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ON A GEOMETRIC CONSTRUCTION OF INTEGRABLE HAMILTONIAN HIERARCHIES FOR CLASSICAL $\mathcal{W}$-ALGEBRAS

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Abstract. A class of classical affine $\mathcal{W}$-algebras is shown to be isomorphic as differential algebras to the coordinate rings of double coset spaces of certain prounipotent proalgebraic groups. As an application, integrable Hamiltonian hierarchies associated with them are constructed geometrically, generalizing the corresponding result of Feigin-Frenkel and Enriquez-Frenkel for the principal cases.

1. Introduction

Since the discovery of the Drinfel'd-Sokolov hierarchy [16], a number of integrable systems have been constructed in the same spirit, e.g. [4, 6, 19, 23, 24]. Those integrable systems are not of finite dimension, which are formulated by Poisson algebras, but of infinite dimension, reflecting classical field theory as their origin.

In [2], Barakat, De Sole, and Kac used *Poisson vertex algebras*, which play the role of Poisson algebras for integrable systems as above, and introduced the notion of *integrable Hamiltonian hierarchies* as a framework of integrability for Poisson vertex algebras. Since then, De Sole, Kac, and Valeri (e.g. [11, 12, 13, 14, 15]) have studied systematically integrable Hamiltonian hierarchies associated with Poisson vertex algebras, called the *classical (affine) $\mathcal{W}$-algebras*, which are obtained as the classical limit of vertex algebras, called the (affine) $\mathcal{W}$-algebras [9, 28], see also [10, 33].

The $\mathcal{W}$-algebras are parametrized by a triple $(\mathfrak{g}, f, k)$ consisting of a finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, a nonzero nilpotent element $f \in \mathfrak{g}$, and a complex number $k \in \mathbb{C}$, called the *level* and are denoted by $\mathcal{W}^k(\mathfrak{g}, f)$.

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The classical $\mathcal{W}$-algebras are also parametrized by the same data, and thus we denote them also by $\mathcal{W}^k(\mathfrak{g}, f)$. De Sole, Kac, and Valeri recovered the earlier results on the integrable Hamiltonian hierarchies mentioned above in their study and obtained a culminating result in [15] on the existence of integrable Hamiltonian hierarchies associated with $\mathcal{W}^k(\mathfrak{g}, f)$, which states: *the classical affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ admits an integrable Hamiltonian hierarchy for any finite-dimensional simple Lie algebra $\mathfrak{g}$ of classical type, nonzero nilpotent element f, and $k \neq 0$.*

Besides the algebraic theory of integrable Hamiltonian hierarchies mentioned above, there is a result of Feigin-Frenkel [20, 21] and Enriquez-Frenkel [18] which constructs the Drinfel'd-Sokolov hierarchies by using a geometric realization of the classical affine $\mathcal{W}$-algebra $\mathcal{W}^{k=1}(\mathfrak{g}, f_{\mathrm{prin}})$, where f_{prin} denotes the principal nilpotent element in $\mathfrak{g}$. More precisely, they proved that $\mathcal{W}^{k=1}(\mathfrak{g}, f_{\mathrm{prin}})$ is isomorphic as a differential algebra to the coordinate ring of a certain double coset space of a prounipotent proalgebraic group and that the associated Drinfel'd-Sokolov hierarchy is induced from a natural group action on this space.

The aim of this paper is to generalize those results to the classical $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{g}, f)$ when f satisfies certain properties (see the condition (F) below) and the level $k \in \mathbb{C}$ is nonzero. We note that the choice of f is a special case of the so-called Type I in [4, 6, 19]. By construction, the integrable Hamiltonian hierarchies obtained in the paper for $k = 1$ coincide with special cases of those considered in [3] and are special cases of the result in [11].

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$, and $f \in \mathfrak{g}$ be a nonzero nilpotent element. We fix an $\mathfrak{sl}_2$-triple $\{e, h = 2x, f\}$ containing f. We denote by $\Gamma : \mathfrak{g} = \oplus \mathfrak{g}_j$ the ad_x-grading and by $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$ a root space decomposition of $\mathfrak{g}$ which is homogeneous with respect to Γ. Then the root system Δ admits an induced grading $\Delta = \sqcup \Delta_j$. Let Π denote the subset of $\Delta_{>0} = \sqcup_{j>0} \Delta_j$ consisting of the elements indecomposable in $\Delta_{>0}$. Let $V^k(\mathfrak{g}_0)$ denote the universal affine Poisson vertex algebra associated with the reductive Lie algebra $\mathfrak{g}_0$ at level k and $F(\mathfrak{g}_{1/2})$ the $\beta\gamma$-system Poisson vertex algebra associated with the symplectic vector space $\mathfrak{g}_{1/2}$. Then the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is defined as the 0-th cohomology of the BRST complex $C^k(\mathfrak{g}, f)$, see Section 3.1 for details.

For generic levels k, the (quantum) $\mathcal{W}$-algebras are realized as the joint kernel of certain screening operators [27]. Our first result (Theorem 3.6) is a Poisson vertex algebra analogue of this result. Namely, for $k \neq 0$, we realize the classical affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ as the Poisson vertex subalgebra of $V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2})$ invariant under the derivations Q^W_α, $(\alpha \in \Pi)$:

$$\mathcal{W}^k(\mathfrak{g}, f) \cong \bigcap_{\alpha \in \Pi} \mathrm{Ker}\Big(Q^W_\alpha : V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2}) \to V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2})\Big). \tag{1.1}$$

See (3.15) and (3.16) for the definition of Q^W_α.

We now suppose that the pair $(\mathfrak{g}, f)$ satisfies the following condition (F):

(F1) The grading Γ is a $\mathbb{Z}$-grading.
(F2) There exists an element $y \in \mathfrak{g}_d$ such that $s = f + yt^{-1} \in \mathfrak{g}[t^{\pm 1}]$ is semisimple.
(F3) The Lie subalgebra $\mathrm{Ker}(\mathrm{ad}_s) \subset \mathfrak{g}[t^{\pm 1}]$ is abelian and $\mathrm{Im}(\mathrm{ad}_s) \cap \mathfrak{g}[t^{\pm 1}]_0 = \mathfrak{g}_0$.

Here $\mathfrak{g}[t^{\pm 1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ and $\mathfrak{g}[t^{\pm 1}] = \oplus_{j \in \mathbb{Z}} \mathfrak{g}[t^{\pm 1}]_j$ is the $\mathbb{Z}$-grading given by $\deg(Xt^n) = j + (d+1)n$, $(X \in \mathfrak{g}_j)$ and d is the maximal Γ-grading on $\mathfrak{g}$. The nilpotent elements f satisfying (F1)-(F2) are called Type I in the literature e.g. [19, 4, 6]. We consider the completion $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t))$. It has subalgebras $L\mathfrak{g}_+$, which is the completion of $\mathfrak{g}[t^{\pm 1}]_{>0}$, and $L\mathfrak{g}_- = \mathfrak{g}[t^{\pm 1}]_{\leq 0}$. We have the corresponding closed subgroups LG_+, LG_- of the loop group LG of G.

Let $\mathfrak{a}$ denote the completion of $\mathrm{Ker}(\mathrm{ad}_s)$ in $L\mathfrak{g}$, and $\mathfrak{a}_\pm = \mathfrak{a} \cap L\mathfrak{g}_\pm$. Let A, (resp. $A_\pm$) be the closed subgroup of LG corresponding to $\mathfrak{a}$, (resp. $\mathfrak{a}_\pm$).

The right A-action $LG_- \backslash LG/A_+ \times A \to LG_- \backslash LG/A_+$, induces a Lie algebra homomorphism $\mathfrak{a} \to \mathrm{Der}(\mathbb{C}[LG_- \backslash LG/A_+])$, and so that

$$\mathfrak{a} \to \mathrm{Der}(\mathbb{C}[LG_+/A_+]) \tag{1.2}$$

since $LG_+/A_+ \subset LG_- \backslash LG/A_+$ is an open subset. We denote this action by $a \mapsto a^R$. In particular, we obtain a differential algebra $(\mathbb{C}[LG_+/A_+], s^R)$. On the other hand, the natural left LG_+-action $LG_+ \times LG_+/A_+ \to LG_+/A_+$ induces a Lie algebra homomorphism $L\mathfrak{g}_+ \to \mathrm{Der}(\mathbb{C}[LG_+/A_+])$, which we denote by $X \mapsto X^L$.

Our second result (Theorem 4.3) is the following geometric interpretation of (1.1) under the condition (F): there exists an isomorphism of differential algebras

$$\Psi_k : V^k(\mathfrak{g}_0) \to \mathbb{C}[LG_+/A_+],$$

such that the derivation Q^W_α is identified with X^L_α for some element $X_\alpha \in \mathfrak{g}_\alpha$, $(\alpha \in \Pi)$. Since such root vectors X_α generate a Lie algebra $\mathfrak{g}_+ = \oplus_{j>0} \mathfrak{g}_j$, it implies that $\mathcal{W}^k(\mathfrak{g}, f)$ is isomorphic to $\mathbb{C}[G_+ \backslash LG_+/A_+]$ as a differential algebra (Corollary 4.4). Here G_+ is the closed subgroup of LG_+ corresponding to $\mathfrak{g}_+$.

The action in (1.2) preserves the subalgebra $\mathbb{C}[G_+ \backslash LG_+/A_+]$. Thus, we obtain a space of mutually commutative derivations of $\mathcal{W}^\kappa(\mathfrak{g}, f)$ as the image of $\mathfrak{a}_-$. We denote this space by $\mathcal{H}^\kappa(\mathfrak{g}, f)$. Our third result (Theorem 5.10) states that $\mathcal{H}^k(\mathfrak{g}, f)$ is an integrable Hamiltonian hierarchy associated with $\mathcal{W}^k(\mathfrak{g}, f)$.

The double coset spaces $G_+ \backslash LG_+/A_+$ considered here are embedded into the abelianized Grassmannians $G[t^{-1}] \backslash LG/A_+$ as a Zariski open subset. The abelianized Grassmannians have been used to construct Drinfeld-Sokolov hierarchies geometrically [3]. As pointed out in *loc.cit.*, such a construction implies a strong compatibility of integrable Hamiltonian systems associated

with classical affine $\mathcal{W}$-algebras and Hitchin systems. The author hopes to investigate the relationship between classical affine $\mathcal{W}$-algebras and Hitchin systems in future work.

The main difficulty in extending the geometric construction to the cases of non-principal $\mathcal{W}$-algebras is that each screening operator involves several root components, rather than a single simple-root direction. The key new ingredient in this article is a further decomposition of these operators that geometrically induces the G_+-action. This construction was subsequently understood in terms of parabolic BGG resolutions in [32]. This perspective suggests a generalization of the AKNS hierarchy constructed for affine Poisson vertex algebras [22] to $\mathcal{W}$-algebras associated with nilpotent elements of semisimple type. A geometric interpretation when the nilpotent elements admit an integrable triple [7, 8] remains an open problem.

2. Poisson vertex superalgebras

2.1. *Poisson vertex superalgebra.* We recall here some basics about Poisson vertex superalgebras and their relation to the theory of integrable systems, following [2, 33]. We remove the prefix "super" whenever we consider the non-super cases.

A *differential* $\mathbb{C}$*-superalgebra* is a pair (V,∂) consisting of a supercommutative $\mathbb{C}$-superalgebra V and an even derivation ∂ on it. We denote by $\bar{a}$ the parity of $a\in V$ and by $\mathrm{Der}(V)$ the set of super-derivations of V. In particular, the *superalgebra of differential polynomials* in the variables u_i ($i\in I:=I_{\bar{0}}\sqcup I_{\bar{1}}$) is a tensor product

$$V=\mathbb{C}[\mathbf{U}_{\bar{0}}]\otimes\bigwedge\mathbf{U}_{\bar{1}}$$

of the polynomial ring in the variables $\mathbf{U}_{\bar{0}}=\{u_i^{(n)}\mid i\in I_{\bar{0}},n\geq 0\}$ and the exterior algebra in the variables $\mathbf{U}_{\bar{1}}=\{u_i^{(n)}\mid i\in I_{\bar{1}},n\geq 0\}$ equipped with a derivation ∂ so that $u_i^{(n)}=\partial^n u_i$ holds. A *Poisson vertex superalgebra* is a triple $(V,\partial,\{\,\cdot\,{}_\lambda\,\cdot\,\})$ consisting of a differential $\mathbb{C}$-superalgebra (V,∂) and an even $\mathbb{C}$-bilinear map

$$\{\,\cdot\,{}_\lambda\,\cdot\,\}\colon V\times V\to V[\lambda],\quad (f,g)\mapsto\{f_\lambda g\}=\sum_{n\geq 0}\frac{\lambda^n}{n!}f_{(n)}g, \tag{2.3}$$

called the λ-bracket, satisfying

$$\{\partial f_\lambda g\}=-\lambda\{f_\lambda g\},\quad \{f_\lambda\partial g\}=(\lambda+\partial)\{f_\lambda g\}, \tag{2.4}$$

$$\{g_\lambda f\}=-(-1)^{\bar{f}\bar{g}}_{\leftarrow}\{f_{-\partial-\lambda}g\}, \tag{2.5}$$

$$\{f_\lambda\{g_\mu h\}\}-(-1)^{\bar{f}\bar{g}}\{g_\mu\{f_\lambda h\}\}=\{\{f_\lambda g\}_{\lambda+\mu}h\}, \tag{2.6}$$

$$\{f_\lambda gh\}=\{f_\lambda g\}h+(-1)^{\bar{g}\bar{h}}\{f_\lambda h\}g, \tag{2.7}$$

$$\{fg_\lambda h\}=(-1)^{\bar{g}\bar{h}}\{f_{\lambda+\partial}h\}_{\to}g+(-1)^{\bar{f}(\bar{g}+\bar{h})}\{g_{\lambda+\partial}h\}_{\to}f, \tag{2.8}$$

for $f, g, h \in V$. Here, the arrows in the brackets indicate where λ acts when λ is replaced by non-commutative operators:

$$\{f_{\lambda+\partial}g\}_{\rightarrow} = \sum \frac{1}{n!} f_{(n)} g(\lambda+\partial)^n, \quad {}_{\leftarrow}\{f_{\lambda+\partial}g\} = \sum (\lambda+\partial)^n \frac{1}{n!} f_{(n)} g.$$

Remark 2.1. Given a superalgebra of differential polynomials V in the variables u_i $(i \in I)$, a linear map $F : \mathrm{span}_{\mathbb{C}}\{u_i\}_{i\in I} \to V[\lambda]$ uniquely extends to $F : V \to V[\lambda]$ by (2.4) and (2.8).

