ON SPECTRAL CONCENTRATION FOR A CLASS OF J-SELFADJOINT OPERATORS

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In this paper we consider the spectral concentration for a class of J-selfadjoint operators which possess spectral decompositions like ordinary selfadjoint operators.

Let $X$ be a Hilbert space with a scalar product $(x, y)$ and the norm $\|x\| = (x, x)^{1/2}$. Let $T$ be a closed operator in $X$ defined on $\mathcal{D}(T)$; by $\varrho(T)$, $\sigma(T)$, $R(\lambda, T)$, $T^*$, $R(T)$ we denote its resolvent set, spectrum, resolvent, adjoint and the range, respectively. If $T$ is selfadjoint, $\sigma_{\text{r}}(T)$ denotes its spectral family (continuous from the right), while $E(\mathcal{A})$ denotes the spectral measure of a Borel set $\mathcal{A}$ from the real line.

In the first part of this paper we generalize a result of R. C. Riddell ([3]). Let $T(\varepsilon)$ ($\varepsilon$ real from some interval containing zero) be a family of closed operators such that $(JT(\varepsilon)x, y) = (Jx, T(\varepsilon)y)$ for $x, y \in \mathcal{D}(T(\varepsilon))$ and for a fixed operator $J = J^* = J^{-1}$ on $X$. Let a real point $\lambda$ be a pole of the first order of $R(\lambda; T)$ and let the corresponding spectral projection $P_0$ have the dimension $m < \infty$ such that the restriction of the form $(Jx, y)$ on $P_0X$ is strictly positive. If there are functions $\psi_1(\varepsilon) \leq \mathcal{D}(T(\varepsilon))$ and real functions $\lambda_j(\varepsilon)$, $j = 1, 2, \ldots, m$ such that for $\varepsilon \to 0$ we have $\|\psi_j(\varepsilon)\| \to K > 0$, $T(\varepsilon) - \lambda_j(\varepsilon) \psi_j(\varepsilon) = o(\varepsilon^n)$, $(1 - P_0) \psi_j(\varepsilon) \to 0$, $(J\psi_j(\varepsilon), \psi_k(\varepsilon)) \to \delta_{jk}$, we call $\psi_j(\varepsilon)$ the $J-p$-asymptotic basis for $T(\varepsilon)$. (cf. [3], [4]). We impose on $T(\varepsilon)$ some further conditions which ensure the existence of spectral decompositions of $T(\varepsilon)$ and their strong convergence when $\varepsilon \to 0$ in the sense of the well-known Rellich-Kato's theorem ([2], p. 432). The main result is: Any such family $T(\varepsilon)$ possessing a $J-p$-asymptotic basis has a spectral concentration of the $p$-th order in the sense of Riddell [3].

In the second part we give a sufficient condition for a $J$-symmetric family $T(\varepsilon)$ to have a $J-p$-asymptotic basis. We call the family $T(\varepsilon)$ $J-p$-smooth (cf. [4]) with respect to the point $\lambda \leq \sigma(T(0))$ if the subspace $D_p$ of vectors $\psi$ for which $T(\varepsilon)\psi$ has the $p$-th derivative is sufficiently large (in the sense to be given more precisely below). The main result is as follows: Any $J-p$-smooth family possesses a $J-p$-asymptotic basis. For $J = 1$ this result is contained in [4]. However, Lemma 3. of [4] contains an
error such that by [4] the \((2p-1)\)-smoothness is needed for the existence of a \(p\)-asymptotic basis. Thus, our paper contains a correct proof of the mentioned result of [4].

The main results of the present work can be applied in studying the spectral concentration for the Klein-Gordon equation, describing the motion of a spinless relativistic particle moving in a potential barrier. This application will be the subject of a subsequent paper.

1. ASSUMPTION. The domains of definition \(\mathcal{D}(T(\epsilon))\) of \(T(\epsilon)\) and \(\mathcal{D}(T(\epsilon)*)\) of \(T(\epsilon)*\) coincide and are dense in \(X\). The operator \(T(0)\) has a real eigenvalue \(\lambda\) which is a pole of the first order for the resolvent of \(T(0)\). The corresponding eigenspace has the dimension \(m < \infty\). The respective eigenprojection is denoted by \(P_0\).

2. DEFINITION. (cf. C. Riddell [3]). Suppose that vector functions \(\epsilon \rightarrow \psi_j(\epsilon)\) and scalar functions \(\epsilon \rightarrow \lambda_j(\epsilon)\), \(j = 1, 2, \ldots, m\), are given on the interval \(I\) such that \(\psi_j(\epsilon) \leq \mathcal{D}(T(\epsilon))\) and

\[
(T(\epsilon) - \lambda_j(\epsilon)) \psi_j(\epsilon) = o(\epsilon^p), \quad |\psi_j(\epsilon)| \rightarrow K > 0, \quad \epsilon \rightarrow 0 \tag{1}
\]

for some \(p > 0\). Then any pair of functions \(\lambda_j(\epsilon), \psi_j(\epsilon), j = 1, \ldots, m\) is called a \(p\)-pair of the family \(T(\epsilon)\) with respect to the point \(\lambda\).

If, in addition, a unitary selfadjoint operator \(J = J* = J^{-1}\) is given such that the restriction of the form \(\langle x, y \rangle = (Jx, y)\) on \(P_0X\) is strictly positive and such that

\[
(1 - P_0) \psi_j(\epsilon) \rightarrow 0, \quad \langle \psi_j(\epsilon), \psi_k(\epsilon) \rangle \rightarrow \delta_{jk}, \quad \epsilon \rightarrow 0, \tag{2}
\]

then the vector functions \(\psi_j(\epsilon)\) are called a \(J - p\)-asymptotic basis for \(T(\epsilon)\). If \(J = 1\), \(T(\epsilon)\) is simply called a \(p\)-asymptotic basis. The functions \(\lambda_j(\epsilon)\) are pseudoeigenvalues.

