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# NOTE ON CONCENTRATION VIA THE CONJUGATE-LINEAR HODGE STAR OPERATOR

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Abstract. We construct conjugate-linear perturbations of twisted spin$^c$ Dirac operators on compact almost hermitian manifolds of dimension congruent to 2 or 6 modulo 8, employing the conjugate-linear Hodge star operator rescaled by unit complex numbers depending on degree. These perturbations satisfy the concentration principle.

## 1. Introduction

One of the most fruitful ideas in geometry is *localization*, which underlies the celebrated Witten deformation. Based on this idea, consider a first-order elliptic operator

$$D : \Gamma(E) \to \Gamma(F)$$

on a compact Riemannian manifold $X$, and a deformation of the form

$$D_s = D + sA,$$

where $s \in \mathbb{R}$ and $A : E \to F$ is a bundle map.

As shown in [4], if the perturbation $A$ satisfies the following algebraic condition:

$$\sigma_{D^*}(\gamma) \circ A + A^* \circ \sigma_D(\gamma) = 0 \quad \forall \gamma \in T^*X, \tag{1.1}$$

where $\sigma_D$ denotes the principal symbol of $D$, then as $s \to \infty$, the kernel of $D_s$ (as well as low eigenspaces of $D_s^* D_s$) becomes concentrated near the *singular set* $Z_A$ of $A$, defined by

$$Z_A := \{x \in X : \ker(A_x) \neq 0\}.$$

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The most well-known example of this *concentration principle* is the Witten deformation [12], where $D$ is the Hodge–de Rham operator. Taubes' localization proof of the Riemann–Roch theorem is another well-known and ingenious example [11].

When $D$ is a Dirac operator, Prokhorenkov and Richardson [8] formulated the concentration principle for complex-linear perturbations $A$ and classified such perturbations. Maridakis [4] later extended the principle to include real elliptic operators as described above, yielding many interesting examples.

The concentration principle has appeared in various contexts; see, for example, [9, 3, 1, 7, 5, 2]. In particular, Nagy [5] constructed conjugate-linear perturbations $A$ satisfying the concentration principle in a general setting, using nondegenerate invariant bilinear forms on spinor modules.

The present work aims to generalize the conjugate-linear perturbations introduced in [3] (see Remark 3.4) by constructing conjugate-linear perturbations $A_\varphi$ of the twisted spin$^c$ Dirac operator

$$D_E : \Gamma(\mathbb{S}^+ \otimes E) \to \Gamma(\mathbb{S}^- \otimes E) \tag{1.2}$$

on a compact almost hermitian manifold $X$ with $\dim X \equiv 2, 6 \pmod 8$. These perturbations are defined via the conjugate-linear Hodge star operator rescaled by unit complex numbers depending on degree, together with a section $\varphi$ of $K_X \otimes (E^*)^2$, and satisfy the condition (1.1) (see Theorem 3.2). The dimension restriction is due to $X$ being an almost complex manifold and the fact that the perturbations $A_\varphi$ interchange $\mathbb{S}^+ \otimes E$ and $\mathbb{S}^- \otimes E$ when $\dim X \equiv 2, 6 \pmod 8$.

The rest of the paper is organized as follows. In Section 2, we briefly review basic well-known facts about twisted spin$^c$ Dirac operators $D_E$, and present observations central to our construction. In Section 3, we construct conjugate-linear perturbations $A_\varphi$ of the operator $D_E$ satisfying (1.1), and provide examples where the singular set $Z_{A_\varphi}$ is empty; in such cases, the index of the operator $D_E$ is zero by the concentration principle (see Corollary 3.6).

## 2. Preliminaries

Throughout this paper, $X$ denotes a compact almost hermitian manifold of dimension $2n$, equipped with an almost complex structure $J$ and a Riemannian metric $g$ compatible with the almost complex structure $J$, while $E$ denotes a hermitian vector bundle over $X$.

This section briefly reviews basis well-known facts relevant to our discussion, fixes notation, and presents key observations to our construction.

