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THE HITCHIN–KOBAYASHI CORRESPONDENCE FOR QUIVER BUNDLES OVER THE NON-COMPACT AFFINE GAUDUCHON MANIFOLD

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Abstract. The objective of this paper is to prove a broader, generalized version of the Hitchin–Kobayashi correspondence for the twisted quiver bundle $\mathcal{R}$ over the non-compact special affine Gauduchon manifold (M, D, g, ν). On the one hand, we prove that the analytic (σ, τ)-stability on $\mathcal{R}$ implies the existence of affine (σ, τ)-Hermite–Einstein metric. On the other hand, we prove that the analytic (σ, τ)-semi-stability on $\mathcal{R}$ implies the existence of approximately affine (σ, τ)-Hermite–Einstein structure. The proof of the theorems relies on the heat flow method, alongside the continuity approach by Uhlenbeck and Yau. To overcome the analytical obstacles brought by the structure of the quiver, we use the maximum and minimum values of some eigenvalues to define a new quantity χ. Based on the method of proof by contradiction, the quantity χ can be used in the discussion of constructing weak quiver subbundles that contradict stability or semi-stability.

1. Introduction

The esteemed Hitchin–Kobayashi correspondence (HK correspondence for short), uncovers a deep linkage between stable bundles and Hermite–Einstein metrics. In the literature, the Hitchin–Kobayashi correspondence is also called the Kobayashi–Hitchin correspondence or the Donaldson–Uhlenbeck–Yau correspondence. In the 1980s, propelled by several prominent mathematicians, research on the HK correspondence surged, as chronicled in publications like [16, 20, 27, 39, 41]. Throughout the last three decades, the correspondence has persistently captivated numerous researchers, attested to by numerous works

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([3, 8, 9, 10, 11, 12, 24, 25, 26, 33, 34, 35, 36, 37, 38, 42, 43, 44, 45, 46, 47] and their cited references).

A twisted quiver bundle $\mathcal{R}$ comprises a set of vector bundles, interrelated through vertices, arrows, and morphisms twisted by the bundles. In 2003, Álvarez-Cónsul and García-Prada [2] demonstrated an HK correspondence for such bundles on standard compact Kähler manifolds. Hu and Huang [22] further explored the HK correspondence for quiver bundles on compact generalized Kähler manifolds. Meanwhile, Loftin [31] established an HK correspondence for flat complex vector bundles on compact affine Gauduchon manifolds, proving the existence of an affine Hermite–Einstein metric for stable bundles. Biswas–Loftin [4], along with Biswas–Loftin–Stemmler [5, 6], subsequently extended these results to principal bundles, flat Higgs bundles, and flat pairs on compact Gauduchon manifolds. Recently, Shen, Zhang, and Zhang [38] generalized these results to Higgs bundles on non-compact affine Gauduchon manifolds. Inspired by these works, we aim to formulate a broader HK correspondence for twisted quiver bundles on non-compact affine Gauduchon manifolds.

Drawing inspiration from [39, 38], we initiate our discussion by presenting three pivotal conditions:

- **Condition 1.** The non-compact affine Gauduchon manifold (M, D, g, ν) possesses a finite volume.
- **Condition 2.** There exists an exhaustion function $\varphi \geq 0$ such that $tr_g \partial\bar{\partial}\varphi$ remains bounded.
- **Condition 3.** Let $\xi : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function with $\xi(0) = 0$ and $\xi(x) = x$ for $x > 1$. If f is a bounded positive function on (M, D, g, ν) satisfying $tr_g \partial\bar{\partial} f \geq -C$, then it holds that
$$\sup_M |f| \leq C \cdot \xi \left(\int_M |f| \frac{\omega_g^n}{\nu} \right).$$
Furthermore, if $tr_g \partial\bar{\partial} f \geq 0$, then $tr_g \partial\bar{\partial} f = 0$ necessarily.

Under the aforementioned conditions, we first establish the following theorem:

THEOREM 1.1. *Consider the non-compact special affine Gauduchon manifold (M, D, g, ν) satisfying Conditions 1-3, with the additional assumption that $|\frac{\partial \omega_g^{n-1}}{\nu}|_g \in L^2(M)$. Let $Q = (Q_0, Q_1)$ denote a quiver, and let $\mathcal{R} = (\mathbf{E}, \widetilde{\mathbf{E}}, Q, \phi)$ be a twisted quiver bundle as per Definition 2.1 over (M, D, g, ν), where $\mathbf{E} = \oplus_{v \in Q_0} E_v$ and $\widetilde{\mathbf{E}} = \oplus_{a \in Q_1} E_a$. Fix a background Hermitian metric $\mathbf{K} = \{K_v\}_{v \in Q_0}$ for $\mathcal{R}$. For each $v \in Q_0$, suppose the metric K_v on the flat bundle E_v fulfills*
$$tr_g F_{K_v} \leq 0, \quad \sup_M |tr_g F_{K_v}|_{K_v} < +\infty, \quad \sup_M |\phi|_{K_v} < +\infty.$$

Furthermore, let $\sigma = \{\sigma_v\}$ *and* $\tau = \{\tau_v\}$ *be two sets of positive real numbers. If* $\mathcal{R} = (\mathbf{E}, \widetilde{\mathbf{E}}, Q, \phi)$ *is analytically* (σ, τ)*-stable with respect to* $\mathbf{K}$*, then there exists an affine* (σ, τ)*-Hermite–Einstein metric* $\mathbf{H} = \{H_v\}_{v\in Q_0}$ *on* $\mathcal{R}$ *such that for every* $v \in Q_0$*, the metric* H_v *on* E_v *satisfies*

$$\sigma_v tr_g F_{H_v} + \sum_{a\in\mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_v} - \sum_{a\in\mathfrak{t}^{-1}(v)} \phi_a^{*H_v} \circ \phi_a = \tau_v \cdot \mathrm{Id}_{E_v}.$$

Remark 1.2. The proof of the theorem hinges upon both the flow method and the continuity method. While these methods bear similarities to those employed in [44], certain modifications necessitate careful consideration. The algebraic framework of the quiver bundle presents significant challenges in the analysis of PDEs, and the proof heavily relies on the arguments of weakly L_1^2 quiver sub-bundles. We introduce a novel quantity χ (6.32), defined by the extrema of eigenvalues of morphisms, distinguishing it from [38, 44].