3. DEFINITION. (cf. Riddell [3]). Let \(T(\epsilon)\) be a family of scalar type operators with real spectra, for \(\epsilon \leq I\). Let \(p \geq 0\) and let \(I'\) be a real interval such that

\[
E_\epsilon(I' \setminus \mathcal{C}(\epsilon)) \rightarrow 0, \quad \mu(\mathcal{C}(\epsilon)) = o(\epsilon^p), \quad \epsilon \rightarrow 0 \tag{3}
\]

where \(E_\epsilon(\cdot)\) denotes the spectral measure for \(T(\epsilon)\), \(\mathcal{C}(\epsilon)\) is a family of real Borel sets and \(\mu\) is the Lebesgue measure. In addition, let

\[
\sup_{\epsilon, t, t'} \|E_\epsilon(t, t')\| < \infty. \tag{4}
\]

Then we say that the part \(\sigma_\epsilon(I') = I' \cap \sigma(T(\epsilon))\) of the spectrum of \(T(\epsilon)\) in \(I'\) is \(p\)-concentrated on \(\{\mathcal{C}(\epsilon)\}\).

C. Riddell ([3]) has proved the fundamental theorem (see also [1]).

THEOREM. Let \(T(\epsilon)\) be a family of selfadjoint operators such that \(T(\epsilon) \rightarrow T(0)\) strongly in the generalized sense (see [1], p. 427) and that it satisfies Assumption 1. Let \(T(\epsilon)\) have \(p\)-pairs \(\lambda_j(\epsilon), \psi_j(\epsilon)\)
such that \( \psi_J(\epsilon) \) is a \( p \)-asymptotic basis. Then \( \sigma_\epsilon(J') \) is \( p \)-concentrated. As »concentration sets« \( C(\epsilon) \) we may take the unions of intervals around \( \lambda_i(\epsilon) \), the length of which does not exceed \( o(\epsilon^p) \).

In what follows we shall prove the same result for a class of \( J \)-selfadjoint families.

In [6], [7] we considered the operators of the form

\[ T = S + V, \quad S = S^*, \quad V \text{ bounded}, \]

and we proved the following: Let \( (-\delta, \delta) \subseteq \sigma(S) \) for some \( \delta > 0 \) and \( J = \text{sign } S \). If \( V \) is \( J \)-symmetric, i.e., \( V = JVJ^* \) and \( \| V \| < \delta/2 \) then \( T \) is a scalar type operator with a real spectrum. Here we consider a family

\[ T(\epsilon) = S(\epsilon) + V(\epsilon) \quad (6) \]

of such operators for which

\[ (-\delta, \delta) \subseteq \sigma(S(\epsilon)), \quad \| V(\epsilon) \| < \delta/2, \]

where \( \delta > 0 \) does not depend on \( \epsilon \). Moreover, let

\[ J = \text{sign } S(\epsilon), \quad V(\epsilon) = JV(\epsilon)^* J, \quad (8) \]

where \( J \) does not depend on \( \epsilon \).

4. THEOREM. Let \( T(\epsilon) \) satisfy (6), (7), (8) and Assumption 1. with \( J \) as in (8) and \( \lambda > 0 \). Moreover, let \( V(\epsilon) \to V(0) \) and let \( S(\epsilon) \to S(0) \) in the generalized sense. Then \( T(\epsilon) \) possesses a spectral concentration in the way described by Riddell's Theorem, provided that \( T(\epsilon) \) has a \( J - p \)-asymptotic basis.

Proof. In [5] we proved that the integral

\[ K(\epsilon) = -\frac{1}{4\pi} s \text{-lim } \int_{-i\beta}^{i\beta} R(\lambda; T(\epsilon)) d\lambda \]

exists and that

\[ K(\epsilon) \to K(0), \quad A(\epsilon) \to A(0), \quad A(\epsilon)^{-1} \to A(0)^{-1}, \quad \epsilon \to 0, \quad (10) \]

where

\[ A(\epsilon) = (JK(\epsilon))^{1/2} \]

are bounded symmetric operators with bounded inverses. Moreover

\[ T(\epsilon) = A(\epsilon) T(\epsilon) A(\epsilon)^{-1} \]

are selfadjoint with

\[ T(\epsilon) \to T(0). \]

* The strong convergence in the generalized sense means just the strong convergence of resolvents (see T. Kato [2], p. 427).
Therefore (see [2])

\[ E(t, \varepsilon) \to E(t, 0) \]  

(14)

for any \( t \) which is not an eigenvalue for \( T(0) \). Here \( E(t, \varepsilon) \) denotes the spectral family of \( T(\varepsilon) \).

Note also that

\[ K(0) P_0 = P_0 K(0) = P_0, \]  

(15)

which is a consequence of \( \lambda > 0 \).