2.1. *Twisted Dirac operator.* The almost complex structure $J$, extended by complex linearity to $TX \otimes \mathbb{C}$ and $T^*X \otimes \mathbb{C}$, induces the decompositions:

$$TX \otimes \mathbb{C} = (TX)^{1,0} \oplus (TX)^{0,1} \quad \text{and} \quad T^*X \otimes \mathbb{C} = (T^*X)^{1,0} \oplus (T^*X)^{0,1},$$

where $(TX)^{1,0}$ and $(TX)^{0,1}$ are the $\pm i$-eigenbundles of $J$, and similarly for $T^*X \otimes \mathbb{C}$.

Let $g_{\mathbb{C}}$ denote the complex bilinear extension of the metric $g$ to $\Lambda^*_{\mathbb{C}}X = \Lambda^*X \otimes \mathbb{C}$. The associated hermitian metric on $\Lambda^*_{\mathbb{C}}X$ is defined by

$$\langle \alpha, \beta \rangle := g_{\mathbb{C}}(\alpha, \bar{\beta}).$$

The almost hermitian manifold $X$ carries a canonical spin$^c$ structure, which comes with a spinor bundle

$$\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^- \quad \text{where} \quad \mathbb{S}^\pm = \Lambda^{0,\text{even/odd}}X, \tag{2.1}$$

and a Clifford multiplication $\mathrm{c} : T^*X \to \mathrm{End}_{\mathbb{C}}(\mathbb{S}^\pm, \mathbb{S}^\mp)$, given by

$$\mathrm{c}(\gamma)\alpha := \sqrt{2}\big(\gamma^{0,1} \wedge \alpha \ - \iota(\gamma^{1,0})\alpha\big), \tag{2.2}$$

where $\iota(\gamma^{1,0})$ denotes contraction by $\gamma^\sharp_{0,1}$, the vector dual to $\gamma^{1,0}$ with respect to $g_{\mathbb{C}}$. Note that

$$\langle \gamma^{0,1} \wedge \alpha, \beta \rangle = \langle \alpha, \iota(\gamma^{1,0})\beta \rangle \quad \text{and} \quad \langle \mathrm{c}(\gamma)\alpha, \beta \rangle = -\langle \alpha, \mathrm{c}(\gamma)\beta \rangle.$$

A spin$^c$ connection $\nabla$ on the spinor bundle $\mathbb{S}$, induced from the Levi-Civita connection on $X$ and a hermitian connection on $K_X^{-1}$, preserves the chiral grading in (2.1). The induced connection $\nabla^+$ on $\mathbb{S}^+$, together with a hermitian connection $\nabla^E$ on the vector bundle $E$ and the Clifford multiplication, defines the twisted Dirac operator (1.2) via the composition

$$D_E : \Gamma(\mathbb{S}^+ \otimes E) \xrightarrow{\nabla^+ \otimes \mathrm{Id}_E + \mathrm{Id}_{\mathbb{S}^+} \otimes \nabla^E} \Gamma(T^*X \otimes \mathbb{S}^+ \otimes E) \xrightarrow{\mathrm{c} \otimes \mathrm{Id}_E} \Gamma(\mathbb{S}^- \otimes E). \tag{2.3}$$

Consequently, the principal symbols of $D_E$ and its adjoint $D_E^*$ are given by

$$\sigma_{D_E}(\gamma) = \mathrm{c}(\gamma) \otimes \mathrm{Id}_E \quad \text{and} \quad \sigma_{D_E^*}(\gamma) = -\sigma_D(\gamma)^* = \mathrm{c}(\gamma) \otimes \mathrm{Id}_E$$

for all $\gamma \in T^*X$. (See, e.g., [6, 10] for more on Dirac operators.)

2.2. *Conjugate-linear Hodge star operator.* The conjugate-linear operator $\bar{*}$, which maps $\Lambda^{p,q}X$ to $\Lambda^{n-p,n-q}X$, is defined by

$$\alpha \wedge \bar{*}(\beta) = \langle \alpha, \beta \rangle dv_g$$

for $\alpha, \beta \in \Omega^{p,q}(X)$, where $dv_g$ denotes the volume form on $X$. The following lemma is a key observation in our construction.