As for the semi-stable case, we establish the following theorem:

Theorem 1.3. *Consider the non-compact special affine Gauduchon manifold* (M, D, g, ν) *satisfying Conditions 1-3, with the additional assumption that* $|\frac{\partial\omega_g^{n-1}}{\nu}|_g \in L^2(M)$*. Let* $Q = (Q_0, Q_1)$ *denote a quiver, and let* $\mathcal{R} = (\mathbf{E}, \widetilde{\mathbf{E}}, Q, \phi)$ *be a twisted quiver bundle as per Definition 2.1 over* (M, D, g, ν)*, where* $\mathbf{E} = \oplus_{v\in Q_0} E_v$ *and* $\widetilde{\mathbf{E}} = \oplus_{a\in Q_1} E_a$*. Fix a background Hermitian metric* $\mathbf{K} = \{K_v\}_{v\in Q_0}$ *for* $\mathcal{R}$*. For each* $v \in Q_0$*, suppose the metric* K_v *on the flat bundle* E_v *fulfills*

$$tr_g F_{K_v} \le 0, \quad \sup_M |tr_g F_{K_v}|_{K_v} < +\infty, \quad \sup_M |\phi|_{K_v} < +\infty.$$

Furthermore, let $\sigma = \{\sigma_v\}$ *and* $\tau = \{\tau_v\}$ *be two sets of positive real numbers. If* $\mathcal{R} = (\mathbf{E}, \widetilde{\mathbf{E}}, Q, \phi)$ *is analytically* (σ, τ)*-semi-stable with respect to* $\mathbf{K}$*, then there exists an approximately affine* (σ, τ)*-Hermite–Einstein structure* $\mathbf{H} = \{H_v\}_{v\in Q_0}$ *on* $\mathcal{R}$ *such that for every* $v \in Q_0$*, the metric* H_v *on* E_v *satisfies*

$$\sup_M |\sigma_v tr_g F_{H_v} + \sum_{a\in\mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_v} - \sum_{a\in\mathfrak{t}^{-1}(v)} \phi_a^{*H_v} \circ \phi_a - \tau_v \cdot \mathrm{Id}_{E_v}|_{H_v} < \varepsilon$$

for any $\varepsilon > 0$.

Remark 1.4. In [38], the authors modified the method used in the limit of the Hermite–Yang–Mills flow instead of a combination of the heat flow method and the continuity method. Their approach might not be directly effective in studying the semi-stable case in Theorem 1.3.

2. Basic notations

2.1. *Flat vector bundle over affine Gauduchon manifold.* In this section, we introduce the essential setup and notation pertaining to affine Gauduchon manifolds, which remain consistent throughout the paper. For a deeper insight into affine Gauduchon manifolds, readers are advised to consult [31].

Consider an n-dimensional affine manifold (M, D), where D signifies a flat, torsion-free connection on the tangent bundle TM. This is equivalent to an affine structure, furnished by an atlas of M with transition functions given by affine transformations of the form

$$x \mapsto Ax + b,$$

with $A \in \mathrm{Gl}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. Throughout this paper, all manifolds are presumed to be both connected and smooth. When an atlas on M comprises solely of affine transformations as transition maps, the associated coordinates $\{x^i\}$ are termed local affine. If $\{x^i\}$ is defined over an open subset $U \subset M$, we denote the fiber coordinates, corresponding to the local trivialization of TM by $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$, as y^i. Consequently, on the open subset $TU \subset TM$, we obtain the holomorphic coordinate functions $z^i = x^i + \sqrt{-1}y^i$, naturally transforming TM into a complex manifold. This n-dimensional complex manifold is designated as $M^{\mathbb{C}}$.

The vector bundle of (p, q)-forms on M is defined as

$$\mathcal{A}^{p,q} = \wedge^p T^*M \otimes \wedge^q T^*M,$$

constituting restrictions of (p, q)-forms from the complex manifold $M^{\mathbb{C}}$. These are differential operators given by

$$\partial := \frac{1}{2}(d \otimes \mathrm{Id}) : \wedge^p T^*M \otimes \wedge^q T^*M \to \wedge^{p+1} T^*M \otimes \wedge^q T^*M,$$

$$\overline{\partial} := (-1)^k \frac{1}{2}(\mathrm{Id} \otimes d) : \wedge^p T^*M \otimes \wedge^q T^*M \to \wedge^p T^*M \otimes \wedge^{q+1} T^*M,$$

which are derived as restrictions from the operators on $M^{\mathbb{C}}$.

An affine manifold (M, D) is deemed special if it possesses a volume form ν that remains covariant constant relative to the flat connection D on TM. Throughout this paper, we operate under the assumption that (M, D, g, ν) is special.

On such a special affine manifold (M, g, ν), the volume form ν induces homomorphisms:

$$\mathcal{A}^{n,q} \to \wedge^q T^*M, \quad \nu \otimes \theta \mapsto (-1)^{\frac{n(n-1)}{2}} \theta,$$

$$\mathcal{A}^{p,n} \to \wedge^p T^*M, \quad \theta \otimes \nu \mapsto (-1)^{\frac{n(n-1)}{2}} \theta,$$

termed as division by ν. When M is compact, integration of an (n, n)-form θ is facilitated through

$$\int_M \frac{\theta}{\nu}.$$

A smooth Riemannian metric g on (M, D) induces a $(1,1)$-form, expressed in local affine coordinates as

$$\omega_g = \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j,$$

which is a restriction of the corresponding $(1,1)$-form on $M^{\mathbb{C}}$ obtained by extending g to $M^{\mathbb{C}}$. The metric g is termed an affine Gauduchon metric when

$$\partial\overline{\partial}\omega_g^{n-1} = 0.$$

As per [31], every conformal class of Riemannian metrics on a compact, connected special affine manifold contains a unique affine Gauduchon metric, up to a positive scalar multiple.

