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DEGENERATE EISENSTEIN SERIES ON THE SYMPLECTIC GROUP OF RANK TWO REVISITED BY A NEW METHOD FOR PROVING HOLOMORPHY

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Abstract. In a recent preprint entitled "Holomorphy of Eisenstein series – a new method and applications in the case of the general linear group", the author has developed a new method for proving holomorphy of degenerate Eisenstein series, based on the Franke filtration of spaces of automorphic forms. In this paper, the method is applied in the case of degenerate Eisenstein series on the symplectic group of rank two. Although the analytic properties of Eisenstein series in that case are already known, the goal is to exhibit the method in a simple setting, in which all additional technical details are peeled off.

1. Introduction

The goal of this short note is to exhibit the strength and scope of the new method for proving holomorphy of degenerate Eisenstein series, developed by the author in [6], in a setting as simple as possible. For that purpose we consider the Eisenstein series on the symplectic group of rank two. In the case of the symplectic group of rank two, the poles of Eisenstein series are completely understood and described in the work of Hanzer and Muić [7]. Therefore, we claim no novelty of the results in this paper, except for the application and presentation of the new method [6].

The development of the method was motivated by the work of Ginzburg and Soudry [4], and the work of Hanzer and Muić [8], who studied the degenerate Eisenstein series on the general linear group by very different methods. Our method relies on the Franke filtration of spaces of automorphic forms.

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These filtrations are already completely described in [5] in the case of the symplectic group of rank two. The residual spectra of the symplectic group of rank two, and of the Levi factors of all of its proper parabolic subgroups, are determined, cf. [9], [11]. This makes the symplectic group of rank two a perfect setting for the presentation of the new method for proving holomorphy of Eisenstein series. All the technical additional details are already known from the previous work, so that the method can be described and applied in its bare core form with all the technical additional details peeled off.

However, we should be very careful not to fall in the trap of making a circular argument. The point is that in constructing the Franke filtration, we require some information on the analytic properties of Eisenstein series. On the other hand, we intend to use the Franke filtration to prove holomorphy of Eisenstein series. This seems as a circular argument, and we now make clear what is the precise problem we address.

PROBLEM 1.1. *Assume that the residual spectrum of the group under consideration and all the Levi factors of its proper parabolic subgroups are understood in terms of induced representations, that is, in terms of the cuspidal support. Under this assumption determine the regions of holomorphy of (degenerate) Eisenstein series on the group under consideration.*

In this paper, we solve this problem in the case of the symplectic group of rank two using our method for proving holomorphy of Eisenstein series based on the Franke filtration of spaces of automorphic forms [6]. We emphasize that the Franke filtration obtained in [5] in the case of the symplectic group of rank two is constructed using only the information on the residual spectrum that are assumed in the statement of the problem above.

Our method of [6] is based on the construction of the Franke filtration, cf. [2], [3]. Given a degenerate Eisenstein series associated with a residual representation of a Levi factor, and the evaluation point in the closure of the positive Weyl chamber, its cuspidal support can be determined. In the Franke filtration of the space of automorphic forms with that fixed cuspidal support, the given Eisenstein series contributes to some quotient of the filtration. The key observation is that, if the given Eisenstein series is not holomorphic at the given evaluation point, then it cannot contribute to the deepest quotient of the filtration, that is, the quotient which is a subrepresentation of the space of automorphic forms with the fixed cuspidal support. This observation is a consequence of the fact that, by the very construction of the Franke filtration, the coefficients in the principal part of the Laurent expansion of the Eisenstein series around the pole must belong to a deeper quotient of the filtration than the Eisenstein series itself. See [2], [3, Sect. 1.4], [6] for more details. In summary, if we can show that the given Eisenstein series evaluated at the given evaluation point contributes to the deepest quotient of the Franke filtration of the appropriate space of automorphic forms, then we may conclude that it is holomorphic at the given evaluation point.

Since the paper is intended to serve as a short note presenting the new method, we often omit details of the notation and basic notions, and refer to [5] and [6] for details. In these references all the necessary prerequisites are carefully introduced, and we make sure to use here exactly the same notation.

The paper is structured as follows. In Section 2 we briefly review the necessary preliminaries, although for details the reader should consult the references therein. Section 3 deals with the degenerate Eisenstein series arising from residual representations of the Siegel parabolic subgroup, while Section 4 considers those arising from the non-Siegel parabolic subgroup of the symplectic group of rank two. In Section 5, we summarize the most interesting findings.