LEMMA 2.1. *For any $\beta \in \Omega^{0,p}(X)$, $\gamma \in \Omega^1(X)$, and $\eta \in \Omega^{n,0}(X)$, we have:*

(a) $|\eta|^2\, \bar{*}\big(\gamma^{0,1} \wedge \beta\big) = (-1)^{n(p+1)+p}\, \eta \wedge \iota(\gamma^{1,0})\bar{*}\big(\eta \wedge \beta\big)$.
(b) $|\eta|^2\, \bar{*}\big(\iota(\gamma^{1,0})\beta\big) = (-1)^{(n+1)(p-1)}\, \eta \wedge \gamma^{0,1} \wedge \bar{*}\big(\eta \wedge \beta\big)$.

Proof. For any $\alpha \in \Omega^{0,p+1}X$, we have:

$$\begin{aligned}\alpha \wedge |\eta|^2\, \bar{*}\big(\gamma^{0,1} \wedge \beta\big) &= |\eta|^2 \big\langle \alpha, \gamma^{0,1} \wedge \beta\big\rangle dv_g = |\eta|^2 \,\big\langle \iota(\gamma^{1,0})\alpha, \beta\big\rangle dv_g \\ &= \big\langle \eta \wedge \iota(\gamma^{1,0})\alpha, \eta \wedge \beta\big\rangle dv_g = \eta \wedge \iota(\gamma^{1,0})\alpha \wedge \bar{*}\big(\eta \wedge \beta\big) \\ &= (-1)^p\, \eta \wedge \alpha \wedge \iota(\gamma^{1,0})\bar{*}\big(\eta \wedge \beta\big) \\ &= \alpha \wedge \Big((-1)^{n(p+1)+p}\, \eta \wedge \iota(\gamma^{1,0})\bar{*}\big(\eta \wedge \beta\big)\Big),\end{aligned}$$

where the third equality follows from the definition of the hermitian metric on $\Lambda^*_{\mathbb{C}}X$, and the sign factors follow since contraction and wedge product are antiderivations. This implies (a). The proof of (b) is similar. ∎

2.2.1. *Rescaling.* For our purposes, we rescale the operator $\bar{*}$ as follows:

$$\tau := \epsilon_k \bar{*} \quad \text{on} \quad \Lambda^k_{\mathbb{C}}X, \quad \text{where} \quad \epsilon_k = i^{k(k-1)+n}. \tag{2.4}$$

The unit complex number $\epsilon_k$, which also appears in the context of the signature operator, plays a useful role in our construction. Observe that

$$\bar{*}^2 = (-1)^k \mathrm{Id} \quad \text{on} \quad \Lambda^k_{\mathbb{C}}X \qquad \text{and} \qquad \tau^2 = (-1)^n \mathrm{Id}. \tag{2.5}$$

The following identity is central to our construction.

Proposition 2.2. *For any $\beta \in \Omega^{0,p}(X)$, $\gamma \in \Omega^1(X)$, and $\eta \in \Omega^{n,0}(X)$, we have*

$$|\eta|^2\, \tau \circ \mathrm{c}(\gamma)(\beta) = (-1)^{\frac{n(n+1)}{2}+1}\, \eta \wedge \mathrm{c}(\gamma) \circ \tau(\eta \wedge \beta).$$

Proof. The proof follows from Lemma 2.1 and the identity:

$$\epsilon_{p+1}\epsilon^{-1}_{n+p}(-1)^{n(p+1)+p} = (-1)^{\frac{n(n+1)}{2}} = \epsilon_{p-1}\epsilon^{-1}_{n+p}(-1)^{(n+1)(p-1)}.$$

∎

2.2.2. *Adjoint.* Since $\tau$ is conjugate-linear, its adjoint $\tau^*$ is defined with respect to the real part of the hermitian metric. By (2.5), we obtain:

Lemma 2.3. $\tau^* = (-1)^n \tau$.