In the realm of affine manifolds, a flat complex vector bundle serves as the appropriate counterpart to a holomorphic vector bundle on a complex manifold. To elucidate, consider a smooth complex vector bundle E over an affine manifold M. Denote the pullback of E to $M^{\mathbb{C}}$ via the natural projection $M^{\mathbb{C}} = TM \to M$ as $E^{\mathbb{C}}$. The transition functions for $E^{\mathbb{C}}$ are derived by extending those of E uniformly along the fibers of TM. A transition function on $M^{\mathbb{C}}$ is holomorphic precisely when the associated transition function for E is locally constant. Hence, $E^{\mathbb{C}}$ constitutes a holomorphic vector bundle over $M^{\mathbb{C}}$ if and only if E is a flat vector bundle over M. Thus, the assignment $E \mapsto E^{\mathbb{C}}$ establishes a bijective relationship between flat vector bundles on M and holomorphic vector bundles on $M^{\mathbb{C}}$ that remain constant across the fibers of TM. Given that $E^{\mathbb{C}}$ is the pullback of a vector bundle on M, the term "constant across the fibers of TM" is unambiguously defined.

Let H be a Hermitian metric on E, which induces a Hermitian metric on $E^{\mathbb{C}}$. Denote by ∇_H the Chern connection associated with this metric on $E^{\mathbb{C}}$. According to the decomposition into $(1,0)$- and $(0,1)$-parts, ∇_H aligns with the pair

$$(\partial_H, \overline{\partial}_E) = (\partial_{H,\nabla}, \overline{\partial}_{E,\nabla}),$$

where $\partial_{H,\nabla} : \Gamma(E) \to \mathcal{A}^{1,0}(E)$ and $\overline{\partial}_{E,\nabla} : \Gamma(E) \to \mathcal{A}^{0,1}(E)$ are smooth differential operators. This pair is referred to as the extended Hermitian connection of (E, H).

For a locally constant frame $\{s_1, \ldots, s_r\}$ on E associated with the flat connection ∇, and denoting $H_{\alpha\bar{\beta}} = H(s_\alpha, s_\beta)$, we define:

- The extended connection form $A_H = H^{-1}\partial H \in \mathcal{A}^{1,0}(\mathrm{End}E)$,
- The extended curvature form $F_H = \overline{\partial} A_H \in \mathcal{A}^{1,1}(\mathrm{End}E)$,
- The extended mean curvature $\mathcal{K}_H = tr_g F_H \in C^\infty(M, \mathrm{End}E)$,
- The extended first Chern form $c_1(E, H) = tr_E F_H \in \mathcal{A}^{1,1}$.

All these forms are restrictions of their counterparts on $E^{\mathbb{C}}$. Note that tr_g signifies the contraction of differential forms using the Riemannian metric g, while tr_E denotes the trace map on the fibers of $\mathrm{End}E$. The degree of the flat vector bundle (E, ∇) over an affine Gauduchon manifold (M, D, g, ν) is defined as

$$\deg_g(E) := \int_M \frac{c_1(E, H) \wedge \omega_g^{n-1}}{\nu},$$

which is well-defined for compact manifolds [31].

2.2. *Quiver bundle over affine Gauduchon manifold.* In this section, we introduce the essential setup and notation for quiver bundles, which are consistently employed throughout the paper. For a thorough comprehension of twisted quiver bundles, please consult [2].

DEFINITION 2.1. *A quiver consists of a pair $Q = (Q_0, Q_1)$ equipped with two maps, $\mathfrak{h}$ and $\mathfrak{t}$, that assign vertices to arrows. The set Q_0 contains vertices, while Q_1 comprises arrows. For each arrow $a \in Q_1$, the vertex $\mathfrak{h}a$ denotes the head, and $\mathfrak{t}a$ denotes the tail.*

A twisted quiver bundle over an affine Gauduchon manifold (M, D, g, ν) is defined as a 4-tuple $(\mathbf{E}, \widetilde{\mathbf{E}}, Q, \phi)$, where:

1. *$\mathbf{E}$ is a collection of flat vector bundles E_v on (M, D, g, ν), each associated with a vertex $v \in Q_0$,*
2. *$\widetilde{\mathbf{E}}$ is a collection of flat vector bundles $\widetilde{E}_a$ on (M, D, g, ν), each corresponding to an arrow $a \in Q_1$,*
3. *ϕ is a collection of morphisms $\phi_a : E_{\mathfrak{t}a} \otimes \widetilde{E}_a \to E_{\mathfrak{h}a}$, with the stipulation that $E_v = 0$ for all vertices $v \in Q_0$ except a finite number, and similarly, $\phi_a = 0$ for all arrows $a \in Q_1$ except a finite number.*

An Hermitian metric $\mathbf{H}$ on a twisted quiver bundle $\mathcal{R} = (\mathbf{E}, \widetilde{\mathbf{E}}, Q, \phi)$ comprises a set of Hermitian metrics H_v assigned to each non-zero vector bundle E_v associated with a vertex $v \in Q_0$. Given collections of real numbers $\sigma = \{\sigma_v\}_{v \in Q_0}$ and $\tau = \{\tau_v\}_{v \in Q_0}$, the bundle $\mathcal{R}$ is said to admit an *affine (σ, τ)-Hermite–Einstein metric* $\mathbf{H} = \{H_v\}_{v \in Q_0}$ if, for all non-zero E_v, the following equation holds:

$$\sigma_v \cdot tr_g F_{H_v} + \sum_{a \in \mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_v} - \sum_{a \in \mathfrak{t}^{-1}(v)} \phi_a^{*H_v} \circ \phi_a = \tau_v \cdot \mathrm{Id}_{E_v}, \tag{2.1}$$

where F_{H_v} is the curvature of the Chern connection ∇_{H_v} on E_v, and $\phi_a^{*H_v}$ denotes the adjoint of ϕ_a with respect to H_v. The bundle $\mathcal{R}$ is said to admit an *approximately affine (σ, τ)-Hermite–Einstein structure* $\mathbf{H} = \{H_v\}_{v \in Q_0}$ if, for every $v \in Q_0$, the metric H_v on E_v satisfies

$$\sup_M |\sigma_v tr_g F_{H_v} + \sum_{a \in \mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_v} - \sum_{a \in \mathfrak{t}^{-1}(v)} \phi_a^{*H_v} \circ \phi_a - \tau_v \cdot \mathrm{Id}_{E_v}|_{H_v} < \varepsilon$$

for any $\varepsilon > 0$.

