Matem. Sbornik **8** (1940), 295—306 (cf. Ref. Ž. Mat. **11 A** 249, **11** (1963), p. 44).

Notice that the polyhedron P admits ε-mappings onto the n-cell I^n, for each $\varepsilon > 0$ and $n \geqq 3$. Hence, the above argument also proves that, for $n \geqq 3$, the continuum X of Kinoshita is an inverse limit of n-cells I^n with bonding mappings onto. Thus, continua X like the n-cell I^n need not have the fixed point property, provided $n \geqq 3$. For $n = 2$, this is a hard unsolved problem.

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PRESLIKAVANJE INVERZNIH SISTEMA

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Sadržaj

U članku se promatraju inverzni sistemi $\{X_a\,;\pi_{aa'}\}$ $(a, a' \leqslant A)$ kompaktnih prostora X_a, te preslikavanja inverznih sistema $F = \{f_\beta\} : \{X_a\,;\pi_{aa'}\} \to \{Y_\beta\,;\varrho_{\beta\beta'}\}$. Svakom inverznom sistemu pripada granični prostor $X = \operatorname{Inv}\lim \{X_a\,;\pi_{aa'}\}$, dok preslikavanju sistema F pripada granično preslikavanje $f = \operatorname{Inv}\lim \{f_\beta\} : X \to Y$ graničnih prostora.

Nadalje se promatra neka klasa Π povezanih poliedara i kaže se da je kompakt X *poput* Π, ako za svaki otvoreni pokrivač u prostora X postoji u-preslikavanje $g_u : X \to P_u$ na neki poliedar $P_u \leqslant \Pi$. Klasa svih kompakata koji su poput Π označava se sa $[\Pi]$, dok se sa (Π) označava potklasa svih *metričkih* kompakata poput Π (ovi pojmovi su uvedeni i promatrani već u [6]).

Cilj je članka da se ispita istinitost ovih dviju izreka:

I z r e k a A. *Neka su* Π *i* Σ *dvije klase povezanih poliedara. Tada za svako preslikavanje* $f: X \to Y$ *metričkog kontinuuma* $X \leqslant \leqslant (\Pi)$ *na metrički kontinuum* $Y \leqslant (\Sigma)$ *postoje inverzni nizovi* $\{X_i; \pi_{ii'}\}$, $\{Y_j; \varrho_{jj'}\}$, *gdje su* $\pi_{ii'}$, $\varrho_{jj'}$ *preslikavanja na, a* X_i *i* Y_j *su poliedri* $X_i \leqslant \Pi$, $Y_j \leqslant \Sigma$. *Nadalje, postoji preslikavanje* $F = \{f_j\}: : \{X_i; \pi_{ii'}\} \to \{Y_j; \varrho_{jj'}\}$ *i homeomorfizmi* $h: X \to X' = \mathrm{Inv\,lim}\,\{X_i; \pi_{ii'}\}$, $k: Y \to Y' = \mathrm{Inv\,lim}\,\{Y_j; \varrho_{jj'}\}$, *takovi da je* $f' h = k f$, *pri čemu je* $f' = \mathrm{Inv\,lim}\,\{f_j\}$.

I z r e k a B. *Neka su* Π *i* Σ *dvije klase povezanih poliedara. Tada, za svako preslikavanje* $f: X \to Y$ *kontinuuma* $X \leqslant [\Pi]$ *na kontinuum* $Y \leqslant [\Sigma]$, *postoje inverzni sistemi* $\{X_\alpha; \pi_{\alpha\alpha'}\}$ $(\alpha, \alpha' \leqslant A)$, $\{Y_\beta; \varrho_{\beta\beta'}\}$ $(\beta, \beta' \leqslant B)$ *metričkih kontinuuma* $X_\alpha \leqslant (\Pi)$, $Y_\beta \leqslant (\Sigma)$, *pri čemu su* $\pi_{\alpha\alpha'}$ *i* $\varrho_{\beta\beta'}$ *preslikavanja na. Također postoji preslikavanje sistema* $F = \{f_\beta\}: \{X_\alpha; \pi_{\alpha\alpha'}\} \to \{Y_\beta; \varrho_{\beta\beta'}\}$, *te homeomorfizmi* $h: X \to \to X' = \mathrm{Inv\,lim}\,\{X_\alpha; \pi_{\alpha\alpha'}\}$, $k: Y \to Y' = \mathrm{Inv\,lim}\,\{Y_\beta; \varrho_{\beta\beta'}\}$ *takovi, da je* $f' h = k f$, *gdje je* $f' = \mathrm{Inv\,lim}\,\{f_\beta\}$. *Nadalje, može se postići da* $(A, \leqq)$ *i* $(B, \leqq)$ *budu usmjereni skupovi sa svojstvom da im svaki element ima samo konačno mnogo prethodnika i da im potencije* $k(A)$ *i* $k(B)$ *ne premašuju težine* $w(X)$, *odnosno* $w(Y)$, *prostora* X *i* Y, *tj. da bude* $k(A) \leqq w(X)$ *i* $k(B) \leqq w(Y)$.

Glavni rezultati članka utvrđuju, da *izreka* A *općenito ne stoji, dok je, naprotiv, izreka* B *istinita.* Istinitost izreke B dobiva se iz teorema 4, koji daje nešto preciznije informacije nego li sama izreka B, te predstavlja poopćenje gotovo svih dosada postignutih autorovih rezultata, koji se odnose na inverzne sisteme nemetričkih prostora.

Primjer, kojim se dokazuje da je izreka A općenito neistinita, daje ujedno i djelomičan odgovor na jedan problem J. M i o d u s z e w s k o g ([8]; vidi i Problem **P 389**, Colloq. Math. **10** (1963), str. 185).

Napokon, u posljednjoj tački **4** članka se pokazuje, da jedan raniji rezultat J. S e g a l a i autora [7], te jedan primjer S. K i n o s h i t e [3] daju negativan odgovor na jedno pitanje J. M i o d u s z e w s k o g i M. R o c h o w s k o g ([9] i [10]) o fiksnim tačkama inverznih limesa.

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