2. Brief review of the preliminaries

In this section we briefly recall the notation regarding the symplectic group of rank two, automorphic forms and representations, and Eisenstein series, required for the rest of the paper. For details, we refer to [5] or the standard references mentioned below.

Let F be an algebraic number field with the ring of adèles $\mathbb{A}$, and the group of idèles $\mathbb{I}$. Let $G = Sp_2$ be the symplectic group of rank two defined over F. The fixed choice of a Borel subgroup of G is denoted by B, and a maximal split torus in B by T. The Siegel parabolic subgroup refers to the standard parabolic subgroup P_1 with the Levi factor L_1 isomorphic to GL_2. Its Levi decomposition is $P_1 = L_1 N_1$, where N_1 is the unipotent radical. The non-Siegel parabolic subgroup, also called the Heisenberg parabolic subgroup, refers to the standard parabolic subgroup P_2 with the Levi factor L_2 isomorphic to $GL_1 \times SL_2$. Its Levi decomposition is $P_2 = L_2 N_2$, where N_2 is the unipotent radical.

Automorphic forms on the adèlic group $G(\mathbb{A})$ are defined as in Borel–Jacquet [1]. By definition, besides other requirements, these are K-finite, where K is a fixed maximal compact subgroup of $G(\mathbb{A})$. However, the K-finiteness is not preserved under the action of $G(\mathbb{A})$ by right translations. Therefore, the considered spaces of automorphic forms on $G(\mathbb{A})$ are not representations of the full group $G(\mathbb{A})$. The problem is at the archimedean places of F, at which only the structure of a Harish-Chandra module is present. Nevertheless, we always refer to these spaces as automorphic representations of $G(\mathbb{A})$, as usual in the theory of automorphic forms, cf. [1, Sect. 4.6].

The discrete spectrum of $G(\mathbb{A})$ is the discrete part in the spectral decomposition of the L^2-space of (classes of) square-integrable measurable functions on $G(F)\backslash G(\mathbb{A})$ with respect to a Haar measure. The discrete spectrum automorphic representations of $G(\mathbb{A})$ are the automorphic representations, in the sense as above, of $G(\mathbb{A})$ on the space of smooth K-finite functions in irreducible summands in the spectral decomposition of the discrete spectrum of $G(\mathbb{A})$.

An automorphic form on $G(\mathbb{A})$ is called cuspidal if its constant terms vanish along all proper parabolic subgroups of G. Cuspidal automorphic representations of $G(\mathbb{A})$ refer to those discrete spectrum representations of $G(\mathbb{A})$ that consist of cuspidal automorphic forms. The non-cuspidal discrete spectrum representations of $G(\mathbb{A})$ are referred to as residual automorphic representations of $G(\mathbb{A})$. The latter form the so-called residual spectrum of $G(\mathbb{A})$.

All these notions related to automorphic forms can also be formulated for the Levi factors of G, but we omit the details. Given an automorhic representation Π in the discrete spectrum of the Levi factor $L(\mathbb{A})$ of one of the three proper standard parabolic subgroups of G, the Eisenstein series associated with Π is denoted by $E(f,\underline{s})$ and defined as in [5, Sect. 3.2], [3], viewed as a function of $g \in G(\mathbb{A})$, where $\underline{s}$ is the complex parameter, and f is a function in the space $\mathcal{W}_\Pi$ associated with Π. For the details of the definition and the fundamental properties of Eisenstein series, see the standard references [13], [12].

Let $\{P\}$ denote the associate class of proper parabolic subgroups of G represented by a standard proper parabolic subgroup P, that is, P is one of the parabolic subgroups B, P_1, P_2. Let $\varphi(\pi)$ be an associate class of twisted cuspidal automorphic representations of the Levi factors of elements of $\{P\}$, represented by a representation π of the Levi factor $L(\mathbb{A})$ of P. Then, $\mathcal{A}_{\{P\},\varphi(\pi)}$ is defined as the space of all automorphic forms on $G(\mathbb{A})$ with the cuspidal support in the associate class $\varphi(\pi)$, cf. [13], [3] for a precise definition.