Fix a background Hermitian metric $\mathbf{K} = \{K_v\}_{v\in Q_0}$ on $\mathcal{R}$ over the affine Gauduchon manifold (M, D, g, ν). The degree of E_v is defined as [39]

$$\deg(E_v, K_v) = \frac{1}{n}\int_M tr_{E_v}(tr_g F_{K_v})\frac{\omega_g^n}{\nu},$$

where F_{K_v} is the curvature of the Chern connection ∇_{K_v} on E_v. According to the Chern–Weil theory [39], for any saturated subsheaf E'_v of E_v, the analytic degree is given by

$$\deg(E'_v, K_v) = \frac{1}{n}\int_M \left(tr_{E_v}(\pi_v tr_g F_{K_v}) - |\overline{\partial}_{E_v}\pi_v|^2_{K_v}\right)\frac{\omega_g^n}{\nu},$$

where π_v denotes the projection onto E'_v with respect to K_v.

The analytic (σ, τ)-degree and (σ, τ)-slope of the twisted quiver bundle $\mathcal{R}$ are defined based on weighted combinations of the degrees and ranks of the vector bundles E_v associated with each vertex v in Q_0. Specifically, the (σ, τ)-degree is expressed as

$$\deg_{\sigma,\tau}(\mathcal{R}, \mathbf{K}) = \sum_{v\in Q_0}(\sigma_v \cdot \deg(E_v, K_v) - \tau_v \cdot \mathrm{rk}(E_v)),$$

where σ_v and τ_v are real numbers corresponding to each vertex v. The (σ, τ)-slope is subsequently defined as the ratio of the (σ, τ)-degree to the total weighted rank:

$$\mathcal{S}_{\sigma,\tau}(\mathcal{R}, \mathbf{K}) = \frac{\deg_{\sigma,\tau}(\mathcal{R}, \mathbf{K})}{\sum_{v\in Q_0}\sigma_v rk(E_v)}.$$

The twisted quiver bundle $\mathcal{R}$ is deemed analytic (σ, τ)-(semi)stable with respect to $\mathbf{K}$ if, for all proper quiver subsheaves $\mathcal{R}'$ of $\mathcal{R}$, the following condition holds:

$$\mathcal{S}_{\sigma,\tau}(\mathcal{R}', \mathbf{K}) < (\leq)\mathcal{S}_{\sigma,\tau}(\mathcal{R}, \mathbf{K}).$$

In the framework of twisted quiver bundles, this definition enables the establishment of moduli spaces of (σ, τ)-stable twisted quiver bundles, which exhibit favorable geometric properties [1]. This condition generalizes the stability criterion for vector bundles, a concept that is pivotal in the investigation of moduli spaces for vector bundles. Over recent years, the exploration of moduli spaces for vector bundles and various geometric objects has garnered significant attention and focus (see [7, 13, 14, 17, 19, 21, 23, 30] and references therein).

3. Preliminary results

3.1. *The perturbed heat flow.* Let $\mathcal{R} = (\mathbf{E}, \widetilde{\mathbf{E}}, Q, \phi)$ denote a twisted quiver bundle over the affine Gauduchon manifold (M, D, g, ν), and let $\mathbf{H}_0 = \{H_{0,v}\}_{v\in Q_0}$

be a Hermitian metric on $\mathcal{R}$. For each $v \in Q_0$ and nonnegative constant ε, we introduce the perturbed heat flow as follows:

$$H_v^{-1}\frac{\partial H_v}{\partial t} = -\frac{4}{\sigma_v}\Phi_{\varepsilon,v}(H_v), \tag{3.2}$$

where $H_v := H_v(t)$ and $\Phi_{\varepsilon,v}(H_v)$ is given by

$$\begin{aligned}\Phi_{\varepsilon,v}(H_v) = \sigma_v tr_g F_{H_v} + \sum_{a\in\mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_v} - \sum_{a\in\mathfrak{t}^{-1}(v)} \phi_a^{*H_v} \circ \phi_a \\ - \tau_v \cdot \mathrm{Id}_{E_v} + \varepsilon\sigma_v \log(H_{0,v}^{-1}H_v).\end{aligned}$$

For simplicity, we define

$$h_v := h_v(t) = H_{0,v}^{-1}H_v(t).$$

Furthermore, we define the complex Laplacian by

$$\widetilde{\Delta} f = 4tr_g\bar{\partial}\partial f = g^{i\bar{j}}\frac{\partial^2 f}{\partial z^i \partial \bar{z}^j},$$

where $(g^{i\bar{j}})$ is the inverse of the metric matrix $(g_{i\bar{j}})$. Additionally, we refer to the Beltrami-Laplacian as Δ. It is well-known that the relationship between these Laplacians is given by

$$(\widetilde{\Delta} - \Delta)f = \langle V, \nabla f\rangle_g,$$

where V is a well-defined vector field on the affine Gauduchon manifold M.

We begin by establishing the following proposition, which will be used in proving the long-time existence of the flow (3.2).

