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SOME NOTES ON *-REGULAR RINGS

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1.

It is known that a *-regular ring R is a ring with a unit element 1 in which

$$(\forall a \leqslant R)(\exists x \leqslant R)(a = a\,x\,a) \tag{1}$$

and in which there is an involutory anti-automorphism $a \to a^*$ with the additional property

$$a^* a = 0 \Rightarrow a = 0\,. \tag{2}$$

An element $a \leqslant R$ for which $a = a^*$ is called Hermitian and an element $e \leqslant R$ which is both Hermitian and idempotent, i. e. $e = e^* = e^2$, is called a projection.

It was shown [3] that the right (left) principal ideal $a\,R$ $(R\,a)$ of each element $a \leqslant R$ is generated by a uniquely defined projection e (f), which, because of $a = e\,a = a\,f$, we call the left (right) projection of the element a.

I. Kaplansky also proved [1]: if e and f are the respective left and right projections of an element $a \leqslant R$, then there exists exactly one element $\bar{a} \leqslant R$ such that $f\,\bar{a} = \bar{a}$ and $a\,\bar{a} = e$. The uniquely determined element $\bar{a}$ is called the relative inverse of the element a. Moreover, it is easy to see that $\bar{a}\,a = f$ and therefore

$$a = a\,\bar{a}\,a\,,\quad \bar{a} = \bar{a}\,a\,\bar{a}\,,\ \bar{\bar{a}} = a \tag{3}$$

Consequently, each element $a \leqslant R$ can be written in the form $a = a\,x\,a$ in such a way, that $a\,x$ and $x\,a$ are the respective left and right projections of a. In wiev of this fact let us examine the general solution of the equation $a = a\,x\,a$.

Suppose that an element $a \leqslant R$ is expressed by

$$a = axa\,,$$

where $e = a\,x$ and $f = x\,a$ are the respective left and right projections of a. Since

$$f\,(f\,x) = f^2\,x = f\,x \quad \text{and} \quad a\,(f\,x) = (a\,f)\,x = a\,x = e\,,$$

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it follows that

$$f x = \bar{a} . \tag{4}$$

Therefore we conclude that

$$f x = \bar{a} = f \bar{a} \Rightarrow f (x - \bar{a}) = 0 \Rightarrow x - \bar{a} = (1 - f) u$$

and hence

$$x = \bar{a} + (1 - f) u \tag{5}$$

for an element $u \leqslant R$. But since it must also be

$$x e = x a x = f x = \bar{a} = \bar{a} a \bar{a} = \bar{a} e ,$$

it follows from (5) that

$$(1 - f) u e = 0 \Rightarrow (1 - f) u = z (1 - e)$$

or

$$(1 - f) u = (1 - f)^2 u = (1 - f) z (1 - e)$$

for an element $z \leqslant R$. So we obtain that x must be of the form

$$x = \bar{a} + (1 - f) z (1 - e) . \tag{6}$$

Conversely, it is easy to verify that for any $z \leqslant R$, the element x defined by (6) does satisfy the prescribed conditions. So we can state:

Theorem 1. *The general solution of the equation $a = a x a$, with $a x = e$ and $x a = f$ being the respective left and right projections of the element $a \leqslant R$, is given by* (6), *where z is any element of R.*

Let us now take a Hermitian element $h \leqslant R$, $h = h^*$. Since in this case the left and right projections of h are equal, say e, (6) takes the form

$$x = \bar{h} + (1 - e) z (1 - e) . \tag{7}$$

Because of $h x = x h = e$, all elements x in (7) commute with h and therefore $\bar{h}$ as well. Accordingly, $h \bar{h} = \bar{h} h = \bar{h}^* h = h \bar{h}^*$ and from $\bar{h} = \bar{h} h \bar{h}$ we get immediately

$$\bar{h}^* = \bar{h}^* h \bar{h}^* = \bar{h} h \bar{h}^* = \bar{h} h \bar{h} = \bar{h} . \tag{8}$$

Hence the relative inverse of an Hermitian element is also Hermitian. But since $\bar{\bar{h}} = h$, it follows:

Corollary 1. *An element $h \leqslant R$ is Hermitian if and only if its relative inverse $\bar{h}$ is Hermitian.*

Consequently, each Hermitian element $h \leqslant R$ can be written in the form

$$h = h x h \text{ with } x = x^* \text{ and } h x = x h \text{ a projection.} \tag{9}$$

Suppose conversely that for an element $h \leqslant R$, (9) is valid. Then it is $h^* = h^* x h^* = h x h^* = h x h = h$ and hence:

An element $h \leqslant R$ is Hermitian if and only if it can be written according to (9).

