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ON SPECTRAL CONCENTRATION FOR A CLASS OF J-SELFADJOINT OPERATORS

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In this paper we consider the spectral concentration for a class of J-selfadjoint operators which possess spectral decompositions like ordinary selfadjoint operators.

Let X be a Hilbert space with a scalar product (x, y) and the norm $\| x \| = (x, x)^{1/2}$. Let T be a closed operator in X defined on $\mathfrak{D}(T)$; by $\varrho(T)$, $\sigma(T)$, $R(\lambda, T)$, T^*, $R(T)$ we denote its resolvent set, spectrum, resolvent, adjoint and the range, respectively. If T is selfadjoint, $E(t)$ denotes its spectral family (continuous from the right), while $E(\Delta)$ denotes the spectral measure of a Borel set Δ from the real line.

In the first part of this paper we generalize a result of R. C. Riddell ([3]). Let $T(\varepsilon)$ (ε real from some interval containing zero) be a family of closed operators such that $(JT(\varepsilon)x, y) = (Jx, T(\varepsilon)y)$ for $x, y \in \mathfrak{D}(T(\varepsilon))$ and for a fixed operator $J = J^* = J^{-1}$ on X. Let a real point λ be a pole of the first order of $R(\lambda; T)$ and let the corresponding spectral projection P_0 have the dimension $m < \infty$ such that the restriction of the form (Jx, y) on $P_0 X$ is strictly positive. If there are functions $\psi_j(\varepsilon) \in \mathfrak{D}(T(\varepsilon))$ and real functions $\lambda_j(\varepsilon)$, $j = 1, 2, \ldots, m$ such that for $\varepsilon \to 0$ we have $\|\psi_j(\varepsilon)\| \to K > 0$, $(T(\varepsilon) - \lambda_j(\varepsilon))\psi_j(\varepsilon) = o(\varepsilon^p)$, $(1 - P_0)\psi_j(\varepsilon) \to 0$, $(J\psi_j(\varepsilon), \psi_k(\varepsilon)) \to \delta_{jk}$, we call $\psi_j(\varepsilon)$ the J—p-asymptotic basis for $T(\varepsilon)$. (cf. [3], [4]). We impose on $T(\varepsilon)$ some further conditions which ensure the existence of spectral decompositions of $T(\varepsilon)$ and their strong convergence when $\varepsilon \to 0$ in the sense of the well-known Rellich-Kato's theorem ([2], p. 432). The main result is: Any such family $T(\varepsilon)$ possessing a J—p-asymptotic basis has a spectral concentration of the p-th order in the sense of Riddell [3].

In the second part we give a sufficient condition for a J-symmetric family $T(\varepsilon)$ to have a J—p-asymptotic basis. We call the family $T(\varepsilon)$ J—p-smooth (cf. [4]) with respect to the point $\lambda \in \sigma(T(0))$ if the subspace D_p of vectors ψ for which $T(\varepsilon)\psi$ has the p-th derivative is sufficiently large (in the sense to be given more precisely below). The main result is as follows: Any J—p-smooth family possesses a J—p-asymptotic basis. For $J = 1$ this result is contained in [4]. However, Lemma 3. of [4] contains an error such that by [4] the $(2p-1)$-smoothness is needed for the existence of a p-asymptotic basis. Thus, our paper contains a correct proof of the mentioned result of [4].

The main results of the present work can be applied in studying the spectral concentration for the Klein-Gordon equation, describing the motion of a spinless relativistic particle moving in a potential barrier. This application will be the subject of a subsequent paper.

1. ASSUMPTION. The domains of definition $\mathfrak{D}(T(\varepsilon))$ of $T(\varepsilon)$ and $\mathfrak{D}(T(\varepsilon)^*)$ of $T(\varepsilon)^*$ coincide and are dense in X. The operator $T(0)$ has a real eigenvalue λ which is a pole of the first order for the resolvent of $T(0)$. The corresponding eigenspace has the dimension $m<\infty$. The respective eigenprojection is denoted by P_0.

2. DEFINITION. (cf. C. Riddell [3]). *Suppose that vector functions $\varepsilon \to \psi_j(\varepsilon)$ and scalar functions $\varepsilon \to \lambda_j(\varepsilon)$, $j=1,2,\ldots,m$, are given on the interval I such that $\psi_j(\varepsilon) \in \mathfrak{D}(T(\varepsilon))$ and*

$$(T(\varepsilon)-\lambda_j(\varepsilon))\,\psi_j(\varepsilon)=o(\varepsilon^p), \qquad |\psi_j(\varepsilon)|\to K>0, \quad \varepsilon\to 0 \tag{1}$$

for some $p>0$. Then any pair of functions $\lambda_j(\varepsilon)$, $\psi_j(\varepsilon)$, $j=1,\ldots,m$ is called a p-pair of the family $T(\varepsilon)$ with respect to the point λ.

If, in addition, a unitary selfadjoint operator $J=J^*=J^{-1}$ is given such that the restriction of the form $\langle x,y\rangle=(Jx,y)$ on P_0X is strictly positive and such that

$$(1-P_0)\,\psi_j(\varepsilon)\to 0, \quad \langle \psi_j(\varepsilon),\psi_k(\varepsilon)\rangle \to \delta_{jk}, \quad \varepsilon\to 0, \tag{2}$$

then the vector functions $\psi_j(\varepsilon)$ are called a J — p-asymptotic basis for $T(\varepsilon)$. If $J=1$, $T(\varepsilon)$ is simply called a p-asymptotic basis. The functions $\lambda_j(\varepsilon)$ are pseudoeigenvalues.