Proof. For any $\alpha \in \Omega^k_{\mathbb{C}}(X)$ and $\beta \in \Omega^{2n-k}_{\mathbb{C}}(X)$, we have

$$\begin{aligned}\langle \beta, \tau\alpha\rangle dv_g &= \bar{\epsilon}_k(-1)^k\, \beta \wedge \alpha = \bar{\epsilon}_k\, \alpha \wedge \beta = \bar{\epsilon}_k\, \epsilon_{2n-k}(-1)^{2n-k}\, \alpha \wedge \bar{*}(\tau\beta) \\ &= (-1)^n \langle \alpha, \tau\beta\rangle dv_g.\end{aligned}$$

This shows that $\mathrm{Re}\langle \tau\alpha, \beta\rangle = \mathrm{Re}\langle \alpha, (-1)^n \tau\beta\rangle$, and hence proves the lemma. ∎

2.2.3. *Operator with values in $E$.* The hermitian metric on $E$, still denoted by $\langle\ ,\ \rangle$, defines a conjugate-linear isomorphism $E \to E^*$ by $\xi \mapsto \xi^* := \langle\ ,\xi\rangle$. This induces the map

$$\bar{*}_E : \Lambda^{p,q}X \otimes E^* \to \Lambda^{n-p,n-q}X \otimes E$$

defined by $\bar{*}_E(\alpha \otimes \xi^*) = \bar{*}(\alpha) \otimes \xi$. Now we set

$$\tau_E := \epsilon_k \bar{*}_E \quad \text{on} \quad \Lambda^k_{\mathbb{C}}X \otimes E.$$

Note that $\tau_E(\alpha \otimes \xi^*) = \tau(\alpha) \otimes \xi$ and $\tau_E^*(\alpha \otimes \xi) = (-1)^n \tau(\alpha) \otimes \xi^*$.

## 3. Conjugate-linear perturbations

3.1. *Concentrating condition.* This subsection constructs conjugate-linear perturbations of the twisted Dirac operator $D$ in (2.3) that satisfy the concentrating condition (1.1).

Definition 3.1. *Given a section $\varphi \in \Gamma\big(K_X \otimes (E^*)^2\big)$, viewed as a complex-linear bundle map $\varphi : E \to K_X \otimes E^*$, we define a conjugate-linear bundle map*

$$A_\varphi : \Lambda^{0,p}X \otimes E \to \Lambda^{0,n-p}X \otimes E$$

*by the composition*

$$A_\varphi : \Lambda^{0,p} \otimes E \xrightarrow{\mathrm{Id}\otimes\varphi} \Lambda^{0,p} \otimes K_X \otimes E^* \cong \Lambda^{n,p} \otimes E^* \xrightarrow{\tau_E} \Lambda^{0,n-p} \otimes E.$$

Noting $A_\varphi : \mathbb{S}^\pm \otimes E \to \mathbb{S}^\mp \otimes E$ when $\dim X \equiv 2, 6 \pmod 8$, and using the decomposition

$$E^* \otimes E^* = \mathrm{Sym}^2 E^* \oplus \Lambda^2 E^*,$$

we obtain concentrating pairs $(\sigma_{D_E}, A_\varphi)$ in the sense of [4]:

Theorem 3.2. *Let $D_E$ be the twisted Dirac operator in (2.3) and $A_\varphi$ as in Definition 3.1. Suppose $\varphi \in \Gamma(K_X \otimes \mathrm{Sym}^2 E^*)$ if $\dim X \equiv 2 \pmod 8$ and $\varphi \in \Gamma(K_X \otimes \Lambda^2 E^*)$ if $\dim X \equiv 6 \pmod 8$. Then $A_\varphi$ satisfies the concentrating condition*

$$\sigma_{D_E^*}(\gamma) \circ A_\varphi + A_\varphi^* \circ \sigma_{D_E}(\gamma) = 0 \quad \forall \gamma \in T^*X. \tag{3.1}$$