3. Degenerate Eisenstein series on the Siegel parabolic subgroup

The Levi factor L_1 of the Siegel parabolic subgroup P_1 is isomorphic to GL_2. The residual spectrum of $GL_2(\mathbb{A})$ consists of characters $\Pi = \chi \circ \det$, where χ ranges over unitary Hecke characters χ of the group of idèles $\mathbb{I}$, cf. [5, Sect. 6.1]. The cuspidal support of Π is represented by the character

$$\chi|\cdot|^{1/2} \otimes \chi|\cdot|^{-1/2},$$

of $GL_1(\mathbb{A}) \times GL_1(\mathbb{A}) \cong \mathbb{I} \times \mathbb{I}$, where $|\cdot|$ is the usual adèlic absolute value.

Hence, all degenerate Eisenstein series associated with the Siegel parabolic subgroup arise from residual representations Π as above viewed as representations of the Levi factor $L_1(\mathbb{A})$. We denote these Eisenstein series by $E(f,s)$, where s is the complex parameter, and f is a function in the space $\mathcal{W}_\Pi$ associated with Π as in [5, Sect. 3.2]. More precisely, $\mathcal{W}_\Pi$ is the space of K-finite smooth complex functions f on $L_1(F)N_1(\mathbb{A})\backslash G(\mathbb{A})$ such that the function on

$L_1(\mathbb{A})$ given by the assignment $l \mapsto f(lg)$ belongs to the space of Π for all $g \in G(\mathbb{A})$.

The space of complex parameters is one-dimensional, identified with characters of $L_1(\mathbb{A}) \cong GL_2(\mathbb{A})$ by the assignment

$$s \mapsto |\det|^s,$$

for $s \in \mathbb{C}$. As explained in [10, p. 121], see also [5, Sect. 3.1], we may and will normalize the representations in such a way that the possible poles of Eisenstein series are real. On the other hand, our method is applicable only for values of complex parameters in the closure of the positive Weyl chamber, which is in this case given by the condition $Re(s) \geq 0$. Hence, in the theorem below, we only consider values of the complex parameter $s = s_0$ that are real and $s_0 \geq 0$.

THEOREM 3.1. *Let* $\Pi = \chi \circ \det$ *be a character of the Levi factor* $L_1(\mathbb{A})$ *of the Siegel parabolic subgroup* P_1 *of* G*, where* χ *is a unitary Hecke character of the group of idèles* $\mathbb{I}$*. Let* $E(f,s)$ *be the degenerate Eisenstein series associated with* Π*. Then,* $E(f,s)$ *is holomorphic at* $s = s_0$ *in the following cases:*

- $s_0 \geq 0$ *and* $s_0 \neq 3/2, 1/2$*,*
- $s_0 = 3/2$ *and* $\chi \neq \mathbf{1}$*,*
- $s_0 = 1/2$ *and* $\chi^2 \neq \mathbf{1}$*,*

where $\mathbf{1}$ *is the trivial character of* $\mathbb{I}$*.*

PROOF. The cuspidal support of the Eisenstein series $E(f,s)$ evaluated at $s = s_0$ is represented by the character

$$\chi|\cdot|^{s_0+1/2} \otimes \chi|\cdot|^{s_0-1/2}$$

of $T(\mathbb{A})$. This follows from the cuspidal support of $\Pi = \chi \circ \det$ given above. However, in [5], we always choose the representative of the cuspidal support in such a way that the complex parameter belongs to the closure of the positive Weyl chamber determined by B. This requirement is equivalent to the inequality

$$s_0 + 1/2 \geq s_0 - 1/2 \geq 0,$$

which is satisfied only if $s_0 \geq 1/2$. Hence, if $0 \leq s_0 < 1/2$ we conjugate the representative above by the element of the Weyl group of G with respect to T that takes the inverse of the second factor in the tensor product. As a result, we make the following choice of the representative

$$\pi = \begin{cases} \chi|\cdot|^{s_0+1/2} \otimes \chi|\cdot|^{s_0-1/2}, & \text{if } s_0 \geq 1/2, \\ \chi|\cdot|^{s_0+1/2} \otimes \chi^{-1}|\cdot|^{1/2-s_0}, & \text{if } 0 \leq s_0 < 1/2, \end{cases}$$

and consider the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms with cuspidal support represented by π, as in [5, Sect. 3.3].