Proposition 3.1. *For each $v \in Q_0$, let $H_v = H_v(t)$ denote a solution of the flow* (3.2)*, then*

$$(\frac{\partial}{\partial t} - \widetilde{\Delta})[\sum_{v\in Q_0}\frac{1}{\sigma_v}|\Phi_{\varepsilon,v}|^2_{H_v}] \leq 0. \tag{3.3}$$

and

$$(\frac{\partial}{\partial t} - \widetilde{\Delta})\{e^{2\varepsilon t}tr_{E_v}(\Phi_{\varepsilon,v})\} = 0. \tag{3.4}$$

Proof. By direct calculation, we have:

(3.5)
$$
\begin{aligned}
\frac{\partial}{\partial t}\Phi_{\varepsilon,v} &= \sigma_v tr_g \overline{\partial}_{E_v}\partial_{H_{0,v}}(h_v^{-1}\frac{\partial h_v}{\partial t}) \\
&- \sum_{a\in h^{-1}(v)} (\phi_a \circ H_{\mathfrak{t}a}^{-1}\frac{\partial H_{\mathfrak{t}a}}{\partial t}\otimes \mathrm{Id}_{\widetilde{E}_a}\circ \phi_a^{*H_a} - \phi_a\circ\phi_a^{*H_a}\circ H_v^{-1}\frac{\partial H_v}{\partial t}) \\
&- \sum_{a\in t^{-1}(v)} (H_v^{-1}\frac{\partial H_v}{\partial t}\phi_a^{*H_a}\circ\phi_a - \phi_a^{*H_a}\circ H_{\mathfrak{h}a}^{-1}\frac{\partial H_{\mathfrak{h}a}}{\partial t}\otimes \mathrm{Id}_{\widetilde{E}_a}\circ\phi_a) \\
&+ \varepsilon\sigma_v\frac{\partial}{\partial t}\log(h_v)
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\Delta}|\Phi_{\varepsilon,v}|^2_{H_v} &= -4tr_g\bar{\partial}\partial tr_{E_v}\{\Phi_{\varepsilon,v}H_v^{-1}\bar{\Phi}^t_{\varepsilon,v}H_v\} \\
&= -4tr_g\bar{\partial}tr_{E_v}\{\partial\Phi_{\varepsilon,v}H_v^{-1}\bar{\Phi}^t_{\varepsilon,v}H_v - \Phi_{\varepsilon,v}H_v^{-1}\partial H_v H_v^{-1}\bar{\Phi}^t_{\varepsilon,v}H_v \\
&\quad + \Phi H_v^{-1}\overline{\bar{\partial}\Phi_{\varepsilon,v}}^t H_v + \Phi_{\varepsilon,v}H_v^{-1}\bar{\Phi}^t_{\varepsilon,v}H_v H_v^{-1}\partial H_v\} \\
&= 2\mathrm{Re}\langle -4tr_g\overline{\partial}_{E_v}\partial_{H_v}\Phi_{\varepsilon,v},\Phi_{\varepsilon,v}\rangle_{H_v} + \langle[4tr_g F_{H_v},\Phi_{\varepsilon,v}],\Phi_{\varepsilon,v}\rangle_{H_v} \\
&\quad + 4|\partial_{H_v}\Phi_{\varepsilon,v}|^2_{H_v} + 4|\overline{\partial}_{E_v}\Phi_{\varepsilon,v}|^2_{H_v}.
\end{aligned}
$$

Using the above formulas, we conclude that

$$
\begin{aligned}
(\frac{\partial}{\partial t}-\widetilde{\Delta})[\sum_{v\in Q_0}\frac{1}{\sigma_v}|\Phi_{\varepsilon,v}|^2_{H_v}] &= -\sum_{v\in Q_0}\frac{4}{\sigma_v}|\nabla_{H_v}\Phi_{\varepsilon,v}|^2_{H_v} \\
&- 4\sum_{a\in Q_1}\Big(|\phi_a^{*H_a}\frac{\Phi_{\varepsilon,\mathfrak{h}a}}{\sigma_{\mathfrak{h}a}}|^2_{H_{\mathfrak{h}a}} + |\frac{\Phi_{\varepsilon,\mathfrak{t}a}}{\sigma_{\mathfrak{t}a}}\phi_a^{*H_a}|^2_{H_{\mathfrak{t}a}} \\
&\qquad - 2\langle\phi_a\circ\frac{\Phi_{\varepsilon,\mathfrak{t}a}}{\sigma_{\mathfrak{t}a}}\otimes\mathrm{Id}_{\widetilde{E}_a}\circ\phi_a^{*H_a},\frac{\Phi_{\varepsilon,\mathfrak{h}a}}{\sigma_{\mathfrak{h}a}}\rangle_{H_{\mathfrak{h}a}\otimes H_{\mathfrak{t}a}}\Big) \\
&- 4\sum_{a\in Q_1}\Big(|\phi_a\frac{\Phi_{\varepsilon,\mathfrak{t}a}}{\sigma_{\mathfrak{t}a}}|^2_{H_{\mathfrak{t}a}} + |\frac{\Phi_{\varepsilon,\mathfrak{h}a}}{\sigma_{\mathfrak{h}a}}\phi_a|^2_{H_{\mathfrak{h}a}} \\
&\qquad - 2\langle\phi_a^{*H_a}\circ\frac{\Phi_{\varepsilon,\mathfrak{h}a}}{\sigma_{\mathfrak{h}a}}\otimes\mathrm{Id}_{\widetilde{E}^*_a}\circ\phi_a,\frac{\Phi_{\varepsilon,\mathfrak{t}a}}{\sigma_{\mathfrak{t}a}}\rangle_{H_{\mathfrak{h}a}\otimes H_{\mathfrak{t}a}}\Big) \\
&+ \sum_{v\in Q_0}\frac{4\varepsilon}{\sigma_v}\langle\frac{\partial}{\partial t}\log(h_v),\Phi_{\varepsilon,v}\rangle_{H_v} \\
&\leq 0,
\end{aligned}
$$

where the last inequality used (3.2) and the following inequality [44]

$$
\langle\frac{\partial}{\partial t}\log(h_v), h_v^{-1}\frac{\partial h_v}{\partial t}\rangle_{H_v}\geq 0.
$$

By taking the trace of both sides of (3.5), we obtain the equality (3.4).
□

3.2. *Donaldson's distance along the flow.* Below, we recall the Donaldson's distance [16, 48] defined on the space of Hermitian metrics.

DEFINITION 3.2. *Given two Hermitian metrics H and K on the bundle E, the Donaldson's distance between them is given by*

$$\sigma(H, K) := tr_E(H^{-1}K) + tr_E(K^{-1}H) - 2\mathrm{rk}(E).$$

For collections of Hermitian metrics $\mathbf{H} = \{H_v\}_{v\in Q_0}$ and $\mathbf{K} = \{K_v\}_{v\in Q_0}$ on the twisted quiver bundle $\mathcal{R}$, we define the Donaldson's distance on $\mathcal{R}$ as

$$\sigma(\mathbf{H}, \mathbf{K}) := \sum_{v\in Q_0} \sigma_v \cdot \sigma(H_v, K_v),$$

where σ_v is a weighting factor associated with each vertex v.