Proof. In unitary frames $\{\phi_i\}$ for $E$ (with dual coframe $\{\phi^i\}$) and $\eta$ for $K_X$ (with dual coframe $\eta^*$), the complex bundle map $\varphi : E \to K_X \otimes E^*$ is given by

$$\varphi = \varphi_{ij}\eta \otimes \phi^i \otimes \phi^j : \xi = \xi_k \phi_k \mapsto \varphi_{ij}\xi_j \eta \otimes \phi^i. \tag{3.2}$$

It follows that for any $\beta \otimes \xi \in \Gamma(\mathbb{S}^+ \otimes E)$ and any $\gamma \in \Omega^1(X)$,

$$\begin{aligned} \sigma_{D^*}(\gamma) \circ A_\varphi(\beta \otimes \xi) &= (\mathrm{c}(\gamma) \otimes \mathrm{Id}_E) \circ \tau_E\big(\eta \wedge \beta \otimes \varphi_{ij}\xi_j \phi^i\big) \\ &= \mathrm{c}(\gamma)\tau(\eta \wedge \beta) \otimes \overline{\varphi}_{ij}\overline{\xi}_j \phi_i. \end{aligned} \tag{3.3}$$

On the other hand, since $|\eta|^2 = 1$, by Lemma 2.3 and Proposition 2.2, we have

$$\begin{aligned}\tau_E^* \circ \sigma_D(\gamma)(\beta \otimes \xi) &= (-1)^n \tau(\mathrm{c}(\gamma)\beta) \otimes \overline{\xi}_k \phi^k \\ &= (-1)^{\frac{n(n+1)}{2}+n+1}\, \eta \wedge \mathrm{c}(\gamma) \circ \tau(\eta \wedge \beta) \otimes \overline{\xi}_k \phi^k.\end{aligned} \tag{3.4}$$

Observe that the adjoint $\varphi^* : K_X \otimes E^* \to E$ of the complex bundle map $\varphi$ is given by

$$\varphi^* = \overline{\varphi}_{ji} \phi_i \otimes \eta^* \otimes \phi_j : \nu = \nu_k \eta \otimes \phi^k \mapsto \overline{\varphi}_{ji} \nu_j \phi_i.$$

With the canonical isomorphism $\Lambda^{n,p} X \cong \Lambda^{0,p} \otimes K_X$, from (3.4) we obtain

$$\begin{aligned}&A_\varphi^* \circ \sigma_D(\gamma)(\beta \otimes \xi) \\ &= (-1)^{\frac{n(n+1)}{2}+n+1}\, \mathrm{Id} \otimes \varphi^*\big(\mathrm{c}(\gamma) \circ \tau(\eta \wedge \beta) \otimes \overline{\xi}_k \eta \otimes \phi^k\big) \\ &= (-1)^{\frac{n(n+1)}{2}+n+1}\, \mathrm{c}(\gamma) \circ \tau(\eta \wedge \beta) \otimes \overline{\varphi}_{ji} \overline{\xi}_j \phi_i.\end{aligned} \tag{3.5}$$

By the assumption on $\varphi$, (3.3), and (3.5), the proof now follows from the identity

$$(-1)^{\frac{n(n+1)}{2}+n+1} = \begin{cases} -1 & \text{if } n \equiv 1 \pmod 4 \\ \;\;\,1 & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

□

The next two remarks are related to the motivation for our approach.

REMARK 3.3. For the twisted Dirac operator $D_E : \Gamma(\mathbb{S}^+ \otimes E) \to \Gamma(\mathbb{S}^- \otimes E)$, there is no complex-linear bundle map $A : \mathbb{S}^+ \otimes E \to \mathbb{S}^- \otimes E$ that satisfies the concentrating condition (3.1). This explains why the Dolbeault and signature operators do not admit such complex perturbations. (See Section 2 of [8].)

REMARK 3.4. When $\dim X = 2$, the manifold $X$ is a compact complex curve, and the operator $D_E = \sqrt{2}\,\bar\partial_E : \Omega^0(E) \to \Omega^{0,1}(E)$ is the usual Cauchy–Riemann operator on $E$. In [3], when $E$ is a holomorphic line bundle and $\varphi$ is a holomorphic section of $K_X \otimes (E^*)^2$, a conjugate-linear bundle map $R : E \to \Lambda^{0,1} X \otimes E$ is defined as the composition $R = \bar{*}_E \circ \varphi$. In this case, $A_\varphi = iR$ and $A_\varphi^* = iR^*$, so our construction generalizes the conjugate-linear bundle map $R$.