Now, let \( \lambda_j(\varepsilon), \psi_j(\varepsilon), j = 1, \ldots, m \) be \( p \)-pairs for \( T(\varepsilon) \) and let \( \psi_j(\varepsilon) \) be a \( J-p \)-asymptotic basis. Putting \( \varphi_j(\varepsilon) = A(\varepsilon) \psi_j(\varepsilon) \) the formula (1) gives

\[ (T(\varepsilon) - \lambda_j(\varepsilon)) \varphi_j(\varepsilon) = o(\varepsilon^p). \]  

(16)

Furthermore, we have

\[
\lim_{\varepsilon \to 0} (1 - A(0) P_0 A(0)^{-1}) A(\varepsilon) \psi_j(\varepsilon) = \\
= \lim_{\varepsilon \to 0} (1 - A(0) P_0 A(0)^{-1}) A(\varepsilon) (P_0 \psi_j(\varepsilon) + (1 - P_0) \psi_j(\varepsilon)) = \\
= A(0) \lim_{\varepsilon \to 0} (1 - P_0) A(0)^{-1} A(\varepsilon) P_0 \psi_j(\varepsilon) = \\
= A(0) \lim_{\varepsilon \to 0} (1 - P_0) \psi_j(\varepsilon) = \\
= A(0) \lim_{\varepsilon \to 0} (1 - P_0) \psi_j(\varepsilon) = 0.
\]

This means

\[ (1 - P_0) \varphi_j(\varepsilon) \to 0, \quad \varepsilon \to 0, \quad P_0 = A(0) P_0 A(0)^{-1}. \]  

(17)

Here we used (2), the finite dimensionality of \( P_0 \) and the fact that \( A(\varepsilon) \to A(0) \).

Finally, we have

\[
\lim_{\varepsilon \to 0} (\varphi_j(\varepsilon), \varphi_k(\varepsilon)) = \lim_{\varepsilon \to 0} (A(\varepsilon)^2 \psi_j(\varepsilon), \psi_k(\varepsilon)) = \\
= \lim_{\varepsilon \to 0} (J(K(0) + K(\varepsilon) - K(0)) (P_0 \psi_j(\varepsilon) + \\
+ (1 - P_0) \psi_j(\varepsilon)), P_0 \psi_k(\varepsilon) + (1 - P_0) \psi_k(\varepsilon)).
\]

Here, using the finite dimensionality of \( P_0 \) and the boundedness of \( |\psi_j(\varepsilon)| \) for \( \varepsilon \to 0 \), all vanishes except possibly

\[
\lim_{\varepsilon \to 0} (J K(0) P_0 \psi_j(\varepsilon), P_0 \psi_k(\varepsilon)) = \lim_{\varepsilon \to 0} (J P_0 \psi_j(\varepsilon), \psi_k(\varepsilon))
\]
where we used the $J$-symmetry of $P_0$ and (15). Using both relations in (2), we obtain

$$
\lim_{\varepsilon \to 0} (\varphi_j(\varepsilon), \varphi_k(\varepsilon)) = \begin{cases} 
0 & j + k \\
K_j > 0, & j = k.
\end{cases}
$$

Relations (16), (17), (18) mean that $\lambda_j(\varepsilon), \varphi_j(\varepsilon)/\|\varphi_j(\varepsilon)\|$ are p-pairs for $T(\varepsilon)$ and that $\varphi_j(\varepsilon)/\|\varphi_j(\varepsilon)\|$ is a p-asymptotic basis. This permits the use of Riddell’s theorem which tells that the spectrum of $T(\varepsilon)$ is p-concentrated.

The spectral families of $T(\varepsilon)$ and $T(\varepsilon)$ are connected by the same similarity relation as in (12). So the p-concentration also follows for the family $T(\varepsilon)$. Q.E.D.

In the following we shall give a sufficient condition for a $J$-symmetric family $T(\varepsilon)$ to have a $J - p$-asymptotic basis. Some proofs are quite analogous to those in [4]. We still include some of them for the sake of completeness.

Let $T(\varepsilon)$ be a family satisfying Assumption 1. For an integer $p \geq 0$ denote by $D_p$ the set of all vectors $\psi \leq X$ such that $D_p \subseteq \mathcal{D}(T(\varepsilon))$ and that the vector function

$$
\varepsilon \to T(\varepsilon)\psi
$$

has the derivative of the order $p$ for $\varepsilon = 0$. Then

$$
D_0 \supseteq D_1 \supseteq \ldots,
$$

where $D_0$ denotes the set of all $\psi \leq X$, for which $\varepsilon \to T(\varepsilon)\psi$ is continuous at $\varepsilon = 0$.

5. LEMMA. The set $D_p$ is a subspace of $X$ and for $\psi \leq D_p$ we have

$$
T(\varepsilon)\psi = T_0\psi + \varepsilon T_1\psi + \ldots + \varepsilon^p T_p\psi + o(\varepsilon^p),
$$

where $T_0, \ldots, T_p$ are linear operators, defined on $D_p$ as

$$
T_\tau\psi = \frac{1}{\tau!} \left( \frac{d^\tau}{d\varepsilon^\tau} T(\varepsilon)\psi \right)_{\varepsilon=0}
$$

Proof. See [4].

In the following we shall require that $D_p$ is sufficiently large. The later considerations will justify the following definition.

6. DEFINITION. Let a family $T(\varepsilon)$ satisfy Assumption 1. We denote by

$$
Z = - \lim_{\nu \to \lambda} (1 - P_0)(\mu - T(0))^{-1}
$$

the reduced resolvent of $T(0)$ in the point $\lambda$. Furthermore, for an integer $p \geq 0$ we denote by $V_p$ the set of all operators of the form
where we have taken

\[ V_0 = \{1\}. \]

We say that the family \( T(\varepsilon) \) is \( p \)-smooth at \( \varepsilon = 0 \) with respect to the point \( \lambda \) if any of the operators from \( V_p \) is defined at least on \( P_0 X \) and maps \( P_0 X \) into \( D_p \).

