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INDUCTION FROM TWO LINKED SEGMENTS WITH ONE HALF BORDER AND CUSPIDAL REDUCIBILITY

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Abstract. In this paper, we determine the composition series of the induced representation $\delta([\nu^{\frac{1}{2}}\rho, \nu^c\rho]) \times \delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma$ where $a, b, c \in \mathbb{Z} + \frac{1}{2}$ such that $\frac{1}{2} \leq a < b < c$, ρ is an irreducible cuspidal unitary representation of a general linear group and σ is an irreducible cuspidal representation of a classical group such that $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ reduces.

1. Introduction

Parabolic induction is an important tool for constructing representations of groups. However, problem of composition series of induced representations is solved only in some special cases, such as [2], [7], [9], [14] and [21], and for some low-rank groups. Motivated by simple results obtained in [3] and its extensions in [4] and [5], here we continue this effort and calculate composition series of certain induced representations with increased complexity, in the sense that we have more reducibilities occuring when inducing from subsets of starting representations.

To explain this we introduce some notation. Let F be a local non-archimedean field of characteristic different than two, $|\ |_F$ its normalized absolute value and $\nu = |\det|_F$. Let ρ be an irreducible cuspidal unitary representation of some $GL(m, F)$, and $x, y \in \mathbb{R}$, such that $y - x + 1 \in \mathbb{Z}_{>0}$. The set $\Delta = [\nu^x\rho, \nu^y\rho] = \{\nu^x\rho, ..., \nu^y\rho\}$ is called a segment. We have a unique

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irreducible subrepresentation

$$\delta = \delta(\Delta) = \delta([\nu^x\rho, \nu^y\rho]) \hookrightarrow \nu^y\rho \times \cdots \times \nu^x\rho,$$

of the parabolically induced representation. Set $e(\delta) = (x+y)/2$. For a sequence of segments, such that $e(\delta_1) \geq \cdots \geq e(\delta_k) > 0$ and an irreducible tempered representation τ, of a symplectic or (full) orthogonal group, denote by

$$L(\delta_1 \times \cdots \times \delta_k \rtimes \tau),$$

the Langlands quotient of the parabolically induced representation.

Now we fix ρ as above, and so we shorten the notation $\delta(x,y) = \delta([\nu^x\rho, \nu^y\rho])$. Further, let σ be an irreducible cuspidal representation of a symplectic or (full) orthogonal group. We assume that $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ reduces. Let $a, b, c \in \mathbb{Z} + \frac{1}{2}$ such that $\frac{1}{2} \leq a < b < c$. In [3], we determined composition series of induced representation

$$\delta(-b,c) \times \delta(\tfrac{1}{2}, a) \rtimes \sigma.$$

We extended this results in [4] to certain induced representations of form

$$\delta_1 \times \cdots \times \delta_k \rtimes \sigma$$

where $\delta_i \times \delta_j$ and $\delta_i \times \widetilde{\delta_j}$, $i \neq j$, are irreducible and $\widetilde{\ }$ stands for the contragredient. Loosing this condition on segments, in [5], we determined composition series of

$$\delta(-a,c) \times \delta(\tfrac{1}{2}, b) \rtimes \sigma.$$

In this paper, we determine composition series of

$$\delta(\tfrac{1}{2}, c) \times \delta(-a,b) \rtimes \sigma.$$

Here, inducing from segments only, both representations $\delta(\frac{1}{2}, c) \times \delta(-a,b)$ and $\delta(\frac{1}{2}, c) \times \delta(-a,b)\widetilde{\ } \cong \delta(\frac{1}{2}, c) \times \delta(-b,a)$ reduce.

Our main methods are Jacquet modules, Mœglin-Tadić classification of discrete series and intertwining operators.

To describe the main result of the paper, we introduce some discrete series, appearing as only irreducible subrepresentations in the following induced representations (see Theorem 3.1, Proposition 3.2, Lemmas 3.4 and 3.5 and Theorem 3.7):

$$\sigma_a \hookrightarrow \delta(\tfrac{1}{2}, a) \rtimes \sigma, \text{ and similarly for } \sigma_b \text{ and } \sigma_c,$$
$$\sigma^+_{b,c} \hookrightarrow \delta(\tfrac{1}{2}, b) \rtimes \sigma_c, \quad \sigma^+_{b,c} + \sigma^-_{b,c} \hookrightarrow \delta(-b,c) \rtimes \sigma, \text{ and similarly for } \sigma^\pm_{a,c},$$
$$\sigma^\pm_{b,c,a} \hookrightarrow \delta(\tfrac{1}{2}, a) \rtimes \sigma^\pm_{b,c}, \quad \sigma^+_{a,b,c} + \sigma^-_{a,b,c} \hookrightarrow \delta(-a,b) \rtimes \sigma_c,$$

where $\sigma^+_{a,b,c} = \sigma^+_{b,c,a}$ denotes the same representation. In terms of Mœglin-Tadić classification ([12],[13]), where a discrete series π is described by an admissible triple $(\mathrm{Jord}(\pi), \pi_{cusp}, \epsilon_\pi)$, we have the second parameter σ in all our cases, and the remaining are

$$\mathrm{Jord}(\sigma_a) = \{(2a+1,\rho)\} \cup Jord(\sigma),$$
$$Jord(\sigma^+_{b,c}) = Jord(\sigma^-_{b,c}) = \{(2b+1,\rho),(2c+1,\rho)\} \cup Jord(\sigma),$$

$\epsilon_{\sigma_a}, \epsilon_{\sigma^+_{b,c}}$, and $\epsilon_{\sigma^-_{b,c}}$ extend ϵ_σ, such that $\epsilon_{\sigma_a}(2a+1,\rho) = 1$,
$\epsilon_{\sigma^+_{b,c}}(2b+1,\rho) = \epsilon_{\sigma^+_{b,c}}(2c+1,\rho) = 1, \epsilon_{\sigma^-_{b,c}}(2b+1,\rho) = \epsilon_{\sigma^-_{b,c}}(2c+1,\rho) = -1$,

$$\begin{aligned}\mathrm{Jord}(\sigma^+_{b,c,a}) =& \mathrm{Jord}(\sigma^-_{b,c,a}) = \mathrm{Jord}(\sigma^-_{a,b,c}) = \\ & \{(2a+1,\rho),(2b+1,\rho),(2c+1,\rho)\} \cup Jord(\sigma),\end{aligned}$$

$\epsilon_{\sigma^+_{b,c,a}}$, $\epsilon_{\sigma^-_{b,c,a}}$ and $\epsilon_{\sigma^-_{a,b,c}}$ extend ϵ_σ such that

$$\begin{array}{lll}\epsilon_{\sigma^+_{b,c,a}}(2a+1,\rho)=1, & \epsilon_{\sigma^+_{b,c,a}}(2b+1,\rho)=1, & \epsilon_{\sigma^+_{b,c,a}}(2c+1,\rho)=1,\\ \epsilon_{\sigma^-_{b,c,a}}(2a+1,\rho)=1, & \epsilon_{\sigma^-_{b,c,a}}(2b+1,\rho)=-1, & \epsilon_{\sigma^-_{b,c,a}}(2c+1,\rho)=-1,\\ \epsilon_{\sigma^-_{a,b,c}}(2a+1,\rho)=-1, & \epsilon_{\sigma^-_{a,b,c}}(2b+1,\rho)=-1, & \epsilon_{\sigma^-_{a,b,c}}(2c+1,\rho)=1.\end{array}$$

Now we have

THEOREM. *Let* $\psi = \delta(\frac{1}{2}, c) \times \delta(-a, b) \rtimes \sigma$ *and define representations*

$$\begin{aligned}
W_1 =& \sigma^+_{b,c,a} + L(\delta(\tfrac{1}{2}, a) \rtimes \sigma^-_{b,c}),\\
W_2 =& L(\delta(\tfrac{1}{2}, a) \rtimes \sigma^+_{b,c}) + L(\delta(\tfrac{1}{2}, b) \rtimes \sigma^-_{a,c}) + L(\delta(-b,c) \rtimes \sigma_a) + L(\delta(-a,b) \rtimes \sigma_c),\\
W_3 =& \sigma^-_{b,c,a} + \sigma^-_{a,b,c} + L(\delta(\tfrac{1}{2}, b) \rtimes \sigma^+_{a,c}) + L(\delta(-a,c) \rtimes \sigma_b) + L(\delta(-b,c) \times \delta(\tfrac{1}{2}, a) \rtimes \sigma),\\
W_4 =& L(\delta(\tfrac{1}{2}, b) \times \delta(-a,c) \rtimes \sigma) + L(\delta(\tfrac{1}{2}, c) \rtimes \sigma^+_{a,b}) + L(\delta(\tfrac{1}{2}, c) \rtimes \sigma^-_{a,b}),\\
W_5 =& L(\psi).
\end{aligned}$$

Then there exists a sequence $\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 \subseteq V_5 = \psi$*, such that*

$$V_i/V_{i-1} \cong W_i, \quad i = 1, \ldots, 5.$$

The content of the paper is as follows. After Preliminaries, we introduce the notation in Section 3 and list some reducibility results. In Section 4 we explain an approach to decompose the induced representation were an important part plays a kernel of certain intertwining operator. In Section 5 we provide some results on tempered representations, that are used in Section 6, when determining discrete series of the induced representations. The search for non-tempered candidates is done by Section 7, and their multiplicities are determined in Sections 8 and 10, while Sections 9 and 11 list composition factors of representations that are considered. Finally, composition series of needed kernel are provided in Section 12, while the main result is proved in Section 13.

2. Preliminaries

Our setting is as in [5], so we briefly recall. We fix a local non-archimedean field F of characteristic different than two. As in [13], we fix a tower of symplectic or orthogonal non-degenerate F vector spaces V_n, $n \geq 0$ where n is the Witt index. Denote by G_n the group of isometries of V_n. It has split rank n. We fix the set of standard parabolic subgroups $\{P_s\}$ in the usual way and have Levi factorization $P_s = M_s N_s$. By Alg G_n we denote smooth representations of G_n, and by Irr G_n irreducible ones. Also, for Alg we use following subscripts to denote smooth representations with certain property: $f.l.$ means finite length, u unitary, and *cusp* cuspidal. Similarly, we also use subscripts u and *cusp* for Irr.

For $\delta_i \in$ Alg $GL(n_i, F)$, $i = 1, ..., k$ and $\tau \in$ Alg G_{n-m} we write

$$\delta_1 \times \cdots \times \delta_k \rtimes \tau = \mathrm{Ind}_{M_s}^{G_n}(\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau)$$

to denote the normalized parabolic induction. If $\sigma \in$ Alg G_n we denote by $\mathrm{r}_s(\sigma) = \mathrm{r}_{M_s}(\sigma) = \mathrm{Jacq}_{M_s}^{G_n}(\sigma)$ the normalized Jacquet module of σ.