Let V be a superalgebra of differential polynomials in the variables u_i, $(i \in I)$. We denote by

$$\frac{\partial}{\partial u_i^{(n)}}, \frac{\partial_R}{\partial_R u_i^{(n)}} \in \mathrm{Der}(V)$$

denote the derivations with respect to $u_i^{(n)}$ from the left and the right, respectively (which coincide if V is non-super). Then the λ-bracket on V is determined by the values $\{u_{i\lambda}u_j\}$ in the following sense.

Theorem 2.2 ([2, 33]). *Let (V, ∂) be a superalgebra of differential polynomials in the variables u_i, $(i \in I)$, and $H_{ij}(\lambda)$, $(i, j \in I)$, an element of $V[\lambda]$ with the same parity as $u_i u_j$. Then there is a unique Poisson vertex superalgebra structure on V satisfying $\{u_{i\lambda}u_j\} = H_{ji}(\lambda)$ if and only if the λ-bracket satisfies* (2.5) *and* (2.6) *for u_i, $(i \in I)$. Moreover, the λ-bracket is given by*

$$\{f_\lambda g\} = \sum_{\substack{i,j\in I,\\ n,m\geq 0}} (-1)^{\bar{f}\bar{g}+\bar{i}\bar{j}} \frac{\partial_R g}{\partial_R u_j^{(n)}} (\lambda+\partial)^n H_{ji}(\lambda+\partial)_{\rightarrow}(-\lambda-\partial)^m \frac{\partial f}{\partial u_i^{(m)}}.$$

Example 2.3. Given a finite-dimensional Lie algebra L over $\mathbb{C}$ and a nondegenerate symmetric invariant bilinear form κ on L, let $V^\kappa(L)$ denote the algebra of differential polynomials in the variables given by a basis of L. Then the λ-bracket $\{\cdot\,{}_\lambda\,\cdot\} : L \times L \to V^\kappa(L)[\lambda]$ given by $\{u_\lambda v\} = [u, v] + \kappa(u, v)\lambda$, $(u, v \in L)$, defines a unique Poisson vertex algebra structure on $V^\kappa(L)$. This is called the universal *affine Poisson vertex algebra* associated with L at level κ.

Given a Poisson vertex superalgebra V, a *Poisson vertex module* over V (cf. [1]) is a vector superspace M which is a V-module as a supercommutative algebra V and endowed with an even $\mathbb{C}$-bilinear map $\{\cdot\,{}_\lambda\,\cdot\} : V \times M \to M[\lambda]$ satisfying

$$\begin{aligned}
&\{\partial f_\lambda m\} = -\lambda\{f_\lambda m\},\\
&\{f_\lambda\{g_\mu m\}\} - (-1)^{\bar{f}\bar{g}}\{g_\mu\{f_\lambda m\}\} = \{\{f_\lambda g\}_{\lambda+\mu} m\},\\
&\{f_\lambda g \cdot m\} = \{f_\lambda g\} \cdot m + (-1)^{\bar{f}\bar{g}} g \cdot \{f_\lambda m\},\\
&\{f g_\lambda m\} = (-1)^{\bar{m}\bar{g}}\{f_{\lambda+\partial} m\}_{\rightarrow} g + (-1)^{\bar{f}(\bar{g}+\bar{m})}\{g_{\lambda+\partial} m\}_{\rightarrow} f
\end{aligned}$$

for $f, g \in V$, $m \in M$. Here, the right V-action $M \times V \to M$ is defined by the action of V as a supercommutative algebra. In this case, we define the λ-bracket

$$\{ \cdot \, _\lambda \, \cdot \} \colon M \times V \to M[\lambda], \quad (m, f) \mapsto \{m_\lambda f\} = -(-1)^{\bar{m}\bar{f}} {}_{\leftarrow}\{f_{-\partial-\lambda} m\}.$$

For $m \in M$, the linear map $\{m_\lambda \cdot \} : V \to M[\lambda]$ is called an *intertwining operator* and m is called the Hamiltonian.

For a Poisson vertex superalgebra V, the vector superspace $\mathrm{Lie}(V) = V/\partial V$ is called *the space of local functionals*. We denote by

$$\int : V \to \mathrm{Lie}(V), \quad f \mapsto \int f$$

the canonical projection.

Proposition 2.4 ([2, 33]).

1. *The bilinear map*

$$\mathrm{Lie}(V) \times \mathrm{Lie}(V) \to \mathrm{Lie}(V), \quad \left(\int f, \int g \right) \mapsto \int \{f_\lambda g\}_{\lambda=0}$$

is well-defined and defines a Lie superalgebra structure. Moreover, if V is even and an algebra of differential polynomials in the variables $\{u_i\}_{i \in I}$, then

$$\left[\int f, \int g \right] = \sum_{i.j \in I} \int \frac{\delta_R g}{\delta_R u_j} \{u_{i\partial} u_j\}_{\to} \frac{\delta f}{\delta u_i},$$

where

$$\frac{\delta f}{\delta u_i} = \sum_{n \geq 0} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}, \quad \frac{\delta_R f}{\delta_R u_i} = \sum_{n \geq 0} (-\partial)^n \frac{\partial_R f}{\partial_R u_i^{(n)}}$$

denote the left and right variational derivative of f with respect to u_i.

2. *The Lie superalgebra $\mathrm{Lie}(V)$ acts on V by*

$$\eta : \mathrm{Lie}(V) \to \mathrm{Der}(V), \quad \int f \mapsto \{f_\lambda \cdot \}|_{\lambda=0}.$$

We use the following lemma below.

Lemma 2.5 (cf. [2, Proposition 1.33]). *Let L be a finite-dimensional Lie algebra over $\mathbb{C}$ equipped with a nondegenerate symmetric invariant bilinear form κ. For the universal affine Poisson vertex algebra $V^\kappa(L)$, the kernel of $\eta : \mathrm{Lie}(V^\kappa(L)) \to \mathrm{Der}(V^\kappa(L))$ is*

$$\mathrm{Ker}(\eta) = \mathrm{span}_{\mathbb{C}} \left\{ \int 1, \int u \mid u \in Z(L) \right\}$$

where $Z(L)$ is the center of L.

PROOF. Let $\{u_i\}_{i\in I}$ be a basis of L. Suppose $\int F \in \mathrm{Ker}(\eta) \subset \mathrm{Lie}(V^\kappa(L))$. By Theorem 2.2, we have

$$\eta\left(\int F\right) = \sum_{i,j,n} \partial^n \left(([u_i,u_j] + \kappa(u_i,u_j)\partial)\frac{\delta F}{\delta u_i}\right)\frac{\partial}{\partial u_j^{(n)}}$$

and thus

$$M_j = \sum_i ([u_i,u_j] + \kappa(u_i,u_j)\partial)\frac{\delta F}{\delta u_i} = 0$$

for all $j \in I$. Define a degree on $V^\kappa(L)$ by $\deg(u_i^{(n)}) = n$ and $\deg(AB) = \deg(A) + \deg(B)$. Then $\delta/\delta u_i$ preserves the degree. Let G_i be the top degree component of $\delta F/\delta u_i$. Then the top component M_j^{top} of M_j is $M_j^{\mathrm{top}} = \sum_i \kappa(u_i,u_j)\partial G_i$. Since κ is nondegenerate, we obtain $\partial G_i = 0$, which implies $G_i \in \mathbb{C}$. Since $V^\kappa(L)$ is an algebra of differential polynomials, we conclude $F \in \mathbb{C} \oplus L$. (See the proof of [2, Proposition 1.5].) Set $F = a + \sum_i b_i u_i$, $(a, b_i \in \mathbb{C})$. Then $M_j = [\sum_i b_i u_i, u_j] = 0$, $(j \in I)$, which implies $\sum_i b_i u_i \in Z(L)$. ∎

Finally, given a Poisson vertex superalgebra V, an element $\int f \in \mathrm{Lie}(V)$ is called *integrable* if there exists an infinite-dimensional abelian Lie subsuperalgebra $\mathcal{H}$ of $\mathrm{Lie}(V)$ which contains $\int f$. In this case, $\mathcal{H}$ is called an *integrable Hamiltonian hierarchy* associated with V.

2.2. *Differential graded Poisson vertex superalgebra.* A *differential graded Poisson vertex superalgebra* (d.g. Poisson vertex superalgebra) is a pair (V, d) consisting of a Poisson vertex superalgebra $V = \oplus_{n\in\mathbb{Z}} V_n$ and a linear map $d : V \to V$, called the differential, satisfying

- V is a $\mathbb{Z}$-graded Poisson vertex superalgebra, i.e., $V = \oplus_{n\in\mathbb{Z}} V_n$ as a vector superspace satisfying

$$V_n \cdot V_m \subset V_{n+m}, \quad \{V_n{}_\lambda V_m\} \subset V_{n+m}[\lambda],$$

- the linear map d is odd and satisfies $d^2 = 0$,

$$d : V_n \to V_{n+1},$$

which satisfies

$$d(ab) = da \cdot b + (-1)^{\bar{d}\bar{a}} a \cdot db, \quad d\{a_\lambda b\} = \{da_\lambda b\} + (-1)^{\bar{d}\bar{a}}\{a_\lambda db\}.$$

The cohomology $H^*(V;d) = \oplus_{n\in\mathbb{Z}} H^n(V;d)$ inherits a $\mathbb{Z}$-graded Poisson vertex algebra structure. Moreover, $H^0(V;d)$ is a Poisson vertex subsuperalgebra and $H^n(V;d)$, $(n \in \mathbb{Z})$, is a Poisson vertex module over $H^0(V;d)$. In the sequel, we also use the notion of a *differential graded vertex superalgebra*. The definition is similar, and therefore, we omit the details.

2.3. *Classical limit.* Let V be a vertex superalgebra over a polynomial ring $\mathbb{C}[\epsilon]$. Suppose that V is free as a $\mathbb{C}[\epsilon]$-module and the λ-bracket satisfies

$$[V_\lambda V] \subset \epsilon V. \tag{2.9}$$

Define the vector superspace $V^{\mathrm{cl}} = V/\epsilon V$ and let $V \to V^{\mathrm{cl}}$, $(f \mapsto \bar{f})$ denote the canonical projection. Then

$$V^{\mathrm{cl}} \times V^{\mathrm{cl}} \to V^{\mathrm{cl}}, \quad (\bar{f}, \bar{g}) \mapsto \overline{f_{(-1)}g}$$

is well-defined and defines an associative supercommutative algebra structure on V^{cl}. Since the translation operator ∂ of V preserves ϵV, it induces a linear map

$$\partial : V^{\mathrm{cl}} \to V^{\mathrm{cl}}, \quad \partial \bar{f} \mapsto \overline{\partial(f)},$$

which is a derivation of V^{cl}. Since the ϵV is an ideal of V by (2.9), the λ-bracket of V induces a bilinear map

$$\{ \cdot \, _\lambda \, \cdot \} : V^{\mathrm{cl}} \times V^{\mathrm{cl}} \to V^{\mathrm{cl}}[\lambda], \quad (\bar{f}\bar{g}) \mapsto \overline{[f_\lambda g]}.$$

The triple $(V^{\mathrm{cl}}, \partial, \{ \cdot \, _\lambda \, \cdot \})$ defines a Poisson vertex superalgebra, called the *classical limit* of V (cf. [9, 25]).

3. Screening operators for classical affine $\mathcal{W}$-algebras

In this section, we describe the classical affine $\mathcal{W}$-algebras by using screening operators. They will be obtained as a classical limit of the screening operators for the affine $\mathcal{W}$-algebras obtained in [27]. We will use the same notation for Poisson vertex algebras as vertex algebras since there will be no confusion.

3.1. *Affine $\mathcal{W}$-algebras.* Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ with the normalized symmetric invariant bilinear form $\kappa = (\,\cdot\,|\,\cdot\,)$. Let $f \in \mathfrak{g}$ be a nonzero nilpotent element, fix an $\mathfrak{sl}_2$-triple $\{e, h = 2x, f\}$ containing f and denote by $\Gamma : \mathfrak{g} = \oplus_{-d \le j \le d} \mathfrak{g}_j$ the $\frac{1}{2}\mathbb{Z}$-grading given by ad_x, with d the largest number such that $\mathfrak{g}_d \neq 0$. We fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ so that $x \in \mathfrak{h}$, $\mathfrak{g}_{>0} = \oplus_{j>0} \mathfrak{g}_j \subset \mathfrak{n}_+$, and $\mathfrak{g}_{<0} = \oplus_{j<0} \mathfrak{g}_j \subset \mathfrak{n}_-$. Let $\mathfrak{g} = \mathfrak{h} \bigoplus \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be a root space decomposition, $\Delta_j = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{g}_j\}$, and $\Delta_{>0} = \sqcup_{j>0} \Delta_j$. Fix a nonzero root vector e_α in $\mathfrak{g}_\alpha$ and a basis e_i, $(i \in I)$, of $\mathfrak{h}$. Then e_α, $(\alpha \in I \sqcup \Delta)$, form a basis of $\mathfrak{g}$. We denote by $\{e_{\bar{\alpha}}\}_{\alpha \in I \sqcup \Delta}$ its dual basis of $\mathfrak{g}$ with respect to κ. Let $c^\gamma_{\alpha,\beta}$ denote the structure constants of $\mathfrak{g}$, i.e., $[e_\alpha, e_\beta] = \sum_{\gamma \in I \sqcup \Delta} c^\gamma_{\alpha,\beta} e_\gamma$.

Let $V^k(\mathfrak{g})$ be the universal affine vertex algebra of $\mathfrak{g}$ at level k, generated by the even elements e_α, $(\alpha \in I \sqcup \Delta)$, with λ-bracket $[e_{\alpha\lambda} e_\beta] = [e_\alpha, e_\beta] + k(e_\alpha|e_\beta)\lambda$. Let $F^{\mathrm{ch}}(\mathfrak{g}_{>0})$ be the charged free fermion vertex superalgebra associated with the symplectic odd vector superspace $\mathfrak{g}_{>0} \oplus \mathfrak{g}^*_{>0}$, generated by the odd elements φ_α, φ^α, $(\alpha \in \Delta_{>0})$, with λ-bracket $[\varphi_{\alpha\lambda} \varphi^\beta] = \delta_{\alpha,\beta}$,

$[\varphi_{\alpha\lambda}\varphi_\beta] = [\varphi^\alpha{}_\lambda\varphi^\beta] = 0$. Let $F(\mathfrak{g}_{1/2})$ be the $\beta\gamma$-system vertex algebra associated with the symplectic vector space $\mathfrak{g}_{1/2}$, generated by Φ_α, $(\alpha \in \Delta_{1/2})$, with λ-bracket $[\Phi_{\alpha\lambda}\Phi_\beta] = \chi([e_\alpha, \beta])$, where $\chi(\ \cdot\ ) = (f \mid\ \cdot\ )$.

The affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ associated with the triple $(\mathfrak{g}, f, k)$, $(k \in \mathbb{C})$, is the vertex algebra defined as the 0-th cohomology of the differential graded vertex algebra

$$C^k(\mathfrak{g}, f) = V^k(\mathfrak{g}) \otimes F^{\mathrm{ch}}(\mathfrak{g}_{>0}) \otimes F(\mathfrak{g}_{1/2}),$$

with differential $d = [Q_\lambda\ \cdot\ ]|_{\lambda=0}$ with

$$Q = \sum_{\alpha\in\Delta_{>0}} \big((e_\alpha + \Phi_\alpha + \chi(e_\alpha)\big)\varphi^\alpha - \frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Delta_{>0}} c^\gamma_{\alpha,\beta}\varphi_\gamma\varphi^\alpha\varphi^\beta,$$

called the *BRST complex*. The grading $C^k(\mathfrak{g}, f) = \oplus_{n\in\mathbb{Z}} C^k_n(\mathfrak{g}, f)$ is given by $\mathrm{gr}(e_\alpha) = \mathrm{gr}(\Phi_\beta) = 0$, $\mathrm{gr}(\varphi^\alpha) = -\mathrm{gr}(\varphi_\alpha) = 1$ with $\mathrm{gr}(AB) = \mathrm{gr}(A) + \mathrm{gr}(B)$ and $\mathrm{gr}(\partial A) = \mathrm{gr}(A)$. Then we have $d : C^k_n(\mathfrak{g}, f) \to C^k_{n+1}(\mathfrak{g}, f)$. We have $H^n(C^k(\mathfrak{g}, f)) = 0$ for $n \neq 0$ (see [29, 30]).

3.2. *Classical limit.* By [29, 30], we have vertex subsuperalgebras $C^k_\pm(\mathfrak{g}, f)$, which gives a decomposition of a complex $C^k(\mathfrak{g}, f) = C^k_-(\mathfrak{g}, f) \otimes C^k_+(\mathfrak{g}, f)$ and satisfies $H^n(C^k_+(\mathfrak{g}, f)) \cong \delta_{n,0}\mathbb{C}$. Thus $H^0(C^k_-(\mathfrak{g}, f)) \cong \mathcal{W}^k(\mathfrak{g}, f)$. As a vertex superalgebra, $C^k_-(\mathfrak{g}, f)$ is generated by

$$J^u = u + \sum_{\beta,\gamma\in\Delta_{>0}} c^\gamma_{u,\beta}\varphi_\gamma\varphi^\beta,\ (u \in \mathfrak{g}_{\leq 0}),$$
$$\Phi_\alpha, (\alpha \in \Delta_{1/2}), \quad \varphi^\alpha, (\alpha \in \Delta_{>0}).$$

Following [27], we introduce the *classical affine $\mathcal{W}$-algebra* as the cohomology of the differential graded Poisson vertex algebra in the classical limit of $C^k_-(\mathfrak{g}, f)$.

Suppose $k + h^\vee \neq 0$. Set $\epsilon = \frac{k'}{k+h^\vee}$, $(k' \in \mathbb{C}\backslash\{0\})$, $\bar{J}^u = \epsilon J^u$, $(u \in \mathfrak{g}_{\leq 0})$, and $\bar{\Phi}_\alpha = \epsilon\Phi_\alpha$, $(\alpha \in \Delta_{1/2})$. Then we have

$$[\bar{J}^u{}_\lambda \bar{J}^v] = \epsilon\big(\bar{J}^{[u,v]} + \big(k'(u|v) + o(\epsilon)\big)\lambda\big), \quad [\varphi^\alpha{}_\lambda \bar{J}^u] = \epsilon\sum_{\beta\in\Delta_{>0}} c^\alpha_{u,\beta}\varphi^\beta, \tag{3.10}$$

$$[\bar{\Phi}_{\alpha\lambda}\bar{\Phi}_\beta] = \epsilon\cdot\chi([e_\alpha, e_\beta]), \quad [\bar{J}^u{}_\lambda\bar{\Phi}_\alpha] = [\varphi^\alpha{}_\lambda\varphi^\beta] = [\varphi^\alpha{}_\lambda\bar{\Phi}_\beta] = 0, \tag{3.11}$$

(cf. [27]). Viewing ϵ as an indeterminate in (3.10), (3.11), we obtain a vertex superalgebra $\tilde{C}^k_-$ over the polynomial ring $\mathbb{C}[\epsilon]$. By Section 2.3, we obtain a Poisson vertex superalgebra $\tilde{C}^k_-/\epsilon\tilde{C}^k_-$, which we denote by $C^{\mathrm{cl}}_{k'} = C^{\mathrm{cl}}_{k'}(\mathfrak{g}, f)$. We have an isomorphism

$$C^{\mathrm{cl}}_{k'} \cong V^{k'}(\mathfrak{g}_{\leq 0}) \otimes F(\mathfrak{g}_{\frac{1}{2}}) \otimes \bigwedge \mathfrak{g}^*_{>0}$$

of a Poisson vertex superalgebra where $V^{k'}(\mathfrak{g}_{\leq 0})$ is the universal affine Poisson vertex algebra generated by $\mathfrak{g}_{\leq 0}$ with λ-bracket $\{u_\lambda v\} = [u, v] + k'(u|v)\lambda$,

$F(\mathfrak{g}_{1/2})$ the $\beta\gamma$-system Poisson vertex algebra generated by Φ_α, $(\alpha \in \Delta_{1/2})$, with λ-bracket $\{\Phi_\alpha{}_\lambda \Phi_\beta\} = (f|[e_\alpha, e_\beta])$, and $\bigwedge \mathfrak{g}^*_{>0}$ the differential $\mathbb{C}$-superalgebra generated by odd elements φ^α, $(\alpha \in \Delta_{>0})$, which satisfy

$$\{\varphi^\alpha{}_\lambda u\} = \sum_{\beta\in\Delta_{>0}} c^\alpha_{u,\beta}\varphi^\beta, \quad \{\varphi^\alpha{}_\lambda \varphi^\beta\} = \{\varphi^\beta{}_\lambda \Phi_\alpha\} = 0.$$

Decompose the differential d as $d = d_{\mathrm{st}} + d_{\mathrm{ne}} + d_\chi$ with $d_\bullet = [Q_{\bullet\lambda}\cdot\,]|_{\lambda=0}$ where

$$Q_{\mathrm{st}} = \sum_{\alpha\in\Delta_{>0}} e_\alpha\varphi^\alpha - \frac{1}{2}\sum_{\substack{\alpha,\beta\in\Delta_{>0}\\ \gamma\in\Delta_{>0}}} c^\gamma_{\alpha,\beta}\varphi_\gamma\varphi^\alpha\varphi^\beta,$$
$$Q_{\mathrm{ne}} = \sum_{\alpha\in\Delta_{\frac{1}{2}}} \Phi_\alpha\varphi^\alpha, \quad Q_\chi = \sum_{\alpha\in\Delta_{>0}} \chi(e_\alpha)\varphi^\alpha.$$

Then we have

$$[Q_{\mathrm{st}\lambda}\bar{J}^u] = -\sum_{\substack{\alpha\in I\sqcup\Delta_{\leq 0}\\ \beta\in\Delta_{>0}}} c^\alpha_{u,\beta}\bar{J}^{e_\alpha}\varphi^\beta + \sum_{\beta\in\Delta_{>0}} (k'(u|v)(\partial+\lambda)+o(\epsilon))\varphi^\beta,$$
$$[Q_{\mathrm{ne}\lambda}\bar{J}^u] = \sum_{\substack{\alpha\in\Delta_{1/2}\\ \beta\in\Delta_{>0}}} c^\alpha_\beta\bar{\Phi}_\alpha\varphi^\beta, \qquad \frac{1}{\epsilon}[Q_{\chi\lambda}\bar{J}^u] = \sum_{\beta\in\Delta_{>0}} \chi([u,e_\beta])\varphi^\beta,$$
$$[Q_{\mathrm{st}\lambda}\varphi^\alpha] = -\frac{1}{2}\sum_{\beta,\gamma\in\Delta_{>0}} c^\alpha_{\beta,\gamma}\varphi^\beta\varphi^\gamma, \quad [Q_{\mathrm{ne}\lambda}\bar{\Phi}_\alpha] = \sum_{\beta\in\Delta_{1/2}} \chi([e_\beta,e_\alpha])\varphi^\beta,$$
$$[Q_{\mathrm{ne}\lambda}\varphi^\alpha] = [Q_{\chi\lambda}\varphi^\alpha] = [Q_{\mathrm{st}\lambda}\bar{\Phi}_\alpha] = [Q_{\chi\lambda}\bar{\Phi}_\alpha] = 0.$$

The differential d is thus given by $d = d_{\mathrm{st}} + d_{\mathrm{ne}} + d_\chi$ with

$$d_\bullet = \{Q_{\bullet\lambda}\cdot\}|_{\lambda=0}\colon C^{\mathrm{cl}}_k(\mathfrak{g},f) \to C^{\mathrm{cl}}_k(\mathfrak{g},f)[\lambda],$$

which satisfy

$$\{Q_{\mathrm{st}\lambda}u\} = -\sum_{\substack{\alpha\in I\sqcup\Delta_{\leq 0}\\ \beta\in\Delta_{>0}}} c^{\alpha}_{u,\beta}e_\alpha\varphi^\beta + \sum_{\beta\in\Delta_{>0}} k(u|e_\beta)(\partial+\lambda)\varphi^\beta,$$

$$\{Q_{\mathrm{st}\lambda}\Phi_\alpha\} = 0, \quad \{Q_{\mathrm{st}\lambda}\varphi^\alpha\} = -\frac{1}{2}\sum_{\beta,\gamma\in\Delta_{>0}} c^{\alpha}_{\beta,\gamma}\varphi^\beta\varphi^\gamma,$$

$$\{Q_{\mathrm{ne}\lambda}u\} = \sum_{\substack{\alpha\in\Delta_{1/2}\\ \beta\in\Delta_{>0}}} c^{\alpha}_{u,\beta}\Phi_\alpha\varphi^\beta,$$

$$\{Q_{\mathrm{ne}\lambda}\Phi_\alpha\} = \sum_{\beta\in\Delta_{1/2}} (f|[e_\beta,e_\alpha])\varphi^\beta, \quad \{Q_{\mathrm{ne}\lambda}\varphi^\alpha\} = 0,$$

$$\{Q_{\chi\lambda}u\} = \sum_{\beta\in\Delta_{>0}} (f|[u,e_\beta])\varphi^\beta, \quad \{Q_{\chi\lambda}\varphi^\alpha\} = \{Q_{\chi\lambda}\Phi_\alpha\} = 0,$$

and are extended to C^{cl}_k by (2.4) and (2.7). The 0-th cohomology of $C^{\mathrm{cl}}_k(\mathfrak{g},f)$, which we denote by $\mathcal{W}^k(\mathfrak{g},f) = H^0(C^{\mathrm{cl}}_k(\mathfrak{g},f))$, is a Poisson vertex algebra called *the classical affine $\mathcal{W}$-algebra associated with* $(\mathfrak{g},f,k)$ ([27]). Note that $H^n(C^{\mathrm{cl}}_k(\mathfrak{g},f)) = 0$ holds for all $n\neq 0$.

3.3. *Screening operators.* Introduce another grading wt on C^{cl}_k by

$$\mathrm{wt}(u) = -2j, (u\in\mathfrak{g}_j), \quad \mathrm{wt}(\Phi_\alpha) = 0,\ (\alpha\in\Delta_{1/2}), \quad \mathrm{wt}(\varphi^\alpha) = 2j\ (\alpha\in\Delta_j),$$

exteded so that

$$\mathrm{wt}(\partial A) = \mathrm{wt}(A), \text{ and } \mathrm{wt}(AB) = \mathrm{wt}(A)+\mathrm{wt}(B).$$

and a decreasing filtration $\{F_pC^{\mathrm{cl}}_k\}_{p\geq 0}$ on C^{cl}_k by

$$F_pC^{\mathrm{cl}}_k = \mathrm{span}\{A\in C^{\mathrm{cl}}_k | \mathrm{wt}(A)\geq p\},$$

This filtration is exhaustive, separated, and compatible with the grading of C^{cl}_k as a complex. The associated spectral sequence $\{E_r,d_r\}_{r\geq 0}$ has the differentials

$$d_0 = d_{\mathrm{st}}, \quad d_1 = d_{\mathrm{ne}}, \quad d_2 = d_\chi, \quad d_r = 0,\ (r\geq 3),$$

and thus converges at $r=3$. We will describe $\mathcal{W}^k(\mathfrak{g},f) = H^0(C^{\mathrm{cl}}_k(\mathfrak{g},f))$ by using it. Since the calculation is straightforward, we omit the details. (The analogous argument for vertex algebras can be found in [27].)

To calculate $E_1 = H^*(C^{\mathrm{cl}}_k, d_{\mathrm{st}})$, note that d_{st} acts trivially on $F(\mathfrak{g}_{1/2})$ and, thus, it suffices to compute the complex

$$\overline{\mathcal{C}}^{\mathrm{cl}}_k := V^k(\mathfrak{g}_{\leq 0})\otimes\bigwedge\mathfrak{g}^*_{>0}$$

equipped with the differential d_0. As discussed in [27, §4.5], we may identify this complex as the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g}_{>0}[\![t]\!]$

with coefficients in the space of algebraic functions over the following space of k-connections:

$$\mathcal{C}onn := k\frac{d}{dt} + \mathfrak{g}_{\geq 0}[\![t]\!]dt$$

with the Adjoint $G_{>0}[\![t]\!]$-action:

$$\begin{aligned} g(t).\left(k\frac{d}{dt} + a(t)dt\right) &= g(t)^{-1}\left(k\frac{d}{dt} + a(t)dt\right)g(t) \\ &= k\frac{d}{dt} + \left(k\frac{d}{dt}X(t) + g(t)^{-1}a(t)g(t)\right)dt, \end{aligned}$$

where $G_{>0}[\![t]\!]$ is the proalgebaic group whose Lie algebra is $\mathfrak{g}_{>0}[\![t]\!]$ so that

$$\mathrm{e}^{\bullet}\colon \mathfrak{g}_{>0}[\![t]\!] \xrightarrow{\simeq} G_{>0}[\![t]\!], \quad X(t) \mapsto g(t) := \exp(X(t)).$$

The identification $V^k(\mathfrak{g}_{\leq 0}) \simeq \mathbb{C}[\mathcal{C}onn]$ is given by the residue pairing:

$$\partial^{(n)}u\left(k\frac{d}{dt} + a(t)dt\right) = \mathrm{Res}_{t=0}\left(ut^{-n-1}|a(t)\right)dt$$

with $\partial^{(n)}u = \frac{1}{n!}\partial^n u$. Note that if $k \neq 0$, then we have an isomorphism

$$G_{>0}[\![t]\!]_+ \times \left(k\frac{d}{dt} + \mathfrak{g}_0[\![t]\!]dt\right) \xrightarrow{\simeq} \mathcal{C}onn$$

induced by the group action, where $G_{>0}[\![t]\!]_+ \subset G_{>0}[\![t]\!]$ whose Lie algebra is $\mathfrak{g}_{>0}[\![t]\!]t \subset \mathfrak{g}_{>0}[\![t]\!]$. Hence, the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(\mathfrak{g}_{>0}; H^q(\mathfrak{g}_{>0}[\![t]\!]t; \mathbb{C}[\mathcal{C}onn])) \Rightarrow H^{\bullet}(\mathfrak{g}_{>0}[\![t]\!]\colon \mathbb{C}[\mathcal{C}onn]).$$

associated with $\mathfrak{g}_{>0}[\![t]\!] \simeq \mathfrak{g}_{>0} \ltimes \mathfrak{g}_{>0}[\![t]\!]t$ implies that

$$E_2^{p,q} \simeq E_\infty^{p,q} \simeq \delta_{q,0}V^k(\mathfrak{g}_0) \otimes H^p(\mathfrak{g}_{>0}; \mathbb{C}). \tag{3.12}$$

Indeed, by setting $\bigwedge_{\mathrm{red}}$ to be the exterier algebra generated by φ^α ($\alpha \in \Delta_{>0}$), we find a subcomplex

$$V^k(\mathfrak{g}_0) \otimes \bigwedge\nolimits_{\mathrm{red}} \subset \overline{\mathcal{C}}_k^{\mathrm{cl}}$$

whose cohomology agrees with (3.12). Thus we obtain the following lemma on the original spectral sequence.