We may briefly say that \( V_p \) contains 1 and any \( r \)-fold product of the factors \( Z, ZT_r \) such that \( r \) varies from 1 to \( p \), and \( r \) does not exceed \( p - r + 1 \).

We see that the \( p \)-smoothness includes the \( s \)-smoothness for \( p \geq s \).

7. LEMMA. The sets \( V_0, V_1, \ldots \) are ordered by inclusion i.e., \( V_0 \subseteq V_1 \subseteq \ldots \). If \( A \subseteq V_n, B \subseteq V_k \), then \( AB \subseteq V_s \), for \( n + k \leq s \).

Proof. See [4].

Let us introduce the subspaces

\[ Y_r = \sum_{A \in V_r} AP_0 X, \quad r = 0, 1, 2, \ldots, \] (26)

with the corresponding orthogonal projections \( R_r \). We see that all \( Y_r \) are finite dimensional and ordered as

\[ Y_0 \subseteq Y_1 \subseteq \ldots. \]

Now, the \( p \)-smoothness implies

\[ Y_r \subseteq D_r, \quad r \leq p. \]

8. LEMMA. Let \( A = X_1 X_2 \ldots X_r \subseteq V_p \) and let \( T(\varepsilon) \) be \( p \)-smooth. Then

\[ X_1 X_2 \ldots X_r | P_0 = R_p X_1' X_2' \ldots X_r' | P_0, \]

where

\[ X_s' = \begin{cases} X_s & \text{if } X_s = Z, 1 \\ X_s R_p & \text{if } X_s = ZT_r \end{cases}, \]

\[ s = 1, 2, \ldots, r. \]

Proof. If \( X_1 X_2 \ldots X_r \subseteq V_p \), then \( X_s X_{s+1} \ldots X_r \subseteq V_p \), \( 1 \leq s \leq r \). Then by the \( p \)-smoothness

\[ X_1 X_2 \ldots X_r | P_0 = R_p X_1' X_2' \ldots X_r' | P_0 = R_p X_1' X_2' \ldots X_r' | P_0. \]

* Here \( A | X_0 \) denotes the restriction of \( A \) on \( X_0 \).
Let us now introduce an operator family

\[ P^{(p)}(c) = \sum_{n=0}^{p} c^n P_n, \]

where

\[ P_n = \sum_{r=1}^{n} (-1)^{r+1} \sum_{k_1 + \ldots + k_r+1 = r, k_i \geq 0, \nu_1 + \ldots + \nu_r = n, \nu_i \geq 1} Z^{(k_1)} T_{\nu_1} Z^{(k_2)} \ldots Z^{(k_r)} T_{\nu_r} Z^{(k_{r+1})} \]

(28)

\[ Z^{(k)} = Z^k, \quad k \geq 1, \quad Z^{(0)} = P_0. \]

(29)

9. **Lemma.** Let \( T(c) \) be \( p \)-smooth with respect to \( \lambda \). Then the operator

\[ P_i P_j T_k, \quad i + j \leq p, \quad j + k \leq p \]

(30)

is defined at least on \( P_0 X \). The range of \( P_k \) (and therefore of (30)) is contained in \( Y_k \) for \( k \leq p \).

**Proof.** The operator \( P_i P_j T_k \) is a linear combination of the operators

\[ Z^{(k_1)} T_{\nu_1} Z^{(k_2)} \ldots T_{\nu_r} Z^{(k_{r+1})} T_k \]

(31)

\[ k_1 + \ldots + k_{r+1} = r, \quad k_i \geq 0, \quad \nu_1 + \ldots + \nu_r = j, \quad \nu_i \geq 1, \quad j + k \leq p. \]

Since \( k_1 + \ldots + k_{r+1} = r \), at least one of the indices \( k_i \) must vanish. Taking into account all \( k_i = 0 \), (31) can be written as

\[ A P_0 T_{\nu_1} B P_0 T_{\nu_2} \ldots F P_0 T_{\nu_j} G, \quad \nu', \nu'', \ldots, \nu^{(a)} \leq j. \]

(32)

Here \( A, B, \ldots, F, G \) are at most \( r \)-fold products of the factors \( Z, ZT \). By \( \nu_1 + \ldots + \nu_r = j \), then index \( \nu \) entering \( A, B, \ldots, F \) does not exceed \( j - r + 1 \), thus \( A, B, \ldots, F \leq V_j \). However, the index \( \nu \), entering \( G \) does not exceed

\[ \max [j - r + 1, p - j] \leq p - r + 1 \]

because of \( r \leq j \leq p \). Thus

\[ G \leq V_p. \]

Similarly, \( P_i \) is a linear combination of members of the form

\[ A_1 P_0 T_{\nu_1} B_1 P_0 T_{\nu_2} \ldots F_1 P_0 T_{\nu_j} G_1, \quad \nu', \nu'', \ldots, \nu^{(a)} \leq i \]

(33)

where

\[ A_1, B_1, \ldots, G_1 \leq V_i. \]

The operators \( P_i P_j T_k \) will then be a linear combination of products of operators of the forms (32) and (33). Since \( G_1 \leq V_i, A \leq V_j, i + j \leq p \) implies \( G_1 A \leq V_p \) (lemma 7), any such product is of the form

* Since \( T_0 P_0 = T(0) P_0 = \lambda P_0 \), it is important that \( \lambda \neq 0 \). This can always be obtained by adding to \( T(c) \) a sufficiently large multiple of the identity.
\[
A_2 P_0 T_{\varepsilon} B_2 P_0 T_{\eta} \ldots F_2 P_0 T_{\varepsilon(\sigma)} G_2
\]
\[
q', q'', \ldots, q^{(\sigma)} \leq p, \\
A_2, \ldots, G_2 \subseteq V_p.
\]

The family \(T(\varepsilon)\) is \(p\)-smooth, which implies that (34) and therefore \(P_i P_j T_k\) is defined on \(P_0\) and maps \(P_0 X\) into \(Y_k \subseteq D_k\).