Let $|\ |_F$ be normalized absolute value of F and $\nu = |\det|_F$. For an irreducible cuspidal unitary representation ρ of some $GL(m, F)$, and $x, y \in \mathbb{R}$, such that $y - x + 1 \in \mathbb{Z}_{>0}$, the set $\Delta = [\nu^x\rho, \nu^y\rho] = \{\nu^x\rho, ..., \nu^y\rho\}$ is called a segment. We have a unique irreducible subrepresentation

$$\delta = \delta(\Delta) = \delta([\nu^x\rho, \nu^y\rho]) \hookrightarrow \nu^y\rho \times \cdots \times \nu^x\rho,$$

of the parabolically induced representation. For $y - x + 1 \in \mathbb{Z}_{>0}$ define $[\nu^x\rho, \nu^y\rho] = \emptyset$ and $\delta(\emptyset)$ is the irreducible representation of the trivial group. Set $e(\delta) = (x + y)/2$. For a sequence of non-empty segments, such that $e(\delta_1) \geq \cdots \geq e(\delta_k) > 0$ and an irreducible tempered representation τ, of a symplectic or (full) orthogonal group, denote by

$$L(\delta_1 \times \cdots \times \delta_k \rtimes \tau),$$

the Langlands quotient of the parabolically induced representation.

If σ is a discrete series representation of G_n then by the Mœglin-Tadić, now unconditional classification ([12],[13]), it is described by an admissible triple $(\mathrm{Jord}, \sigma_{cusp}, \epsilon)$.

Let $R(G_n)$ be the free Abelian group generated by classes of irreducible representations of G_n. Put $R(G) = \oplus_{n \geq 0} R(G_n)$. Let $R_0^+(G)$ be a $\mathbb{Z}_{\geq 0}$ subspan of classes of irreducible representations. For $\pi_1, \pi_2 \in R(G)$ we define $\pi_1 \leq \pi_2$ if $\pi_2 - \pi_1 \in R_0^+(G)$. Similarly define $R(GL) = \oplus_{n \geq 0} R(GL(n, F))$. We have the map $\mu^* : R(G) \to R(GL) \otimes R(G)$ defined by

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} \mathrm{s.s.}(r_{(k)}(\sigma)),\ \sigma \in R(G_n).$$

where s.s. denotes the semisimplification. The following result derives from Theorems 5.4 and 6.5 of [17], see also section 1. in [13]. They are based on Geometrical Lemma (2.11 of [1]). We use $\tilde{}$ to denote the contragredient.

THEOREM 2.1. *Let σ be a smooth representation of finite length a classical group and $[\nu^x\rho,\nu^y\rho]\neq\emptyset$ a segment. Then*

(2.1)

$$\mu^*(\delta([\nu^x\rho,\nu^y\rho])\rtimes\sigma)=\sum_{\delta'\otimes\sigma'\leq\mu^*(\sigma)}\sum_{i=0}^{y-x+1}\sum_{j=0}^{i}$$
$$\delta([\nu^{i-y}\widetilde{\rho},\nu^{-x}\widetilde{\rho}])\times\delta([\nu^{y+1-j}\rho,\nu^{y}\rho])\times\delta'\otimes\delta([\nu^{y+1-i}\rho,\nu^{y-j}\rho])\rtimes\sigma'$$

Now we write some formulae for Jacquet modules. Details can be found in [10] and corrections of typographical errors, that we state below, in the Introduction in [11]. Let ρ be an irreducible cuspidal representations of a general linear group, σ an irreducible cuspidal representations of a classical group and $c,d\in\mathbb{R}, c+d\in\mathbb{Z}_{\geq 0}$. Assume that $\alpha\in\frac{1}{2}\mathbb{Z}_{\geq 0}$ is such that $\nu^\alpha\rho\rtimes\sigma$ reduces. Such α is unique and ρ is selfdual. Consider induced representation

$$\pi=\delta([\nu^{-c}\rho,\nu^{d}\rho])\rtimes\sigma\overset{R(G)}{=}\delta([\nu^{-d}\rho,\nu^{c}\rho])\rtimes\sigma.$$

Three terms are defined: $\delta([\nu^{-c}\rho,\nu^{d}\rho]_+;\sigma)$, $\delta([\nu^{-c}\rho,\nu^{d}\rho]_-;\sigma)$ and $L_\alpha(\delta([\nu^{-c}\rho,\nu^{d}\rho]);\sigma)$. Each of them is either an irreducible representation or zero. We have in $R(G)$:

$$\begin{aligned}\delta([\nu^{-c}\rho,\nu^{d}\rho])\rtimes\sigma=&\delta([\nu^{-c}\rho,\nu^{d}\rho]_+;\sigma)+\delta([\nu^{-c}\rho,\nu^{d}\rho]_-;\sigma)+\\&L_\alpha(\delta([\nu^{-c}\rho,\nu^{d}\rho]);\sigma).\end{aligned}\tag{2.2}$$

We have

(2.3)

$$\mu^*(\delta([\nu^{-c}\rho,\nu^{d}\rho]_\pm;\sigma))=$$
$$\sum_{i=-c-1}^{d-1}\sum_{j=i+1}^{d}\delta([\nu^{-i}\rho,\nu^{c}\rho])\times\delta([\nu^{j+1}\rho,\nu^{d}\rho])\otimes\delta([\nu^{i+1}\rho,\nu^{j}\rho]_\pm;\sigma)+$$
$$\sum_{\substack{-c-1\leq i\leq c-1\\ i+j<-1}}\sum_{i+1\leq j\leq c}\delta([\nu^{-i}\rho,\nu^{c}\rho])\times\delta([\nu^{j+1}\rho,\nu^{d}\rho])\otimes L_\alpha(\delta([\nu^{i+1}\rho,\nu^{j}\rho]);\sigma)$$
$$+\sum_{i=-c-1}^{\pm\alpha-1}\delta([\nu^{-i}\rho,\nu^{c}\rho])\times\delta([\nu^{i+1}\rho,\nu^{d}\rho])\otimes\sigma.$$

The above formula has corrected two typographical errors which exist in [10]. First, the upper limit in the first sum of the second row needs to be $d-1$ (instead of c, as it is in the published version). Then, the limits of the first sum in the third row are $-c-1\leq i\leq c-1$ (instead of $-c-1\leq i\leq c$;

the index c does not give any contribution). The same corrections applies to corresponding formulas in Corollaries 4.3, 5.4 and 6.4 of [10].

And for $c < \alpha$ or $\alpha \le c < d$ we have

(2.4)

$$\mu^*(L_\alpha(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma)) =$$
$$\sum_{-c-1 \le i \le d-1} \sum_{\substack{i+1 \le j \le d \\ 0 \le i+j}} L(\delta([\nu^{-i}\rho, \nu^c\rho]), \delta([\nu^{j+1}\rho, \nu^d\rho])) \otimes L_\alpha(\delta([\nu^{i+1}\rho, \nu^j\rho]); \sigma)$$
$$+ \sum_{i=\alpha}^{d} L(\delta([\nu^{-i}\rho, \nu^c\rho]), \delta([\nu^{i+1}\rho, \nu^d\rho])) \otimes \sigma$$

Also, the above formula has corrected a typographical error existing [10]: the limits in the first sum in the second row are $-c-1 \le i \le d-1$ (instead of $-c-1 \le i \le d$; the index d does not contribute in the formula). The same correction applies to corresponding formulas in Corollaries 4.3, 5.4 and 6.4. in [10].

In this paper we consider the case $\alpha = \frac{1}{2}$. Subquotients of π are as follows.

If $\frac{1}{2} < -c$, then π is irreducible and $L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma) = \pi$.

If $-c \le \frac{1}{2}$, then π reduces. By Lemma 3.3 of [10], π has a unique irreducible subquotient that has in its minimal standard Jacquet module at least one irreducible subquotient whose all exponents are non-negative. We denote it by $\delta([\nu^{-c}\rho, \nu^d\rho]_+; \sigma)$.

If $-c = \frac{1}{2}$, then π is of length two, $\delta([\nu^{-c}\rho, \nu^d\rho]_+; \sigma)$ is a discrete series subrepresentation and $L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma)$ is the Langlands quotient of π.

If $-c \le -\frac{1}{2}$, and $c = d$, then π is a direct sum of two tempered representations, $\delta([\nu^{-c}\rho, \nu^d\rho]_+; \sigma)$ and $\delta([\nu^{-c}\rho, \nu^d\rho]_-; \sigma)$.

If $-c \le -\frac{1}{2}$, and $c \ne d$, then π is of length three. It has two discrete series representations, $\delta([\nu^{-c}\rho, \nu^d\rho]_+; \sigma)$ and $\delta([\nu^{-c}\rho, \nu^d\rho]_-; \sigma)$, and $L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma)$ is the Langlands quotient.

3. Notation and basic reducibilities

Now we fix the notation and write some reducibility results. Let ρ be an irreducible unitary cuspidal representation of $GL(m_\rho, F)$ and σ an irreducible cuspidal representation of G_n such that $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ reduces. By Proposition 2.4 of [18] ρ is self-dual. We consider

$$\frac{1}{2} \le a, b, c \in \mathbb{Z} + \frac{1}{2},$$

that need not be fixed. If all three a, b and c appear in a formula, we have $a < b < c$. If only two appear, depending on which do appear, we have $a < b$,

$b < c$ or $a < c$. We want to decompose the representation

$$\psi = \delta(\tfrac{1}{2}, c) \times \delta(-a, b) \rtimes \sigma.$$

First, using (2.2) we define:

$$\sigma_a = \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]_+; \sigma), \quad \sigma^-_{b,c} = \delta([\nu^{-b}\rho, \nu^c\rho]_-; \sigma), \quad \sigma^+_{b,c} = \delta([\nu^{-b}\rho, \nu^c\rho]_+; \sigma),$$
$$\tau^+_{a,a} = \delta([\nu^{-a}\rho, \nu^a\rho]_+; \sigma), \quad \tau^-_{a,a} = \delta([\nu^{-a}\rho, \nu^a\rho]_-; \sigma).$$

Here σ_a, $\sigma^+_{b,c}$ and $\sigma^-_{b,c}$ are discrete series whose classification, in terms of Mœglin-Tadić classification, is given by the following theorem. Further $\tau^+_{a,a}$ and $\tau^-_{a,a}$ are irreducible tempered representations such that we have a direct sum

$$\delta(-a, a) \rtimes \sigma = \tau^+_{a,a} + \tau^-_{a,a}$$

where $\tau^+_{a,a}$ is the only one that has in its minimal standard Jacquet module at least one irreducible subquotient whose all exponents are non-negative.

Observe that by (2.3) there does exist an irreducible representation π of G_n for some n, such that $\mu^*(\sigma^+_{b,c}) \geq \delta(\frac{1}{2}, b) \otimes \pi$, but such representation does not exist for $\sigma^-_{b,c}$. Now, using (2) of Theorem 1.3 of [20], we derive the following result from Theorem 2.3 of [14].