Indeed, $A_\varphi = iR$ satisfies the following equation, which is stronger than the concentrating condition:

$$D_E^* \circ A_\varphi + A_\varphi^* \circ D_E = 0.$$

(See Lemma 2.1 of [3].) When $E$ is a theta characteristic (i.e., $E^2 \cong K_X$), this implies that $A_\varphi$ restricts to a conjugate-linear isomorphism $\ker D_E \to \ker D_E^*$, which leads to a proof of the Atiyah–Mumford theorem: $h^0(E)$ ($= \dim\ker D_E$) mod 2 is *deformation invariant* (see Section 3 of [3]).

3.2. *Concentration principle.* Let $D_E, A_\varphi : \Gamma(\mathbb{S}^+ \otimes E) \to \Gamma(\mathbb{S}^- \otimes E)$ be as in Theorem 3.2 and consider the deformation of the twisted Dirac operator $D_E$ given by

$$D_s = D_E + sA_\varphi.$$

The following concentration principle shows that as $s \to \infty$, the kernel of $D_s$ (as well as low eigenspaces of $D_s^* D_s$) becomes concentrated near the singular set

$$Z_\varphi := Z_{A_\varphi} = \{x \in X : \det(\varphi_x) = 0\}.$$

THEOREM 3.5. *Let $D_s = D_E + A_\varphi$ be as above. For each $\delta > 0$ and $C \geq 0$, there exits a constant $C' = C'(\delta, A_\varphi, C) > 0$ such that if $\zeta \in \Gamma(\mathbb{S}^+ \otimes E)$ satisfies $||\zeta||_2 = 1$ and $||D_s \zeta||_2^2 \leq C|s|$, where $||\cdot||_2$ denotes the $L^2$-norm, then we have*

$$\int_{X \setminus Z_\varphi(\delta)} |\zeta|^2 dv_g < \frac{C'}{|s|},$$

*where $Z_\varphi(\delta)$ is the $\delta$-neighborhood of the singular set $Z_\varphi$.*

PROOF. Since $A_\varphi$ satisfies the concentrating condition (3.1), the theorem follows directly from Proposition 2.4 of [4]. ∎

In applying this principle, the singular set plays a crucial role. If $Z_\varphi = X$, no concentration occurs. For example, this happens when

$$\dim X \equiv 6 \pmod 8 \qquad \text{and} \qquad \operatorname{rank}(E) \text{ is odd},$$

since $\varphi_{ij} = -\varphi_{ji}$ in (3.2), and thus $\det(\varphi_{ij}) = 0$. On the other hand, when $Z_\varphi = \emptyset$, we have:

COROLLARY 3.6. *Let $D_E$ and $A_\varphi$ be as in Theorem 3.2. If $Z_\varphi = \emptyset$, then* $\operatorname{ind} D_E = 0$.

PROOF. By Theorem 3.5, the singular set $Z_\varphi = \emptyset$ implies that for sufficiently large $s$, the operator $D_s$ has trivial kernel. It follows that

$$\operatorname{ind} D_E = \tfrac{1}{2}\operatorname{ind}_{\mathbb{R}} D_s \leq 0,$$

where the second term denotes the index of $D_s$ as a real operator, and the equality follows since $D_s$ is a compact perturbation of $D_E$. Applying the same argument to the operator $D_E^* + sA_\varphi$ yields $\operatorname{ind} D_E^* = -\operatorname{ind} D_E \leq 0$, so the claim follows. ∎