Q.E.D.

10. LEMMA. Let \(T(\varepsilon)\) be \(p\)-smooth with respect to \(\lambda\). Then

\[
(P_0 P_n + \ldots + P_n P_0) \varphi = P_n \varphi, \quad \varphi \leq \sum_{k=0}^{p-n} T_k P_0 X,
\]

(35)

\[
(T_n P_0 + \ldots + T_0 P_n) q = (P_0 T_n + \ldots + P_n T_0) q, \quad q \leq P_0 X,
\]

(36)

for \(n = 0, 1, \ldots, p\).

Proof. Set

\[
T'(\varepsilon) = T_0' + \varepsilon T_1' + \ldots + \varepsilon^n T_n', \quad T_0' = T(0), \quad T_k' = T_k R_p, \quad k \geq 1.
\]

The operators \(T_k' = T_k R_p, k \geq 1\) are bounded, since the projection \(R_p\) is finite (the products \(T_k R_p\) are well defined, since \(T_k\) is defined on \(D_p\) and \(Y_p \subseteq D_p\)). Thus, the family \(T'(\varepsilon)\) is a holomorphic family of type \((A)\) for all complex \(\varepsilon\) (see T. Kato, [2], p. 375). Since \(T'(0) = T(0)\), and \(\lambda\) is an isolated point of \(\sigma(T(0)) = \sigma(T'(0))\), there is a family of bounded projections

\[
P'(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k P_k', \quad P'(0) = P_0' = P_0,
\]

which is analytic in some neighbourhood of \(\varepsilon = 0\), such that \(P(\varepsilon)\) projects on the root space belonging to the group of eigenvalues coming by perturbations from the point \(\lambda\). Since \(\lambda\) is a pole of the first order for the function \(\mu \to (\mu - T(0))^{-1}\) we have (cf. Kato [2], p. 76)

\[
P_k' = \sum_{r=1}^{p} \frac{(-1)^{r+1}}{k_1 + \ldots + k_{r+1} = r, \quad k_i \geq 0} \sum Z(k_1) T_{r_1} Z(k_2) \ldots T_{r_r} Z(k_{r+1}),
\]

\[
k \leq p.
\]

Furthermore, \(P'(\varepsilon)^2 = P'(\varepsilon)\) implies

\[
P_0' P_0' + \ldots + P'_k P_0' = P'_k, \quad k = 0, 1, 2, \ldots.
\]

(37)

On the other hand, the left-hand side of equality (35) consists of summands of the form

\[
P_s P_t P_j \varphi, \quad s + t \leq p, \quad t + j \leq p.
\]
Formula (34) together with Lemma 8 shows that

$$P_s'P_i'T_j'\psi = P_sP_iT_j\psi \text{ for } \psi \leq \sum_{k=0}^{p-n} T_kP_0X.$$  

Also, formula (31) with Lemma 8 shows that $$P_k'\psi = P_k\psi,$$  

$$\psi \leq \sum_{j=0}^{p-k} T_jP_0X.$$  

Thus, (35) follows.

To prove (36), we introduce the operator family

$$P'(\varepsilon)T'(\varepsilon)P'(\varepsilon) = \frac{1}{2\pi i} \int_{\Gamma} \mu(\mu - T'(\varepsilon))^{-1} d\mu = P'(\varepsilon)T'(\varepsilon) = T'(\varepsilon)P(\varepsilon),$$

where $\Gamma$ is a small circle around $\lambda$. The function $\varepsilon \rightarrow T'(\varepsilon)P'(\varepsilon)$ is bounded analytic in some neighbourhood of zero.

On the other hand, for $\varphi \leq \mathcal{D}(T(0))$ the function $T'(\varepsilon)\varphi$ is analytic on $(-\infty, \infty)$. Hence, for $\varphi, \psi \leq \mathcal{D}(T(0))$ we have

$$(P'(\varepsilon)T'(\varepsilon)\varphi,\psi) = (T'(\varepsilon)\varphi, P'(\varepsilon)^*\psi) = \sum_{k=0}^{\infty} \varepsilon^k z_k,$$

$$z_k = (T(0)\varphi, P_k^*\psi) + \ldots + (T_k'\varphi, P_0^*\psi) = ((P_k' T_0' + \ldots + P_0' T_k') \varphi, \psi).$$

Since

$$(T'(\varepsilon)P'(\varepsilon)\varphi,\psi) = \sum_{k=0}^{p} \varepsilon^k ((T_k'P_0' + \ldots + T_0' P_k')\varphi, \psi) + o(\varepsilon^p),$$

we have

$$(P_k' T_0' + \ldots + P_0' T_k') \varphi = (T_0' P_k' + \ldots + T_k' P_0') \varphi,$$

$$k = 0, 1, 2, \ldots, p,$$

for $\varphi \leq P_0X$. Now, $P_jT_i = P_j'T_i'$, $T_iP_j = T_i'P_j'$, $i + j \leq p$ (Lemma 8) implies (36). Q.E.D.

