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A REMARK ON THE PAPER „ON SOME FUNCTIONAL EQUATIONS" by S. Kurepa

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Professor János Aczél has called my attention to the paper »On some functional equations« by S. Kurepa (Glasnik mat. fiz. i astr. **11** (1956), 3—5). Dealing with the functional equation

$$f(x+y, z) + f(x, y) = f(y, z) + f(x, y+z) \tag{1}$$

it was pointed out that any function of the form

$$f(x, y) = g(x+y) - g(x) - g(y) \tag{2}$$

satisfies (1) and it was proved that all differentiable solutions of (1) are of the form (2). The question, whether (2) is the general solution of (1) or not, was raised too.

Now, the aim of this note is to show that any continuous solution of (1) is of the form (2). On the other hand we shall see that (2) is not the general solution of (1).

First we prove the following criterion.

A solution of (1) is of the form (2) if and only if it is a symmetric function (i. e. $f(x, y) = f(y, x)$ *holds for any* x, y).

By the theory of O. Schreier on group extensions, there exists a one-to-one correspondence between all extensions of the additive group R of real numbers by itself (apart from equivalent extensions) on one side, and all symmetric solutions of (1) if we do not make distinction between solutions whose difference is of the form (2) on the other side. (By a group we mean always an abelian group!) In this way the direct sum $R+R$ corresponds to the class of all functions (2). Now, R is a divisible group (i. e. $nR = R$ holds for any integer n), thus, by a theorem of R. Baer, any extension of R contains R as a direct summand. So any symmetric solution of (1) has the form (2). Conversely, it is evident, that any function (2) is symmetric. This completes the proof.

It is easy to prove by group-theoretical considerations that *any continuous solution of (1) is a symmetric function.*

Another proof of this last proposition is due to Professor J. Aczél. His proof, which will be presented below, shows that continuity can be replaced by weaker conditions. Let the function $h(x, y)$ be defined as follows:

$$h(x, y) = f(y, x) - f(x, y).$$

Clearly, we have

$$h(x, y) = -h(y, x). \tag{3}$$

On the other hand, the relation

$$h(x+z, y) = h(x, y) + h(z, y) \tag{4}$$

holds if $f(x, y)$ is a solution of (1). This can be verified in the following way. Changing the role of x and y in (1) we get

$$f(x+y, z) + f(y, x) = f(x, z) + f(y, x+z). \tag{1'}$$

Similarly, changing z and y in (1), we get

$$f(x+z, y) + f(x, z) = f(z, y) + f(x, y+z). \tag{2'}$$

Subtracting (1) from the sum of (1') and (2'), we obtain the required relation (4). Now, we observe that for any fixed y, (4) is Cauchy's functional equation. Thus, by the continuity condition,

$$h(x, y) = c(y)\, x$$

with suitable function $c(y)$. Making use of (3),

$$c(x)\, y = -c(y)\, x.$$

This implies

$$\frac{c(x)}{x} = -\frac{c(y)}{y} = \text{constant} = 0.$$

Therefore

$$h(x, y) = 0,$$

i. e.

$$f(x, y) = f(y, x),$$

as it was stated.

It is a consequence of the preceding two statements that *any continuous solution of (1) is of the form (2).*

Finally, the following example shows that *(2) is not the general solution of (1).*

Let b_1, b_2, b_ν ($\nu \in \Gamma$) form a Hamel basis of real numbers. We define the function $f(x, y)$ as follows. If

$$x = \xi_1 b_1 + \xi_2 b_2 + \sum_{\nu \in \Gamma} \xi_\nu b_\nu$$

and

$$y = \eta_1 b_1 + \eta_2 b_2 + \sum_{\nu \in \Gamma} \eta_\nu b_\nu$$

are the expressions of x and y relative to the Hamel basis under consideration (the ξ_ν' and η_ν' are rational numbers; they are all $= 0$, but for a finite number of exceptions), then let

$$f(x, y) = \xi_1 \eta_2 - \xi_2 \eta_1.$$

It is easy to see that the relations

$$f(x+y, z) = f(x, z) + f(y, z)$$

and

$$f(x, y+z) = f(x, y) + f(x, z)$$

hold, therefore, $f(x, y)$ satisfies (1). On the other hand, $f(x, y)$ is not a symmetric function, so it is not of the form (2).

PRIMJEDBA NA ČLANAK „O NEKIM FUNKCIONALNIM JEDNADŽBAMA" od S. Kurepe

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Sadržaj

U članku »On some functional equations« (Glasnik mat. fiz. i astr. **11**, (1956), 3—5) dokazano je, da svaka funkcija oblika (2) zadovoljava funkcionalnu jednadžbu (1) i da svaka derivabilna funkcija f, koja zadovoljava funkcionalnu jednadžbu (1) ima oblik (2).

U ovom članku su dokazani ovi teoremi:

Rješenje funkcionalne jednadžbe (1) je oblika (2) onda i samo onda, ako je funkcija f *simetrična.*

Svaka neprekidna funkcija f, *koja zadovoljava funkcionalnu jednadžbu (1), je simetrična.*

Na taj način je pokazano, da svaka neprekidna funkcija f, koja zadovoljava funkcionalnu jednadžbu (1), ima oblik (2).

Nadalje je dan primjer funkcije f, koja zadovoljava funkcionalnu jednadžbu (1), a koja nije simetrična.

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