3. DEFINITION. (cf. Riddell [3]). *Let $T(\varepsilon)$ be a family of scalar type operators with real spectra, for $\varepsilon \in I$. Let $p\geq 0$ and let I' be a real interval such that*

$$E_\varepsilon(I'\setminus \mathfrak{C}(\varepsilon)) \underset{s}{\to} 0, \quad \mu(\mathfrak{C}(\varepsilon))=o(\varepsilon^p), \quad \varepsilon\to 0, \tag{3}$$

where $E_\varepsilon(\cdot)$ denotes the spectral measure for $T(\varepsilon)$, $\mathfrak{C}(\varepsilon)$ is a family of real Borel sets and μ is the Lebesgue measure. In addition, let

$$\sup_{\varepsilon,t,t'} \|E_\varepsilon(t,t')\|<\infty. \tag{4}$$

Then we say that the part $\sigma_\varepsilon(I')=I'\cap\sigma(T(\varepsilon))$ of the spectrum of $T(\varepsilon)$ in I' is p-concentrated on $\{\mathfrak{C}(\varepsilon)\}$.

C. Riddell ([3]) has proved the fundamental theorem (see also [1]).

THEOREM. *Let $T(\varepsilon)$ be a family of selfadjoint operators such that $T(\varepsilon)\to T(0)$ strongly in the generalized sense* (see [1], p. 427) *and that it satisfies Assumption 1. Let $T(\varepsilon)$ have p-pairs $\lambda_j(\varepsilon)$, $\psi_j(\varepsilon)$*

such that $\psi_j(\varepsilon)$ *is a p-asymptotic basis. Then* $\sigma_\varepsilon(J')$ *is p-concentrated. As »concentration sets«* $\mathfrak{C}(\varepsilon)$ *we may take the unions of intervals around* $\lambda_j(\varepsilon)$, *the length of which does not exceed* $o(\varepsilon^p)$.

In what follows we shall prove the same result for a class of J-selfadjoint families.

In [6], [7] we considered the operators of the form

$$T = S + V, \quad S = S^*, \quad V \text{ bounded}, \tag{5}$$

and we proved the following: Let $(-\delta, \delta) \subseteq \varrho(S)$ for some $\delta > 0$ and $J = \operatorname{sign} S$. If V is J-symmetric, i. e., $V = JVJ^*$ and $\| V \| < \delta/2$ then T is a scalar type operator with a real spectrum. Here we consider a family

$$T(\varepsilon) = S(\varepsilon) + V(\varepsilon) \tag{6}$$

of such operators for which

$$(-\delta, \delta) \subseteq \varrho(S(\varepsilon)), \quad \| V(\varepsilon) \| < \delta/2, \tag{7}$$

where $\delta > 0$ does not depend on ε. Moreover, let

$$J = \operatorname{sign} S(\varepsilon), \quad V(\varepsilon) = JV(\varepsilon)^* J, \tag{8}$$

where J does not depend on ε.

4. THEOREM. *Let* $T(\varepsilon)$ *satisfy* (6), (7), (8) *and Assumption* 1. *with* J *as in* (8) *and* $\lambda > 0$. *Moreover, let* $V(\varepsilon) \underset{s}{\to} V(0)$ *and let* $S(\varepsilon) \to S(0)$ *in the generalized sense.** *Then* $T(\varepsilon)$ *possesses a spectral concentration in the way described by Riddell's Theorem, provided that* $T(\varepsilon)$ *has a* J — *p-asymptotic basis.*

Proof. In [5] we proved that the integral

$$K(\varepsilon) = -\frac{1}{i\pi}\ s\text{-}\lim_{\beta\to\infty} \int_{-i\beta}^{i\beta} R(\lambda; T(\varepsilon))\, d\lambda \tag{9}$$

exists and that

$$K(\varepsilon) \underset{s.}{\to} K(0), \quad A(\varepsilon) \underset{s.}{\to} A(0), \quad A(\varepsilon)^{-1} \to A(0)^{-1}, \quad \varepsilon \to 0, \tag{10}$$

where

$$A(\varepsilon) = (JK(\varepsilon))^{1/2} \tag{11}$$

are bounded symmetric operators with bounded inverses. Moreover

$$\mathbf{T}(\varepsilon) = A(\varepsilon)\, T(\varepsilon)\, A(\varepsilon)^{-1} \tag{12}$$

are selfadjoint with

$$\mathbf{T}(\varepsilon) \underset{s}{\to} \mathbf{T}(0). \tag{13}$$

* The strong convergence in the generalized sense means just the strong convergence of resolvents (see T. Kato [2], p. 427).

Therefore (see [2])

$$\mathbf{E}(t, \varepsilon) \underset{s}{\rightarrow} \mathbf{E}(t, 0) \tag{14}$$

for any t which is not an eigenvalue for $\mathbf{T}(0)$. Here $\mathbf{E}(t, \varepsilon)$ denotes the spectral family of $T(\varepsilon)$.

Note also that

$$K(0) P_0 = P_0 K(0) = P_0, \tag{15}$$

which is a consequence of $\lambda > 0$.