It is evident that $\sigma(\mathbf{H}, \mathbf{K})$ is non-negative and vanishes if and only if $\mathbf{H} = \mathbf{K}$. Furthermore, a sequence of metrics $\mathbf{H}(t)$ converges to a limiting metric $\mathbf{H}$ in the C^0 sense if and only if $\sup \sigma(\mathbf{H}(t), \mathbf{H}) \to 0$ as t approaches the limit.

PROPOSITION 3.3. *Let $\mathbf{H}(t) = \{H_v(t)\}_{v\in Q_0}$ and $\mathbf{K}(t) = \{K_v(t)\}_{v\in Q_0}$ denote two sets of Hermitian metrics on the twisted quiver bundle $\mathcal{R}$. Assuming $H_v(t)$ and $K_v(t)$ satisfy the flow equation (3.2) for each $v \in Q_0$, it follows that*

$$\left(\frac{\partial}{\partial t} - \widetilde{\Delta}\right) \sigma(\mathbf{H}(t), \mathbf{K}(t)) \leq 0.$$

PROOF. For brevity, we denote by

$$h_v := K_v(t)^{-1} H_v(t).$$

By direct calculations, we have

$$
\begin{aligned}
&(\frac{\partial}{\partial t}-\widetilde{\Delta})\Big(\sum_{v\in Q_0}\sigma_v(tr_{E_v}h_v+tr_{E_v}h_v^{-1})\Big)\\
&=-4\sum_{v\in Q_0}\sigma_v\Big(tr_{E_v}(-tr_g\overline{\partial}_{E_v}h_vh_v^{-1}\partial_{K_v}h_v)+tr_{E_v}(-tr_g\overline{\partial}_{E_v}h_v^{-1}h_v\partial_{K_v}h_v^{-1})\Big)\\
&\quad-4\sum_{a\in Q_1}tr_{E_v}\Big(\phi_a^{*K_a}\circ\phi_a\circ h_{\mathfrak{t}a}+h_{\mathfrak{t}a}\circ\phi_a\circ h_{\mathfrak{t}a}^{-1}\otimes\mathrm{Id}_{\widetilde{E}_a}\circ\phi_a^{*K_a}\circ h_{\mathfrak{h}a}\\
&\quad-\phi_a^{*K_a}\circ h_{\mathfrak{h}a}\otimes\mathrm{Id}_{\widetilde{E}_a^*}\circ\phi_a-\phi_a\circ\phi_a^{*K_a}\circ h_{\mathfrak{h}a}\Big)\\
&\quad-4\sum_{a\in Q_1}tr_{E_v}\Big(\phi_a^{*H_a}\circ\phi_a\circ h_{\mathfrak{t}a}^{-1}+h_{\mathfrak{t}a}^{-1}\circ\phi_a\circ h_{\mathfrak{t}a}\otimes\mathrm{Id}_{\widetilde{E}_a}\circ\phi_a^{*H_a}\circ h_{\mathfrak{h}a}^{-1}\\
&\quad-\phi_a^{*H_a}\circ h_{\mathfrak{h}a}^{-1}\otimes\mathrm{Id}_{\widetilde{E}_a^*}\circ\phi_a-\phi_a\circ\phi_a^{*H_a}\circ h_{\mathfrak{h}a}^{-1}\Big)\\
&\quad+4\varepsilon\sum_{v\in Q_0}tr_{E_v}\Big\{h_v\big(\log(H_{0,v}^{-1}H_v)-\log(H_{0,v}^{-1}K_v)\big)\\
&\quad+h_v^{-1}\big(\log(H_{0,v}^{-1}K_v)-\log(H_{0,v}^{-1}H_v)\big)\Big\}\\
&\leq 0,
\end{aligned}
$$

where we used the inequalities [16, 38]

$$tr_{E_v}(-tr_g\overline{\partial}_{E_v}h_vh_v^{-1}\partial_{K_v}h_v)\geq 0,\quad tr_{E_v}(-tr_g\overline{\partial}_{E_v}h_v^{-1}h_v\partial_{K_v}h_v^{-1})\geq 0,$$

the summations on $a\in Q_0$ are non-negative [48], and the following inequality [44]

$$tr_{E_v}\{h_v(\log(H_{0,v}^{-1}H_v)-\log(H_{0,v}^{-1}K_v))+h_v^{-1}(\log(H_{0,v}^{-1}K_v)-\log(H_{0,v}^{-1}H_v))\}\geq 0.$$

□

We omit the proof of the ensuing proposition, since it bears resemblance to the proof provided for Proposition 3.3.

Proposition 3.4. *Let $\mathbf{H}=\{H_v\}_{v\in Q_0}$ and $\mathbf{K}=\{K_v\}_{v\in Q_0}$ denote two sets of Hermitian metrics on the twisted quiver bundle $\mathcal{R}$. Provided that each H_v and K_v satisfies* (2.1) *for all $v\in Q_0$, it follows that*

$$\widetilde{\Delta}\sigma(\mathbf{H},\mathbf{K})\geq 0.$$

3.3. *An inequality used for C^0-estimate.* The ensuing proposition acts as a bridge connecting the stability of the bundle and the C^0-estimate. Relying heavily on [44], we shall only outline the proof here.