Theorem 3.1. *With discrete series being subrepresentations, we have in* $R(G)$

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma = \sigma_a + L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma),$$
$$\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma = \sigma^+_{b,c} + \sigma^-_{b,c} + L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma).$$

Here

$$Jord(\sigma_a) = \{(2a+1, \rho)\} \cup Jord(\sigma),$$
$$Jord(\sigma^+_{b,c}) = Jord(\sigma^-_{b,c}) = \{(2b+1, \rho), (2c+1, \rho)\} \cup Jord(\sigma).$$

Further, $\epsilon_{\sigma_a}, \epsilon_{\sigma^+_{b,c}}$, *and* $\epsilon_{\sigma^-_{b,c}}$ *extend* ϵ_σ, *such that* $\epsilon_{\sigma_a}(2a+1, \rho) = 1$, *and* $\epsilon_{\sigma^+_{b,c}}(2b+1, \rho) = \epsilon_{\sigma^+_{b,c}}(2c+1, \rho) = 1, \epsilon_{\sigma^-_{b,c}}(2b+1, \rho) = \epsilon_{\sigma^-_{b,c}}(2c+1, \rho) = -1.$

The next proposition finishes our notation. As in Theorem 3.1, the sign above σ should denote value ± 1 of the epsilon function on Jordan blocks corresponding to the first two indices. We add the third index, and the epsilon function has value 1 on the block corresponding to it. The result follows from Theorem 2.1 of [14].

Proposition 3.2. *We use* $\sigma^+_{b,c,a} = \sigma^+_{a,b,c}$, $\sigma^-_{b,c,a}$ *and* $\sigma^-_{a,b,c}$ *to denote non-isomorphic discrete series subrepresentations, such that in* $R(G)$ *we have*

$$\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_a = \sigma^+_{b,c,a} + \sigma^-_{b,c,a} + L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_a),$$
$$\delta(-a, b) \rtimes \sigma_c = \sigma^+_{b,c,a} + \sigma^-_{a,b,c} + L(\delta(-a, b) \rtimes \sigma_c).$$

These discrete series appear as subrepresentations in induced representations. Also

$$Jord(\sigma^+_{b,c,a}) = Jord(\sigma^-_{b,c,a}) = Jord(\sigma^-_{a,b,c}) = \{(2a+1,\rho),(2b+1,\rho),(2c+1,\rho)\} \cup Jord(\sigma)$$

and $\epsilon_{\sigma^+_{b,c,a}}$, $\epsilon_{\sigma^-_{b,c,a}}$ *and* $\epsilon_{\sigma^-_{a,b,c}}$ *extend* ϵ_σ *such that*

$$\epsilon_{\sigma^+_{b,c,a}}(2a+1,\rho)=1,\quad \epsilon_{\sigma^+_{b,c,a}}(2b+1,\rho)=1,\quad \epsilon_{\sigma^+_{b,c,a}}(2c+1,\rho)=1,$$
$$\epsilon_{\sigma^-_{b,c,a}}(2a+1,\rho)=1,\quad \epsilon_{\sigma^-_{b,c,a}}(2b+1,\rho)=-1,\quad \epsilon_{\sigma^-_{b,c,a}}(2c+1,\rho)=-1,$$
$$\epsilon_{\sigma^-_{a,b,c}}(2a+1,\rho)=-1,\quad \epsilon_{\sigma^-_{a,b,c}}(2b+1,\rho)=-1,\quad \epsilon_{\sigma^-_{a,b,c}}(2c+1,\rho)=1.$$

Next is a consequence of Theorem 6.3 of [19], see also section 3 there.

PROPOSITION 3.3. *We have in* $R(G)$*, with multiplicity one:*

$$\nu^a\rho \times \cdots \times \nu^{\frac{1}{2}} \rtimes \sigma \geq \sigma_a.$$

The next lemma follows from Theorem 5.1 of [14], ii) and Lemma 5.2 of [5].

LEMMA 3.4. *We have in* $R(G)$*, with discrete series being a subrepresentation*

$$\delta(\tfrac{1}{2},a) \rtimes \sigma_b = \sigma^+_{a,b} + L(\delta(\tfrac{1}{2},a)\rtimes\sigma_b), \tag{3.1}$$

$$\delta(\tfrac{1}{2},b) \rtimes \sigma_a = \sigma^+_{a,b} + L(\delta(-a,b)\rtimes\sigma) + L(\delta(\tfrac{1}{2},a)\rtimes\sigma_b) + L(\delta(\tfrac{1}{2},b)\rtimes\sigma_a). \tag{3.2}$$

By Proposition 2.4 of [3] we have

LEMMA 3.5. *We have in* $R(G)$

$$\delta(\tfrac{1}{2},a) \rtimes \sigma^\pm_{b,c} = \sigma^\pm_{b,c,a} + L(\delta(\tfrac{1}{2},a)\rtimes\sigma^\pm_{b,c}),\ \text{so} \tag{3.3}$$

$$\mu^*(\sigma^\pm_{b,c,a}) \geq \delta(\tfrac{1}{2},a)\otimes\sigma^\pm_{b,c}. \tag{3.4}$$

By Propositions 11.6 and 11.8 of [5] we have composition series of $\delta(\frac{1}{2},b)\times\sigma^\pm_{a,c}$.

LEMMA 3.6. *Representations* $\delta(\frac{1}{2},b)\times\sigma^+_{a,c}$ *and* $\delta(\frac{1}{2},b)\times\sigma^-_{a,c}$ *have filtrations*

$$L(\delta(\tfrac{1}{2},a)\rtimes\sigma^+_{b,c}) + L(\delta(-a,b)\rtimes\sigma_c) \hookrightarrow \delta(\tfrac{1}{2},b)\rtimes\sigma^+_{a,c}/\sigma^+_{a,b,c} \twoheadrightarrow L(\delta(\tfrac{1}{2},b)\rtimes\sigma^+_{a,c}),$$
$$L(\delta(\tfrac{1}{2},a)\rtimes\sigma^-_{b,c}) \hookrightarrow \delta(\tfrac{1}{2},b)\rtimes\sigma^-_{a,c} \twoheadrightarrow L(\delta(\tfrac{1}{2},b)\rtimes\sigma^-_{a,c}).$$

By Propositions 3.2 of [8] and 13.7 of [5] we have

THEOREM 3.7. *Representation* $\delta(-a,c)\rtimes\sigma_b$ *has a filtration*

$$L(\delta(-b,c)\rtimes\sigma_a) + L(\delta(-a,b)\rtimes\sigma_c) \hookrightarrow \delta(-a,c)\rtimes\sigma_b/\sigma^+_{a,b,c} \twoheadrightarrow L(\delta(-a,c)\rtimes\sigma_b).$$

Proof. Composition factors and discrete series being a subrepresentation follow from Propositions 3.2 of [8] and its proof. Proposition 13.7 of [5] relies on Proposition 13.1 there, and for the sake of completness we write full proof of the latter, obtaining both results.

We have embeddings into a multiplicity one representation (Theorem 10.3 of [5])

$$\delta(-a,c)\rtimes\sigma_b\hookrightarrow\delta(\tfrac{1}{2},b)\times\delta(-a,c)\rtimes\sigma\hookleftarrow\delta(\tfrac{1}{2},b)\times\sigma^+_{a,c}$$

Now filtration of $\delta(\frac{1}{2},b)\times\sigma^+_{a,c}$ (Propositions 11.6 of [5]) show that $L(\delta(-a,b)\rtimes\sigma_c)$ needs quotient $\delta(-a,c)\rtimes\sigma_b/\sigma^+_{b,c,a}$ to embed.

Next, consider following two compositions of an embedding and an epimorphism

$$\delta(-a,c)\rtimes\sigma_b\hookrightarrow\delta(\tfrac{1}{2},c)\times\delta(-a,-\tfrac{1}{2})\rtimes\sigma_b\twoheadrightarrow\delta(\tfrac{1}{2},c)\times\sigma^+_{a,b},$$
$$\delta(-b,c)\times\sigma_a/\sigma^-_{b,c,a}\hookrightarrow\delta(\tfrac{1}{2},c)\times\delta(-b,-\tfrac{1}{2})\times\sigma_a/\sigma^-_{b,c,a}\twoheadrightarrow\delta(\tfrac{1}{2},c)\times\sigma^+_{a,b}.$$

By Lemma 9.1 of [5] and (3.4) we have $\delta(\frac{1}{2},a)\otimes L(\delta(-b,c)\rtimes\sigma)\le\mu^*(L(\delta(-b,c)\rtimes\sigma_a))$ and $\delta(\frac{1}{2},a)\otimes\sigma^+_{b,c}\le\mu^*(\sigma^+_{b,c,a})$ and one can check that they both appear with multiplicity one in all representations in above two compositions. Thus both compositions are embeddings and they induce an embedding

$$\delta(-b,c)\times\sigma_a/\sigma^-_{b,c,a}\hookrightarrow\delta(-a,c)\rtimes\sigma_b.$$

This shows that $L(\delta(-b,c)\rtimes\sigma_a)$ needs quotient $\delta(-a,c)\rtimes\sigma_b/\sigma^+_{b,c,a}$ to embed. □

The last two statements are Proposition 12.1 of [5] and the main result there.

Theorem 3.8. *There exists a filtration* $\{V_i\}$ *of* $\delta(-c,b)\times\delta(\frac{1}{2},a)\rtimes\sigma$ *such that*

$$\begin{aligned}V_1&\cong L(\delta(-b,c)\rtimes\sigma_a),\\ V_2/V_1&\cong\sigma^+_{b,c,a}+\sigma^-_{b,c,a}+L(\delta(-b,c)\times\delta(\tfrac{1}{2},a)\rtimes\sigma),\\ V_3/V_2&\cong L(\delta(\tfrac{1}{2},a)\rtimes\sigma^+_{b,c})+L(\delta(\tfrac{1}{2},a)\rtimes\sigma^-_{b,c}).\end{aligned}$$

Theorem 3.9. *There exists a filtration* $\{V_i\}$ *of* $\delta(-a,c)\times\delta(\frac{1}{2},b)\rtimes\sigma$ *such that*

$$\begin{aligned}V_1&\cong\sigma^+_{b,c,a}+L(\delta(\tfrac{1}{2},a)\rtimes\sigma^-_{b,c}),\\ V_2/V_1&\cong L(\delta(\tfrac{1}{2},a)\rtimes\sigma^+_{b,c})+L(\delta(-a,b)\rtimes\sigma_c)+L(\delta(\tfrac{1}{2},b)\rtimes\sigma^-_{a,c})+L(\delta(-b,c)\rtimes\sigma_a),\\ V_3/V_2&\cong L(\delta(\tfrac{1}{2},b)\rtimes\sigma^+_{a,c})+L(\delta(-a,c)\rtimes\sigma_b)+\sigma^-_{b,c,a}+L(\delta(-b,c)\times\delta(\tfrac{1}{2},a)\rtimes\sigma),\\ V_4/V_3&\cong L(\delta(-a,c)\times\delta(\tfrac{1}{2},b)\rtimes\sigma).\end{aligned}$$