In the construction of the Franke filtration of $\mathcal{A}_{\{B\},\varphi(\pi)}$, the Eisenstein series $E(f,s)$ evaluated at $s = s_0$ is obtained from the triple $(P_1, \chi \circ \det, s_0)$, cf. [5, Sect. 4.2]. As explained above, our method would imply holomorphy at $s = s_0$, if $E(f, s)$ contributed to the deepest quotient of the filtration, or equivalently, if this triple were a maximal element in the partial order introduced by Franke, cf. [5, Sect. 4.2]. This can be read off from the proofs of the appropriate theorems in [5, Sect. 7.2, Sect. 8.2]. In the notation of *loc. cit.*, the representative of the cuspidal support is the character

$$\pi = \chi_1 |\cdot|^{s_{0,1}} \otimes \chi_2 |\cdot|^{s_{0,2}},$$

of $T(\mathbb{A})$, where χ_1 and χ_2 are unitary Hecke characters, and $s_{0,1} \geq s_{0,2} \geq 0$ are real numbers. Hence, in our case we have

- $\chi_1 = \chi_2 = \chi$ and $s_{0,1} = s_0 + 1/2$, $s_{0,2} = s_0 - 1/2$, if $s_0 \geq 1/2$,
- $\chi_1 = \chi_2^{-1} = \chi$ and $s_{0,1} = s_0 + 1/2$, $s_{0,2} = 1/2 - s_0$, if $0 \leq s_0 < 1/2$,

and look at Thm. 7.2 (Case 1–1c), Thm. 7.4 (Case 1–3b), Thm. 7.6 (Cases 2d, 2e, 2f, 2g) and Thm. 7.7 (Cases 4d, 4f, 4h, 4j, 4k) of [5], and the corresponding steps in the proof, namely, Step 2, Step 4, Step 6, Step 9, Step 11. We refer to these steps in the rest of the proof below.

More precisely, if $s_0 \neq 3/2, 1/2, 0$, then it follows from Step 2.2 and Step 4.2 that the triple $(P_1, \chi \circ \det, s_0)$ is maximal for every unitary Hecke character χ, and thus $E(f, s)$ is holomorphic at $s = s_0$. The same conclusion in the case of $s_0 = 0$ follows from Steps 11.3 and 11.4. Thus, the first claim is proved.

Let now $s_0 = 3/2$. Step 6.2 implies that $(P_1, \chi \circ \det, 3/2)$ is maximal if $\chi \neq \mathbf{1}$, and thus, $E(f, s)$ is holomorphic at $s = 3/2$ if $\chi \neq \mathbf{1}$. We remark that in the case of the trivial character $\chi = \mathbf{1}$, the triple $(P_1, \mathbf{1} \circ \det, 3/2)$ is not a maximal element in the partial order, as obtained in Step 6.4, so that our method does not apply.

Finally, let $s_0 = 1/2$. If $\chi^2 \neq \mathbf{1}$, then Step 9.3 implies that the triple $(P_1, \chi \circ \det, 1/2)$ is a maximal element. Thus, $E(f, s)$ is holomorphic at $s = 1/2$ if $\chi^2 \neq \mathbf{1}$. We again remark that in the case of $\chi^2 = \mathbf{1}$ our method does not apply, because $(P_1, \chi \circ \det, 1/2)$ is not a maximal element in the partial order according to Steps 9.7 and 9.8. ∎

4. Degenerate Eisenstein series on the non-Siegel parabolic subgroup

In this section we follow closely the exposition of the previous Section 3. The Levi factor L_2 of the non-Siegel parabolic subgroup P_2 is isomorphic to $GL_1 \times SL_2$. The residual spectrum of $SL_2(\mathbb{A})$ consists only of constants, that is, only the trivial character $\mathbf{1}_{SL_2(\mathbb{A})}$ belongs to the residual spectrum of $SL_2(\mathbb{A})$, cf. [5, Sect. 6.1]. Hence, all residual representations of the Levi factor $L_2(\mathbb{A})$ are of the form

$$\Pi = \chi \otimes \mathbf{1}_{SL_2(\mathbb{A})},$$

where χ is a unitary Hecke character of $GL_1(\mathbb{A}) \cong \mathbb{I}$. The cuspidal support of Π is represented by the character

$$\chi \otimes |\cdot|,$$

of $GL_1(\mathbb{A}) \times GL_1(\mathbb{A}) \cong \mathbb{I} \times \mathbb{I}$, where $|\cdot|$ stands for the usual adèlic absolute value.