Lemma 3.1. *For $k \neq 0$, we have an isomorphism of graded vector superspaces*

$$V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2}) \otimes H^{\bullet}(\mathfrak{g}_{>0}; \mathbb{C}) \xrightarrow{\simeq} E_1^{\bullet},$$

which restricts to an isomorphism of Poisson vertex algebras

$$V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2}) \xrightarrow{\simeq} E_1^{(0)}.$$

Note that each cohomology $E_1^{(n)}$ is a Poisson vertex modules over $E_1^{(0)} \simeq V^k(\mathfrak{g}_0)\otimes F(\mathfrak{g}_{1/2})$. Let us describe it more explicitly. Recall that the coadjoint representation on $\mathfrak{g}_{>0}^*$ of $\mathfrak{g}_0$ induces a representation of $\mathfrak{g}_0$ on $H^n(\mathfrak{g}_{>0},\mathbb{C})$ as described as follows (cf. [31, Chapter 3]). Let W denote the Weyl group of $\mathfrak{g}$ and set $I_0 = \{i \in I \mid \alpha_i \in \Delta_0\}$, $W_0' = \{w \in W \mid w\Delta_0^+ \subset \Delta^+\}$. Let $* : W \times \mathfrak{h}^* \to \mathfrak{h}^*$ denote the shifted action of W and $l : W \to \mathbb{Z}_{\geq 0}$ the length function. Then there is an isomorphism of $\mathfrak{g}_0$-modules

$$H^n(\mathfrak{g}_{>0};\mathbb{C}) \cong \bigoplus_{\substack{w\in W_0' \\ l(w)=n}} L_0(w^{-1}*0),$$

where $L_0(w^{-1}*0)$ is the integrable highest weight $\mathfrak{g}_0$-module with highest weight $w^{-1}*0$.

For a $\mathfrak{g}_0$-module M, set $\mathbb{M}^k = V^k(\mathfrak{g}_0)\otimes_{\mathbb{C}} M$. The space $\mathbb{M}^k$ has a unique Poisson vertex module over $V^k(\mathfrak{g}_0)$ such that $V^k(\mathfrak{g}_0)$ acts as a commutative algebra by multiplication on the first component and the λ-bracket $\{\cdot\,{}_\lambda\,\cdot\} : V^k(\mathfrak{g}_0)\otimes\mathbb{M}^k \to \mathbb{M}^k[\lambda]$ satisfies $\{a_\lambda b\otimes m\} = \{a_\lambda b\}\otimes m + b\otimes a\cdot m$ for $a \in \mathfrak{g}_0$, $b \in V^k(\mathfrak{g}_0)$, and $m \in M$. We denote by $\mathbb{L}_0^k(w^{-1}*0)$ the Poisson vertex module obtained from $L_0(w^{-1}*0)$.

LEMMA 3.2. *There is an isomorphism*

$$E_1^{(n)} \cong \bigoplus_{\substack{w\in W_0' \\ l(w)=n}} \mathbb{L}_0^k(w^{-1}*0)\otimes F(\mathfrak{g}_{1/2}),$$

as a Poisson vertex module over $V^k(\mathfrak{g}_0)\otimes F(\mathfrak{g}_{1/2})$.

In particular, we have $E_1^{(1)} \cong \bigoplus_{i\in I\setminus I_0} \mathbb{L}_0^k(-\alpha_i)\otimes F(\mathfrak{g}_{1/2})$ and the subspace $L_0(-\alpha_i)$ is identified as

$$L_0(-\alpha_i) \cong \bigoplus_{\beta\in[\alpha_i]} \mathbb{C}\varphi^\beta \subset E_1^{(1)},$$

where $[\alpha_i] = \Delta_{>0}\cap(\alpha_i + Q_0)$ and Q_0 denotes the root lattice of Δ_0. For $\alpha \in [\alpha_i]$ with $i \in I\setminus I_0$, we have

$$\partial\varphi^\alpha = \frac{1}{k}\sum_{\substack{\beta\in[\alpha],\\ \gamma\in I\sqcup\Delta_0}} c_{\beta,\gamma}^{\alpha} e_{\bar\gamma}\varphi^\beta. \tag{3.13}$$

Let us describe the differentials on E_1 induced from Q_{ne} and Q_χ. Consider the intertwining operators $Q_i^W : V^k(\mathfrak{g}_0)\otimes F(\mathfrak{g}_{1/2}) \to \mathbb{L}_0^k(-\alpha_i)\otimes F(\mathfrak{g}_{1/2})$ given by

$$Q_i^W = \begin{cases} \sum_{\beta\in[\alpha_i]} \{\Phi_\beta \varphi^\beta{}_\lambda \cdot\}|_{\lambda=0}, & (i \in I_{1/2}), \\ \sum_{\beta\in[\alpha_i]} \{(f|e_\beta)\varphi^\beta{}_\lambda \cdot\}|_{\lambda=0}, & (i \in I_1). \end{cases}$$

Then we have:

LEMMA 3.3. *The differentials on E_1 induced by d_{ne} and d_χ are given by*

$$d_{\mathrm{ne}} = \sum_{i\in I_{1/2}} Q_i^W \text{ and } d_\chi = \sum_{i\in I_1} Q_i^W.$$

Recall that the complex $C_k^{\mathrm{cl}} = C_k^{\mathrm{cl}}(\mathfrak{g}, f)$ is $\mathbb{Z}_{\geq 0}$-graded and that

$$H^n(C_k^{\mathrm{cl}}(\mathfrak{g}, f)) \cong \delta_{n,0}\mathcal{W}^k(\mathfrak{g}, f).$$

Then we see that $\mathcal{W}^k(\mathfrak{g}, f)$ is a subalgebra of the 0-th degree $C_{k,0}^{\mathrm{cl}} = V^k(\mathfrak{g}_{\leq 0}) \otimes F(\mathfrak{g}_{1/2})$. Since the ideal $\mathcal{I}$ generated by $\partial^n u$ ($u \in \mathfrak{g}_{<0}$, $n \geq 0$) is also a Poisson vertex ideal of $C_{k,0}^{\mathrm{cl}}$, we obtain a homomorphism of Poisson vertex algebras

$$\mathcal{W}^k(\mathfrak{g}, f) \to C_{k,0}^{\mathrm{cl}}/\mathcal{I} \cong V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2}).$$

It is injective, and the image is described by using the differentials d_{ne} and d_χ as in the following theorem.

THEOREM 3.4. *For $k \neq 0$, there is an isomorphism*

$$j : \mathcal{W}^k(\mathfrak{g}, f) \cong \bigcap_{i\in I_{1/2}\sqcup I_1} \mathrm{Ker}\Big(Q_i^W : V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2}) \to \mathbb{L}_0^k(-\alpha_i) \otimes F(\mathfrak{g}_{1/2})\Big),$$

of Poisson vertex algebras.

The operators $\{Q_i^W\}_{i\in I_{1/2}\sqcup I_1}$ are called the *screening operators* for $\mathcal{W}^k(\mathfrak{g}, f)$. The inclusion $j : \mathcal{W}^k(\mathfrak{g}, f) \to V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2})$ in Theorem 3.4 induces a Lie algebra homomorphism between their spaces of local functionals

$$j_* : \mathrm{Lie}\big(\mathcal{W}^k(\mathfrak{g}, f)\big) \to \mathrm{Lie}\big(V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2})\big).$$

(See Proposition 2.4.)

LEMMA 3.5. *The Lie algebra homomorphism j_* is injective.*

PROOF. It is easy to see

$$\mathrm{Ker}\Big(\partial : \mathbb{L}_0^k(-\alpha_i) \otimes F(\mathfrak{g}_{1/2}) \to \mathbb{L}_0^k(-\alpha_i) \otimes F(\mathfrak{g}_{1/2})\Big) = 0 \tag{3.14}$$

for $i \in I_{1/2} \sqcup I_1$. Now, take an element $f \in \mathcal{W}^k(\mathfrak{g}, f)$ such that $\int f \in \mathrm{Ker}\, j_*$. Then there exists an element g in $V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2})$ such that $j(f) = \partial g$. Let H_i^W denote the Hamiltonian of Q_i^W. Then we have

$$0 = Q_i^W j(f) = \{H_i^W{}_\lambda j(f)\}|_{\lambda=0} = \{H_i^W{}_\lambda \partial g\}|_{\lambda=0} = \partial\{H_i^W{}_\lambda g\}|_{\lambda=0}.$$

It follows $\{H_i^W{}_\lambda g\}|_{\lambda=0}=0$ by (3.14) and thus $g\in j(\mathcal{W}^k(\mathfrak{g},f))$ by Theorem 3.4. Therefore, $\int f=0$. ∎

Let $Q_\alpha^W : V^k(\mathfrak{g}_0)\otimes F(\mathfrak{g}_{1/2})\to V^k(\mathfrak{g}_0)\otimes F(\mathfrak{g}_{1/2})$, $(\alpha\in\Pi)$, be the derivation determined by

$$Q_\alpha^W e_\beta=\begin{cases}\sum\limits_{\gamma\in[\alpha]}c_{\beta\alpha}^\gamma\Phi_\gamma\\ (f|[e_\beta,e_\alpha]),\end{cases}\qquad Q_\alpha^W\Phi_\beta=\begin{cases}(f|[e_\alpha,e_\beta]) & (\alpha\in\Pi_{1/2})\\ 0 & (\alpha\in\Pi_1),\end{cases}\tag{3.15}$$

and

$$[Q_\alpha^W,\partial]=\frac{1}{k}\sum_{\substack{\beta\in I\sqcup\Delta_0\\ \gamma\in[\alpha]}}c_{\alpha,\beta}^\gamma e_{\bar\beta}Q_\gamma^W.\tag{3.16}$$

THEOREM 3.6. *For $k\neq 0$, the classical affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g},f)$ is isomorphic to the Poisson vertex subalgebra*

$$\mathcal{W}^k(\mathfrak{g},f)\cong\bigcap_{\alpha\in\Pi}\mathrm{Ker}\Big(Q_\alpha^W : V^k(\mathfrak{g}_0)\otimes F(\mathfrak{g}_{1/2})\to V^k(\mathfrak{g}_0)\otimes F(\mathfrak{g}_{1/2})\Big),\tag{3.17}$$

of $V^k(\mathfrak{g}_0)\otimes F(\mathfrak{g}_{1/2})$ invariant under the derivations Q_α^W, $(\alpha\in\Pi)$.

PROOF. Since $Q_i^W : V^k(\mathfrak{g}_0)\otimes F(\mathfrak{g}_{1/2})\to\mathbb{L}_0^k(-\alpha_i)\otimes F(\mathfrak{g}_{1/2})$ acts by derivation, it decomposes as $Q_i^W=\sum_{\alpha\in[\alpha_i]}\varphi^\alpha Q_\alpha^W$, where $Q_\alpha^W\in\mathrm{Der}(V^k(\mathfrak{g}_0)\otimes F(\mathfrak{g}_{1/2}))$ and φ^α is the multiplication by φ^α. We check that Q_α^W satisfies (3.16) and (3.15). By direct calculation, (3.15) follows from the definition of Q_i^W. To show (3.16), recall that $V^k(\mathfrak{g}_0)$ has a Virasoro element $L=\frac{1}{2k}\sum_{\alpha\in I\sqcup\Delta_0}e_\alpha e_{\bar\alpha}$, i.e., it satisfies $\{L_\lambda L\}=(\partial+\lambda)L+c_k\lambda^3$ for some $c_k\in\mathbb{C}$ and $\partial=L_{(0)}$. Then we have

$$\begin{aligned}[Q_i^W,\partial]&=[\{H_i^W{}_\lambda\,\cdot\,\}|_{\lambda=0},\{L_\mu\,\cdot\,\}|_{\mu=0}]\\ &=\{\{H_i^W{}_\lambda L\}_{\lambda+\mu}\,\cdot\,\}|_{\lambda=\mu=0}=-\{\partial H_i^W{}_\lambda\,\cdot\,\}|_{\lambda=0}=0.\end{aligned}$$

where H_i^W is the Hamiltonian of Q_i^W. Here, we have used (2.6) in the second, (2.5) in the third, and (2.4) in the last equality. Therefore,

$$\sum_{\alpha\in[\alpha_i]}\big(\varphi^\alpha[Q_\alpha^W,\partial]-(\partial\varphi^\alpha)Q_\alpha^W\big)=[Q_i^W,\partial]=0.$$

and, thus, (3.16) by (3.13). ∎

4. Geometric Realization of $\mathcal{W}^k(\mathfrak{g},f)$

4.1. *Double coset space.* Consider the Lie algebra $\mathfrak{g}[t^{\pm1}]=\mathfrak{g}\otimes\mathbb{C}[t^{\pm1}]$. By abuse of notation, we denote by $\kappa=(\,\cdot\,|\,\cdot\,)$ the invariant bilinear form on $\mathfrak{g}[t^{\pm1}]$ given by $(at^n|bt^m)=(a|b)\delta_{n+m,0}$, which extends the one on $\mathfrak{g}$. We extend the grading Γ on $\mathfrak{g}$ to $\mathfrak{g}[t^{\pm1}]$ by setting $\deg(at^n)=\deg(a)+(d+1)n$ (see Section 3.1) and fix a homogeneous basis e_α, $(\alpha\in I\sqcup\hat\Delta)$, extending the

basis e_α, ($\alpha \in I \sqcup \Delta$), of $\mathfrak{g}$. We denote $|\alpha| = \deg(e_\alpha)$ for simplicity and $c^\gamma_{\alpha,\beta}$ the structure constants, i.e., $[e_\alpha, e_\beta] = \sum_\gamma c^\gamma_{\alpha,\beta} e_\gamma$. Let $e_{\bar{\alpha}}$, ($\alpha \in I \sqcup \hat{\Delta}$), denote the dual basis of e_α, ($\alpha \in I \sqcup \hat{\Delta}$), with respect to κ.