4. An estimate for $\delta(\frac{1}{2},c)\times\delta(-a,b)\rtimes\sigma$

We start our decomposition by considering some standard intertwining operators.

$$\delta(\tfrac{1}{2},c)\times\delta(-a,b)\rtimes\sigma \xrightarrow{f_1} \delta(-a,b)\times\delta(\tfrac{1}{2},c)\rtimes\sigma \xrightarrow{f_2} \delta(-a,b)\times\delta(-c,-\tfrac{1}{2})\rtimes\sigma$$
$$\xrightarrow{f_3} \delta(-c,-\tfrac{1}{2})\times\delta(-a,b)\rtimes\sigma \xrightarrow{f_4} \delta(-c,-\tfrac{1}{2})\times\delta(-b,a)\rtimes\sigma.$$

For all $i\geq 1$ denote $K_i = \mathrm{Ker}\ f_i$. By Theorem 3.1 and the composition series of $\delta_1\times\delta_2$ we have

$$K_1\cong\delta(\tfrac{1}{2},b)\times\delta(-a,c)\rtimes\sigma,\quad K_2\cong\delta(-a,b)\rtimes\sigma_c,$$
$$K_3\cong\delta(-c,b)\times\delta(\tfrac{1}{2},a)\rtimes\sigma,\quad K_4\cong\delta(-c,-\tfrac{1}{2})\rtimes\sigma^+_{a,b}+\delta(-c,-\tfrac{1}{2})\rtimes\sigma^-_{a,b}.$$

We have in $R(G)$:

$$\begin{aligned}
&\delta(\tfrac{1}{2},c)\times\delta(-a,b)\rtimes\sigma = Dom(f_1) = Ker(f_1)+Im(f_1)\\
&= K_1 + Dom(f_2|_{Imf_1}) = K_1 + Ker(f_2|_{Imf_1}) + Im(f_2|_{Imf_1})\\
&\leq K_1+K_2+Im(f_2\circ f_1) = K_1+K_2+Dom(f_3|_{Im(f_2\circ f_1)})\\
&= K_1+K_2+Ker(f_3|_{Im(f_2\circ f_1)})+Im(f_3|_{Im(f_2\circ f_1)})\\
&\leq K_1+K_2+K_3+Im(f_3\circ f_2\circ f_1)\\
&\leq\ldots\leq K_1+K_2+K_3+K_4+Im(f_4\circ f_3\circ f_2\circ f_1).
\end{aligned}\tag{4.1}$$

The Langlands quotient $L(\delta(\frac{1}{2},c)\times\delta(-a,b)\rtimes\sigma)$ is the unique irreducible quotient of $\delta(\frac{1}{2},c)\times\delta(-a,b)\rtimes\sigma$, and the unique irreducible subrepresentation of $\delta(-c,-\frac{1}{2})\times\delta(-b,a)\rtimes\sigma$ appearing in both representations with multiplicity one. So in $R(G)$: $Im(f_4\circ f_3\circ f_2\circ f_1)\leq L(\delta(\frac{1}{2},c)\times\delta(-a,b)\rtimes\sigma)$.

We denoted $\psi=\delta(\frac{1}{2},c)\times\delta(-a,b)\rtimes\sigma$ and obtain the estimate, in $R(G)$:

$$\forall i\quad K_i\leq\psi\leq K_1+K_2+K_3+K_4+L(\psi).\tag{4.2}$$

By Section 3, it remains to decompose K_4 to obtain all composition factors of ψ.

5. On some irreducible tempered repesentations

Here we provide some results that we use for multiplicities of discrete series and their Jacquet modules in Section 6. By [6], we have decomposition of the induced representation

$$\delta(-a,a)\rtimes\sigma_c = T^+ + T^-\tag{5.1}$$

into a direct sum of non-equivalent tempered representations. They are important for discrete series $\sigma^{\pm}_{a,b,c}$ since by [13], see also Theorem 1.1 of [14], there exist unique $\epsilon,\eta\in\{\pm\}$, $\epsilon\neq\eta$, such that we have embeddings

$$\sigma^+_{a,b,c}\to\delta(a+1,b)\rtimes T^{\epsilon},\quad \sigma^-_{a,b,c}\to\delta(a+1,b)\rtimes T^{\eta}.\tag{5.2}$$

Similar to Definition 4.6 of [20], we use T^+ to denote the unique irreducible subquotient such that

$$\mu^*(T^+) \geq \delta(\tfrac{1}{2}, a) \times \delta(\tfrac{1}{2}, a) \otimes \pi,$$

for some irreducible representation π. Use (2.1) and (2.3) to calculate

$$\mu^*(T^+) \geq \delta(\tfrac{1}{2}, a) \times \delta(\tfrac{1}{2}, a) \otimes \sigma_c. \tag{5.3}$$

Thus

$$\mu^*(\delta(a+1, b) \rtimes T^+) \geq \delta(\tfrac{1}{2}, a) \times \delta(\tfrac{1}{2}, b) \otimes \sigma_c$$

The following lemma gives some more information.

Lemma 5.1. *Writting out all irreducible subquotients of form $\delta(\frac{1}{2}, a) \otimes \pi$, for some irreducible representation π, with maximum multiplicities, we have*

$$\mu^*(T^+) \geq \delta(\tfrac{1}{2}, a) \otimes L(\delta(\tfrac{1}{2}, a) \rtimes \sigma_c) + 2 \cdot \delta(\tfrac{1}{2}, a) \otimes \sigma_{a,c}^+,$$
$$\mu^*(T^-) \geq \delta(\tfrac{1}{2}, a) \otimes L(\delta(\tfrac{1}{2}, a) \rtimes \sigma_c).$$

Proof. Looking for $\delta(\frac{1}{2}, a) \otimes \pi$ in (5.1) we obtain

$$\mu^*(\delta(-a, a) \rtimes \sigma_c) \geq 2 \cdot \delta(\tfrac{1}{2}, a) \otimes \delta(\tfrac{1}{2}, a) \rtimes \sigma_c,$$

where by (3.1), in $R(G)$: $\delta(\frac{1}{2}, a) \rtimes \sigma_c = \sigma_{a,c}^+ + L(\delta(\frac{1}{2}, a) \rtimes \sigma_c)$. Since $\mu^*(\sigma_{a,c}^+) \geq \delta(\frac{1}{2}, a) \otimes \sigma_c$, transitivity of Jacquet module and (5.3) imply $\mu^*(T^+) \geq 2 \cdot \delta(\frac{1}{2}, a) \otimes \sigma_{a,c}^+$. The rest of the claim follows as

$$T^+ + T^- = \delta(-a, a) \rtimes \sigma_c \hookrightarrow \delta(\tfrac{1}{2}, a) \times \delta(-a, -\tfrac{1}{2}) \times \sigma_c$$

implies $\mu^*(T^{\pm}) \geq \delta(\frac{1}{2}, a) \otimes \delta(-a, -\frac{1}{2}) \otimes \sigma_c$ and $\mu^*(\sigma_{a,c}^+) \not\geq \delta(-a, -\frac{1}{2}) \otimes \sigma_c$. ∎

Proposition 5.2. *Writting with maximum multiplicities we have in $R(G)$:*

$$\delta(\tfrac{1}{2}, c) \times \delta(-a, a) \rtimes \sigma \geq 1 \cdot T^+ + 1 \cdot T^-, \tag{5.4}$$

where $\delta(-a, a) \rtimes \sigma_c = T^+ + T^-$ and $\mu^*(T^+) \geq \delta(\frac{1}{2}, a) \times \delta(\frac{1}{2}, a) \otimes \sigma_c$.

Proof. Since $\delta(-a, a) \rtimes \sigma_c \leq \delta(\frac{1}{2}, c) \times \delta(-a, a) \rtimes \sigma$, we look for $\delta(\frac{1}{2}, a) \otimes \pi$, for some irreducible representation π, in $\mu^*(\delta(\frac{1}{2}, c) \times \delta(-a, a) \rtimes \sigma)$ and obtain

$$2 \cdot \delta(\tfrac{1}{2}, a) \otimes \delta(\tfrac{1}{2}, a) \times \delta(\tfrac{1}{2}, c) \rtimes \sigma.$$

By Lemma 8.3 of [5], $L(\delta(\frac{1}{2}, a) \rtimes \sigma_c)$ appears once in $\delta(\frac{1}{2}, a) \times \delta(\frac{1}{2}, c) \rtimes \sigma$, so Lemma 5.1 gives (5.4). ∎

6. Discrete series subquotients

Here we determine discrete series subquotients in the following induced representations

$$\delta(\tfrac{1}{2}, c) \times \delta(-a, b) \rtimes \sigma \geq \delta(\tfrac{1}{2}, c) \rtimes \sigma^{+}_{a,b} + \delta(\tfrac{1}{2}, c) \rtimes \sigma^{-}_{a,b}. \tag{6.1}$$

Note that by Lemma 13.4 of [5] we know $\sigma^{+}_{b,c,a} \hookrightarrow \delta(\frac{1}{2}, c) \rtimes \sigma^{+}_{a,b}$.

Proposition 6.1. *Writting all discrete series with maximum multiplicities we have*

$$\delta(\tfrac{1}{2}, c) \times \delta(-a, b) \rtimes \sigma \geq 1 \cdot \sigma^{+}_{b,c,a} + 1 \cdot \sigma^{-}_{b,c,a} + 1 \cdot \sigma^{-}_{a,b,c}, \tag{6.2}$$

$$\delta(\tfrac{1}{2}, c) \rtimes \sigma^{+}_{a,b} \geq 1 \cdot \sigma^{+}_{b,c,a} + 1 \cdot \sigma^{-}_{b,c,a}, \tag{6.3}$$

$$\delta(\tfrac{1}{2}, c) \rtimes \sigma^{-}_{a,b} \geq 1 \cdot \sigma^{-}_{a,b,c}. \tag{6.4}$$

Proof. As in Lemma 5.1 of [5], we obtain that $\sigma^{+}_{b,c,a} = \sigma^{+}_{a,b,c,}$, $\sigma^{-}_{b,c,a}$ and $\sigma^{-}_{a,b,c}$ are only possible discrete subquotients in all equations. Proposition 3.2 and Section 4 imply that they do appear in (6.2).

To check multiplicity one of $\sigma^{\pm}_{b,c,a}$ in (6.2), use Lemma 5.2 of [5] and (3.4) to see multiplicity one of $\delta(\frac{1}{2}, a) \otimes \sigma^{\pm}_{b,c}$ in the appropriate Jacquet module. Now, multiplicity of $\sigma^{\pm}_{b,c,a}$ is one in (6.3), by Lemma 3.4, since $\mu^*(\sigma^{+}_{a,b}) \geq \delta(\frac{1}{2}, a) \otimes \sigma_b$. Thus multiplicity of $\sigma^{\pm}_{b,c,a}$ is zero in (6.4).