It is also obvious that one can take for x in (9) any element defined by (7), which is Hermitian.

A special type of Hermitian elements are elements of the form $a^* a$. Let it be $a = a x a$ with $a x = e$ and $x a = f$ as the respective left and right projections of a. From

$$a^* a = a^* a f = f a^* a \text{ and } f = a^* x^* a^* x^* = (a^* a) x x^* = x x^* (a^* a),$$

it follows that f is at the same time both the left and the right projection of the Hermitian element $a^* a$. If $\bar{a}$ is the relative inverse of a, it is easy to see, that $\bar{a} \bar{a}^*$ is the relative inverse of $a^* a$. In fact

$$f (\bar{a} \bar{a}^*) = \bar{a} \bar{a}^* \quad \text{and}$$

$$(a^* a) \bar{a} \bar{a}^* = a^* (a \bar{a}) \bar{a}^* = a^* e \bar{a}^* = a^* \bar{a}^* = (\bar{a} a)^* = f^* = f.$$

But since $\bar{a} \bar{a}^* = (\bar{a}^*)^* \bar{a}^*$, we see that the relative inverse of an element of the form $a^* a$ has the same form. Because of the reciprocity of the relative inverse, we obtain

Corollary 2. *A Hermitian element is of the form $a^* a$ if and only if its relative inverse is of the same form.*

Hence, any Hermitian element of the form $a^* a$ can be written in the form

$$a^* a = a^* a x a^* a \text{ with } x = u^* u \text{ and } a^* a x = x a^* a \text{ a projection.}$$

In this case one can obviously take for x any element defined by

$$x = \bar{a} \bar{a}^* + (1 - f) z (1 - f) \tag{10}$$

which can be put into the form $u^* u$. It is a routine matter to verify that this is always the case if we take in (10)

$$z = t^* (1 - e) t,$$

where e is the left projection of a and t any element of R. Then we get

$$u = \bar{a}^* + (1 - e) t (1 - f).$$

Assume now conversely that for an element $b \leqslant R$ we have

$$b = b x b \text{ with } x = u^* u \text{ and } b x = x b \text{ being a projection.} \tag{11}$$

Then it is $b = b u^* u b = b^* u^* u b = (u b)^* u b$ and hence:

An element $b \leqslant R$ is of the form $a^* a$ if and only if it can be written according to (11).

Since for every projection $p \in R$ it is $p = p^* p$, let us examine the converse case, when an element of the form $a^* a$ represents a projection. Assume that for an element $a \in R$

$$f = a^* a \tag{12}$$

is a projection. Since

$$(a f - a)^* (a f - a) = (f a^* - a^*) (a f - a) = \\ = f (a^* a) f - (a^* a) f - f (a^* a) + a^* a = 0,$$

we conclude by virtue of (2) that $a f - a = 0$ and hence

$$a = a a^* a. \tag{13}$$

Since $(a a^*)^2 = (a a^* a) a^* = a a^*$, the Hermitian element

$$e = a a^* \tag{14}$$

is a projection too and it is

$$a = e a = a f. \tag{15}$$

From (12), (14) and (15) it follows, that e and f are the respective left and right projections of the element a. Moreover, we see from $f a^* = a^* a a^* = a^*$, that

$$a^* = \bar{a}, \tag{16}$$

which means that the relative inverse of the element a is identical to its involutoric image a^*.

Conversely it follows immediately from (16), that $a^* a$ and $a a^*$ are the respective right and left projections of the element a. So we obtain

Corollary 3. *An element of the form $a^* a$ is a projection if and only if $a^* = \bar{a}$. In this case $a^* a$ and $a a^*$ are the respective right and left projections of the element a.*

2.