Now, let $\lambda_j(\varepsilon)$, $\psi_j(\varepsilon)$, $j = 1, \ldots, m$ be p-pairs for $T(\varepsilon)$ and let $\psi_j(\varepsilon)$ be a J — p-asymptotic basis. Putting $\varphi_j(\varepsilon) = A(\varepsilon)\psi_j(\varepsilon)$ the formula (1) gives

$$(\mathbf{T}(\varepsilon) - \lambda_j(\varepsilon))\varphi_j(\varepsilon) = o(\varepsilon^p). \tag{16}$$

Furthermore, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(1 - A(0) P_0 A(0)^{-1}) A(\varepsilon)\psi_j(\varepsilon) = \\
&= \lim_{\varepsilon\to 0}(1 - A(0) P_0 A(0)^{-1}) A(\varepsilon)(P_0\psi_j(\varepsilon) + (1 - P_0)\psi_j(\varepsilon)) = \\
&= A(0)\lim_{\varepsilon\to 0}(1 - P_0) A(0)^{-1} A(\varepsilon) P_0\psi_j(\varepsilon) = \\
&= A(0)\lim_{\varepsilon\to 0}(1 - P_0) P_0\psi_j(\varepsilon) = \\
&= A(0)\lim_{\varepsilon\to 0}(1 - P_0)\psi_j(\varepsilon) = 0.
\end{aligned}$$

This means

$$(1 - \mathbf{P}_0)\varphi_j(\varepsilon) \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad \mathbf{P}_0 = A(0) P_0 A(0)^{-1}. \tag{17}$$

Here we used (2), the finite dimensionality of P_0 and the fact that $A(\varepsilon) \underset{s}{\rightarrow} A(0)$.

Finally, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(\varphi_j(\varepsilon), \varphi_k(\varepsilon)) = \lim_{\varepsilon\to 0}(A(\varepsilon)^2\psi_j(\varepsilon), \psi_k(\varepsilon)) = \\
&= \lim_{\varepsilon\to 0}(J(K(0) + K(\varepsilon) - K(0))(P_0\psi_j(\varepsilon) + \\
&+ (1 - P_0)\psi_j(\varepsilon)), P_0\psi_k(\varepsilon) + (1 - P_0)\psi_k(\varepsilon)).
\end{aligned}$$

Here, using the finite dimensionality of P_0 and the boundedness of $|\psi_j(\varepsilon)|$ for $\varepsilon \rightarrow 0$, all vanishes except possibly

$$\lim_{\varepsilon\to 0}(JK(0) P_0\psi_j(\varepsilon), P_0\psi_k(\varepsilon)) = \lim_{\varepsilon\to 0}(JP_0\psi_j(\varepsilon), \psi_k(\varepsilon))$$

where we used the J-symmetry of P_0 and (15). Using both relations in (2), we obtain

$$\lim_{\varepsilon\to 0} (\varphi_j(\varepsilon), \varphi_k(\varepsilon)) = \begin{cases} 0 & j \neq k \\ K_j > 0, & j = k. \end{cases} \tag{18}$$

Relations (16), (17), (18) mean that $\lambda_j(\varepsilon)$, $\varphi_j(\varepsilon)/\|\varphi_j(\varepsilon)\|$ are p-pairs for $\mathbf{T}(\varepsilon)$ and that $\varphi_j(\varepsilon)/\|\varphi_j(\varepsilon)\|$ is a p-asymptotic basis. This permits the use of Riddell's theorem which tells that the spectrum of $\mathbf{T}(\varepsilon)$ is p-concentrated.

The spectral families of $\mathbf{T}(\varepsilon)$ and $T(\varepsilon)$ are connected by the same similarity relation as in (12). So the p-concentration also follows for the family $T(\varepsilon)$. Q.E.D.

In the following we shall give a sufficient condition for a J-symmetric family $T(\varepsilon)$ to have a $J-p$-asymptotic basis. Some proofs are quite analogous to those in [4]. We still include some of them for the sake of completeness.

Let $T(\varepsilon)$ be a family satisfying Assumption 1. For an integer $p \geq 0$ denote by D_p the set of all vectors $\psi \in X$ such that $D_p \subseteq \subseteq \mathfrak{D}(T(\varepsilon))$ and that the vector function

$$\varepsilon \to T(\varepsilon)\psi$$

has the derivative of the order p for $\varepsilon = 0$. Then

$$D_0 \supseteq D_1 \supseteq \ldots, \tag{19}$$

where D_0 denotes the set of all $\psi \in X$, for which $\varepsilon \to T(\varepsilon)\psi$ is continuous at $\varepsilon = 0$.

5. LEMMA. *The set D_p is a subspace of X and for $\psi \in D_p$ we have*

$$T(\varepsilon)\psi = T_0\psi + \varepsilon T_1\psi + \ldots + \varepsilon^p T_p\psi + o(\varepsilon^p), \tag{20}$$

where $T_0, \ldots, T_p$ are linear operators, defined on D_p as

$$T_r\psi = \frac{1}{r!}\left(\frac{d^r}{d\varepsilon^r} T(\varepsilon)\psi\right)_{\varepsilon=0} \tag{21}$$

Proof. See [4].

In the following we shall require that D_p is sufficiently large. The later considerations will justify the following definition.

6. DEFINITION. *Let a family $T(\varepsilon)$ satisfy Assumption* 1. *We denote by*

$$Z = -\lim_{\mu\to\lambda} (1 - P_0)(\mu - T(0))^{-1} \tag{23}$$

the reduced resolvent of $T(0)$ in the point λ. Furthermore, for an integer $p \geq 0$ we denote by V_p the set of all operators of the form

$$
\begin{array}{ll}
X & X \leqslant \{1, Z, ZT_1, \ldots, ZT_p\} \\
X_1 X_2 & X_1, X_2 \leqslant \{Z, ZT_1, \ldots, ZT_{p-1}\} \\
\multicolumn{2}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
X_1 X_2 \ldots X_p & X_k \leqslant \{Z, ZT_1\}, \quad k = 1, 2, \ldots, p,
\end{array}
\tag{24}
$$

where we have taken

$$V_0 = \{1\}. \tag{25}$$

We say that the family $T(\varepsilon)$ is p-smooth at $\varepsilon = 0$ with respect to the point λ if any of the operators from V_p is defined at least on $P_0 X$ and maps $P_0 X$ into D_p.