To check multiplicity of $\sigma^{\pm}_{a,b,c}$, we search for $\delta(a+1, b) \otimes \pi$, for some irreducible representation π, in $\mu^*(\delta(\frac{1}{2}, c) \times \delta(-a, b) \rtimes \sigma)$ and obtain

$$\delta(a+1, b) \otimes \delta(\tfrac{1}{2}, c) \times \delta(-a, a) \rtimes \sigma.$$

Proposition 5.2 implies multiplicity one of $\sigma^{\pm}_{a,b,c}$ in (6.2). Here we had contribution

$$\begin{aligned}\mu^*(\delta(-a, b) \rtimes \sigma) &\geq \delta(a+1, b) \otimes \delta(-a, a) \rtimes \sigma \\ &= \delta(a+1, b) \otimes \tau^{+}_{a,a} + \delta(a+1, b) \otimes \tau^{-}_{a,a}.\end{aligned}$$

Since $\mu^*(\sigma^{\pm}_{a,b}) \geq \delta(a+1, b) \otimes \tau^{\pm}_{a,a}$, the contribution comes from $\sigma^{+}_{a,b} + \sigma^{-}_{a,b}$. Using $\delta(\frac{1}{2}, c) \rtimes \sigma^{+}_{a,b} \geq \sigma^{+}_{b,c,a}$, we have $\delta(\frac{1}{2}, c) \rtimes \sigma^{\pm}_{a,b} \geq \sigma^{\pm}_{a,b,c}$, proving (6.3) and (6.4). ∎

7. Non-tempered candidates

As noted in Section 4, we search for non-tempered subquotients in

$$\delta(\tfrac{1}{2}, c) \rtimes \sigma^{+}_{a,b} \quad \text{and} \quad \delta(\tfrac{1}{2}, c) \rtimes \sigma^{-}_{a,b}.$$

LEMMA 7.1. *With maximum multiplicity of $\sigma^{\pm}_{b,c}$ being one, and $\epsilon, \eta \in \{\pm\}$ we have*

$$\sigma^{\epsilon}_{b,c} \leq \delta(a+1,c) \rtimes \sigma^{\eta}_{a,b} \quad \iff \epsilon = \eta,$$
$$\sigma^{+}_{b,c} + \sigma^{-}_{b,c} \leq \delta(a+1,c) \times \delta(-b,a) \rtimes \sigma.$$

PROOF. Denote the last representation by π. We have embeddings

$$\sigma^{\epsilon}_{b,c} \to \delta(-b,c) \rtimes \sigma \to \pi$$

By (2.3), $\delta(a+1,c) \times \delta(a+1,b) \otimes \tau^{\epsilon}_{a,a}$ appears once in $\mu^*(\sigma^{\epsilon}_{b,c})$. Check the same for $\mu^*(\pi)$. The first claim follows as $\mu^*(\sigma^{\epsilon}_{a,b}) \geq \delta(a+1,b) \otimes \tau^{\epsilon}_{a,a}$. □

Now we determine non-tempered candidates.

PROPOSITION 7.2. *Fix $\epsilon = +$ or $-$. If π is a non-tempered subquotient of $\delta(\frac{1}{2},c) \rtimes \sigma^{\epsilon}_{a,b}$, different from its Langlands quotient, then π can be*

$$L(\delta(\tfrac{1}{2},a) \rtimes \sigma^{\epsilon}_{b,c}) \quad \text{or} \quad L(\delta(\tfrac{1}{2},b) \rtimes \sigma^{\epsilon}_{a,c}),$$

and moreover if $\epsilon = +$, then π can also be

$$L(\delta(-a,c) \rtimes \sigma_b), \quad L(\delta(-a,b) \rtimes \sigma_c) \quad \text{or} \quad L(\delta(-b,c) \rtimes \sigma_a).$$

PROOF. We use Lemma 2.2 of [14] (in terms of that lemma $\pi \leq \delta(-l_1,l_2) \rtimes \sigma$, $-l_1 = \frac{1}{2}$, $l_2 = c$ and $\sigma = \sigma^{\epsilon}_{a,b}$). Write π as a Langlands subrepresentation

(7.1) $$\pi \to \delta(-\alpha_1,\beta_1) \times \delta(-\alpha_2,\beta_2) \times \cdots \times \delta(-\alpha_k,\beta_k) \rtimes \pi_t,$$

where $\alpha_i, \beta_i \in \frac{1}{2} + \mathbb{Z}$, $i = 1, \dots, k$,

(7.2) $$-\alpha_1 + \beta_1 \leq -\alpha_2 + \beta_2 \leq \cdots \leq -\alpha_k + \beta_k < 0,$$

and π_t is tempered. If $k = 1$ define $\pi' = \pi_t$. Else, if $k \geq 2$, let π' be a unique Langlands subrepresentation

$$\pi' \hookrightarrow \delta(-\alpha_2,\beta_2) \times \cdots \times \delta(-\alpha_k,\beta_k) \rtimes \pi_t.$$

Similarly, if π' is not tempered, π'' is defined. We have an embedding

(7.3) $$\pi \hookrightarrow \delta(-\alpha_1,\beta_1) \rtimes \pi'.$$

Again, if π' is not tempered,

(7.4) $$\pi' \hookrightarrow \delta(-\alpha_2,\beta_2) \rtimes \pi''.$$

By the lemma, there exists an irreducible representation σ_1 such that

(7.5) $$\begin{cases} \mu^*(\sigma^{\epsilon}_{a,b}) \geq \delta(\frac{1}{2},\beta_1) \otimes \sigma_1, \\ \pi' \leq \delta(\alpha_1+1,c) \rtimes \sigma_1, \end{cases}$$

and we must have

(7.6) $$c \geq \alpha_1 > \beta_1 \geq -\tfrac{1}{2}.$$

We have two possible cases:

a) $\beta_1 = -\frac{1}{2}$. Now $\sigma_1 = \sigma^{\epsilon}_{a,b}$ and

$$\pi' \leq \delta(\alpha_1 + 1, c) \rtimes \sigma^{\epsilon}_{a,b}. \tag{7.7}$$

- Assume that π' is tempered. By *ii)* of Lemma 2.2 of [14], we have $2\alpha_1 + 1 \in \mathrm{Jord}_\rho(\sigma^{\epsilon}_{a,b})$. So
  1) $\alpha_1 = a$. Then
  $$\pi' \leq \delta(a + 1, c) \rtimes \sigma^{\epsilon}_{a,b}.$$
  Lemma 8.1 of [13] implies $\pi' = \sigma^{+}_{b,c}$ or $\sigma^{-}_{b,c}$. By Lemma 7.1 we have $\pi' \cong \sigma^{\epsilon}_{b,c}$. Thus $\pi \cong L(\delta(\frac{1}{2}, a) \rtimes \sigma^{\epsilon}_{b,c})$.
  2) $\alpha_1 = b$. Then $\pi' \leq \delta(b + 1, c) \rtimes \sigma^{\epsilon}_{a,b}$. Lemma 8.1 of [13] implies $\pi' = \sigma^{+}_{a,c}$ or $\sigma^{-}_{a,c}$. By Section 8. of [20], we have $\pi' = \sigma^{\epsilon}_{a,c}$. So $\pi \cong L(\delta(\frac{1}{2}, b) \rtimes \sigma^{\epsilon}_{a,c})$.
- If π' is not tempered ($k \geq 2$), by Lemma 2.2 of [14], $2\beta_2 + 1 \in \mathrm{Jord}_\rho(\sigma^{\epsilon}_{a,b})$ and $(2\beta_2 + 1)_- = 2(\beta_2)_- + 1 \in \mathrm{Jord}_\rho(\sigma^{\epsilon}_{a,b}) = \{2a + 1, 2b + 1\}$ is defined. So $(\beta_2)_- = a$ and $\beta_2 = b$. Thus, (7.4) is
  $$\pi' \hookrightarrow \delta(-\alpha_2, b) \rtimes \pi''. \tag{7.8}$$
  Further, the lemma (in terms of the lemma $\alpha_1 \leq (\beta_2)_- < \beta_2 < \alpha_2 \leq l_2$) and (7.6) give
  $$\tfrac{1}{2} \leq \alpha_1 \leq a < b < \alpha_2 \leq c. \tag{7.9}$$
  We want to determine π''. The embedding (7.8) and (7.7) imply $\delta(-\alpha_2, b) \otimes \pi'' \leq \mu^*(\delta(\alpha_1 + 1, c) \rtimes \sigma^{\epsilon}_{a,b})$. So there exist $0 \leq i \leq j \leq c - \alpha_1$ and $\delta'' \otimes \sigma_2 \leq \mu^*(\sigma^{\epsilon}_{a,b})$ such that
  $$\begin{cases} \delta(-\alpha_2, b) \leq \delta(i - c, -\alpha_1 - 1) \times \delta(c + 1 - j, c) \times \delta'' \\ \pi'' \leq \delta(c + 1 - i, c - j) \rtimes \sigma_2. \end{cases} \tag{7.10}$$
  We have $j = 0$, $i = c - \alpha_2$ and thus $\delta'' \cong \delta(-\alpha_1, b)$. Now
  $$\delta'' \otimes \sigma_2 \cong \delta(-\alpha_1, b) \otimes \sigma_2 \leq \mu^*(\sigma^{\epsilon}_{a,b}) \leq \mu^*(\delta(-a, b) \rtimes \sigma), \tag{7.11}$$
  $$\pi'' \leq \delta(\alpha_2 + 1, c) \rtimes \sigma_2. \tag{7.12}$$
  Use (2.1) or (2.3) to search for σ_2 in (7.11). There are two posibilities
  i) $\alpha_1 = a, \quad \sigma_2 \cong \sigma$,
  ii) $\alpha_1 < a, \quad \sigma_2 \cong \delta(-a, -\alpha_1 - 1) \rtimes \sigma$.
  Assume *ii)*, the proof is similar for *i)*. By (7.12), we have
  $$\pi'' \leq \delta(-c, -\alpha_2 - 1) \times \delta(-a, -\alpha_1 - 1) \rtimes \sigma$$
  By (7.9) and [16], [15] and [21], the induced representation is irreducible. So $\alpha_3, \alpha_4, \beta_3$ and β_4 are defined ($k = 4$) and
  $$-\alpha_2 + \beta_2 = -\alpha_2 + b \not\leq -c - \alpha_2 - 1 = -\alpha_3 + \beta_3,$$

contradicting (7.2).

b) $\beta_1 > -\frac{1}{2}$. Then, by the lemma, $2\beta_1+1 \in \mathrm{Jord}_\rho(\sigma^\epsilon_{a,b}) = \{2a+1, 2b+1\}$. Here we have two options:

- $\beta_1 = a$. Then $\mu^*(\sigma^\epsilon_{a,b}) \geq \delta(\frac{1}{2}, a) \otimes \sigma_1$. Now (2.3) implies $\epsilon = +$, $\sigma_1 = \sigma_b$ and $\pi' \leq \delta(\alpha_1+1, c) \rtimes \sigma_b$ for some $a < \alpha_1 \leq c$; $\alpha_1 \in \mathbb{Z} + \frac{1}{2}$. By Proposition 3.1 of [14], depending on α_1, π' can be:

$$\pi' \cong \begin{cases} L(\delta(\alpha_1+1, b) \rtimes \sigma_c) \text{ or } L(\delta(\alpha_1+1, c) \rtimes \sigma_b), & a < \alpha_1 < b, \\ \sigma_c \text{ or } L(\delta(b+1, c) \rtimes \sigma_b), & \alpha_1 = b, \\ L(\delta(\alpha_1+1, c) \rtimes \sigma_b), & b < \alpha_1 < c, \\ \sigma_b, & \alpha_1 = b. \end{cases}$$

  Assume any of the cases for which $\pi' \cong L(\delta(\alpha_1+1, c) \rtimes \sigma_b)$, where $a < \alpha_1 < c$. Introducing Φ, we have embeddings

$$\pi \hookrightarrow \delta(-\alpha_1, a) \times \delta(-c, -\alpha_1 - 1) \rtimes \sigma_b =: \Phi,$$
$$L(\delta(-a, c) \rtimes \sigma_b) \hookrightarrow \delta(-c, a) \rtimes \sigma_b \hookrightarrow \Phi.$$

  By 9.1 of [21], and transitivity of Jacquet modules, for every

$$\delta(-\alpha_1, a) \otimes \delta(-c, -\alpha_1 - 1) \otimes \sigma_b \leq \mu^*(\Phi)$$

  there also exists one

$$\delta(-c, a) \otimes \sigma_b \leq \mu^*(\Phi).$$

  It is not hard to check that the last multiplicity is one.
  Thus $\pi \cong L(\delta(-a, c) \rtimes \sigma_b)$, as in the case $\pi' \cong \sigma_b$. Other cases similarly give $\pi \cong L(\delta(-a, b) \rtimes \sigma_c)$.
- $\beta_1 = b$. Then $\mu^*(\sigma^\epsilon_{a,b}) \geq \delta(\frac{1}{2}, b) \otimes \sigma_1$. Now (2.3) implies $\epsilon = +$, $\sigma_1 = \sigma_a$ and $\pi' \leq \delta(\alpha_1+1, c) \rtimes \sigma_a$ for some $b < \alpha_1 \leq c$, $\alpha_1 \in \mathbb{Z} + \frac{1}{2}$, which is irreducible by Proposition 3.1 of [14]. Thus

$$\pi \hookrightarrow \delta(-\alpha_1, b) \times \delta(-c, -\alpha_1 - 1) \rtimes \sigma_a.$$

  Similarly as above, we obtain $\pi \cong L(\delta(-b, c) \rtimes \sigma_a)$.

□

8. Non-tempered subquotients and their multiplicities

Now we show multiplicity one for non-tempered candidates from Proposition 7.2.

8.1. *Multiplicity of* $L(\delta(\frac{1}{2},a) \rtimes \sigma^{\pm}_{b,c})$.

LEMMA 8.1.1. *Writting with maximum multiplicity, we have in* $R(G)$*:*

$$\mu^*(\delta(\tfrac{1}{2},c) \times \delta(-a,b) \rtimes \sigma) \geq 1 \cdot \delta(-a,-\tfrac{1}{2}) \otimes \sigma^{\pm}_{b,c},$$
$$\mu^*(\delta(\tfrac{1}{2},c) \rtimes \sigma^{\pm}_{a,b}) \geq 1 \cdot \delta(-a,-\tfrac{1}{2}) \otimes \sigma^{\pm}_{b,c}.$$

PROOF. Looking for $\delta(-a,-\frac{1}{2}) \otimes \pi$, for some irreducible representation π, in $\mu^*(\delta(\frac{1}{2},c) \times \delta(-a,b) \rtimes \sigma)$ we obtain

$$\delta(-a,-\tfrac{1}{2}) \otimes \delta(a+1,c) \times \delta(-a,b) \rtimes \sigma,$$

with contribution $\mu^*(\delta(-a,b) \rtimes \sigma) \geq 1 \otimes \delta(-a,b) \rtimes \sigma$. We prove both claims using Lemma 7.1. ☐

PROPOSITION 8.1.2. *Writting with maximum multiplicity, we have in* $R(G)$*:*

$$\delta(\tfrac{1}{2},c) \times \delta(-a,b) \rtimes \sigma \geq 1 \cdot L(\delta(\tfrac{1}{2},a) \rtimes \sigma^{\pm}_{b,c}),$$
$$\delta(\tfrac{1}{2},c) \rtimes \sigma^{\pm}_{a,b} \geq 1 \cdot L(\delta(\tfrac{1}{2},a) \rtimes \sigma^{\pm}_{b,c}).$$

8.2. *Multiplicity of* $L(\delta(\frac{1}{2},b) \rtimes \sigma^{\pm}_{a,c})$.

LEMMA 8.2.1. *Writting with maximum multiplicity, we have in* $R(G)$*:*

$$1 \cdot \sigma^{\pm}_{b,c} \leq \delta(b+1,c) \rtimes \sigma^{\pm}_{a,b},$$
$$1 \cdot \sigma^{+}_{b,c} + 1 \cdot \sigma^{-}_{b,c} \leq \delta(b+1,c) \times \delta(-a,b) \rtimes \sigma.$$

PROOF. This follows from Section 8. of [20] and (2.1). ☐

LEMMA 8.2.2. *Writting with maximum multiplicity, we have in* $R(G)$*:*

$$\mu^*(\delta(\tfrac{1}{2},c) \times \delta(-a,b) \rtimes \sigma) \geq 1 \cdot \delta(-b,-\tfrac{1}{2}) \otimes \sigma^{\pm}_{a,c},$$
$$\mu^*(\delta(\tfrac{1}{2},c) \rtimes \sigma^{\pm}_{a,b}) \geq 1 \cdot \delta(-b,-\tfrac{1}{2}) \otimes \sigma^{\pm}_{a,c}.$$

PROOF. Looking for $\delta(-b,-\frac{1}{2}) \otimes \pi$, for some irreducible representation π, in $\mu^*(\delta(\frac{1}{2},c) \times \delta(-a,b) \rtimes \sigma)$ we obtain

$$\delta(-b,-\tfrac{1}{2}) \otimes \delta(b+1,c) \times \delta(-a,b) \rtimes \sigma,$$

with contribution $\mu^*(\delta(-a,b) \rtimes \sigma) \geq 1 \otimes \delta(-a,b) \rtimes \sigma$. We prove both claims using Lemma 8.2.1. ☐

PROPOSITION 8.2.3. *Writting with maximum multiplicity, we have in* $R(G)$*:*

$$\delta(\tfrac{1}{2},c) \times \delta(-a,b) \rtimes \sigma \geq 1 \cdot L(\delta(\tfrac{1}{2},b) \rtimes \sigma^{\pm}_{a,c}),$$
$$\delta(\tfrac{1}{2},c) \rtimes \sigma^{\pm}_{a,b} \geq 1 \cdot L(\delta(\tfrac{1}{2},b) \rtimes \sigma^{\pm}_{a,c}).$$

8.3. *Multiplicity of $L(\delta(-a,c) \rtimes \sigma_b)$.*

LEMMA 8.3.1. *Writting with maximum multiplicity, we have in $R(G)$:*

$$\mu^*(\delta(\tfrac{1}{2},c) \times \delta(-a,b) \rtimes \sigma) \geq 1 \cdot \delta(-c,a) \otimes \sigma_b,$$
$$\mu^*(\delta(\tfrac{1}{2},c) \rtimes \sigma^+_{a,b}) \geq 1 \cdot \delta(-c,a) \otimes \sigma_b.$$

PROOF. Looking for $\delta(-c,a) \otimes \pi$, for some irreducible representation π, in $\mu^*(\delta(\frac{1}{2},c) \times \delta(-a,b) \rtimes \sigma)$ we obtain

$$\delta(-c,-\tfrac{1}{2}) \times \delta(\tfrac{1}{2},a) \otimes \delta(\tfrac{1}{2},b) \rtimes \sigma,$$

with contribution $\mu^*(\delta(-a,b) \rtimes \sigma) \geq \delta(\frac{1}{2},a) \otimes \delta(\frac{1}{2},b) \rtimes \sigma$. The first claims followby Theorem 3.1 and the second by $\mu^*(\sigma^+_{a,b}) \geq \delta(\frac{1}{2},a) \otimes \sigma_b$. □

PROPOSITION 8.3.2. *Writting with maximum multiplicity, we have in $R(G)$:*

$$\delta(\tfrac{1}{2},c) \times \delta(-a,b) \rtimes \sigma \geq 1 \cdot L(\delta(-a,c) \rtimes \sigma_b),$$
$$\delta(\tfrac{1}{2},c) \rtimes \sigma^+_{a,b} \geq 1 \cdot L(\delta(-a,c) \rtimes \sigma_b).$$

8.4. *Multiplicities of $L(\delta(-a,b) \rtimes \sigma_c)$ and $L(\delta(-b,c) \rtimes \sigma_a)$.*

LEMMA 8.4.1. *Writting with maximum multiplicity, we have in $R(G)$:*

$$\mu^*(\delta(\tfrac{1}{2},c) \times \delta(-a,b) \rtimes \sigma) \geq 1 \cdot \delta(\tfrac{1}{2},a) \otimes L(\delta(\tfrac{1}{2},b) \rtimes \sigma_c) + 1 \cdot \delta(\tfrac{1}{2},a) \otimes L(\delta(-b,c) \rtimes \sigma).$$
$$\mu^*(\delta(\tfrac{1}{2},c) \rtimes \sigma^+_{a,b}) \geq 1 \cdot \delta(\tfrac{1}{2},a) \otimes L(\delta(\tfrac{1}{2},b) \rtimes \sigma_c) + 1 \cdot \delta(\tfrac{1}{2},a) \otimes L(\delta(-b,c) \rtimes \sigma).$$

PROOF. Looking for $\delta(\frac{1}{2},a) \otimes \pi$, for some irreducible representation π, in $\mu^*(\delta(\frac{1}{2},c) \times \delta(-a,b) \rtimes \sigma)$ we obtain

$$\delta(\tfrac{1}{2},a) \otimes \delta(\tfrac{1}{2},c) \times \delta(\tfrac{1}{2},b) \rtimes \sigma,$$

with contribution $\mu^*(\delta(-a,b) \rtimes \sigma) \geq \delta(\frac{1}{2},a) \otimes \delta(\frac{1}{2},b) \rtimes \sigma$. The first claims follows by Lemmas 5.2 and 8.3 of [5] and the second by $\mu^*(\sigma^+_{a,b}) \geq \delta(\frac{1}{2},a) \otimes \sigma_b$. □

PROPOSITION 8.4.2. *Writting with maximum multiplicity, we have in $R(G)$:*

$$\delta(\tfrac{1}{2},c) \times \delta(-a,b) \rtimes \sigma \geq 1 \cdot L(\delta(-a,b) \rtimes \sigma_c) + 1 \cdot L(\delta(-b,c) \rtimes \sigma_a)$$
$$\delta(\tfrac{1}{2},c) \rtimes \sigma^+_{a,b} \geq 1 \cdot L(\delta(-a,b) \rtimes \sigma_c) + 1 \cdot L(\delta(-b,c) \rtimes \sigma_a).$$

PROOF. Both claims follow by Lemmas 8.4 and 9.1 of [5]. □

9. Composition factors of $\delta(\frac{1}{2}, c) \rtimes \sigma^+_{a,b}$ and $\delta(\frac{1}{2}, c) \rtimes \sigma^-_{a,b}$.