Hence, all degenerate Eisenstein series associated with the non-Siegel parabolic subgroup arise from residual representations Π as above viewed as representations of the Levi factor $L_2(\mathbb{A})$. We denote these Eisenstein series by $E(f,s)$, where s is the complex parameter, and f is a function in the space $\mathcal{W}_\Pi$ associated with Π as in [5, Sect. 3.2]. In analogy with Section 3, here $\mathcal{W}_\Pi$ is the space of K-finite smooth complex functions f on $L_2(F)N_2(\mathbb{A})\backslash G(\mathbb{A})$ such that the function on $L_2(\mathbb{A})$ given by the assignment $l \mapsto f(lg)$ belongs to the space of Π for all $g \in G(\mathbb{A})$.

The space of complex parameters is one-dimensional, identified with characters of $L_2(\mathbb{A}) \cong GL_1(\mathbb{A}) \times SL_2(\mathbb{A})$ by the assignment

$$s \mapsto |\cdot|^s \otimes \mathbf{1}_{SL_2(\mathbb{A})},$$

for $s \in \mathbb{C}$. As explained above in Section 3, the representations are normalized in such a way that the possible poles of Eisenstein series are real. On the other hand, the closure of the positive Weyl chamber is in this case again given by the condition $Re(s) \geq 0$. Hence, in the theorem below, we only consider values of the complex parameter $s = s_0$ that are real and $s_0 \geq 0$.

THEOREM 4.1. *Let $\Pi = \chi \otimes \mathbf{1}_{SL_2(\mathbb{A})}$ be a character of the Levi factor $L_2(\mathbb{A})$ of the non-Siegel parabolic subgroup P_2 of G, where χ is a unitary Hecke character of the group of idèles $\mathbb{I}$, and $\mathbf{1}_{SL_2(\mathbb{A})}$ is the trivial character of $SL_2(\mathbb{A})$. Let $E(f,s)$ be the degenerate Eisenstein series associated with Π. Then, $E(f,s)$ is holomorphic at $s = s_0$ in the following cases:*

- *$s_0 \geq 0$ and $s_0 \neq 2$,*
- *$s_0 = 2$ and $\chi \neq \mathbf{1}$,*

where $\mathbf{1}$ is the trivial character of $\mathbb{I}$.

PROOF. The cuspidal support of the Eisenstein series $E(f,s)$ evaluated at $s = s_0$ is represented by the character

$$\chi|\cdot|^{s_0} \otimes |\cdot|$$

of $T(\mathbb{A})$. This follows from the cuspidal support of $\Pi = \chi \otimes \mathbf{1}_{SL_2(\mathbb{A})}$ given above. However, in [5], we always choose the representative of the cuspidal support in such a way that the complex parameter belongs to the closure of the positive Weyl chamber. This requirement is equivalent to the inequality

$$s_0 \geq 1 \geq 0,$$

which is satisfied only if $s_0 \geq 1$. Hence, if $0 \leq s_0 < 1$ we conjugate the representative above by the Weyl group element that flips the two factors in the tensor product. As a result, we make the following choice of the representative

$$\pi = \begin{cases} \chi|\cdot|^{s_0} \otimes |\cdot|, & \text{if } s_0 \geq 1, \\ |\cdot| \otimes \chi|\cdot|^{s_0}, & \text{if } 0 \leq s_0 < 1, \end{cases}$$

and consider the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms with cuspidal support represented by π, as in [5, Sect. 3.3].

In the construction of the Franke filtration of $\mathcal{A}_{\{B\},\varphi(\pi)}$, the Eisenstein series $E(f,s)$ evaluated at $s = s_0$ is obtained from the triple $(P_2, \chi \otimes \mathbf{1}_{SL_2(\mathbb{A})}, s_0)$, cf. [5, Sect. 4.2]. As explained above, our method would imply holomorphy at $s = s_0$, if $E(f,s)$ contributed to the deepest quotient of the filtration, or equivalently, if this triple were a maximal element in the partial order introduced by Franke, cf. [5, Sect. 4.2]. This can be read off from the proofs of the appropriate theorems in [5, Sect. 7.2, Sect. 8.2]. In the notation of *loc. cit.*, the representative of the cuspidal support is the character

$$\pi = \chi_1|\cdot|^{s_{0,1}} \otimes \chi_2|\cdot|^{s_{0,2}},$$

of $T(\mathbb{A})$, where χ_1 and χ_2 are unitary Hecke characters, and $s_{0,1} \geq s_{0,2} \geq 0$ are real numbers. Hence, in our case we have

- $\chi_1 = \chi$, $\chi_2 = \mathbf{1}$, and $s_{0,1} = s_0$, $s_{0,2} = 1$, if $s_0 \geq 1$,
- $\chi_1 = \mathbf{1}$, $\chi_2 = \chi$, and $s_{0,1} = 1$, $s_{0,2} = s_0$, if $0 \leq s_0 < 1$,

and look at Thm. 7.3 (Case 1–2c), Thm. 7.5 (Cases 1–4c and 1–4d) and Thm. 7.7 (Cases 4e, 4g, 4i, 4j, 4k) of [5], and the corresponding steps in the proof, namely, Step 3, Step 5, Step 6, Step 9, Step 10. We refer to these steps in the rest of the proof below.