We may briefly say that V_p contains 1 and any r-fold product of the factors Z, ZT_ν such that r varies from 1 to p, and ν does not exceed $p - r + 1$.

We see that the p-smoothness includes the s-smoothness for $p \geq s$.

7. LEMMA. *The sets $V_0, V_1 \ldots$ are ordered by inclusion i. e., $V_0 \subseteq V_1 \subseteq \ldots$.* If $A \leqslant V_n$, $B \leqslant V_k$, *then* $AB \leqslant V_s$, *for* $n + k \leq s$.

Proof. See [4].

Let us introduce the subspaces

$$Y_r = \sum_{A \in V_r} A P_0 X, \quad r = 0, 1, 2, \ldots, \tag{26}$$

with the corresponding orthogonal projections R_r. We see that all Y_r are finite dimensional and ordered as

$$Y_0 \subseteq Y_1 \subseteq \ldots.$$

Now, the p-smoothness implies

$$Y_r \subseteq D_r, \quad r \leq p.$$

8. LEMMA *Let $A = X_1 X_2 \ldots X_r \leqslant V_p$ and let $T(\varepsilon)$ be p-smooth. Then*

$$X_1 X_2 \ldots X_r \mid P_0 = R_p X_1' X_2' \ldots X_r' \mid P_0,^*$$

where

$$X_s' = \begin{Bmatrix} X_s & \text{if} \quad X_s = Z, 1 \\ X_s R_p & \text{if} \quad X_s = ZT_\nu \end{Bmatrix},$$

$$s = 1, 2, \ldots, r.$$

Proof. If $X_1 X_2 \ldots X_r \leqslant V_p$, then $X_s X_{s+1} \ldots X_r \leqslant V_p$, $1 \leq s \leq r$. Then by the p-smoothness

$$X_1 X_2 \ldots X_r \mid P_0 = R_p X_1 \cdot X_2 \ldots X_r \mid P_0 =$$

$$R_p X_1' X_2 \ldots X_r \mid P_0 = \ldots = R_p X_1' X_2' \ldots X_r' \mid P_0.$$

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* Here $A \mid X_0$ denotes the restriction of A on X_0.

Let us now introduce an operator family

$$P^{(p)}(\varepsilon) = \sum_{n=0}^{p} \varepsilon^n P_n, \tag{27}$$

where

$$P_n = \sum_{r=1}^{n} (-1)^{r+1} \sum_{\substack{k_1 + \ldots + k_{r+1} = r,\ k_i \geq 0 \\ \nu_1 + \ldots + \nu_r = n,\ \nu_i \geq 1}} Z^{(k_1)} T_{\nu_1} Z^{(k_2)} \ldots Z^{(k_r)} T_{\nu_r} Z^{(k_{r+1})} \tag{28}$$

$$Z^{(k)} = Z^k, \quad k \geq 1, \quad Z^{(0)} = P_0. \tag{29}$$

9. LEMMA. *Let $T(\varepsilon)$ be p-smooth with respect to λ. Then the operator*

$$P_i P_j T_k, \quad i + j \leq p, \quad j + k \leq p \tag{30}$$

*is defined at least on $P_0 X$. The range of P_k (and therefore of (30)) is contained in Y_k for $k \leq p$.**

Proof. The operator $P_j T_k$ is a linear combination of the operators

$$Z^{(k_1)} T_{\nu_1} Z^{(k_2)} \ldots T_{\nu_r} Z^{(k_{r+1})} T_k \tag{31}$$

$$k_1 + \ldots + k_{r+1} = r, \quad k_i \geq 0, \quad \nu_1 + \ldots + \nu_r = j, \quad \nu_i \geq 1, \quad j + k \leq p.$$

Since $k_1 + \ldots + k_{r+1} = r$, at least one of the indices k_i must vanish. Taking into account all $k_i = 0$, (31) can be written as

$$A P_0 T_{\nu'} B P_0 T_{\nu''} \ldots F P_0 T_{\nu^{(a)}} G, \quad \nu', \nu'', \ldots, \nu^{(a)} \leq j. \tag{32}$$

Here $A, B, \ldots, F, G$ are at most r-fold products of the factors $Z, Z T_\nu$. By $\nu_1 + \ldots + \nu_r = j$, then index ν entering $A, B, \ldots, F$ does not exceed $j - r + 1$, thus $A, B, \ldots, F \leqslant V_j$. However, the index ν, entering G does not exceed

$$\max\,[j - r + 1, p - j] \leq p - r + 1$$

because of $r \leq j \leq p$. Thus

$$G \leqslant V_p.$$

Similarly, P_i is a linear combination of members of the form

$$A_1 P_0 T_{\nu'} B_1 P_0 T_{\nu''} \ldots F_1 P_0 T_{\nu^{(b)}} G_1, \quad \nu', \nu'', \ldots, \nu^{(b)} \leq i \tag{33}$$

where

$$A_1, B_1, \ldots, G_1 \leqslant V_i.$$

The operators $P_i P_j T_k$ will then be a linear combination of products of operators of the forms (32) and (33). Since $G_1 \leqslant V_1$, $A \leqslant V_j$, $i + j \leq p$ implies $G_1 A \leqslant V_p$ (lemma 7), any such product is of the form

* Since $T_0 P_0 = T(0) P_0 = \lambda P_0$, it is important that $\lambda \neq 0$. This can always be obtained by adding to $T(\varepsilon)$ a sufficiently large multiple of the identity.

$$A_2 P_0 T_{\varrho'} B_2 P_0 T_{\eta''} \dots F_2 P_0 T_{\varrho^{(c)}} G_2 \tag{34}$$
$$\varrho', \varrho'', \dots, \varrho^{(c)} \leq p,$$
$$A_2, \dots, G_2 \leqslant V_p .$$

The family $T(\varepsilon)$ is p-smooth, which implies that (34) and therefore $P_i P_j T_k$ is defined on P_0 and maps $P_0 X$ into $Y_k \subseteq D_k$.
Q.E.D.