11. LEMMA. Let $D$ be a subspace of a normed space $N$ and let $P_0, P_1, \ldots, P_p, T_0, T_1, \ldots, T_p$ be linear operators in $N$ such that

I $P_lP_kT_j$ is defined on $D$ for $l + k \leq p$, $k + j \leq p$

II the operator $T_i$ is bounded on $D$

the operator $P_k$ is bounded on $T_jD$, $j + k \leq p$

the operator $P_l$ is bounded on $P_kT_jD$, $k + j \leq p$, $k + l \leq p$

III $(P_0P_n + \ldots + P_nP_0)\psi = P_n\psi$, $n = 0, 1, 2, \ldots, p$, for $\psi \leq \sum_{j=0}^{p-n} T_jD$.
Then for any vector function $\epsilon \to \chi(\epsilon)$, which is bounded in norm when $\epsilon \to 0$, the implication

$$P_0 \sum_{k=0}^{p} (P_0 T_k + \ldots + P_k T_0) \epsilon^k \chi(\epsilon) = o(\epsilon^p) \Rightarrow$$

$$P_0 \sum_{k=0}^{p} (P_0 T_k + \ldots + P_k T_0) \epsilon^k \chi(\epsilon) = o(\epsilon^p),$$

holds.

**Proof.** For $p = 0$ the assertion is trivially true by $P_0^2 \chi(\epsilon) = P_0 \chi(\epsilon)$. For $p = 1$ the equalities

$$P_0 (P_0 T_0 + \epsilon (P_1 T_0 + P_0 T_1)) \chi(\epsilon) = o(\epsilon)$$

$$\epsilon (P_1 T_0 + P_0 T_1) \chi(\epsilon) = o(\epsilon^0)$$

imply

$$P_0 T_0 \chi(\epsilon) = o(\epsilon^0).$$

(Notice that $P_1 T_0 + P_0 T_1$ is bounded on $D$.)

Hence,

$$o(\epsilon) = P_0 (P_0 T_0 + \epsilon (P_0 T_1 + P_1 T_0)) \chi(\epsilon) =$$

$$= P_0 T_0 + \epsilon (P_0 T_1 + (P_1 - P_1 P_0) T_0)) \chi(\epsilon) =$$

$$= [P_0 T_0 + (P_0 T_1 + P_1 T_0) \epsilon] \chi(\epsilon) - \epsilon P_1 P_0 T_0 \chi(\epsilon).$$

By $P_0 T_0 \chi(\epsilon) = o(\epsilon^0)$ and the boundedness of $P_1$ we have $\epsilon P_1 P_0 T_0 \chi(\epsilon) = o(\epsilon)$. The assertion is, therefore, true if $p = 1$. By induction, suppose that the assertion is true if $p = 0, 1, 2, \ldots, s$ and that

$$o(\epsilon^{s+1}) = P_0 \sum_{k=0}^{s+1} (P_0 T_k + \ldots + P_k T_0) \epsilon^k \chi(\epsilon) =$$

$$= P_0 \sum_{k=0}^{k'} \epsilon^k (P_0 T_k + \ldots + P_k T_0) \chi(\epsilon) +$$

$$+ \epsilon^{k'+1} \sum_{k=k'+1}^{s+1} \epsilon^{k-k'-1} (P_0 T_k + \ldots + P_k T_0) \chi(\epsilon).$$

Since, by supposition, $P_0 (P_0 T_k + \ldots + P_k T_0)$ is bounded on $D$, the second term of the sum is $o(\epsilon^{k'})$, $k' = 0, 1, \ldots, k$. Hence, by the assumption of induction we have

$$\sum_{k=0}^{k'} \epsilon^k (P_0 T_k + \ldots + P_k T_0) \chi(\epsilon) = o(\epsilon^{k'}), \ k' = 0, \ldots, s.$$

Furthermore, using III. we have

$$o(\epsilon^{s+1}) = \sum_{k=1}^{s+1} \epsilon^k (P_0^2 T_k + \ldots + P_0 P_k T_0) \chi(\epsilon) =$$

$$= \sum_{k=0}^{s+1} \epsilon^k (P_0 T_k + (P_1 - P_1 P_0) T_{k-1} + \ldots$$
\[ + (P_k - P_1 P_{k-1} - \ldots - P_k P_0) T_0 \chi (\varepsilon) = \]
\[ = \sum_{k=0}^{s+1} \varepsilon^k (P_0 T_k + P_1 T_{k-1} + \ldots + P_k T_0) \chi (\varepsilon) - \]
\[ - P_1 \sum_{k=0}^{s+1} \varepsilon^k (P_0 T_{k-1} + \ldots + P_{k-1} T_0) \chi (\varepsilon) - \]
\[ - P_{s+1} P_0 T_0 \chi (\varepsilon) = \sum_{k=0}^{s+1} \varepsilon^k (P_0 T_k + \ldots + P_k T_0) \chi (\varepsilon) - \]
\[ - \varepsilon P_1 \sum_{k=0}^{s} \varepsilon^k (P_0 T_k + \ldots + P_k T_0) \chi (\varepsilon) - \]
\[ - \varepsilon^{s+1} P_{s+1} P_0 T_0 \chi (\varepsilon) = \sum_{k=0}^{s+1} \varepsilon^k (P_0 T_k + \ldots + P_k T_0) \chi (\varepsilon) + \]
\[ + \varepsilon^0 (\varepsilon^0) + \varepsilon^2 (\varepsilon^{s-1}) + \ldots + \varepsilon^{s+1} (\varepsilon^0), \]
which proves the assertion for \( p = s + 1 \). Here we used the fact that \( P_k P_l T_j \) is bounded on \( D \) for \( k + l \leq s + 1 \), \( l + j \leq s + 1 \).