Proposition 3.5. *Let $\mathcal{R}$ denote a twisted quiver bundle endowed with a fixed Hermitian metric $\mathbf{K}=\{K_v\}_{v\in Q_0}$ on the non-compact affine Gauduchon manifold (M,D,g,ν). Given a collection of Hermitian metrics $\mathbf{H}=\{H_v\}_{v\in Q_0}$*

on $\mathcal{R}$, set $s_v := \log(K_v^{-1}H_v)$. Suppose the base manifold admits an exhaustion function φ satisfying $\int_M |\widetilde{\Delta}\varphi|\frac{\omega_g^n}{\nu} < +\infty$. Furthermore, assume $\|\frac{\partial\omega_g^{n-1}}{\nu}\|_{L^2(M)} < +\infty$, s_v is bounded, and $\|\overline{\partial}_{E_v}s_v\|_{L^2(M)} < +\infty$. Then, the subsequent inequality holds:

$$\begin{aligned}&\sum_{v\in Q_0}\Big(\int_M tr_{E_v}(\Phi_v(K_v)s_v)\frac{\omega_g^n}{\nu}+\int_M \sigma_v\langle\Psi(s)(\overline{\partial}_{E_v}s),\overline{\partial}_{E_v}s_v\rangle_{K_v}\frac{\omega_g^n}{\nu}\Big)\\ &\leq \int_M tr_{E_v}(\Phi_v(H_v)s_v)\frac{\omega_g^n}{\nu},\end{aligned} \tag{3.6}$$

where

$$\Phi_v(K_v)=\sigma_v tr_g F_{K_v}+\sum_{a\in\mathfrak{h}^{-1}(v)}\phi_a\circ\phi_a^{*K_v}-\sum_{a\in\mathfrak{t}^{-1}(v)}\phi_a^{*K_v}\circ\phi_a-\tau_v\cdot\mathrm{Id}_{E_v}$$

and

$$\Psi(x,y)=\begin{cases}\frac{e^{y-x}-1}{y-x}, & x\neq y;\\ 1, & x=y.\end{cases}$$

PROOF. By direct calculations, we have

$$\begin{aligned}&\sum_{v\in Q_0}\int_M \mathrm{tr}_{E_v}\Big(\Phi_v(H_v)-\Phi_v(K_v))s_v\Big)\frac{\omega_g^n}{\nu}\\ &\geq\sum_{v\in Q_0}\int_M\sigma_v\Big\langle tr_g\overline{\partial}_{E_v}(h_v^{-1}\partial_{K_v}h_v),s_v)\Big\rangle_{K_v}\frac{\omega_g^n}{\nu}\\ &=\sum_{v\in Q_0}\sigma_v\int_M\langle\Psi(s_v)(\overline{\partial}_{E_v}s_v),\overline{\partial}_{E_v}s_v\rangle_{K_v}\frac{\omega_g^n}{\nu}.\end{aligned} \tag{3.7}$$

To derive the first inequality in (3.7), we employed the following fact (see [2, Lemma 3.5]):

$$\sum_{v\in Q_0}\langle\sum_{a\in\mathfrak{h}^{-1}(v)}(\phi_a\circ\phi_a^{*H_v}-\phi_a\circ\phi_a^{*K_v})-\sum_{a\in\mathfrak{t}^{-1}(v)}(\phi_a^{*H_v}\circ\phi_a-\phi_a^{*K_v}\circ\phi_a),s_v\rangle\geq 0 \tag{3.8}$$

The second equality in (3.7) is a direct consequence of [38, Proposition 4.3].

□

REMARK 3.6. It should be mentioned that the proof of [38, Proposition 4.3] rather relies on $\|\frac{\partial\omega_g^{n-1}}{\nu}\|_{L^2(M)} < +\infty$ and the Gauduchon condition $\partial\bar{\partial}\omega_g^{n-1}=0$.

4. The perturbed heat flow on affine Gauduchon manifolds

4.1. *Long-time existence for compact case.* In this section, we investigate the existence of long-term solutions for the perturbed heat flow (3.2) of the twisted quiver bundle $\mathcal{R}$ on an affine Gauduchon manifold (M, D, g, ν), which may or may not have a boundary. In the case where M is a manifold without boundary, we consider the following perturbed heat flow:

$$\begin{cases} H_v^{-1}\frac{\partial H_v}{\partial t} = -\frac{4}{\sigma_v}\Phi_{\varepsilon,v}(H_v), \\ H_v(0) = H_{0,v}. \end{cases} \tag{4.9}$$

For a compact manifold M with a smooth, non-empty boundary, we examine the Dirichlet boundary value problem with a fixed collection of Hermitian metrics $\widetilde{\mathbf{H}} = \{\widetilde{H}_v\}_{v\in Q_0}$ defined on ∂M:

$$\begin{cases} H_v^{-1}\frac{\partial H_v}{\partial t} = -\frac{4}{\sigma_v}\Phi_{\varepsilon,v}(H_v), \\ H_v(0) = H_{0,v}, \\ H_v|_{\partial M} = H_{0,v}. \end{cases} \tag{4.10}$$

Due to the parabolic characteristics of the flow (3.2), the well-established parabolic theory ensures the existence of a solution for a short period of time.

Proposition 4.1. *For any sufficiently small $T > 0$, both (4.9) and (4.10) possess a smooth, well-defined solution $\mathbf{H}(t) = \{H_v(t)\}_{v\in Q_0}$ within the interval $0 \le t < T$.*

Building upon the arguments in [16, Lemma 19], our goal is to prove the continual existence of the perturbed heat flow.

Lemma 4.2. *Suppose a smooth solution $\mathbf{H}(t) = \{H_v(t)\}_{v\in Q_0}$ of either (4.9) or (4.10) is defined on the interval $0 \le t < T < +\infty$. Then, as $t \to T$, the metric $\mathbf{H}(t)$ converges in the C^0 sense to a continuous, non-degenerate metric $\mathbf{H}(T)$ on the quiver bundle $\mathcal{R}$.*

Proof. By the continuity at $t = 0$, for any $\epsilon > 0$, there exists a δ such that $\sup_M \sigma(\mathbf{H}(t_0), \mathbf{H}(t_0')) < \epsilon$ whenever $t_0, t_0' \in (0, \delta)$. Utilizing Proposition 3.3 and the maximum principle, we deduce that $\sup_M \sigma(\mathbf{H}(t), \mathbf{H}(t')) < \epsilon$ for all $t, t' > T - \delta$. This implies that $\mathbf{H}(t)$ is uniformly Cauchy, so $\mathbf{H}(t) \to \mathbf{H}(T)$, where $\mathbf{H}(T)$ is continuous.