10. LEMMA. *Let $T(\varepsilon)$ be p-smooth with respect to λ. Then*

$$(P_0 P_n + \dots + P_n P_0)\psi = P_n \psi, \quad \psi \leqslant \sum_{k=0}^{p-n} T_k P_0 X , \tag{35}$$

$$(T_n P_0 + \dots + T_0 P_n)\varphi = (P_0 T_n + \dots + P_n T_0)\varphi, \quad \varphi \leqslant P_0 X, \tag{36}$$

for $n = 0, 1, \dots, p$.

Proof. Set

$$T'(\varepsilon) = T_0' + \varepsilon T_1' + \dots + \varepsilon^p T_p', \quad T_0' = T(0), \quad T_k' = T_k R_p, \quad k \geq 1 .$$

The operators $T_k' = T_k R_p$, $k \geq 1$ are bounded, since the projection R_p is finite (the products $T_k R_p$ are well defined, since T_k is defined on D_p and $Y_p \subseteq D_p$). Thus, the family $T'(\varepsilon)$ is a holomorphic family of type (A) for all complex ε (see T. Kato, [2], p. 375). Since $T'(0) = T(0)$, and λ is an isolated point of $\sigma(T(0)) = \sigma(T'(0))$, there is a family of bounded projections

$$P'(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k P_k' , \quad P'(0) = P_0' = P_0 ,$$

which is analytic in some neighbourhood of $\varepsilon = 0$, such that $P(\varepsilon)$ projects on the root space belonging to the group of eigenvalues coming by perturbations from the point λ. Since λ is a pole of the first order for the function $\mu \to (\mu - T(0))^{-1}$ we have (cf. Kato [2], p. 76)

$$P_k' = \sum_{r=1}^{p} (-1)^{r+1} \sum_{\substack{k_1 + \dots + k_{r+1} = r,\ k_i \geq 0 \\ \nu_1 + \dots + \nu_r = n}} Z^{(k_1)} T_{\nu_1}' Z^{(k_2)} \dots T_{\nu_r} Z^{(k_{r+1})} ,$$
$$k \leq p .$$

Furthermore, $P'(\varepsilon)^2 = P'(\varepsilon)$ implies

$$P_0' P_k' + \dots + P_k' P_0' = P_k' , \quad k = 0, 1, 2, \dots . \tag{37}$$

On the other hand, the left-hand side of equality (35) consists of summands of the form

$$P_s P_t P_j \psi , \quad s + t \leq p , \quad t + j \leq p .$$

Formula (34) together with Lemma 8 shows that

$$P_s' P_i' T_j' \psi = P_s P_t T_j \psi \quad \text{for} \quad \psi \in \sum_{k=0}^{p-n} T_k P_0 X .$$

Also, formula (31) with Lemma 8 shows that $P_k' \psi = P_k \psi$, $k \leq p$,

$$\psi \in \sum_{j=0}^{p-k} T_j P_0 X .$$

Thus, (35) follows.

To prove (36), we introduce the operator family

$$P'(\varepsilon) T'(\varepsilon) P'(\varepsilon) =$$

$$\frac{1}{2\pi i} \int_\Gamma \mu (\mu - T'(\varepsilon))^{-1} d\mu = P'(\varepsilon) T'(\varepsilon) = T'(\varepsilon) P(\varepsilon) ,$$

where Γ is a small circle around λ. The function $\varepsilon \to T'(\varepsilon) P'(\varepsilon)$ is bounded analytic in some neighbourhood of zero.

On the other hand, for $\varphi \in \mathfrak{D}(T(0))$ the function $T'(\varepsilon)\varphi$ is analytic on $(-\infty, \infty)$. Hence, for $\varphi, \psi \in \mathfrak{D}(T(0))$ we have

$$(P'(\varepsilon) T'(\varepsilon) \varphi, \psi) = (T'(\varepsilon) \varphi, P'(\varepsilon)^* \psi) = \sum_{k=0}^{\infty} \varepsilon^k z_k ,$$

$$z_k = (T(0) \varphi, P_k'^* \psi) + \ldots + (T_k' \varphi, P_0'^* \psi) =$$

$$= ((P_k' T_0' + \ldots + P_0' T_k') \varphi, \psi) .$$

Since

$$(T'(\varepsilon) P'(\varepsilon) \varphi, \psi) = \sum_{k=0}^{p} \varepsilon^k ((T_k' P_0' + \ldots + T_0' P_k') \varphi, \psi) + o(\varepsilon^p) ,$$

we have

$$(P_k' T_0' + \ldots + P_0' T_k') \varphi = (T_0' P_k' + \ldots + T_k' P_0') \varphi ,$$

$$k = 0, 1, 2, \ldots, p ,$$

for $\varphi \in P_0 X$. Now, $P_j T_i = P_j' T_i'$, $T_i P_j = T_i' P_j'$, $i + j \leq p$ (Lemma 8) implies (36). Q.E.D.