Consider the completion $L\mathfrak{g} = \varprojlim_{n>0}(\mathfrak{g}[t^{\pm 1}]/\mathfrak{g}[t^{\pm 1}]_{>n})$, which we call the *loop algebra* of $\mathfrak{g}$. The grading on $\mathfrak{g}[t^{\pm 1}]$ induces a decomposition $L\mathfrak{g} = L\mathfrak{g}_+ \oplus L\mathfrak{g}_-$ where $L\mathfrak{g}_+ = \varprojlim_{n>0}(\mathfrak{g}[t^{\pm 1}]_{>0}/\mathfrak{g}[t^{\pm 1}]_{>n})$ and $L\mathfrak{g}_- = \mathfrak{g}[t^{\pm 1}]_{\leq 0}$. We decompose $X \in L\mathfrak{g}$ as $X = X_+ + X_- \in L\mathfrak{g}_+ \oplus L\mathfrak{g}_-$. The subspace $L\mathfrak{g}_+$ is an affine scheme of infinite type. By setting z_α the coordinates of the basis e_α, ($\alpha \in \hat{\Delta}_{>0} \subset \hat{\Delta}$), of $L\mathfrak{g}_+$, we have $\mathbb{C}[L\mathfrak{g}_+] = \mathbb{C}[z_\alpha \mid \alpha \in \hat{\Delta}_{>0}]$.

Let G be the connected simply-connected algebraic group whose Lie algebra is $\mathfrak{g}$ and LG the loop group of G, whose Lie algebra is $L\mathfrak{g}$. We have closed subgroups $LG_\pm$ of LG whose Lie algebra is $L\mathfrak{g}_\pm$.

Let us consider the quotient space $LG_-\backslash LG$, which is an ind-scheme. It has an open subscheme LG_+. The $\mathbb{C}$-points of LG_+ is identified with the LG_+-orbits of the image of the identity in $LG_-\backslash LG$. Since LG_+ is a prounipotent proalgebraic group, the exponential map $\exp : L\mathfrak{g}_+ \to LG_+$ is an isomorphism of schemes. Thus the coordinate ring $\mathbb{C}[LG_+]$ is identified with $\mathbb{C}[L\mathfrak{g}_+] = \mathbb{C}[z_\alpha \mid \alpha \in \hat{\Delta}_{>0}]$.

The left multiplication $LG_+ \times LG_+ \to LG_+$, $((g_1, g_2) \mapsto g_1^{-1} g_2)$ induces a $L\mathfrak{g}_+$-action on $\mathbb{C}[LG_+]$ as derivations

$$\xi^L : L\mathfrak{g}_+ \to \mathrm{Der}(\mathbb{C}[LG_+]), \quad \phi \mapsto \phi^L. \tag{4.18}$$

Since LG_+ is embedded into $LG_-\backslash LG$ as an open subscheme, the right multiplication $LG_-\backslash LG \times LG_- \to LG_-\backslash LG$, $(([g_1], g_2) \mapsto [g_1 g_2])$, induces a $L\mathfrak{g}_-$-action on $\mathbb{C}[LG_+]$ as derivations

$$\xi^R : L\mathfrak{g}_- \to \mathrm{Der}(\mathbb{C}[LG_+]), \quad \phi \mapsto \phi^R. \tag{4.19}$$

The commutator of these actions ξ^L, ξ^R can be expressed by using the distinguished element $K = \exp(\sum_{\alpha \in \hat{\Delta}_{>0}} z_\alpha e_\alpha) \in LG_+(\mathbb{C}[LG_+])$. Note that this element is coordinate independent since it is identified with the identity morphism under the correspondence

$$LG_+(\mathbb{C}[LG_+]) = \mathrm{Hom}(\mathbb{C}[LG_+], \mathbb{C}[LG_+]).$$

LEMMA 4.1. *For $u \in L\mathfrak{g}_+$ and $v \in L\mathfrak{g}$, $[u^L, v^R] = [u, (KvK^{-1})_-]^L_+$ holds.*

PROOF. We use the notation $\mathrm{e}^X = \exp(X)$ for brevity. Let ϵ_i be dual numbers which satisfy $\epsilon_i^2 = 0$. For $F \in \mathbb{C}[LG_+]$ and $g \in LG_+$, we have

$$\begin{aligned}(v^R F)(g) &= \text{the } \epsilon_2\text{-linear term of } F(g\,\mathrm{e}^{\epsilon_2 v}) \\ &= \text{the } \epsilon_2\text{-linear term of } F((\mathrm{e}^{\epsilon_2 (gvg^{-1})_+})g),\end{aligned}$$

and

$$\begin{aligned}(u^L v^R F)(g) &= \text{the } \epsilon_1\text{-linear term of } (v^R F)(\mathrm{e}^{-\epsilon_1 u}\, g)\\ &= \text{the } \epsilon_1\epsilon_2\text{-linear term of } F((\mathrm{e}^{\epsilon_2(\mathrm{e}^{-\epsilon_1 u}\, gvg^{-1}\, \mathrm{e}^{\epsilon_1 u})_+})\, \mathrm{e}^{-\epsilon_1 u}\, g)\\ &= \text{the } \epsilon_1\epsilon_2\text{-linear term of } F(g_1(\epsilon_1,\epsilon_2)),\end{aligned}$$

where

$$g_1(\epsilon_1,\epsilon_2) = \mathrm{e}^{-u\epsilon_1+(gvg^{-1})_+\epsilon_2-([u,gvg^{-1}]_++(gvg^{-1})_+u)\epsilon_1\epsilon_2}\, g.$$

Similarly, we obtain

$$(v^R u^L)(F)(g) = \text{the } \epsilon_1\epsilon_2\text{-linear term of } F(g_2(\epsilon_1,\epsilon_2))$$

where

$$g_2(\epsilon_1,\epsilon_2) = \mathrm{e}^{-u\epsilon_1+(gvg^{-1})_+\epsilon_2-u(gvg^{-1})_+\epsilon_1\epsilon_2}\, g.$$

Since

$$g_1(\epsilon_1,\epsilon_2) = \mathrm{e}^{-[u,(gvg^{-1})_-]_+\epsilon_1\epsilon_2}\, g_2(\epsilon_1,\epsilon_2),$$

we obtain $[u^L, v^R]F(g) = [u,(gvg^{-1})_-]^L_+ F(g)$. □

In the sequel, we assume that $(\mathfrak{g}, f)$ satisfies the condition (F) introduced in Section 1:

(F1) The grading Γ is a $\mathbb{Z}$-grading.
(F2) There exists an element $y \in \mathfrak{g}_d$ such that $s = f + yt^{-1} \in \mathfrak{g}[t^{\pm 1}]$ is semisimple.
(F3) The Lie subalgebra $\mathrm{Ker}(\mathrm{ad}_s) \subset \mathfrak{g}[t^{\pm 1}]$ is abelian and $\mathrm{Im}(\mathrm{ad}_s) \cap \mathfrak{g}[t^{\pm 1}]_0 = \mathfrak{g}_0$.

EXAMPLE 4.2. *The following pairs $(\mathfrak{g}, f)$ satisfy (F).*

1. *$(\mathfrak{g}, f)$ with f principal nilpotent element.*
2. *$(\mathfrak{g}(C_n), f_{(2^n)})$ and $(\mathfrak{g}(C_{2n}), f_{(4^n)})$, $(n \geq 2)$.*
3. *$(\mathfrak{g}(X_n), f)$ with $X = E, F, G$ listed in Table 1.*

Here we have used the classification of nilpotent elements of $\mathfrak{g}(C_n)$ by symplectic partitions and that of $\mathfrak{g}(X_n)$, $(X = E, F, G)$, by the weighted Dynkin diagram, (see [5].)

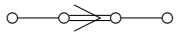
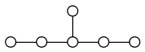
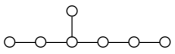
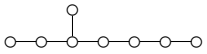
Let $\mathfrak{a}$ denote the completion of $\mathrm{Ker}(\mathrm{ad}_s)$ in $L\mathfrak{g}$. It is abelian by (F3). Let $A \subset LG$, (resp. $A_\pm \subset LG_\pm$) be the closed subgroup of LG whose Lie algebra is $\mathfrak{a}$, (resp. $\mathfrak{a}_\pm = \mathfrak{a} \cap L\mathfrak{g}_\pm$). We consider the quotient space LG_+/A_+, which is an open subset in the double coset $LG_-\backslash LG/A_+$. We have a left LG_+-action

$$LG_+ \times LG_+/A_+ \to LG_+/A_+, \quad (g, [h]) \mapsto [g^{-1}h],$$

and a right A_--action

$$LG_+/A_+ \times A_- \to LG_-\backslash LG/A_+, \quad ([h], g) \mapsto [hg].$$

TABLE 1. Weighted Dynkin diagrams in the exceptional cases

Lie type	Dynkin diagram	Weights of vertices	
G_2		2 0	2 2
F_4		0 2 0 0	0 2 0 2
		2 2 0 2	2 2 2 2
E_6		$\overset{0}{2\ 0\ 2\ 0\ 2}$	$\overset{2}{2\ 2\ 0\ 2\ 2}$
		$\overset{2}{2\ 2\ 2\ 2\ 2}$	
E_7		$\overset{2}{2\ 2\ 0\ 2\ 2\ 2}$	$\overset{2}{2\ 2\ 2\ 2\ 2\ 2}$
E_8		$\overset{0}{0\ 0\ 2\ 0\ 0\ 2\ 0}$	$\overset{0}{2\ 0\ 2\ 0\ 2\ 0\ 2}$
		$\overset{2}{2\ 2\ 0\ 2\ 0\ 2\ 2}$	$\overset{2}{2\ 2\ 2\ 2\ 2\ 2\ 2}$

The right A_--action is well-defined since A is abelian. These actions induce infinitesimal actions

$$\xi^L : L\mathfrak{g}_+ \to \operatorname{Der}\mathbb{C}[LG_+/A_+], \quad (\phi \mapsto \phi^L),$$
$$\xi^R : \mathfrak{a}_- \to \operatorname{Der}\mathbb{C}[LG_+/A_+], \quad (\phi \mapsto \phi^R),$$

respectively. In particular, $\mathbb{C}[LG_+/A_+]$ becomes a differential algebra by letting s^R be the differential. We will see that the $L\mathfrak{g}_+$-action on the differential algebra $\mathbb{C}[LG_+/A_+]$ agrees with those of the screening operators Q^W_α in Theorem 3.17 by identifying $\mathbb{C}[LG_+/A_+]$ and $V^k(\mathfrak{g}_0)$ established below, while the commutative $\mathfrak{a}_-$-action induces an integrable Hamiltonian hierarchy.

Set

$$E_\alpha = k(e_\alpha | KsK^{-1}) \in \mathbb{C}[LG_+], \quad (\alpha \in I \sqcup \Delta_0). \tag{4.20}$$

We have $E_\alpha \in \mathbb{C}[LG_+/A_+]$ since

$$\begin{aligned} a^R(e_\alpha | KsK^{-1}) &= \text{the } \epsilon\text{-linear term of } (e_\alpha | K\,\mathrm{e}^{\epsilon a}\, s(K\,\mathrm{e}^{\epsilon a})^{-1}) \\ &= (e_\alpha | K[a,s]K^{-1}) = 0, \end{aligned}$$

for $a \in \mathfrak{a}_+$. Since the Poisson vertex algebra $V^k(\mathfrak{g}_0)$ is an algebra of differential polynomials, there exists a unique homomorphism of differential algebras

$$\Psi_k : V^k(\mathfrak{g}_0) \to \mathbb{C}[LG_+/A_+], \quad e_\alpha \mapsto E_\alpha.$$

THEOREM 4.3. *Suppose that $(\mathfrak{g}, f)$ satisfies (F) and $k \neq 0$. Then Ψ_k is an isomorphism of differential algebras and satisfies*

$$\Psi_k Q_\alpha^W \Psi_k^{-1} = -\frac{1}{k} e_\alpha^L, \quad \alpha \in \Pi. \tag{4.21}$$

PROOF. Since $V^k(\mathfrak{g}_0)$ and $\mathbb{C}[LG_+/A_+]$ are polynomial rings, it suffices to show that the linear map $d_0\Psi_k : T_0^*\mathrm{Spec}V^k(\mathfrak{g}_0) \to T_{[e]}^* LG_+/A_+$ between the cotangent spaces is an isomorphism. By the identification $L\mathfrak{g}_+^* \cong L\mathfrak{g}_{<0}$ and $\mathfrak{a}_+^* \cong \mathfrak{a}_-$ induced by κ, we obtain $T_{[e]}^* LG_+/A_+ \cong (L\mathfrak{g}_+/\mathfrak{a}_+)^* \cong L\mathfrak{g}_{<0}/\mathfrak{a}_-$. This isomorphism is given by $dz_\alpha \mapsto e_{\bar{\alpha}}$. Under this isomorphism, we have $d_0\Psi_k(\partial^n e_\alpha) = k\,\mathrm{ad}_s^{n+1} e_\alpha$, as proved by inductive use of the $n = 0$ case:

$$\begin{aligned} d_0\Psi_k(e_\alpha) &= kd(e_\alpha|KsK^{-1}) = kd(e_\alpha|[\sum_{|\beta|=1} z_\beta e_\beta, s]) \\ &= \sum_{|\beta|=1} k([s, e_\alpha]|e_\beta)dz_\beta = k\sum_{|\beta|=1} ([s, e_\alpha]|e_\beta)e_{\bar{\beta}} \\ &= k[s, e_\alpha]. \end{aligned}$$

It follows from (F2) and (F3) that Ψ_k is an isomorphism.