Q.E.D.

Note an important fact, which will be used in the following: the boundedness condition II is automatically fulfilled if \( D \) is finite dimensional.

In the following we introduce the \( J \)-symmetry.

12. ASSUMPTION. The operators \( T (\varepsilon) \), \( \varepsilon \leq I \) are \( J \)-symmetric for some fixed \( J \cdot J^* = J^{-1} \), i. e.,

\[ (J T (\varepsilon) x, y) = (J x, T (\varepsilon) y) \quad x, y \leq \mathcal{D} (T (\varepsilon)), \tag{38} \]
and for \( x \leq P_0 X \) we have

\[ (J x, x) \geq 0; \quad (x, x) = 0 \Rightarrow x = 0. \tag{39} \]

13. LEMMA. The operators \( Z, P_k, T_k P_0 + \ldots + T_0 P_k, k = 0, 1, \ldots, p \), for a \( p \)-smooth family \( T (\varepsilon) \) which satisfies Assumption 12. are \( J \)-symmetric. For the domain of \( P_k, T_k P_0 + \ldots + T_0 P_k \) we take

\[ \sum_{i=0}^{p-k} T_i P_0 X, \quad P_0 X, \]
respectively.

Proof. Put \( R (\mu) = (\mu - T (0))^{-1} \). Let \( \Gamma \) be a circle with the centre \( \lambda \), isolating \( \lambda \) from the rest of \( \sigma (T (0)) \). Since \( \lambda = \lambda \) and \( \Gamma = \overline{\Gamma} \) we have
JR (μ) = R (μ)J,  μ ≤ φ (T (0)).

Hence

\[
(JP_0 x, y) = \frac{1}{2\pi i} \int \limits_I (JR(z)x, y) \, dz = (Jy, P_0 x).
\]

Thus, \( P_0 \) is J-symmetric.

Furthermore, \( \lambda = \overline{\lambda} \) implies

\[
(JZx, y) = -\lim_{z \to \lambda} (JR(z)(1 - P_0)x, y) = \\
= -\lim_{z \to \lambda} (Jx, (1 - P_0) R(z)y) = -\lim_{z \to \lambda} (Jx, (1 - P_0) R(z)y) = \\
= (Jx, Zy).
\]

Thus, \( Z \) is J-symmetric. The J-symmetry of \( T(\varepsilon) \) together with (21) implies

\[
(JT_k \psi, \varphi) = (J\psi, T_k \varphi), \quad \psi, \varphi \leq D_p.
\] (40)

The J-symmetry of \( P_k \) follows from the fact that \( T_k, Z, P_0 \) are J-symmetric and that expression (28) for \( P_k \) is invariant under the permutation of the indices \( k_1, \ldots, k_r+1 \).

Thus

\[
(JP_k \psi, \varphi) = (J\psi, P_k \varphi), \quad \psi, \varphi \leq \sum_{i=0}^{p-k} T_i P_0 X.
\] (41)

Finally, for \( \psi, \varphi \leq P_0 X \) formula (36) gives

\[
(J(T_k P_0 + \ldots + T_0 P_k) \psi, \varphi) = (J\psi, (T_k P_0 + \ldots + T_0 P_k) \varphi),
\] (42)

where we have used the J-symmetry of \( T_k, P_k \).

Q.E.D.

14. THEOREM. Any J-symmetric family \( T(\varepsilon) \), \( p \)-smooth in \( \varepsilon = 0 \), with respect to the point \( \lambda \), such that the restriction of the form \( (x, y) = (Jx, y) \) on \( P_0 X \) is strictly positive, possesses a \( J \)-- \( p \)-asymptotic basis.

Proof. In the finite dimensional space \( P_0 X \) consider the generalized eigenvalue problem

\[
(A^{(p)}(\varepsilon) - \lambda_j^{(p)}(\varepsilon) B^{(p)}(\varepsilon)) \varphi_j^{(p)}(\varepsilon) = 0, \quad j = 1, 2, \ldots, m,
\] (43)

where

\[
A^{(p)}(\varepsilon) = P_0 \sum_{k=0}^{p} e^k (P_k T_0 + \ldots + P_0 T_k) | P_0 X,
\] (44)

\[
B^{(p)}(\varepsilon) = P_0 \sum_{k=0}^{p} e^k P_k | P_0 X.
\] (45)

The space \( P_0 X \) is a unitary finite dimensional space with the scalar product \( \langle \cdot, \cdot \rangle \), which is positive definite on \( P_0 X \) by supposition. The operators \( A^{(p)}(\varepsilon), B^{(p)}(\varepsilon) \) are polynomials in \( \varepsilon \) and are symmetric. Since \( B^{(p)}(0) = 1 | P_0 X \), for sufficiently small \( \varepsilon \) the operator \( B^{(p)} \) will be strictly positive definite.
Since
\[ A^{(p)}(0) = \lambda |P_0X|, \]
the solutions \( \lambda_j'(\varepsilon), \varphi_j'(\varepsilon) \) of the problem (43) are analytic at \( \varepsilon = 0 \) and
\[ \lambda_j'(0) = \lambda. \]
(See [2], p. 419). The functions \( \varphi_j'(\varepsilon) \) can be chosen such that
\[ (J\varphi_j'(0), \varphi_{k'}'(0)) = \delta_{jk}. \]  
(46)

Put
\[ \varphi_j'(\varepsilon) = \sum_{k=0}^{p} \varphi_j^{(k)} \varepsilon^k, \quad \lambda_j'(\varepsilon) = \sum_{k=0}^{p} \lambda_j^{(k)} \varepsilon^k. \]