Suppose that in a *-regular ring R there exists a set of n equivalent, pairwise orthogonal projections $e_{11}, e_{22}, \ldots, e_{nn}$ with

$$e_{11} + e_{22} + \ldots + e_{nn} = 1. \tag{17}$$

Because of the orthogonality, it is

$$e_{ii} e_{kk} = \begin{cases} e_{ii}, & \text{if } i = k \\ 0, & \text{if } i \neq k, \end{cases} \tag{18}$$

and owing to the equivalence there are elements

$$e_{1i} \in e_{11} R e_{ii} \text{ and } e_{i1} \in e_{ii} R e_{11} \tag{19}$$

such that

$$e_{11} = e_{1i}\, e_{i1} \quad \text{and} \quad e_{ii} = e_{i1}\, e_{1i} \tag{20}$$

for each $i = 1, 2, \ldots, n$. If we set

$$e_{ij} = e_{i1}\, e_{1j}, \quad i, j = 1, 2, \ldots, n \tag{21}$$

we see at once, that

$$e_{ij}\, e_{kh} = \begin{cases} e_{ih}, & \text{if } j = k \\ 0, & \text{if } j \neq k \end{cases}, \quad i, j, k, h = 1, 2, \ldots, n. \tag{22}$$

It follows from (17) and (22) [2] that the elements e_{ij}, with $i, j = 1, 2, \ldots, n$, form a system of matrix units of the order n. Since the element e_{11} is a projection, we know [1], that the set

$$e_{11}\, R\, e_{11}$$

is a *-regular subring of the ring R with e_{11} as the unit element. And finally, it was shown [2] that the ring R is isomorphic to the ring

$$(e_{11}\, R\, e_{11})_n$$

of all square matrices of the order n with elements in $e_{11}\, R\, e_{11}$, under the mapping

$$x \in R \rightarrow (x_{ij}), \quad \text{where} \quad x_{ij} = e_{1i}\, x\, e_{j1}. \tag{23}$$

Since the subring $e_{11}\, R\, e_{11}$ is regular, the ring of matrices $(e_{11}\, R\, e_{11})_n$ is regular too [2]. But on the basis of the isomorphism (23) it is obvious, that the map induced in the ring $(e_{11}\, R\, e_{11})_n$ by the involutory anti-automorphism of the ring R is at the same time an involutory anti-automorphism in $(e_{11}\, R\, e_{11})_n$ with the additional property (2). Let us find out the form of this map.

For this purpose we form the elements

$$t_i = e_{i1}{}^*\, e_{i1}, \quad i = 1, 2, \ldots, n, \tag{24}$$

which are obviously Hermitian elements of the subring $e_{11}\, R\, e_{11}$. Moreover, all these elements are in $e_{11}\, R\, e_{11}$ regular, with

$$t_i^{-1} = e_{1i}\, e_{1i}{}^*, \quad i = 1, 2, \ldots, n. \tag{25}$$

In fact it is

$$t_i\, t_i^{-1} = e_{i1}{}^*\, e_{i1}\, e_{1i}\, e_{1i}{}^* = e_{i1}{}^*\, e_{ii}\, e_{1i}{}^* = (e_{1i}\, e_{ii}\, e_{i1})^* = e_{11}$$

and

$$t_i^{-1}\, t_i = e_{1i}\, e_{1i}{}^*\, e_{i1}{}^*\, e_{i1} = e_{1i}\, (e_{i1}\, e_{1i})^*\, e_{i1} = e_{1i}\, e_{ii}\, e_{i1} = e_{11}.$$

Further, we see also that

$$t_1 = t_1^{-1} = e_{11}. \tag{26}$$

Now, we represent the situation in the form of a diagram

$$\begin{array}{ccc} R & & (e_{11} R e_{11})_n \\ x & \longrightarrow & (x_{ij}) \\ \downarrow & & \downarrow \\ x^* & \longrightarrow & ((x^*)_{ij}), \end{array}$$

where the dotted arrow represents the map in question. By (23), (22), (19), (24) and (25) we see, that this map is defined by

$$(x^*)_{ij} = e_{1i} x^* e_{j1} = (e_{j1}{}^* x e_{1i}{}^*)^* = (e_{j1}{}^* e_{j1} e_{1j} x e_{i1} e_{1i} e_{1i}{}^*)^* =$$
$$= (t_j x_{ji} t_i^{-1})^* = t_i^{-1} x_{ji}{}^* t_j ,$$

so that we obtain

$$(x_{ij}) \dashrightarrow (t_i^{-1} x_{ji}{}^* t_j) . \tag{27}$$

If we denote the diagonal matrices $(x_{ii} = t_i)$ and $(x_{ii} = t_i^{-1})$ simply by (t_i) and (t_i^{-1}) and take into account that $(t_i^{-1}) = (t_i)^{-1}$, it is easily seen that

$$(t_i^{-1} x_{ji}{}^* t_j) = (t_i)^{-1} (x_{ji}{}^*) (t_i) ,$$

so that (27) can be put into the form

$$(x_{ij}) \dashrightarrow (t_i)^{-1} (x_{ji}{}^*) (t_i) . \tag{28}$$

It follows that the induced involutory anti-automorphism in the ring of all square matrices $(e_{11} R e_{11})_n$ is expressible as the composite of two well-known matrix operations: »taking the adjoint« and »transforming into a similar matrix«. Therefore, we have:

Theorem 2. *Every *-regular ring R, having a set of n equivalent, pairwise arthogonal projections* $e_{11}, e_{22}, \ldots, e_{nn}$ *with* $e_{11} + e_{22} + \ldots + e_{nn} = 1$, *is isomorphic to the *-regular ring of all square matrices of the order n with elements in the subring* $e_{11} R e_{11}$, *where the induced involutory anti-automorphism is defined by* (28).

The regular element (t_i) is evidently Hermitian. Now, let us examine still the question: under which additional conditions is the induced involutory anti-automorphism in the ring $(e_{11} R e_{11})_n$ given only by the operation of »taking the adjoint«? This means, that for any element (x_{ij}) there must be $(t_i)^{-1} (x_{ji}{}^*) (t_i) = (x_{ji}{}^*)$, which is equivalent to

$$(x_{ij}) (t_i) = (t_i) (x_{ij}) . \tag{29}$$

From (29) it follows

$$x_{ij} t_j = t_i x_{ij} , \quad i, j = 1, 2, \ldots, n \tag{30}$$

for any element $x_{ij} \leqslant e_{11} R e_{11}$. If we take $x_{ij} = e_{11}$, we obtain $t_j = t_i$ and by (26) it follows that

$$e_{11} = t_i = e_{i1}^* e_{i1}, \quad i = 1, 2, \ldots, n \tag{31}$$

and therefore

$$e_{1i} = e_{11} e_{1i} = e_{i1}^* e_{i1} e_{1i} = e_{i1}^* e_{ii} = e_{i1}^*, \quad i = 1, 2, \ldots, n. \tag{32}$$

Conversly, (31) follows immediately from (32).

These conditions being evidently also sufficient, we have

Corollary 4. *The induced involutory anti-automorphism in the ring $(e_{11} R e_{11})_n$ is reduced to the only operation of »taking the adjoint« if and only if the elements e_{1i} and e_{i1} used in establishing the equivalence of the projections e_{11} and e_{ii} are mutually adjoint.*

3.

I. Vidav showed [4], that if a *-regular ring R satisfies instead of (2) the stronger requirement

$$a_1^* a_1 + a_2^* a_2 + \ldots + a_p^* a_p = 0 \Rightarrow a_1 = a_2 = \ldots = a_p = 0 \tag{33}$$

for any $p \leqslant N$, then this ring R is an algebra over the field of rational numbers. Moreover, it is possible to define in a natural way the notion of boundedness of an element $a \leqslant R$ and to assign to each bounded element a norm which has nearly the same properties as the norm in a C*-algebra.

In view of this fact it is reasonable to investigate the conditions on which a *-regular ring R fulfills the requirement (33). We wish to give in the sequel only some preliminary remarks in connection with this problem.

Suppose, that there exist, contrary to (33), in a *-regular ring R p non-null elements $a_1, a_2, \ldots, a_p$ such that

$$a_1^* a_1 + a_2^* a_2 + \ldots + a_p^* a_p = 0 \tag{34}$$

with $p \geqq 2$.Since, by (2), from $a_i \neq 0$ it follows $a_i^* a_i \neq 0$, all members on the left side in (34) are non-null. If we write

$$-a_p^* a_p = a_1^* a_1 + a_2^* a_2 + \ldots + a_{p-1}^* a_{p-1} \tag{35}$$

and choose, in agreement with Theorem 1, an element $x \leqslant R$ such that

$$a_p = a_p x a_p \quad \text{and} \quad a_p x = x^* a_p^* = e,$$

where e is the left projection of a_p, then, by multiplying both sides of (35) first with x^* from the left and then with x from the right, we find

$$-e = x^* a_1^* a_1 x + x^* a_2^* a_2 x + \ldots + x^* a_{p-1}^* a_{p-1} x.$$

If we put $b_i = a_i x$, we obtain

$$-e = b_1^* b_1 + b_2^* b_2 + \ldots + b_{p-1}^* b_{p-1}.$$

Since $e \neq 0$, all b_i are certainly not null and it can be concluded that:

If in a *-regular ring R the equality (34) is fulfilled for p non-null elements, then there exist less than p non-null elements $b_1, b_2, \ldots, b_m$ of this ring such that

$$b_1^* b_1 + b_2^* b_2 + \ldots + b_m^* b_m = -e, \tag{36}$$

where e is a non-null projection.