To show (4.21), it suffices to show

$$u_\alpha^L E_\beta = (f|[e_\beta, e_\alpha]),\ [u_\alpha^L, s^R] = \frac{1}{k}\sum_{\substack{\rho\in I\sqcup\Delta_0 \\ \gamma\in[\alpha]}} c_{\alpha,\rho}^\gamma E_{\bar{\rho}} u_\gamma^L, \quad \alpha \in \Pi_1,\ \beta \in I \sqcup \Delta_0,$$

where $u_\alpha = -\frac{1}{k}e_\alpha$ by (3.16), (3.15). For the first one, we have

$$\begin{aligned} u_\alpha^L E_\beta &= \text{the } \epsilon\text{-linear term of } k(e_\beta|\,\mathrm{e}^{-\epsilon e_\alpha/k} KsK^{-1}\,\mathrm{e}^{\epsilon e_\alpha/k}) \\ &= k(e_\beta|[-\frac{1}{k}e_\alpha, KsK^{-1}]) = ([e_\beta, e_\alpha]|KsK^{-1}) \\ &= ([e_\beta, e_\alpha]|s) = ([e_\beta, e_\alpha]|f). \end{aligned}$$

The second one follows from Lemma 4.1 since

$$[u_\alpha^L, s^R] = [u_\alpha, (KsK^{-1})_0]_+^L = \frac{-1}{k^2}\sum_{\substack{\rho\in I\sqcup\Delta_0 \\ \gamma\in[\alpha]}} c_{\alpha,\rho}^\gamma E_{\bar{\rho}} e_\gamma^L = \frac{1}{k}\sum_{\substack{\rho\in I\sqcup\Delta_0 \\ \gamma\in[\alpha]}} c_{\alpha,\rho}^\gamma E_{\bar{\rho}} u_\gamma^L.$$

□

Note that the Lie subalgebra $\mathfrak{g}_+ = \oplus_{j>0}\mathfrak{g}_j$ is generated by the subspace $\oplus_{\alpha\in\Pi}\mathfrak{g}_\alpha$. Let $G_+ \subset LG_+$ denote the closed subgroup whose Lie algebra is $\mathfrak{g}_+$.

COROLLARY 4.4. *The isomorphism Ψ_k restricts to an isomorphism of differential algebras $\mathcal{W}^k(\mathfrak{g}, f) \cong \mathbb{C}[G_+\backslash LG_+/A_+]$.*

PROOF. The claim follows from Theorem 4.3 since

$$\mathcal{W}^k(\mathfrak{g}, f) \cong \bigcap_{\alpha\in\Pi_1} \mathrm{Ker}\big(Q_\alpha^W : V^k(\mathfrak{g}_0) \to V^k(\mathfrak{g}_0)\big),$$

by Theorem 3.6 and (F1). ∎

4.2. *Mutually commutative derivations on $\mathcal{W}^k(\mathfrak{g}, f)$.* By (F3), we have a Lie algebra homomorphism $\xi^R : \mathfrak{a}_- \to \mathrm{Der}(\mathbb{C}[LG_+/A_+]),\ (a \mapsto a^R)$.

Proposition 4.5. *The action ξ^R preserves the subspace*

$$\mathbb{C}[G_+\backslash LG_+/A_+] \subset \mathbb{C}[LG_+/A_+].$$

Proof. By Lemma 4.1, we have, for $e_\alpha \in \mathfrak{g}$, $(\alpha \in \Pi_1)$, and $a \in \mathfrak{a}_-$,

$$[e_\alpha^L, a^R] = [e_\alpha, (KaK^{-1})_-]_+^L = [e_\alpha, (KaK^{-1})_0]^L = \sum_{\beta \in I \sqcup \Delta_0} F_\beta [e_\alpha, e_\beta]^L,$$

for some polynomials $F_\beta \in \mathbb{C}[LG_+/A_+]$. Hence, for $G \in \mathbb{C}[G_+\backslash LG_+/A_+]$, we have

$$e_\alpha^L(a^R G) = [e_\alpha^L, a^R]G = \sum_{\beta \in I \sqcup \Delta_0} F_\beta [e_\alpha, e_\beta]^L G = 0.$$

∎

5. Integrable Hamiltonian hierarchy

In this section, we always assume that the condition (F) holds and $k \neq 0$, and identify $V^k(\mathfrak{g}_0)$ with $\mathbb{C}[LG_+/A_+]$ by Ψ_k, (Theorem 4.3).

5.1. *Hierarchy.* In the previous section, we have proven an isomorphism of differential algebras

$$\mathcal{W}^k(\mathfrak{g}, f) \cong \mathbb{C}[G_+\backslash LG_+/A_+]$$

in Corollary 4.4 and showed that it is preserved by the family of commutative derivations $\xi^R(\mathfrak{a}_-)$ by Proposition 4.5.

Definition 5.1. We denote by $\mathcal{H}^k(\mathfrak{g}, f)$ the family of commutative derivations $\xi^R(\mathfrak{a}_-)$ on $\mathcal{W}^k(\mathfrak{g}, f)$ and call it a geometric hierarchy on the classical affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$.

When f is principal, this hierarchy was constructed by Feigin - Frenkel, and indeed provides a geometric realization of the Drinfeld–Sokolov hierarchy [18, 20, 21].

We will prove that $\mathcal{H}^k(\mathfrak{g}, f)$ is indeed an integrable Hamiltonian hierarchy associated with the classical affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$.

5.2. *Construction of Hamiltonians.* Let $\Omega_{\mathrm{dR}}(LG_+) = \mathbb{C}[LG_+] \otimes \bigwedge L\mathfrak{g}_+^*$ denote the algebraic de Rham complex of LG_+. Here $L\mathfrak{g}_+^*$ is the vector space dual to the space $L\mathfrak{g}_+$ of the right invariant vector fields $L\mathfrak{g}_+^L$ on LG_+. We take a basis φ^α, $(\alpha \in \hat{\Delta}_{>0})$, of $L\mathfrak{g}_+^*$ so that $\varphi^\alpha(e_\beta) = \delta_{\alpha,\beta}$, $(\alpha, \beta \in \hat{\Delta}_{>0})$, holds. Note that the complex $\Omega_{\mathrm{dR}}(LG_+)$ coincides with the Chevalley-Eilenberg complex of the Lie algebra $L\mathfrak{g}_+$ with coefficients in $(\mathbb{C}[LG_+], \xi^L)$. Similarly, let $\Omega_{\mathrm{dR}}(A_+) = \mathbb{C}[A_+] \otimes \bigwedge \mathfrak{a}_+^*$ denote the algebraic de Rham complex of A_+. Then

the inclusion $A_+ \hookrightarrow LG_+$ induces the projection $\Omega_{\mathrm{dR}}(LG_+) \to \Omega_{\mathrm{dR}}(A_+)$. It restricts to $\mathfrak{a}_+^R$-invariant subcomplexes:

$$\pi : \mathbb{C}[LG_+/A_+] \otimes \bigwedge L\mathfrak{g}_+^* \to \mathbb{C} \otimes \bigwedge \mathfrak{a}_+^*. \tag{5.22}$$

As a $L\mathfrak{g}_+$-module, $\mathbb{C}[LG_+/A_+]$ is isomorphic to the $L\mathfrak{g}_+$-module $\mathrm{Coind}_{\mathfrak{a}_+}^{L\mathfrak{g}_+}\mathbb{C}$ coinduced from the trivial $\mathfrak{a}_+$-module $\mathbb{C}$. Then π induces an isomorphism

$$H^*(L\mathfrak{g}_+; \mathbb{C}[LG_+/A_+]) \cong H^*\big(L\mathfrak{g}_+; \mathrm{Coind}_{\mathfrak{a}_+}^{L\mathfrak{g}_+}\mathbb{C}\big) \cong H^*(\mathfrak{a}_+; \mathbb{C})$$

by Shapiro's lemma (cf. [26]). The right-hand side is isomorphic to $\bigwedge \mathfrak{a}_+^*$ since $\mathfrak{a}_+$ is abelian. The action ξ^R of $\mathfrak{a}_-$ induces $\mathfrak{a}_-$-actions on the complexes $\Omega_{\mathrm{dR}}(LG_+)$, $\Omega_{\mathrm{dR}}(A_+)$, which commute with π. Since $\mathfrak{a}$ is abelian, $\mathfrak{a}_-$ acts on $\Omega_{\mathrm{dR}}(A_+)$ trivially. Thus $\mathfrak{a}_-$ acts on the cohomology $H^*(L\mathfrak{g}_+; \mathbb{C}[LG_+/A_+])$ trivially. In particular, s acts on $H^*(L\mathfrak{g}_+; \mathbb{C}[LG_+/A_+])$ trivially. By abuse of notation, we write the above $\mathfrak{a}_-$-actions by ξ^R.

Consider the double complex $\mathscr{C}$

$$\mathbb{C} \xrightarrow{\ \iota\ } \Omega_{\mathrm{dR}}(LG_+) \xrightarrow{\ \pm s^R\ } \Omega_{\mathrm{dR}}(LG_+) \xrightarrow{\ \epsilon\ } \mathbb{C}.$$

Here ι is the unit morphism and ϵ the counit morphism. It has the following shape:

$$\begin{array}{ccccccc}
 & & \mathbb{C}_{(0,2)} & & & & \\
 & & \uparrow{\scriptstyle \epsilon} & & & & \\
0_{(-1,1)} & \longrightarrow & \mathbb{C}[LG_+/A_+]_{(0,1)} & \xrightarrow{d} & \mathbb{C}[LG_+/A_+] \otimes L\mathfrak{g}_{+\ (1,1)}^* & \xrightarrow{d} & \cdots \\
 & & \uparrow{\scriptstyle \partial = s^R} & & \uparrow{\scriptstyle -s^R} & & \\
0_{(-1,0)} & \longrightarrow & \mathbb{C}[LG_+/A_+]_{(0,0)} & \xrightarrow{d} & \mathbb{C}[LG_+/A_+] \otimes L\mathfrak{g}_{+\ (1,0)}^* & \xrightarrow{d} & \cdots, \\
 & & \uparrow{\scriptstyle \iota} & & & & \\
 & & \mathbb{C}_{(0,-1)} & & & &
\end{array}$$

where the subscript (i, j) denotes the bidegree of $\mathscr{C}$. Then the calculation of the cohomology $H^*(\mathscr{C})$ via spectral sequences gives the isomorphisms

$$\mathfrak{a}_+^* \cong H^1(\mathscr{C}) \cong \mathrm{Ker}\Big(d : \frac{\mathbb{C}[LG_+/A_+]}{\mathbb{C} \oplus \mathrm{Im}(s^R)} \to \frac{\mathbb{C}[LG_+/A_+] \otimes L\mathfrak{g}_+^*}{\mathrm{Im}(s^R)}\Big). \tag{5.23}$$

By Theorem 4.3, we have an isomorphism

$$\mathbb{C}[LG_+/A_+]/\mathbb{C} \oplus \mathrm{Im}(s^R) \cong \mathrm{Lie}(V^k(\mathfrak{g}_0))/\mathbb{C}$$

as vector spaces. Since $V^k(\mathfrak{g}_0)$ is an algebra of differential polynomials, $\int : \mathbb{C} \to \mathrm{Lie}\big(V^k(\mathfrak{g}_0)\big)$ is injective. Thus we may replace elements $\int f$ in $\mathrm{Lie}(V^k(\mathfrak{g}_0))/\mathbb{C}$ so that it is without the constant term $f(0) = 0$ if necessary. Identifying $\mathfrak{a}_-$ with $\mathfrak{a}_+^*$ by κ, we obtain an isomorphism

$$\mathfrak{a}_- \cong \mathfrak{a}_+^* \cong \left\{ \int f \in \mathrm{Lie}(V^k(\mathfrak{g}_0)) | f(0) = 0,\ d\int f = 0 \right\}, \quad a \mapsto \int H(a).$$

The local functionals $\int H(a)$ will provide the Hamiltonians for the geometric hierarchy $\mathcal{H}^k(\mathfrak{g}, f)$. Let us denote the space of Hamiltonians as

$$\int H(\mathfrak{a}_-) := \left\{ \int H(a) \mid a \in \mathfrak{a}_- \right\} \subset \mathrm{Lie}(V^k(\mathfrak{g}_0)).$$

Thanks to Lemma 3.5, the following proposition makes sense.

PROPOSITION 5.2. *The image* $\int H(\mathfrak{a}_-)$ *lies in* $\mathrm{Lie}(\mathcal{W}^k(\mathfrak{g}, f))$.

PROOF. We may assume that the basis $\{e_\alpha\}_{\alpha \in \hat{\Delta}_{>0}}$ of $L\mathfrak{g}_+$ respects the decomposition $L\mathfrak{g}_+ = \mathrm{Im}(\mathrm{ad}_s) \oplus \mathrm{Ker}(\mathrm{ad}_s)$. For $a \in \mathfrak{a}_-$, the element $\int H(a) \in \mathrm{Lie}(V^k(\mathfrak{g}_0))$ satisfies, by construction, $dH(a) = s^R A$ for some element $A = \sum_{\alpha \in \hat{\Delta}_{>0}} F_\alpha \otimes \varphi^\alpha \in \mathbb{C}[LG_+/A_+] \otimes L\mathfrak{g}_+^*$. Then,

$$\begin{aligned} dH(a) = s^R A &= s^R \sum_{\alpha \in \hat{\Delta}_{>0}} F_\alpha \otimes \varphi^\alpha \\ &= \sum_{\alpha \in \hat{\Delta}_{>0}} \partial(F_\alpha) \otimes \varphi^\alpha + \sum_{\alpha \in \hat{\Delta}_{>0}} F_\alpha \otimes s^R(\varphi^\alpha) \\ &= \sum_{\alpha \in \Pi_1} \left(\partial F_\alpha + \sum_{\substack{\beta \in \Pi_1 \\ \gamma \in \sqcup \Delta_0}} F_\beta c_{\alpha,\gamma}^\beta E_{\bar{\gamma}} \right) \otimes \varphi^\alpha + \sum_{\alpha \in \hat{\Delta}_{>0} \setminus \Pi_1} \widetilde{F_\alpha^n} \otimes \varphi^\alpha, \end{aligned}$$

for some element $\widetilde{F_\alpha^n} \in \mathbb{C}[LG_+/A_+]$. In the last equality, we have used Lemma 5.3 below. By the identification $\mathbb{C}[LG_+/A_+] \cong V^k(\mathfrak{g}_0)$ and $\mathbb{C}[LG_+/A_+]\varphi^\alpha \cong \mathbb{L}_0^k(-\alpha)$, $(\alpha \in \Pi_1)$, we obtain

$$\sum_{\alpha \in \Pi_1} \left(\partial F_\alpha + \sum_{\substack{\beta \in \Pi_1 \\ \gamma \in I \sqcup \Delta_0}} F_\beta c_{\alpha,\gamma}^\beta E_{\bar{\gamma}} \right) \otimes \varphi^\alpha = \partial \left(\sum_{\alpha \in \Pi_1} F_\alpha \otimes \varphi^\alpha \right) = 0.$$

On the other hand, by the definition of d, we have $dH(a) = \sum_{\alpha \in \hat{\Delta}_{>0}} e_\alpha^L H(a) \otimes \varphi^\alpha$. It follows that $e_\alpha^L \int H(a) = 0$ for all $\alpha \in \Pi_1$, which implies $\int H(a) \in \mathrm{Lie}(\mathcal{W}^k(\mathfrak{g}, f))$ by Theorem 4.3. ∎

LEMMA 5.3. *The following formula holds*

$$s^R \varphi^\alpha = \frac{1}{k} \sum_{\substack{\beta \in \hat{\Delta}_{>0} \\ \gamma \in I \sqcup \Delta_0}} E_{\bar{\gamma}} c_{\beta,\gamma}^\alpha \varphi^\beta + \sum_{\beta \in \hat{\Delta}_{>0}} c_{\beta,s}^\alpha \varphi^\beta, \quad \alpha \in \hat{\Delta}_{>0}. \tag{5.24}$$

PROOF. By Lemma 4.1, we have

$$(5.25)\qquad [s^R, e_\alpha^L] = -[e_\alpha, (KsK^{-1})_-]_+^L = -\frac{1}{k}\sum_{\beta\in I\sqcup\Delta_0} E_{\bar\beta}[e_\alpha, e_\beta]^L - [e_\alpha, s]_+^L.$$

Then, from the definition of $L\mathfrak{g}_+^*$, we have

$$\left(s^R\varphi^\alpha\right)(e_\beta) = -\varphi^\alpha(\partial e_\beta) = \frac{1}{k}\sum_{\gamma\in I\sqcup\Delta_0} E_{\bar\gamma}c_{\beta,\gamma}^\alpha + c_{\beta,s}^\alpha.$$

The claim (5.24) follows from it immediately. ∎

5.3. *Poisson vertex superalgebra structure on* $\mathbb{C}[LG_+/A_+]\otimes\bigwedge L\mathfrak{g}_+^*$. We extend the Poisson vertex algebra structure on $\mathbb{C}[LG_+/A_+]$ given by Theorem 4.3, to the whole differential superalgebra $(\mathbb{C}[LG_+/A_+]\otimes\bigwedge L\mathfrak{g}_+^*, s^R)$.