Here we take in consideration Propositions 6.1 and 7.2 as well as all corollaries of Section 8 to immediately obtain composition factors of above representations.

Theorem 9.1. *We have in $R(G)$*

$$\begin{aligned}\delta(\tfrac{1}{2}, c) \rtimes \sigma^+_{a,b} =& L(\delta(-a, c) \rtimes \sigma_b) + L(\delta(-a, b) \rtimes \sigma_c) + L(\delta(-b, c) \rtimes \sigma_a) + \\ & L(\delta(\tfrac{1}{2}, c) \rtimes \sigma^+_{a,b}) + L(\delta(\tfrac{1}{2}, a) \rtimes \sigma^+_{b,c}) + L(\delta(\tfrac{1}{2}, b) \rtimes \sigma^+_{a,c}) + \\ & \sigma^+_{b,c,a} + \sigma^-_{b,c,a},\end{aligned}$$

$$\begin{aligned}\delta(\tfrac{1}{2}, c) \rtimes \sigma^-_{a,b} =& L(\delta(\tfrac{1}{2}, c) \rtimes \sigma^-_{a,b}) + L(\delta(\tfrac{1}{2}, a) \rtimes \sigma^-_{b,c}) + L(\delta(\tfrac{1}{2}, b) \rtimes \sigma^-_{a,c}) + \\ & \sigma^-_{a,b,c}.\end{aligned}$$

10. Multiplicity of $L(\delta(-b, c) \times \delta(\frac{1}{2}, a) \rtimes \sigma)$

Here we consider multiplicity of the above subquotient in $\delta(\frac{1}{2}, c) \times \delta(-a, b) \rtimes \sigma$, since it appears in two kernels in the decomposition in Section 4.

Lemma 10.1. *We have with maximum multiplicity:*

$$\mu^*(L(\delta(-b, c) \times \delta(\tfrac{1}{2}, a) \rtimes \sigma)) \geq 1 \cdot \delta(-a, b) \otimes \sigma_c$$

Proof. We can check multiplicity one of $\delta(-a, b) \otimes \sigma_c$ in $\mu^*(\delta(\frac{1}{2}, a) \rtimes L(\delta(-b, c) \rtimes \sigma))$, using (2.4). By Corollary 4.1, we have in $R(G)$:

$$\delta(\tfrac{1}{2}, a) \rtimes L(\delta(-b, c) \rtimes \sigma) = L(\delta(-b, c) \rtimes \sigma_a) + L(\delta(-b, c) \times \delta(\tfrac{1}{2}, a) \rtimes \sigma).$$

To see that $\delta(-a, b) \otimes \sigma_c \not\leq \mu^*(L(\delta(-b, c) \rtimes \sigma_a))$, check multiplicity one of $\delta(-a, b) \otimes \sigma_c$ in $\delta(-b, c) \rtimes \sigma_a$ and observe that it comes from $\sigma^+_{a,b,c}$. □

Using (2.1) and (2.3), it is not hard to check the following

Lemma 10.2. *We have with maximum multiplicity:*

$$\mu^*(\delta(-b, c) \times \delta(\tfrac{1}{2}, a) \rtimes \sigma)) \geq 2 \cdot \delta(-a, b) \otimes \sigma_c, \tag{10.1}$$

$$\mu^*(\delta(-a, c) \times \delta(\tfrac{1}{2}, b) \rtimes \sigma)) \geq 2 \cdot \delta(-a, b) \otimes \sigma_c, \tag{10.2}$$

$$\mu^*(\delta(-a, b) \times \delta(\tfrac{1}{2}, c) \rtimes \sigma)) \geq 4 \cdot \delta(-a, b) \otimes \sigma_c, \tag{10.3}$$

$$\mu^*(\delta(\tfrac{1}{2}, c) \rtimes \sigma^+_{a,b}) \geq 2 \cdot \delta(-a, b) \otimes \sigma_c, \tag{10.4}$$

Lemma 10.3. *We have with maximum multiplicity:*

$$\mu^*(L(\delta(\tfrac{1}{2}, c) \rtimes \sigma^+_{a,b})) \geq 1 \cdot \delta(-a, b) \otimes \sigma_c$$

Proof. By Proposition 9.1 and Theorem 3.9 all irreducible subquotients of $\delta(\frac{1}{2}, c) \rtimes \sigma^+_{a,b}$, except $L(\delta(\frac{1}{2}, c) \rtimes \sigma^+_{a,b})$ and $\sigma^-_{a,b,c}$, are contained in $\delta(-a, c) \times \delta(\frac{1}{2}, b) \rtimes \sigma$. Now (10.2) and (10.4) imply the claim, since exactly $\sigma^+_{a,b,c}$ and, by Lemma 10.1, $L(\delta(-b,c)\times\delta(\frac{1}{2},a)\rtimes\sigma)\not\leq\delta(\frac{1}{2},c)\rtimes\sigma^+_{a,b}$ contribute with $\delta(-a,b)\otimes\sigma_c$ in $\mu^*(\delta(-a,c)\times\delta(\frac{1}{2},b)\rtimes\sigma)$. ∎

Using (10.4) and Lemmas 10.1 and 10.3 have

LEMMA 10.4. *Let π be an irreducible subquotient of $\delta(\frac{1}{2},c)\times\delta(-a,b)\rtimes\sigma$. Then $\mu^*(\pi)\geq\delta(-a,b)\otimes\sigma_c$ if and only if π is one of the following*

$$\sigma^+_{a,b,c},\quad \sigma^-_{a,b,c},\quad L(\delta(-b,c)\times\delta(\tfrac{1}{2},a)\rtimes\sigma),\quad L(\delta(\tfrac{1}{2},c)\rtimes\sigma^+_{a,b}).$$

PROPOSITION 10.5. *We have in $R(G)$ with maximum multiplicity*

$$\delta(\tfrac{1}{2},c)\times\delta(-a,b)\rtimes\sigma\geq 1\cdot L(\delta(-b,c)\times\delta(\tfrac{1}{2},a)\rtimes\sigma)$$

PROOF. Follows directly from (10.3) and Lemma 10.4

∎

11. COMPOSITION FACTORS OF $\delta(\frac{1}{2},c)\times\delta(-a,b)\rtimes\sigma$ AND THE FIRST FILTRATION

Finally we determine composition factors of the induced representation.

THEOREM 11.1. *We have in $R(G)$*

$$\begin{aligned}\delta(\tfrac{1}{2},c)\times\delta(-a,b)\rtimes\sigma =&L(\delta(\tfrac{1}{2},a)\times\delta(-b,c)\rtimes\sigma)+\\&L(\delta(\tfrac{1}{2},b)\times\delta(-a,c)\rtimes\sigma)+\\&L(\delta(\tfrac{1}{2},c)\times\delta(-a,b)\rtimes\sigma)\\&+\\&L(\delta(-a,c)\rtimes\sigma_b)+L(\delta(-a,b)\rtimes\sigma_c)+L(\delta(-b,c)\rtimes\sigma_a)\\&+\\&L(\delta(\tfrac{1}{2},a)\rtimes\sigma^+_{b,c})+L(\delta(\tfrac{1}{2},b)\rtimes\sigma^+_{a,c})+L(\delta(\tfrac{1}{2},c)\rtimes\sigma^+_{a,b})+\\&L(\delta(\tfrac{1}{2},a)\rtimes\sigma^-_{b,c})+L(\delta(\tfrac{1}{2},b)\rtimes\sigma^-_{a,c})+L(\delta(\tfrac{1}{2},c)\rtimes\sigma^-_{a,b})\\&+\\&\sigma^+_{a,b,c}+\sigma^-_{a,b,c}+\sigma^-_{b,c,a}.\end{aligned}$$

PROOF. Theorems 3.7, 3.8, 3.9 and 9.1 describe kernels K_i in (4.2), and thus determine all composition factors. Multiplicity one for discrete series is proved in Propostiton 6.1. For the other subquotients, appearing in two or more kernels K_i, multiplicity one is proved in Propostions 8.1.2, 8.2.3, 8.3.2, 8.4.2 and 10.5. ∎

For the sake of the completeness, we say here some more information about kernels from Section 4. Using notation there, let us denote

$$k_1=K_1,\quad k_i=K_i\cap\operatorname{Im}(f_{i-1}\circ\cdots\circ f_1),\quad i=2,3,4.$$

Lemma 11.2. *We have*

$$k_2 \cong \sigma^-_{a,b,c}, \quad k_3 \cong \{0\}, \quad k_4 \cong L(\delta(\tfrac{1}{2},c) \rtimes \sigma^+_{a,b}) + L(\delta(\tfrac{1}{2},c) \rtimes \sigma^-_{a,b}).$$

Proof. Theorem 3.9 gives composition series of K_1. Proposition 3.2 and Theorems 3.8, 9.1 and 11.1 give composition factors of K_2, K_3, K_4 and ψ. All factors of K_2, except $\sigma^-_{a,b,c}$, are in K_1 giving the first equation. All factors in K_3 are in either K_1 or K_2, giving the second equation. Similarly goes for the last equation. □

This gives us a filtration of ψ, but in Section 13 we obtain a more precise result.

Corollary 11.3. *There exists a filtration $\{V_i\}$ of $\delta(\frac{1}{2},c) \times \delta(-a,b) \rtimes \sigma$ such that*

$$\begin{aligned}
V_1 \cong& \sigma^+_{b,c,a} + L(\delta(\tfrac{1}{2},a) \rtimes \sigma^-_{b,c}),\\
V_2/V_1 \cong& L(\delta(\tfrac{1}{2},a) \rtimes \sigma^+_{b,c}) + L(\delta(-a,b) \rtimes \sigma_c) + L(\delta(\tfrac{1}{2},b) \rtimes \sigma^-_{a,c}) + L(\delta(-b,c) \rtimes \sigma_a),\\
V_3/V_2 \cong& L(\delta(\tfrac{1}{2},b) \rtimes \sigma^+_{a,c}) + L(\delta(-a,c) \rtimes \sigma_b) + \sigma^-_{b,c,a} + L(\delta(-b,c) \times \delta(\tfrac{1}{2},a) \rtimes \sigma),\\
V_4/V_3 \cong& L(\delta(-a,c) \times \delta(\tfrac{1}{2},b) \rtimes \sigma),\\
V_5/V_4 \cong& \sigma^-_{a,b,c},\\
V_6/V_5 \cong& L(\delta(\tfrac{1}{2},c) \rtimes \sigma^+_{a,b}) + L(\delta(\tfrac{1}{2},c) \rtimes \sigma^-_{a,b}).
\end{aligned}$$

12. Composition series of $\delta(\frac{1}{2},c) \rtimes \sigma^+_{a,b}$ and $\delta(\frac{1}{2},c) \rtimes \sigma^-_{a,b}$.