3.2.1. *Compact spin almost hermitan manifolds.* Suppose $c_1(X) \equiv 0 \pmod 2$, so that $X$ admits a spin structure. Fix a spin structure $\sigma$ on $X$, which is equivalent to choosing a square root $N_\sigma$ of the canonical bundle $K_X$, i.e., $N_\sigma^2 \cong K_X$. In this case, the complex spinor bundle $\mathbb{S}_\sigma$ of the spin structure $\sigma$ is given by

$$\mathbb{S}_\sigma = \mathbb{S} \otimes N_\sigma.$$

Let $E = N_\sigma \otimes F$, where $F$ is a hermitian vector bundle $F$ over $X$. A section $\psi \in \Gamma\big((F^*)^2\big)$ induces a section $\varphi \in \Gamma\big(K_X \otimes (E^*)^2\big)$ via the isomorphism $(F^*)^2 \cong K_X \otimes (E^*)^2$, which restricts to yield

$$\mathrm{Sym}^2 F^* \cong K_X \otimes \mathrm{Sym}^2 E^* \quad \text{and} \quad \Lambda^2 F^* \cong K_X \otimes \Lambda^2 E^*.$$

When $\dim X \equiv 2, 6 \pmod 8$, we set

$$A_\psi := A_\varphi : \mathbb{S}^\pm \otimes E = \mathbb{S}^\pm_\sigma \otimes F \to \mathbb{S}^\mp \otimes E = \mathbb{S}^\mp_\sigma \otimes F,$$

where $A_\varphi$ is the conjugate-linear bundle map as in Theorem 3.2. The singular set of this bundle map is then given by

$$Z_\psi := Z_\varphi = \{x \in X : \det(\psi_x) = 0\}.$$

Below are examples of bundle maps $A_\psi$ whose singular set $Z_\psi = \emptyset$.

(a) Let $\dim X \equiv 2 \pmod 8$. The following are typical examples of nondegenerate symmetric complex bilinear forms $\psi \in \Gamma(\mathrm{Sym}^2 F^*)$ on $F$:
- $F = TX \otimes \mathbb{C}$ with $\psi = g_\mathbb{C}$, the complex bilinear extension of the metric $g$ to $TX \otimes \mathbb{C}$, or
- $F = \mathrm{End}_\mathbb{C}(W)$, where $W$ is a hermitian vector bundle over $X$, and $\psi$ is the natural trace pairing given by $\psi(A, B) = \mathrm{tr}(A \circ B)$.

(b) Let $\dim X \equiv 6 \pmod 8$. A complex vector bundle $F$ is called a *complex symplectic vector bundle* if it is equipped with a nondegenerate skew-symmetric complex bilinear form $\psi \in \Gamma(\Lambda^2 F^*)$. Standard examples include:
- $F = TX \otimes \mathbb{C}$ with $\psi = \omega_\mathbb{C}$, where $\omega_\mathbb{C}(u, v) = g_\mathbb{C}(u, Jv)$, or
- $F = W \oplus W^*$ with $\psi$ defined by $\psi\big((w_1, f_1), (w_2, f_2)\big) = f_2(w_1) - f_1(w_2)$.

In all of the above examples $(F, \psi)$, the bundle map $A_\psi \,(= A_\varphi)$ satisfies the concentrating condition, and the singular set $Z_\psi = \emptyset$. Hence, by Corollary 3.6, the twisted Dirac operator

$$D_{N_\sigma \otimes F} : \Gamma(\mathbb{S}^+_\sigma \otimes F) \to \Gamma(\mathbb{S}^-_\sigma \otimes F)$$

has index $\mathrm{ind}\, D_{N_\sigma \otimes F} = 0$.

Remark 3.7. The vanishing of the index also follows easily from the Atiyah-Singer index theorem:

$$\mathrm{ind}\, D_{N_\sigma \otimes F} = \int_X \mathrm{ch}(F) \hat{\mathrm{A}}(X),$$

where $\hat{\mathrm{A}}(X)$ is a polynomial in the Pontryagin classes $p_k(X) = (-1)^k c_{2k}(X)$ and $\mathrm{ch}(F)$ is a polynomial in the even Chern classes $c_{2\ell}(F)$ (since $F \cong F^*$ as a complex vector bundle). As $\frac{1}{2}\dim X$ is odd, there is no top-degree term in the integrand, and therefore the integral vanishes.

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