11. LEMMA. *Let D be a subspace of a normed space N and let $P_0, P_1, \ldots, P_p$, $T_0, T_1, \ldots, T_p$ be linear operators in N such that*

I *$P_l P_k T_j$ is defined on D for $l + k \leq p$, $k + j \leq p$*

II *the operator T_j is bounded on D*
the operator P_k is bounded on $T_j D$, $j + k \leq p$
the operator P_l is bounded on $P_k T_j D$, $k + j \leq p$, $k + l \leq p$

III $(P_0 P_n + \ldots + P_n P_0) \psi = P_n \psi$, $n = 0, 1, 2, \ldots, p$, *for* $\psi \in$
$\in \sum_{j=0}^{p-n} T_j D .$

Then for any vector function $\varepsilon \to \chi(\varepsilon)$, which is bounded in norm when $\varepsilon \to 0$, the implication

$$P_0 \sum_{k=0}^{p} (P_0 T_k + \ldots + P_k T_0)\, \varepsilon^k \chi(\varepsilon) = o(\varepsilon^p) \Rightarrow$$

$$\sum_{k=0}^{p} (P_0 T_k + \ldots + P_k T_0)\, \varepsilon^k \chi(\varepsilon) = o(\varepsilon^p),$$

holds.

Proof. For $p = 0$ the assertion is trivially true by $P_0^2 \chi(\varepsilon) = P_0 \chi(\varepsilon)$. For $p = 1$ the equalities

$$P_0 (P_0 T_0 + \varepsilon (P_1 T_0 + P_0 T_1))\, \chi(\varepsilon) = o(\varepsilon)$$

$$\varepsilon (P_1 T_0 + P_0 T_1)\, \chi(\varepsilon) = o(\varepsilon^0)$$

imply

$$P_0 T_0 \chi(\varepsilon) = o(\varepsilon^0).$$

(Notice that $P_1 T_0 + P_0 T_1$ is bounded on D.)

Hence,

$$\begin{aligned} o(\varepsilon) &= P_0 (P_0 T_0 + \varepsilon (P_0 T_1 + P_1 T_0))\, \chi(\varepsilon) = \\ &= P_0 T_0 + \varepsilon (P_0 T_1 + (P_1 - P_1 P_0) T_0))\, \chi(\varepsilon) = \\ &= [P_0 T_0 + (P_0 T_1 + P_1 T_0)\, \varepsilon]\, \chi(\varepsilon) - \varepsilon P_1 P_0 T_0 \chi(\varepsilon). \end{aligned}$$

By $P_0 T_0 \chi(\varepsilon) = o(\varepsilon^0)$ and the boundedness of P_1 we have $\varepsilon P_1 P_0 T_0 \chi(\varepsilon) = o(\varepsilon)$. The assertion is, therefore, true if $p = 1$. By induction, suppose that the assertion is true if $p = 0, 1, 2, \ldots s$ and that

$$\begin{aligned} o(\varepsilon^{s+1}) &= P_0 \sum_{k=0}^{s+1} (P_0 T_k + \ldots + P_k T_0)\, \varepsilon^k \chi(\varepsilon) = \\ &= P_0 \sum_{k=0}^{k'} \varepsilon^k (P_0 T_k + \ldots + P_k T_0)\, \chi(\varepsilon) + \\ &+ \varepsilon^{k'+1} \sum_{k=k'+1}^{s+1} \varepsilon^{k-k'-1} (P_0 T_k + \ldots + P_k T_0)\, \chi(\varepsilon). \end{aligned}$$

Since, by supposition, $P_0 (P_0 T_k + \ldots + P_k T_0)$ is bounded on D, the second term of the sum is $o(\varepsilon^{k'})$, $k' = 0, 1, \ldots, k$. Hence, by the assumption of induction we have

$$\sum_{k=0}^{k'} \varepsilon^k (P_0 T_k + \ldots + P_k T_0)\, \chi(\varepsilon) = o(\varepsilon^{k'}), \quad k' = 0, \ldots, s.$$

Furthermore, using III. we have

$$\begin{aligned} o(\varepsilon^{s+1}) &= \sum_{k=1}^{s+1} \varepsilon^k (P_0^2 T_k + \ldots + P_0 P_k T_0)\, \chi(\varepsilon) = \\ &= \sum_{k=0}^{s+1} \varepsilon^k (P_0 T_k + (P_1 - P_1 P_0)\, T_{k-1} + \ldots \end{aligned}$$

$$+ (P_k - P_1 P_{k-1} - \ldots - P_k P_0) T_0) \chi(\varepsilon) =$$

$$= \sum_{k=0}^{s+1} \varepsilon^k (P_0 T_k + P_1 T_{k-1} + \ldots + P_k T_0) \chi(\varepsilon) -$$

$$- P_1 \sum_{k=0}^{s+1} \varepsilon^k (P_0 T_{k-1} + \ldots + P_{k-1} T_0) \chi(\varepsilon) -$$

....

....

$$- P_{s+1} P_0 T_0 \chi(\varepsilon) = \sum_{k=0}^{s+1} \varepsilon^k (P_0 T_k + \ldots + P_k T_0) \chi(\varepsilon) -$$

$$- \varepsilon P_1 \sum_{k=0}^{s} \varepsilon^k (P_0 T_k + \ldots + P_k T_0) \chi(\varepsilon) -$$

....