PROPOSITION 5.4. *The differential superalgebra* $(\mathbb{C}[LG_+/A_+]\otimes\bigwedge L\mathfrak{g}_+^*, s^R)$ *admits a unique Poisson vertex superalgebra structure, which satisfies*

1. $\{u_\lambda v\} = [u,v] + k(u|v)\lambda, \quad (u,v\in\mathfrak{g}_0)$,
2. $\{\varphi^\alpha{}_\lambda\varphi^\beta\} = 0, \quad (\varphi^\alpha,\varphi^\beta\in L\mathfrak{g}_1^*\cup\mathfrak{a}_+^*)$,
3. $\{u_\lambda\varphi^\alpha\} = -\sum c_{u,\beta}^\alpha\varphi^\beta, \quad (\varphi^\alpha\in L\mathfrak{g}_1^*\cup\mathfrak{a}_+^*, u\in\mathfrak{g}_0)$.

It satisfies

$$(5.26)\qquad \{u_\lambda\varphi^\alpha\} = -\sum c_{u,\beta}^\alpha\varphi^\beta, \quad (\varphi^\alpha\in L\mathfrak{g}^*, u\in\mathfrak{g}_0).$$

PROOF. It follows from Lemma 5.3 that $\mathbb{C}[LG_+/A_+]\otimes\bigwedge L\mathfrak{g}_+^*$ is an algebra of differential polynomials in the variables given by the union of bases of $\mathfrak{g}_0$ and $L\mathfrak{g}_1^*\cup\mathfrak{a}_+^*$. By Theorem 2.2, it suffices to show (2.5), (2.6) for these variables in order to prove that (1)-(3) define a Poisson vertex superalgebra on $\mathbb{C}[LG_+/A_+]\otimes\bigwedge L\mathfrak{g}_+^*$. The only non-trivial ones are (2.6) for $\{e_\alpha, e_\beta, \varphi^\gamma\}$. By (2.5), they reduce to the special case

$$(5.27)\qquad \{e_{\alpha\lambda}\{e_{\beta\mu}\varphi^\gamma\}\} - \{e_{\beta\mu}\{e_{\alpha\lambda}\varphi^\gamma\}\} = \{\{e_{\alpha\lambda}e_\beta\}_{\lambda+\mu}\varphi^\gamma\}.$$

Since it coincides with the Jacobi identity

$$(\mathrm{ad}^*(e_\alpha)\,\mathrm{ad}^*(e_\beta) - \mathrm{ad}^*(e_\beta)\,\mathrm{ad}^*(e_\alpha))\varphi^\gamma = \mathrm{ad}^*([e_\alpha,e_\beta])\varphi^\gamma$$

of the coadjoint $\mathfrak{g}_0$-action on $L\mathfrak{g}_+^*$, (5.27) holds.

Next, we show (5.26). Since ad_s is an isomorphism on $\mathrm{Im}(\mathrm{ad}_s)\subset L\mathfrak{g}$, we have its inverse ad_s^{-1} on $\mathrm{Im}(\mathrm{ad}_s)$ and on its dual space $\mathrm{Im}(\mathrm{ad}_s)^*$. Then we have a decomposition $L\mathfrak{g}_{>0} = \mathrm{ad}_s^{-1}\,\mathrm{Im}(\mathrm{ad}_s)_{>0}\oplus(L\mathfrak{g}_1\cup\mathfrak{a}_+)$. We show (5.26) for $\varphi^\alpha\in\mathrm{Im}(\mathrm{ad}_s)$ by induction on degree. From Lemma 5.3, we have

$$\mathrm{ad}_s^{-1}\,\varphi^a = \partial\varphi^a - \frac{1}{k}\sum_{\alpha,b} c_{b,\alpha}^a e_{\bar\alpha}\varphi^b, \quad \varphi^\alpha\in\mathrm{Im}(\mathrm{ad}_s)_1.$$

Here the sum with respect to Greek letters (resp. Roman letters) is taken over $I \sqcup \Delta_0$ (resp. $\hat{\Delta}_{>0}$). We use the same rule below. Then, for $e_\alpha \in \mathfrak{g}_0$ and $\varphi^a \in \mathrm{Im}(\mathrm{ad}_s)_1$, we have

$$\{e_{\alpha\lambda}(\mathrm{ad}_s^{-1})^n\varphi^a\} = -\sum_{|b|=|a|} c^a_{\alpha,b}(\mathrm{ad}_s^{-1})^n\varphi^b,$$

which is proved by the inductive use of the $n=1$ case:

$$\begin{aligned}
&\{e_{\alpha\lambda}\,\mathrm{ad}_s^{-1}\,\varphi^a\} = \{e_{\alpha\lambda}\partial\varphi^a\} - \frac{1}{k}\sum_{\beta,b} c^a_{b,\beta}\{e_{\alpha\lambda}e_{\bar\beta}\varphi^b\}\\
&= (\partial+\lambda)\{e_{\alpha\lambda}\partial\varphi^a\} - \frac{1}{k}\sum_{\beta,b} c^a_{b,\beta}(\{e_{\alpha\lambda}e_{\bar\beta}\}\varphi^b + \{e_{\alpha\lambda}\varphi^b\}e_{\bar\beta})\\
&= -(\partial+\lambda)\sum_b c^a_{\alpha b}\varphi^b - \frac{1}{k}\sum_{\alpha,b} c^a_{b,\alpha}\left(([e_\alpha,e_{\bar\beta}] + k(e_\alpha|e_{\bar\beta})\lambda)\varphi^b - \sum_c c^b_{\alpha,c}e_{\bar\beta}\varphi^c\right)\\
&= -\lambda\left(\sum_b c^a_{\alpha,b}\varphi^b + \sum_{\beta,b} c^a_{b,\beta}(e_\alpha|e_{\bar\beta})\varphi^b\right)\\
&\qquad - \left(\sum_b c^a_{\alpha,b}\partial\varphi^b + \frac{1}{k}\sum_{\beta,b} c^a_{b,\beta}\big([e_\alpha,e_{\bar\beta}]\varphi^b - \sum_c c^b_{\alpha,c}e_{\bar\beta}\varphi^c\big)\right)\\
&= -\sum_b c^a_{\alpha,b}\,\mathrm{ad}_s^{-1}\,\varphi^b.
\end{aligned}$$

In the last equality, we used $(e_\alpha|e_{\bar\beta}) = \delta_{\alpha,\beta}$ for the λ-linear term and use Lemma 5.3, $c^c_{a,b} = c^{\bar b}_{\bar c,a}$, and the Jacobi identity of the Lie bracket for the constant term. This implies (5.26) for $\varphi^\alpha \in \mathrm{Im}(\mathrm{ad}_s)^*$. Then (5.26) follows from this and (3). ∎

The de Rham differential $d = \sum_{\alpha\in\hat{\Delta}_{>0}} e^L_\alpha \otimes \varphi^\alpha$ on $\mathbb{C}[LG_+/A_+]$ can be described in terms of the Poisson vertex superalgebra structure in Proposition 5.4.

Lemma 5.5. *The differential d is determined uniquely by*

1. $dE_\beta = -k\sum_{\alpha\in\hat{\Delta}_{>0}}([e_\beta,s]|e_\alpha)\otimes\varphi^\alpha, \quad (\beta \in I\sqcup\Delta_0)$,
2. $[d,\partial] = 0$.

Proof. (1) follows immediately from the formula $e^L_{\bar\beta}E_\alpha = -k([e_\alpha,s]|e_\beta)$ (see also (4.20)). (2) follows from (5.25) and Lemma 5.3. ∎

Let $s^* \in L\mathfrak{g}^*_+$ denote the element corresponding to $s \in L\mathfrak{g}_-$ by the identification $\kappa = (\,\cdot\,|\,\cdot\,) : L\mathfrak{g}_- \cong L\mathfrak{g}^*_+$ and $\bar s \in L\mathfrak{g}_+$ the element corresponding to s^*.

PROPOSITION 5.6. *Under the isomorphism* $\Psi_k : V^k(\mathfrak{g}_0) \cong \mathbb{C}[LG_+/A_+]$,

$$d = -k\{s^*{}_\lambda -\}|_{\lambda=0},$$

holds on $\mathbb{C}[LG_+/A_+]$.

PROOF. It suffices to show that $-k\{s^*{}_\lambda \cdot\}|_{\lambda=0}$ satisfies (1) and(2) in Lemma 5.5: (1) follows from the definition of s^* and Proposition 5.4 (3). Then, (2) follows from (2.4). □

The following property of the derivation $\eta_a = \eta(\int H(a)) \in \mathrm{Der}(\mathbb{C}[LG_+/A_+])$, $(a \in \mathfrak{a}_-)$, will be used.

LEMMA 5.7. *For* $a \in \mathfrak{a}_-$, *there exist some polynomials* $F_\alpha(a) \in \mathbb{C}[LG_+/A_+]$, $(\alpha \in I \sqcup \Delta_0)$, *which satisfies*

$$[u^L, \eta_a] = \sum_{\alpha \in I \sqcup \Delta_0} F_\alpha(a)[u, e_\alpha]^L, \quad u \in L\mathfrak{g}_1.$$

PROOF. By construction of $\int H(a)$, we have $d\int H(a) = \int dH(a) = 0 \in \mathrm{Lie}(\mathbb{C}[LG_+/A_+] \otimes \bigwedge L\mathfrak{g}_+^*)$ and thus $\{dH(a)_\lambda \cdot\}|_{\lambda=0} = 0$. By (2.6), we have

$$\begin{aligned}\{dH(a)_\lambda \cdot\}|_{\lambda=0} &= \Big[\{s^*{}_\lambda \cdot\}|_{\lambda=0}, \{H(a)_\mu \cdot\}|_{\mu=0}\Big] = \Big[\sum_{\alpha \in \hat{\Delta}_{>0}} e_\alpha^L \otimes \varphi^\alpha, \eta_a\Big] \\ &= \sum_{\alpha \in \hat{\Delta}_{>0}} \Big([e_\alpha^L, \eta_a] \otimes \varphi^\alpha - e_\alpha^L \otimes \eta_a(\varphi^\alpha)\Big).\end{aligned}$$

It follows from them that

$$[e_\alpha^L, \eta_a] = \sum_{\beta \in \hat{\Delta}_{>0}} \langle e_\alpha \mid \eta_a(\varphi^\beta)\rangle e_\beta^L,$$

where $\langle \cdot \mid \cdot \rangle : L\mathfrak{g}_+ \times L\mathfrak{g}_+^* \to \mathbb{C}$ denotes the canonical pairing. By Proposition 5.4, we have

$$\eta_a(\varphi^\beta) = \{H(a)_\lambda \varphi^\beta\}|_{\lambda=0} = \sum_{\gamma \in I \sqcup \Delta_0} \{e_\gamma \partial \varphi^\beta\}_\to \frac{\delta H(a)}{\delta e_\gamma} = -\sum_{\substack{\gamma \in I \sqcup \Delta_0 \\ \rho \in \hat{\Delta}_{>0}}} c_{\gamma,\rho}^\beta \varphi^\rho \frac{\delta H(a)}{\delta e_\gamma}.$$

Hence, for $\alpha \in L\mathfrak{g}_1$ we obtain

$$\begin{aligned}[e_\alpha^L, \eta_a] &= \sum_{\beta \in \hat{\Delta}_{>0}} \langle e_\alpha \mid \eta_a(\varphi^\beta)\rangle e_\beta^L = -\sum_{\beta \in \hat{\Delta}_{>0}} \sum_{\substack{\gamma \in I \sqcup \Delta_0 \\ \rho \in \hat{\Delta}_{>0}}} c_{\gamma,\rho}^\beta \frac{\delta H(a)}{\delta e_\gamma} \langle e_\alpha \mid \varphi^\rho\rangle e_\beta^L \\ &= \sum_{\gamma \in I \sqcup \Delta_0} \frac{\delta H(a)}{\delta e_\gamma} [e_\alpha, e_\gamma]^L.\end{aligned}$$

This completes the proof. □

5.4. *Integrable Hamiltonian hierarchy.* Let $\mathrm{Vect}(LG_+)$ denote the Lie algebra of the algebraic vector fields on LG_+. Recall the infinitesimal $L\mathfrak{g}_+$-actions (4.18), (4.19).

LEMMA 5.8. *We have* $\mathrm{Vect}(LG_+)^{L\mathfrak{g}_+^L} = L\mathfrak{g}_+^R$ *and* $\mathrm{Vect}(LG_+)^{L\mathfrak{g}_+^R} = L\mathfrak{g}_+^L$.