Since e is a non-null projection if and only if $1 - e$ is a projection different from 1, we can also say that

$$1 + b_1^* b_1 + b_2^* b_2 + \ldots + b_m^* b_m \tag{36a}$$

is a projection different from 1.

The converse is immediate, since any projection e can be written as $e^* e$. Therefore, we obtain

Theorem 3. *The additional requirement* (33) *is valid in exactly those *-regular rings, in which no element of the form* $b_1^* b_1 + b_2^* b_2 + \ldots + b_m^* b_m$ *is the opposite of a non-null projection.*

For that reason in particular, for no element a of such a ring R the product $a^* a$ can be the opposite of a non-null projection. But by an argument quite similar to the one used in the proof of Corollary 3 in **1.**, it can be shown that $a^* a$ is the opposite of a projection if and only if $a^* = -\bar{a}$. Hence,

Corollary 5. *A necessary condition for* (33) *is given by* $a^* \neq -\bar{a}$, *which must hold for every non-null element* a *of the ring* R.

Concerning the case studied in **2.**, let us first state that because of the isomorphism (23) an element of the form $a^* a$ is represented by

$$((a^* a)_{ij}) = (e_{1i}\, a^* a\, e_{j1}) = (e_{1i}\, e_{ii}^*\, a^* a\, e_{j1}) = (e_{1i}\, (e_{i1}\, e_{1i})^*\, a^* a\, e_{j1}) =$$
$$(e_{1i}\, e_{1i}^*\, e_{i1}^*\, a^* a\, e_{j1}) = (t_i^{-1}\, (a\, e_{i1})^*\, (a\, e_{j1})) = (t_i)^{-1}\, ((a\, e_{i1})^*\, (a\, e_{j1})). \tag{37}$$

From this we see that the diagonal elements in the matrix $((a\, e_{i1})^* (a\, e_{j1}))$ are of the form $u_i^* u_i$, where $u_i = a\, e_{i1}$. Since $e_{i1} = e_{i1}\, e_{11}$ and $e_{11} = e_{1i}\, e_{i1}$, it is $R e_{i1} = R e_{11}$ and so it follows that the elements u_i belong to the left ideal of the projection e_{11}. Therefore we have

Corollary 6. *In a *-regular ring R, having a set of n equivalent, pairwise orthogonal projections* $e_{11}, e_{22}, \ldots, e_{nn}$ *with* $e_{11} + e_{22} + \ldots + e_{nn} = 1$, *the requirement* (33) *is fulfilled as soon as it is fulfilled for the elements of the left ideal of the projection* e_{11}.

In fact it follows immediately from (37) that the equality

$$a_1^* a_1 + a_2^* a_2 + \ldots + a_p^* a_p = 0,$$

after multiplying it from the left with (t_i), can be written as

$$\left(\sum_{k=1}^{p} (a_k e_{i1})^* (a_k e_{j1}) \right) = 0.$$

Because of the hypothesis we get at once from the diagonal elements of this matrix $a_k e_{j1} = 0$ for $k = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, n$, which means, owing to (23), that $a_1 = a_2 = \ldots = a_p = 0$.

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NEKE ZABELEŠKE O *-REGULARNIM PRSTENIMA

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Sadržaj

1. Izvodi se opšte rešenje jednačine $a = a x a$ u *-regularnim prstenima, pod uslovom, da su $a x$ i $x a$ levi i desni projektor elementa a. Ovaj rezultat primenjuje se za karakterizaciju hermitskih elemenata i projektora.

2. Svaki *-regularan prsten sa n ekvivalentnih i ortogonalnih projektora, kojih je zbir jednak 1, izomorfan je *-regularnom prstenu kvadratnih matrica reda n. Za taj primer se izvodi matrična reprezentacija involutornog anti-automorfizma.

3. Date su neke opšte napomene u vezi sa dodatnim zahtevom Vidava (33).