....

$$- \varepsilon^{s+1} P_{s+1} P_0 T_0 \chi(\varepsilon) = \sum_{k=0}^{s+1} \varepsilon^k (P_0 T_k + \ldots + P_k T_0) \chi(\varepsilon) +$$

$$+ \varepsilon\, o(\varepsilon^s) + \varepsilon^2 o(\varepsilon^{s-1}) + \ldots + \varepsilon^{s+1} o(\varepsilon^0),$$

which proves the assertion for $p = s + 1$. Here we used the fact that $P_k P_l T_j$ is bounded on D for $k + l \leq s + 1$, $l + j \leq s + 1$.

Q.E.D.

Note an important fact, which will be used in the following: the boundedness condition II is automatically fulfilled if D is finite dimensional.

In the following we introduce the J-symmetry.

12. ASSUMPTION. The operators $T(\varepsilon)$, $\varepsilon \in I$ are J-symmetric for some fixed $J \cdot J^* = J^{-1}$, i. e.,

$$(JT(\varepsilon) x, y) = (Jx, T(\varepsilon) y) \qquad x, y \in \mathfrak{D}(T(\varepsilon)), \tag{38}$$

and for $x \in P_0 X$ we have

$$(Jx, x) \geq 0; \quad (x, x) = 0 \Rightarrow x = 0. \tag{39}$$

13. LEMMA. *The operators Z, P_k, $T_k P_0 + \ldots + T_0 P_k$, $k = 0, 1, \ldots, p$, for a p-smooth family $T(\varepsilon)$ which satisfies Assumption 12. are J-symmetric. For the domain of P_k, $T_k P_0 + \ldots + T_0 P_k$ we take*

$$\sum_{i=0}^{p-k} T_i P_0 X, \quad P_0 X,$$

respectively.

Proof. Put $R(\mu) = (\mu - T(0))^{-1}$. Let Γ be a circle with the centre λ, isolating λ from the rest of $\sigma(T(0))$. Since $\lambda = \bar{\lambda}$ and $\Gamma = \bar{\Gamma}$ we have

$$JR(\bar{\mu}) = R(\mu)^* J, \quad \mu \in \varrho(T(0)).$$

Hence

$$(JP_0 x, y) = \frac{1}{2\pi i} \int_\Gamma (JR(z)x, y)\,dz = (Jy, P_0 x).$$

Thus, P_0 is J-symmetric.

Furthermore, $\lambda = \bar{\lambda}$ implies

$$(JZx, y) = -\lim_{z\to\lambda} (JR(z)(1-P_0)x, y) =$$

$$= -\lim_{z\to\lambda} (Jx, (1-P_0)R(\bar{z})y) = -\lim_{\bar{z}\to\lambda} (Jx, (1-P_0)R(\bar{z})y) =$$

$$= (Jx, Zy).$$

Thus, Z is J-symmetric. The J-symmetry of $T(\varepsilon)$ together with (21) implies

$$(JT_k\psi, \varphi) = (J\psi, T_k\varphi), \quad \psi, \varphi \in D_p. \tag{40}$$

The J-symmetry of P_k follows from the fact that T_k, Z, P_0 are J-symmetric and that expression (28) for P_k is invariant under the permutation of the indices $k_1, \ldots, k_{r+1}$.

Thus

$$(JP_k\psi, \varphi) = (J\psi, P_k\varphi), \quad \psi, \varphi \in \sum_{i=0}^{p-k} T_i P_0 X. \tag{41}$$

Finally, for $\psi, \varphi \in P_0 X$ formula (36) gives

$$(J(T_k P_0 + \ldots + T_0 P_k)\psi, \varphi) = (J\psi, (T_k P_0 + \ldots + T_0 P_k)\varphi), \tag{42}$$

where we have used the J-symmetry of T_k, P_k. Q.E.D.

14. THEOREM. *Any J-symmetric family $T(\varepsilon)$, p-smooth in $\varepsilon = 0$, with respect to the point λ, such that the restriction of the form $\langle x, y\rangle = (Jx, y)$ on $P_0 X$ is strictly positive, possesses a $J-p$-asymptotic basis.*

Proof. In the finite dimensional space $P_0 X$ cosider the generalized eigenvalue problem

$$(A^{(p)}(\varepsilon) - \lambda_j'(\varepsilon) B^{(p)}(\varepsilon))\varphi_j'(\varepsilon) = 0, \; j = 1, 2, \ldots, m, \tag{43}$$

where

$$A^{(p)}(\varepsilon) = P_0 \sum_{k=0}^{p} \varepsilon^k (P_k T_0 + \ldots + P_0 T_k) \,|\, P_0 X, \tag{44}$$

$$B^{(p)}(\varepsilon) = P_0 \sum_{k=0}^{p} \varepsilon^k P_k \,|\, P_0 X. \tag{45}$$

The space $P_0 X$ is a unitary finite dimensional space with the scalar product $\langle \cdot, \cdot \rangle$, which is positive definite on $P_0 X$ by supposition. The operators $A^{(p)}(\varepsilon)$, $B^{(p)}(\varepsilon)$ are polynomials in ε and are symmetric. Since $B^{(p)}(0) = 1 \,|\, P_0 X$, for sufficiently small ε the operator $B^{(p)}$ will be strictly positive definite.