PROOF. Although this is well-known, we give a proof for the completeness of the paper. The pairing

$$\mathcal{U}(L\mathfrak{g}_+) \times \mathbb{C}[LG_+] \to \mathbb{C}, \quad (X_1X_2\cdots X_n, F) \mapsto (X_1^L X_2^L \cdots X_n^L F)(e),$$

defined by is $L\mathfrak{g}_+$-invariant and nondegenerate. Here $\mathcal{U}(L\mathfrak{g}_+)$ denotes the universal enveloping algebra of $L\mathfrak{g}_+$. Hence, we have

$$\mathbb{C}[LG_+]^{L\mathfrak{g}_+^L} \cong \mathcal{U}(L\mathfrak{g}_+)/L\mathfrak{g}_+\mathcal{U}(L\mathfrak{g}_+) \cong \mathbb{C}.$$

Since $L\mathfrak{g}_+^L$ commutes with $L\mathfrak{g}_+^R$ and $\mathrm{Vect}(LG_+) \cong \mathbb{C}[LG_+] \otimes_{\mathbb{C}} L\mathfrak{g}_+^R$, we obtain

$$\mathrm{Vect}(LG_+)^{L\mathfrak{g}_+^L} \cong (\mathbb{C}[LG_+] \otimes_{\mathbb{C}} L\mathfrak{g}_+^R)^{L\mathfrak{g}_+^L} = \mathbb{C}[LG_+]^{L\mathfrak{g}_+^L} \otimes_{\mathbb{C}} L\mathfrak{g}_+^R \cong L\mathfrak{g}_+^R.$$

The latter claim is proved similarly. □

We say that an element $X \in \mathrm{Vect}(LG_+)$ satisfies (P) if

$$[u^L, X] \in \sum_{\alpha \in I \sqcup \Delta_0} \mathbb{C}[LG_+][u, e_\alpha]^L, \quad u \in L\mathfrak{g}_1.$$

Let $\mathcal{L}$ denote the set of elements in $\mathrm{Vect}(LG_+)$ satisfying (P). It is straightforward to show that $\mathcal{L} \subset \mathrm{Vect}(LG_+)$ forms a Lie subalgebra. The Witt algebra $\mathrm{Witt} = \mathbb{C}((t))\frac{\partial}{\partial t}$ acts on $L\mathfrak{g}$ by derivations with respect to t. Since the subalgebra $\mathrm{Witt}_- = \mathbb{C}[t^{-1}]t\frac{\partial}{\partial t}$ preserves $L\mathfrak{g}_{\leq 0}$, it acts on $B\backslash G((t))$ infinitesimally and therefore on the open dense subset LG_+. The induced Witt_--action on $\mathbb{C}[LG_+]$ is given by

$$L_n \exp\left(\sum_{\alpha \in \hat{\Delta}_{>0}} z_\alpha e_\alpha\right) = \text{the } \epsilon\text{-linear term of } \exp\left(\sum_{\alpha \in \hat{\Delta}_{>0}} z_\alpha e_\alpha\right) e^{\epsilon L_n},$$

where $L_n = -t^{n+1}\frac{\partial}{\partial t} \in \mathrm{Witt}_-$.

LEMMA 5.9. *We have* $\mathcal{L} = \mathrm{Witt}_- \ltimes L\mathfrak{g}^R$.

PROOF. By Lemma 4.1, we have $L\mathfrak{g}^R \subset \mathcal{L}$. In particular, this implies that $\mathcal{L}$ is a $L\mathfrak{g}$-module. The inclusion $\mathrm{Witt}_- \subset \mathcal{L}$ is shown in the same way as in the proof of Lemma 4.1. It is obvious that these two actions give an action of their semidirect product $\mathrm{Witt}_- \ltimes L\mathfrak{g}^R \subset \mathcal{L}$.

To show its equality, consider the quotient $L\mathfrak{g}$-module $M := \mathcal{L}/\mathrm{Witt}_- \ltimes L\mathfrak{g}^R$. Let us show that M belongs to the category $\mathcal{O}$ of the affine Lie algebra $\widetilde{\mathfrak{g}} = \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}C \oplus \mathbb{C}L_0$ at level 0. Let $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}_-$ be the standard triangular decomposition of $\widetilde{\mathfrak{g}}$. Indeed, the action of the Cartan subalgebra $\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}C \oplus L_0$ on $\mathrm{Vect}(LG_+)$ is diagonalizable and, moreover, preserves $\mathcal{L}$

and $\mathrm{Witt}_- \ltimes L\mathfrak{g}^R$. Hence, it is diagonalizable on $M = \mathcal{L}/\mathrm{Witt}_- \ltimes L\mathfrak{g}^R$. Let us show that the dimension $\dim \mathcal{U}(\widetilde{\mathfrak{n}}_+)w$ is finite for any $w \in M$. Let $\tilde{w} \in \mathcal{L}$ denote a lift of w. The decomposition $\widetilde{\mathfrak{n}}_+ = \mathfrak{g}_0^+ \ltimes \widetilde{\mathfrak{n}}_{>0}$ where $\mathfrak{g}_0^+ = \bigoplus_{\alpha\in\Delta_{0+}} \mathfrak{g}_{0\alpha}$ and $\widetilde{\mathfrak{n}}_{>0} = \widetilde{\mathfrak{n}}_+ \cap L\mathfrak{g}_{>0}$ induces a decomposition of the universal enveloping algebra $\mathcal{U}(\widetilde{\mathfrak{n}}_+) \cong \mathcal{U}(\mathfrak{g}_0^+) \otimes \mathcal{U}(\widetilde{\mathfrak{n}}_{>0})$ as vector spaces. By definition of $\mathcal{L}$, $\tilde{w}$ satisfies, for any $u \in L\mathfrak{g}_1$,

$$[u^L, w] = \sum_{\alpha\in I\sqcup\Delta_0} F_\alpha(w)[e_\alpha, u]^L \text{ for some element } F_\alpha(w) \in \mathbb{C}[LG_+].$$

Since $\widetilde{\mathfrak{n}}_{>0}^L$ and $\widetilde{\mathfrak{n}}_{>0}^R$ commute, for any $Z \in \mathcal{U}(\widetilde{\mathfrak{n}}_{>0})$,

$$[u^L, Z^R(w)] = Z^R([u^L, w]) = \sum_{\alpha\in I\sqcup\Delta_0} Z^R\big(F_\alpha(w)\big)[e_\alpha, u]^L.$$

If $Z^R(F_\alpha(w)) = 0$ for all $\alpha \in I \sqcup \Delta_0$ and $u \in L\mathfrak{g}_1$, then by Lemma 5.8, we have $[Z^R, w] \in \widetilde{\mathfrak{n}}_{>0}^R$, which is zero in M. Considering the $\widetilde{\mathfrak{h}}$-weights of $Z^R(F_\alpha(w))$, we conclude that $\dim \mathcal{U}(\widetilde{\mathfrak{n}}_{>0})^R(F_\alpha(w)) < \infty$ for each α and hence $\dim \mathcal{U}(\widetilde{\mathfrak{n}}_{>0})^R w < \infty$. Thus it remains to show $\dim \mathcal{U}(\mathfrak{g}_0^+)w < \infty$. Take $u \in \widetilde{\mathfrak{g}}_1$ and $v \in \mathfrak{g}_0^+$, we have $[u^L, v^R] = [u, v]^L$ by Lemma 4.1. Then by setting $[u^L, w] = \sum_{\alpha\in I\sqcup\Delta_0} G_\alpha(w)[u, e_\alpha]^L$ for some element $G_\alpha \in \mathbb{C}[LG_+]$, we have

$$\begin{aligned}[u^L, [v^R, w]] &= [[u, v]^L, w] + [v^R, [u^L, w]] \\ &= \sum_{\alpha\in I\sqcup\Delta_0} G_\alpha(w)[[u, v], e_\alpha]^L + \sum_{\alpha\in I\sqcup\Delta_0} v^R(G_\alpha(w))[u, e_\alpha]^L.\end{aligned}$$

By induction, we obtain

$$[u^L, [v_1^R, [v_2^R, \cdots, [v_n^R, w]\cdots] = \sum_{\substack{\alpha\in I\sqcup\Delta_0 \\ A\sqcup B=\{1,2,\cdots,n\}}} (v_A^R G_\alpha(w))[[u, v_B], e_\alpha]^L$$

for $v_1, v_2, \cdots, v_n \in \mathfrak{g}_0^+$, where $v_A^R = v_{a_1}^R v_{a_2}^R \cdots v_{a_{\max}}^R$ for $A = \{a_1, a_2, \cdots, a_{\max}\}$, $(a_1 < a_2 < \cdots < a_{\max})$, and $[u, v_B] = [\cdots [u, v_{b_1}], v_{b_2}], \cdots], v_{b_{\max}}]$ for $B = \{b_1, b_2, \cdots, b_{\max}\}$, $(b_1 < b_2 < \cdots < b_{\max})$. Since

$$\mathrm{ad}_{\mathfrak{g}_0^+}^N L\mathfrak{g}_1 = 0, \quad ((\mathfrak{g}_0^+)^R)^N G_\alpha(w) = 0$$

for N large enough, it follows that $[u^L, [v_1^R, [v_2^R, \cdots, [v_n^R, w]\cdots] = 0$ for n sufficiently large and hence $[v_1^R, [v_2^R, \cdots, [v_n^R, w], \cdots] \in \widetilde{\mathfrak{n}}_{>0}^R$, which is zero in M. Therefore, we obtain $\dim \mathcal{U}(\mathfrak{g}_0^+)^R w < \infty$. Thus, we conclude that M belongs to the category $\mathcal{O}$.

Suppose $M \neq 0$. Then it contains a nonzero highest weight vector w. Let $\tilde{w} \in \mathcal{L}$ denote a lift of w. Then $\mathrm{Witt}_- \ltimes L\mathfrak{g}^R + \mathbb{C}\tilde{w}$ is an extension of $\widetilde{\mathfrak{n}}$-module $\mathrm{Witt}_- \ltimes L\mathfrak{g}^R$ by a trivial 1-dimensional $\widetilde{\mathfrak{n}}$-module $\mathbb{C}w$, which is nontrivial by construction. Thus, this extension defines a nonzero element in $H^1(\widetilde{\mathfrak{n}}; \mathrm{Witt}_- \ltimes L\mathfrak{g}^R)$. Since the action $L\mathfrak{g}^R$ on $\mathbb{C}[LG_+]$ is faithful, $L\mathfrak{g}^R \cong L\mathfrak{g}$

as $\widetilde{\mathfrak{n}}$-modules. On the other hand, the cohomology $H^n(\widetilde{\mathfrak{n}}; \mathrm{Witt}_- \ltimes L\mathfrak{g}^R)$ is describes as

$$H^i(\widetilde{\mathfrak{n}}; \mathrm{Witt}_- \ltimes L\mathfrak{g}) \cong H^{i-1}(\widetilde{\mathfrak{n}}; \mathbb{C}) \otimes \mathrm{Witt}_{>0},$$

where $\mathrm{Witt}_{>0} = \mathbb{C}[[t]]t^2\frac{\partial}{\partial t}$ ([20]). This isomorphism is $\widetilde{\mathfrak{h}}$-equivariant. In particular, all the degrees of the homogeneous elements with respect to the action of $x + (d+1)L_0$ in the first cohomology $H^1(\widetilde{\mathfrak{n}}; \mathrm{Witt}_- \ltimes L\mathfrak{g}) \cong \mathrm{Witt}_{>0}$ are positive. (See Section 1 for the definition of x and d.) On the other hand, the cocycle corresponding to $\mathbb{C}w$ is negative due to the definition of $\mathcal{L}$, which is a contradiction. Therefore, we conclude $M = 0$. This completes the proof □

THEOREM 5.10. *Suppose that $(\mathfrak{g}, f)$ satisfies (F) and $k \neq 0$. Then, $\mathcal{H}^k(\mathfrak{g}, f)$ is an integrable Hamiltonian hierarchy associated with $\mathcal{W}^k(\mathfrak{g}, f)$.*

PROOF. By abuse of notation, let $\eta_a \in \mathrm{Der}(V^k(\mathfrak{g}_0))$ also denote the corresponding element in $\mathrm{Vect}(LG_+/A_+)$ by Ψ_k in Theorem 4.3. Since $LG_+ \cong (LG_+/A_+) \times A_+$ as affine schemes in an A_+^R-equivariant way, the elements of $\mathrm{Vect}(LG_+/A_+)$ naturally lift to elements of $\mathrm{Vect}(LG_+)$ which commute with $\mathfrak{a}_+^R$. Again, let η_a denote the lift of η_a in $\mathrm{Vect}(LG_+)$. Then by Lemma 5.7, the lift η_a lies in $\mathcal{L}$ and, moreover, $[u^L, \eta_a] = \sum_{\alpha I \sqcup \Delta_0} F_\alpha(\eta_a)[u, e_\alpha]^L$ with $F_\alpha(\eta_a) \in \mathbb{C}[LG_+/A_+]$ for $u \in L\mathfrak{g}_1$. Since $\bar{s} \in \mathfrak{a}_+$, we have, for $u \in L\mathfrak{g}_1$,

$$[u^L, [\bar{s}^R, \eta_a]] = [\bar{s}^R, [u^L, \eta_a]] = \Big[\bar{s}^R, \sum_\alpha F_\alpha(\eta_a)[u, e_\alpha]^L\Big] = \sum_\alpha \bar{s}^R(F_\alpha(\eta_a))[u, e_\alpha]^L$$
$$= 0.$$

Since $L\mathfrak{g}_1$ generates $L\mathfrak{g}_+$, we obtain $[\bar{s}^R, \eta_a] \in \mathrm{Vect}(LG_+)^{L\mathfrak{g}_+^L} = L\mathfrak{g}_+^R$. The degree of $[\bar{s}^R, \eta_a]$ is non-positive, and the degrees of the elements in $L\mathfrak{g}_+^R$ are positive with respect to the action of $x + (d+1)L_0$. This implies $[\bar{s}^R, \eta_a] = 0$, i.e., $\eta_a \in \mathcal{L}^{\bar{s}^R}$. On the other hand, we have $\mathcal{L}^{\bar{s}^R} = \mathrm{Witt}_-^{\bar{s}^R} \ltimes (L\mathfrak{g}^{\bar{s}})^R$ since $[\bar{s}^R, \mathrm{Witt}_-] \subset [\mathfrak{a}, \mathrm{Witt}_-]^R \subset \mathfrak{a}^R$ and $[\bar{s}^R, L\mathfrak{g}^R] \cap \mathfrak{a}^R = 0$. Since $\bar{s} \in L\mathfrak{g}$ is not an element in $\mathfrak{g}$, we have $\mathrm{Witt}_-^{\bar{s}^R} = 0$. From the definition of $\bar{s}$ (and s), we obtain $L\mathfrak{g}^{\bar{s}^R} = \mathfrak{a}^R$. Thus $\mathcal{L}^{\bar{s}^R} = \mathfrak{a}^R$ holds. Therefore, η_a lies in $\mathfrak{a}^R$, which means $\int H(\mathfrak{a}_-) \subset \mathfrak{a}^R$. Considering the degree given by the action of $x + (d+1)L_0$, we see $\int H(\mathfrak{a}_-) \subset \mathfrak{a}_-^R$. By Lemma 2.5, it is easy to show that the map $\int$ on $H(\mathfrak{a}_-)$ is injective. Since the map $a \mapsto \int H(a)$ preserves the degree with respect to the action of $x + (d+1)L_0$, we obtain $\int H(\mathfrak{a}_-) = \mathfrak{a}_-^R$. □

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