Here we determine composition series of kernel K_4 from Section 4.

Theorem 12.1. *There exists a filtration $\{V_i\}$ of $\delta(\frac{1}{2},c) \rtimes \sigma^+_{a,b}$ where*

$$\begin{aligned}
V_1 =& \sigma^+_{b,c,a},\\
V_2/V_1 \cong& L(\delta(\tfrac{1}{2},a) \rtimes \sigma^+_{b,c}) + L(\delta(-a,b) \rtimes \sigma_c) + L(\delta(-b,c) \rtimes \sigma_a),\\
V_3/V_2 \cong& L(\delta(\tfrac{1}{2},b) \rtimes \sigma^+_{a,c}) + L(\delta(-a,c) \rtimes \sigma_b) + \sigma^-_{b,c,a},\\
V_4/V_3 \cong& L(\delta(\tfrac{1}{2},c) \rtimes \sigma^+_{a,b}).
\end{aligned}$$

Proof. By Theorem 11.1 we have multiplicity one representations:

$$\delta(\tfrac{1}{2},c) \rtimes \sigma^+_{a,b} \hookrightarrow \delta(\tfrac{1}{2},c) \times \delta(-a,b) \rtimes \sigma \hookleftarrow \delta(\tfrac{1}{2},b) \times \delta(-a,c) \rtimes \sigma.$$

By Theorems 3.9 and 9.1 all irreducible subquotents of $\delta(\frac{1}{2},c) \rtimes \sigma^+_{a,b}$, except its Langlands quotient, appear as irreducible subquotients in $\delta(\frac{1}{2},b) \times \delta(-a,c) \rtimes \sigma$. We conclude that the unique maximal submodule M of $\delta(\frac{1}{2},c) \rtimes \sigma^+_{a,b}$ is isomorphic to a subrepresentation of $\delta(\frac{1}{2},b) \times \delta(-a,c) \rtimes \sigma$. Further, by filtrations of $\delta(\frac{1}{2},b) \times \delta(-a,c) \rtimes \sigma$ (Theorem 14.1 of [5]) and $\delta(\frac{1}{2},b) \rtimes \sigma^-_{a,c}$ (Proposition 10.2. of [5]) we have

$$M \hookrightarrow \delta(\tfrac{1}{2},b) \times \delta(-a,c) \rtimes \sigma / \delta(\tfrac{1}{2},b) \rtimes \sigma^-_{a,c}$$

and the claim follows. □

Now we consider composition series of $\delta(\frac{1}{2},c) \rtimes \sigma^-_{a,b}$. First we need a lemma.

LEMMA 12.2. *We have an embedding*

$$\delta(\tfrac{1}{2},b) \rtimes \sigma^-_{a,c} \hookrightarrow \delta(\tfrac{1}{2},c) \rtimes \sigma^-_{a,b}. \tag{12.1}$$

PROOF. By Theorem 11.1 we have multiplicity one representations:

$$\begin{aligned}\delta(\tfrac{1}{2},b) \rtimes \sigma^-_{a,c} &\hookrightarrow \delta(\tfrac{1}{2},b) \times \delta(-a,c) \rtimes \sigma \\ &\hookrightarrow \delta(\tfrac{1}{2},c) \times \delta(-a,b) \rtimes \sigma \hookleftarrow \delta(\tfrac{1}{2},c) \rtimes \sigma^-_{a,b}.\end{aligned}$$

By Lemma 3.6 and Theorem 9.1, all irreducible subquotients of $\delta(\frac{1}{2},b) \rtimes \sigma^-_{a,c}$ appear in composition factors of $\delta(\frac{1}{2},c) \rtimes \sigma^-_{a,b}$, and the claim follows. □

By Lemmas 3.6 and 12.2 we have composition series

THEOREM 12.3. *There exists a filtration* $\{V_i\}$ *of* $\delta(\frac{1}{2},c) \rtimes \sigma^-_{a,b}$ *where*

$$\begin{aligned}&V_1 \cong L(\delta(\tfrac{1}{2},a) \rtimes \sigma^-_{b,c}), && V_2/V_1 \cong L(\delta(\tfrac{1}{2},b) \rtimes \sigma^-_{a,c}),\\ &V_3/V_2 \cong \sigma^-_{a,b,c}, && V_4/V_3 \cong L(\delta(\tfrac{1}{2},c) \rtimes \sigma^-_{a,b}).\end{aligned}$$

13. COMPOSITION SERIES OF $\delta(\frac{1}{2},c) \times \delta(-a,b) \rtimes \sigma$

Now we have the main result.

THEOREM 13.1. *Let* $\psi = \delta(\frac{1}{2},c) \times \delta(-a,b) \rtimes \sigma$ *and define representations*

$$\begin{aligned}W_1 =& \sigma^+_{b,c,a} + L(\delta(\tfrac{1}{2},a) \rtimes \sigma^-_{b,c}),\\ W_2 =& L(\delta(\tfrac{1}{2},a) \rtimes \sigma^+_{b,c}) + L(\delta(\tfrac{1}{2},b) \rtimes \sigma^-_{a,c}) + L(\delta(-b,c) \rtimes \sigma_a) + L(\delta(-a,b) \rtimes \sigma_c),\\ W_3 =& \sigma^-_{b,c,a} + \sigma^-_{a,b,c} + L(\delta(\tfrac{1}{2},b) \rtimes \sigma^+_{a,c}) + L(\delta(-a,c) \rtimes \sigma_b) + L(\delta(-b,c) \times \delta(\tfrac{1}{2},a) \rtimes \sigma),\\ W_4 =& L(\delta(\tfrac{1}{2},b) \times \delta(-a,c) \rtimes \sigma) + L(\delta(\tfrac{1}{2},c) \rtimes \sigma^+_{a,b}) + L(\delta(\tfrac{1}{2},c) \rtimes \sigma^-_{a,b}),\\ W_5 =& L(\psi).\end{aligned}$$

Then there exists a sequence $\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 \subseteq V_5 = \psi$*, such that*

$$V_i/V_{i-1} \cong W_i, \quad i = 1, \dots, 5.$$

PROOF. Since

$$\delta(\tfrac{1}{2},b) \times \delta(-a,c) \rtimes \sigma \hookrightarrow \psi,$$

its filtration, Theorem 3.9, implies the existence of V_1 and V_2. Additionaly, composition factors of $\delta(\frac{1}{2},c) \rtimes \sigma^-_{a,b}$, Theorem 9.1, and $\delta(\frac{1}{2},c) \rtimes \sigma^-_{a,b} \hookrightarrow \psi$ imply

$$\sigma^-_{a,b,c} \hookrightarrow \psi/V_2 \hookleftarrow \delta(\tfrac{1}{2},b) \times \delta(-a,c) \rtimes \sigma/V_2.$$

This shows $W_3 \hookrightarrow \psi/V_2$ proving existence of V_3. Finally, Theorems 3.9, 9.1 and

$$\delta(\tfrac{1}{2}, c) \rtimes \sigma^{\pm}_{a,b} \hookrightarrow \psi \hookleftarrow \delta(\tfrac{1}{2}, b) \times \delta(-a, c) \rtimes \sigma.$$

show $W_4 \hookrightarrow \psi/V_3$, proving existence of V_4. □

References

[1] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups. I,* Ann. Sci. École Norm. Sup. **(4) 10** (1977), no. 4, 441–472.

[2] B. Bošnjak, *Representations induced from cuspidal and ladder representations of classical p-adic groups,* Proc. Amer. Math. Soc. **149** (2021), no. 12, 5081–5091.

[3] I. Ciganović, *Composition series of a class of induced representations, a case of one half cuspidal reducibility,* Pacific J. Math. **296** (2018), no. 1, 21–30.

[4] I. Ciganović, *Composition series of a class of induced representations built on discrete series,* Manuscripta Math. **170** (2023), no. 1-2, 1–18.

[5] I. Ciganović, *Parabolic induction from two segments, linked under contragredient, with a one half cuspidal reducibility, a special case*, Glas. Mat. Ser. III **59(79)** (2024), no. 1, 77–105.

[6] D. Goldberg, *Reducibility of induced representations for $Sp(2n)$ and $SO(2n)$,* Amer. J. Math, **116** (1994), no. 5, 1101–1151.

[7] Y. Kim, B. Liu, and I. Matić, *Degenerate principal series for classical and odd GSpin groups in the general case,* Represent. Theory **24** (2020), 403–434.

[8] I. Matić, *On discrete series subrepresentations of the generalized principal series,* Glas. Mat. Ser.III **51(71)** (2016), no. 1, 125–152.

[9] I. Matić, *Representations induced from the Zelevinsky segment and discrete series in the half-integral case,* Forum Math. **33** (2021), no. 1, 193–212.

[10] I. Matić, M. Tadić, *On Jacquet modules of representations of segment type,* Manuscripta Math. **147** (2015), no. 3-4, 437–476.

[11] I. Matić, M. Tadić, *On Jacquet modules of representations of segment type,* `http://www.hazu.hr/~tadic/53-JM-segment.pdf`

[12] C. Mœglin, *Sur la classification des séries discrètes des groupes classiques p-adiques: paramètres de Langlands et exhaustivité.* J. Eur. Math. Soc. (JEMS) **4** (2002), no. 2, 143–200.

[13] C. Mœglin, M. Tadić, *Construction of discrete series for classical p-adic groups,* J. Amer. Math. Soc. **15** (2002), no. 3, 715–786.

[14] G. Muić, *Composition series of generalized principal series; the case of strongly positive discrete series,* Israel J. Math. **140** (2004), 157–202.

[15] G. Muić, *Reducibility of generalized principal series,* Canad. J. Math. **57** (2005), no. 3, 616–647.

[16] G. Muić, *Reducibility of standard representations,* Pacific J. Math. **222** (2005), no. 1, 133–168.

[17] M. Tadić, *Structure arising from induction and Jacquet modules of representations of classical p-adic groups,* J. Algebra **177** (1995), no. 1, 1–33.

[18] M. Tadić, *On reducibility of parabolic induction*, Israel J. Math. **107** (1998), 29–91.

[19] M. Tadić, *On regular square integrable representations of p-adic groups,* Amer. J. Math. **120** (1998), no. 1, 159–210.

[20] M. Tadić, *On tempered and square integrable representations of classical p-adic groups,* Sci. China Math. **56** (2013), no. 11, 2273–2313.

[21] A. V. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of* $GL(n)$, Ann. Sci. École Norm. Sup. **(4) 13** (1980), no. 2, 165–210.

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