Since

$$A^{(p)}(0) = \lambda \,|\, P_0 X\,,$$

the solutions $\lambda_j'(\varepsilon)$, $\varphi_j'(\varepsilon)$ of the problem (43) are analytic at $\varepsilon = 0$ and

$$\lambda_j'(0) = \lambda\,.$$

(See [2], p. 419). The functions $\varphi_j'(\varepsilon)$ can be chosen such that

$$(J\varphi_j'(0), \varphi_k'(0)) = \delta_{jk}\,. \tag{46}$$

Put

$$\varphi_j'(\varepsilon) = \sum_{k=0}^{\infty} \varphi_j^{(k)} \varepsilon^k, \quad \lambda_j'(\varepsilon) = \sum_{k=0}^{\infty} \lambda_j^{(k)} \varepsilon^k.$$

The function

$$\varepsilon \to A^{(p)}(\varepsilon) - \lambda_j'(\varepsilon)\, B^{(p)}(\varepsilon)$$

is a power series, whose coefficient of the k-th order is

$$P_0\,(P_0\,(T_k - \lambda_j^{(k)}) + \ldots + P_k\,(T_0 - \lambda)) \,|\, P_0 X, \quad k = 0, \ldots, p\,.$$

Omitting the powers higher than p, we obtain

$$P_0 \sum_{k=0}^{p} (P_0\,(T_k - \lambda_j^{(k)}) + \ldots + P_k\,(T_0 - \lambda))\, \varepsilon^k \varphi_j(\varepsilon) = o\,(\varepsilon^p)\,, \tag{47}$$

where

$$\varphi_j(\varepsilon) = \sum_{k=0}^{p} \varepsilon^k \varphi_j^{(k)}\,.$$

By (35) the operators P_k, $T_k - \lambda_j^{(k)}$, the functions $\varphi_j(\varepsilon) \leqslant P_0 X$ and the subspace $D = P_0 X$ satisfy the conditions of Lemma 11 ($P_0 X$ is finite dimensional!). Thus (47) implies

$$\sum_{k=0}^{p} (P_0\,(T_k - \lambda_j^{(k)}) + \ldots + P_k\,(T_0 - \lambda))\, \varepsilon^k \varphi_j(\varepsilon) = o\,(\varepsilon^p)\,.$$

Furthermore, (36) implies

$$\sum_{k=0}^{p} ((T_k - \lambda_j^{(k)})\, P_0 + \ldots + (T_0 - \lambda)\, P_k)\, \varepsilon^k \varphi_j(\varepsilon) = o\,(\varepsilon^p)\,.$$

Since

$$\sum_{k=0}^{p} ((T_k - \lambda_j^{(k)})\, P_0 + \ldots + (T_0 - \lambda)\, P_k)\, \varepsilon^k \varphi_j(\varepsilon) =$$

$$= [\sum_{k=0}^{p} (T_k - \lambda_j^{(k)})\, \varepsilon^k]\, [\sum_{k=0}^{p} P_k\, \varepsilon^k \varphi_j(\varepsilon)] + o\,(\varepsilon^p)\,,$$

we have

$$\sum_{k=0}^{p} (T_k - \lambda_j^{(k)})\, \varepsilon^k \psi_j'(\varepsilon) = o\,(\varepsilon^p)\,, \tag{48}$$

where

$$\psi_j'(\varepsilon) = P^{(p)}(\varepsilon)\,\varphi_j(\varepsilon). \tag{49}$$

Note that $\psi_j'(\varepsilon)$ is a polynomial in ε, whose order does not exceed $2p$.

Also, $\psi_j'(\varepsilon) \in Y_p$ independently of ε. Omitting all powers higher than p in $\psi_j'(\varepsilon)$, we obtain the polynomial

$$\psi_j(\varepsilon) = \sum_{k=0}^{p} \varepsilon^k \psi_j^{(k)}, \quad \psi_j^{(k)} \in Y_p \subseteq D_p,$$

where the last inclusion is implied by the p-smoothness. Lemma 5. gives

$$T(\varepsilon)\,\psi_j^{(k)} = \sum_{i=0}^{p} T_i\,\varepsilon^i \psi_j^{(k)} + o(\varepsilon^p),$$

which together with (48) implies

$$(T(\varepsilon) - \lambda_j^{(p)}(\varepsilon))\,\psi_j(\varepsilon) = o(\varepsilon^p), \tag{50}$$

where

$$\lambda_j^{(p)}(\varepsilon) = \sum_{j=0}^{p} \varepsilon^k \lambda_j^{(k)}. \tag{51}$$

Since $\psi_j(0) = \psi_j'(0) = P_0\,\varphi_j(0) = \varphi_j(0) = \varphi_j'(0)$, (46) implies

$$(J\psi_j(\varepsilon), \psi_k(\varepsilon)) \to \delta_{jk}, \quad \varepsilon \to 0 \tag{52}$$

and obviously

$$(1 - P_0)\,\psi_k(\varepsilon) \to 0, \quad \varepsilon \to 0. \tag{53}$$

The space $P_0 X$ is finite dimensional and the norms generated by $(\cdot,\cdot)$ and $\langle \cdot,\cdot \rangle$ are equivalent. Therefore

$$|\psi_j(\varepsilon)| \to |\psi_j(0)| \neq 0, \quad \varepsilon \to 0. \tag{54}$$

According to Definition 2, formulae (50), (52), (53), (54) imply that $\{\psi_j(\varepsilon)\}$ is a J — p-asymptotic basis for $T(\varepsilon)$ with the eigenvalues $\lambda_j^{(p)}(\varepsilon)$. Q.E.D.

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O SPEKTRALNOJ KONCENTRACIJI ZA JEDNU KLASU J-HERMITSKIH OPERATORA

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Sadržaj

U članku se poopćuju raniji rezultati R. C. Riddella [3] i autora [4].

U prvom dijelu dokazuje se veza spektralne koncentracije i asimptotskih baza za klasu J-hermitskih operatora promatranu u [6], [7], dok se u drugom dijelu dokazuje postojanje asimptotskih baza za neke klase J-hermitskih operatorskih familija.