The function
\[ \varepsilon \to A^{(p)}(\varepsilon) - \lambda_j'(\varepsilon) B^{(p)}(\varepsilon) \]
is a power series, whose coefficient of the \( k \)-th order is
\[ P_0 \left( P_0 (T_k - \lambda_j^{(k)}) + \ldots + P_k (T_0 - \lambda) \right) P_0 X, \quad k = 0, \ldots, p. \]

Omitting the powers higher than \( p \), we obtain
\[ P_0 \sum_{k=0}^{p} \left( P_0 (T_k - \lambda_j^{(k)}) + \ldots + P_k (T_0 - \lambda) \right) \varepsilon^k \varphi_j(\varepsilon) = o(\varepsilon^p), \]  
(47)

where
\[ \varphi_j(\varepsilon) = \sum_{k=0}^{p} \varepsilon^k \varphi_j^{(k)}. \]

By (35) the operators \( P_k, T_k - \lambda_j^{(k)} \), the functions \( \varphi_j(\varepsilon) \leq P_0 X \) and the subspace \( D = P_0 X \) satisfy the conditions of Lemma 11 (\( P_0 X \) is finite dimensional!). Thus (47) implies
\[ \sum_{k=0}^{p} \left( P_0 (T_k - \lambda_j^{(k)}) + \ldots + P_k (T_0 - \lambda) \right) \varepsilon^k \varphi_j(\varepsilon) = o(\varepsilon^p). \]

Furthermore, (36) implies
\[ \sum_{k=0}^{p} \left( (T_k - \lambda_j^{(k)}) P_0 + \ldots + (T_0 - \lambda) P_k \right) \varepsilon^k \varphi_j(\varepsilon) = o(\varepsilon^p). \]

Since
\[ \sum_{k=0}^{p} \left( (T_k - \lambda_j^{(k)}) P_0 + \ldots + (T_0 - \lambda) P_k \right) \varepsilon^k \varphi_j(\varepsilon) = \]
\[ = \left[ \sum_{k=0}^{p} (T_k - \lambda_j^{(k)}) \varepsilon^k \right] \left[ \sum_{k=0}^{p} P_k \varepsilon^k \varphi_j(\varepsilon) \right] + o(\varepsilon^p), \]
we have
\[ \sum_{k=0}^{p} (T_k - \lambda_j^{(k)}) \varepsilon^k \psi_j'(\varepsilon) = o(\varepsilon^p), \]  
(48)
where
\[ \psi_j'(\varepsilon) = \mathcal{P}^{(p)}(\varepsilon) \varphi_j(\varepsilon). \quad (49) \]

Note that \( \psi_j'(\varepsilon) \) is a polynomial in \( \varepsilon \), whose order does not exceed \( 2p \).

Also, \( \psi_j'(\varepsilon) \leq Y_p \) independently of \( \varepsilon \). Omitting all powers higher than \( p \) in \( \psi_j'(\varepsilon) \), we obtain the polynomial
\[
\psi_j(\varepsilon) = \sum_{k=0}^{p} \varepsilon^k \psi_j^{(k)}, \quad \psi_j^{(k)} \leq Y_p \subseteq D_p,
\]
where the last inclusion is implied by the \( p \)-smoothness. Lemma 5. gives
\[
T(\varepsilon) \psi_j^{(k)} = \sum_{i=0}^{p} T_i \varepsilon^i \psi_j^{(k)} + o(\varepsilon^p),
\]
which together with (48) implies
\[
(T(\varepsilon) - \lambda_j^{(p)}(\varepsilon)) \psi_j(\varepsilon) = o(\varepsilon^p), \quad (50)
\]
where
\[
\lambda_j^{(p)}(\varepsilon) = \sum_{j=0}^{p} \varepsilon^k \lambda_j^{(k)}. \quad (51)
\]

Since \( \psi_j(0) = \psi_j'(0) = \mathcal{P}_0 \varphi_j(0) = \varphi_j(0) = \varphi_j'(0) \), (46) implies
\[
(J\psi_j(\varepsilon), \psi_k(\varepsilon)) \rightarrow \delta_{jk}, \quad \varepsilon \rightarrow 0. \quad (52)
\]

and obviously
\[
(1 - P_0) \psi_k(\varepsilon) \rightarrow 0, \quad \varepsilon \rightarrow 0. \quad (53)
\]

The space \( P_0 X \) is finite dimensional and the norms generated by \( (\cdot, \cdot) \) and \( \langle \cdot, \cdot \rangle \) are equivalent. Therefore
\[
\| \psi_j(\varepsilon) \| \rightarrow \| \psi_j(0) \| \neq 0, \quad \varepsilon \rightarrow 0. \quad (54)
\]

According to Definition 2, formulae (50), (52), (53), (54) imply that \( \{ \varphi_j(\varepsilon) \} \) is a \( J - p \)-asymptotic basis for \( T(\varepsilon) \) with the eigenvalues \( \lambda_j^{(p)}(\varepsilon) \).

Q.E.D.

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O SPEKTRALNOJ KONCENTRACIJI ZA JEDNU KLASU \( J \)-HERMITSKIH OPERATORA

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Sadržaj


U prvom dijelu dokazuje se veza spektralne koncentracije i asimptotskih baza za klasu \( J \)-hermitskih operatora promatranu u [6], [7], dok se u drugom dijelu dokazuje postojanje asimptotskih baza za neke klase \( J \)-hermitskih operatorskih familija.