If $s_0 \neq 2, 1, 0$, then it follows from Step 3.2 and Step 5.2 that the triple $(P_2, \chi \otimes \mathbf{1}_{SL_2(\mathbb{A})}, s_0)$ is maximal for every unitary Hecke character χ, and thus $E(f,s)$ is holomorphic at $s = s_0$. The same conclusion in the case of $s_0 = 1$ follows from Steps 10.3 and 10.4, and in the case of $s_0 = 0$ from Steps 9.5, 9.6 and 9.8. Thus, the first claim of the theorem is proved.

Let now $s_0 = 2$. Step 6.3 implies that $(P_2, \chi \otimes \mathbf{1}_{SL_2(\mathbb{A})}, 2)$ is maximal if $\chi \neq \mathbf{1}$, and thus, $E(f,s)$ is holomorphic at $s = 2$ if $\chi \neq \mathbf{1}$. We remark that in the case of the trivial character $\chi = \mathbf{1}$, the triple $(P_2, \mathbf{1} \otimes \mathbf{1}_{SL_2(\mathbb{A})}, 2)$ is not a maximal element in the partial order, as obtained in Step 6.4, so that our method does not apply.

□

5. Final remarks

As already mentioned in the proofs of Theorem 3.1 and Theorem 4.1, our method is not applicable whenever the triple associated with the considered Eisenstein series is not a maximal element in the partial order required for the construction of the Franke filtration. In the theorems these are precisely the cases in which the holomorphy of the Eisenstein series could not have been concluded. From the results of Hanzer and Muić [7], we can observe that my method is sharp in the case of the symplectic group of rank two, that is, the remaining points are precisely the poles of the degenerate Eisenstein series in question.

From the point of view of our method and the Franke filtration, the most interesting phenomenon occurs for the degenerate Eisenstein series associated with the residual representation $\mathbf{1} \circ \det$ of the Levi factor $L_1(\mathbb{A}) \cong GL_2(\mathbb{A})$ of P_1 evaluated at $s = 1/2$, and the degenerate Eisenstein series associated with the residual representation $\mathbf{1} \otimes \mathbf{1}_{SL_2(\mathbb{A})}$ of the Levi factor $L_2(\mathbb{A}) \cong GL_1(\mathbb{A}) \times SL_2(\mathbb{A})$ of P_2 evaluated at $s = 0$. Denote the former Eisenstein series by $E_1(f, s)$, and the latter by $E_2(f, s)$. These two Eisenstein series, evaluated at the given evaluation points, share the same cuspidal support represented by the character

$$\pi = |\cdot| \otimes \mathbf{1}$$

of $T(\mathbb{A})$, and thus both contribute to the Franke filtration of the space of automorphic forms $\mathcal{A}_{\{B\},\varphi(\pi)}$. However, $E_2(f, s)$ is holomorphic at $s = 0$, as we obtained in Theorem 4.1, although observe that this is already well-known since $s = 0$ is on the imaginary axis, cf. [12]. For the other Eisenstein series $E_1(f, s)$, evaluated at $s = 1/2$, our method does not apply. The problem is exactly the triple associated with $E_2(f, s)$ evaluated at $s = 0$, which is greater than the triple associated with $E_1(f, s)$ at $s = 1/2$, and thus prevents us from applying our method.

From the results of [7], and already of [9], we know the underlying reasons for this phenomenon. The Eisenstein series $E_1(f, s)$ has a pole at $s = 1/2$, but the residues are not square-integrable. That is the reason why we need another triple, greater than the one associated with $E_1(f, s)$ at $s = 1/2$, but not arising from a residual representation of $G(\mathbb{A})$, as such representation does not exist with cuspidal support represented by π. The required triple is exactly the one associated with $E_2(f, s)$ are $s = 0$. For more details see [5, Sect. 9.4]

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