Alternatively, by Proposition 3.1, $|\Phi_{\varepsilon,v}(H_v)|_{H_v}$ is uniformly bounded. Since

$$\left|\frac{\partial}{\partial t}(\log tr_{E_v} h_v)\right|_{H_v} \le 2|\Phi_{\varepsilon,v}(H_v)|_{H_v}$$

and

$$\left|\frac{\partial}{\partial t}(\log tr_{E_v} h_v^{-1})\right|_{H_v} \le 2|\Phi_{\varepsilon,v}(H_v)|_{H_v},$$

we infer that $\sigma(\mathbf{H}(t), \mathbf{H}(0))$ is uniformly bounded on $M \times [0, T)$. Therefore, the metric $\mathbf{H}(T)$ is non-degenerate. □

By employing a similar argument as in [16, Lemma 19], the subsequent lemma is straightforward to prove.

LEMMA 4.3. *Let (M, D, g, ν) be a compact affine Gauduchon manifold, either without boundary or with a non-empty boundary. Consider the collection of Hermitian metrics $\mathbf{H}(t) = \{H_v(t)\}_{v \in Q_0}$ for $0 \leq t < T$ on the twisted quiver bundle $\mathcal{R}$ over M (subject to Dirichlet boundary conditions). Assume $\mathbf{H}_0 = \{H_{0,v}\}_{v \in Q_0}$ is the initial data on $\mathcal{R}$. If, as $t \to T$, $\mathbf{H}(t)$ converges in C^0 to a non-degenerate continuous metric $\mathbf{H}(T)$ on $\mathcal{R}$, and if $\sup |tr_g F_{H_v(t)}|_{H_{0,v}}$ is uniformly bounded for all t, then $H_v(t)$ is bounded in C^1 and L_2^p (for any $1 < p < +\infty$) for all t.*

We now demonstrate the existence of the flow for extended periods.

PROPOSITION 4.4. *Equations* (4.9) *and* (4.10) *possess a unique solution* $\mathbf{H}(t)$ *that persists for all time.*

PROOF. Proposition 4.1 establishes short-term existence. Assume a solution $\mathbf{H}(t)$ exists for $0 \leq t < T < +\infty$. By Lemma 4.2, $\mathbf{H}(t)$ converges in C^0 to a non-degenerate, continuous $\mathbf{H}(T)$ on $\mathcal{R}$ as $t \to T$. Since T is finite, (3.3) implies $\sup_M |tr_g F_{H_v(t)}|_{H_{0,v}}$ is uniformly bounded on $[0, T)$. Further, by Lemma 4.3, $H_v(t)$ is uniformly bounded in C^1 and L_2^p (for any $1 < p < +\infty$) for all t. Applying Hamilton's methodology [18], we infer $H_v(t) \to H_v(T)$ in C^∞, extending $\mathbf{H}(t)$ beyond T. Thus, (4.9) and (4.10) admit a solution $\mathbf{H}(t)$ for all time. Uniqueness follows from the maximum principle and Proposition 3.3. □

4.2. *Long-time existence for non-compact case.* In the remainder of this section, we focus on the persistent presence of the perturbed heat flow (3.2) for the twisted quiver bundle $\mathcal{R}$ over a non-compact affine Gauduchon manifold (M, D, g, ν). We postulate an exhaustion function $\varphi \geq 0$ with bounded $tr_g \partial\bar{\partial}\varphi$, satisfying Condition 2 for M. For a fixed ρ, let M_ρ denote the compact subspace $\{x \in M \mid \varphi(x) \leq \rho\}$ with boundary ∂M_ρ. Given the initial metric $\mathbf{H}_0$ on $\mathcal{R}$ over M, we consider the Dirichlet boundary condition:

$$\mathbf{H}(t)|_{\partial M_\rho} = \mathbf{H}_0|_{\partial M_\rho}. \tag{4.11}$$

By Proposition 4.4, for each M_ρ, the flow (3.2) with this boundary condition and initial metric $\mathbf{H}_0$ has a unique long-term solution $\mathbf{H}(t)$ for $0 \leq t < +\infty$.

PROPOSITION 4.5. *Assuming $\mathbf{H}(t)$ is a long-term solution to the perturbed heat flow (3.2) on M_ρ that satisfies the Dirichlet boundary condition (4.11),* *we have*

(4.12)

$$|\log h_v|_{H_{0,v}}(x,t) \le \frac{C_1}{\varepsilon}\sum_{v\in Q_0}\max_{M_\rho}|\Phi_v(H_{0,v})|_{H_{0,v}},\quad \forall (x,t)\in M_\rho\times[0,+\infty),$$

where C_1 is a constant independent of ε.

PROOF. By direct computations, we have

$$\begin{aligned}
&\sum_{v\in Q_0}\langle H_v^{-1}\frac{\partial H_v}{\partial t},\log h_v\rangle_{H_{0,v}} = \sum_{v\in Q_0}\langle -\frac{4}{\sigma_v}\Phi_{\varepsilon,v}(H_v),\log h_v\rangle_{H_{0,v}}\\
&=\sum_{v\in Q_0}\langle -\frac{4}{\sigma_v}\Phi_v(H_{0,v}),\log h_v\rangle_{H_{0,v}}\\
&\quad+\sum_{v\in Q_0}\langle -\frac{4}{\sigma_v}(\Phi_{\varepsilon,v}(H_v)-\Phi_v(H_{0,v})),\log h_v\rangle_{H_{0,v}}\\
&\le \sum_{v\in Q_0}\frac{4}{\sigma_v}|\Phi_v(H_{0,v})|_{H_v}|\log h_v|_{H_v}\\
&\quad+\sum_{v\in Q_0}\Big\langle 4\,\mathrm{tr}_g\big(\overline{\partial}_{E_v}(h_v^{-1}\partial_{H_{0,v}}h_v)\big)+\varepsilon\sigma_v\log h_v,\log h_v\Big\rangle_{H_{0,v}},
\end{aligned}$$

where we have used the inequality (3.8).

Alternatively, one can easily verify that

$$\sum_{v\in Q_0}\langle H_v^{-1}\frac{\partial H_v}{\partial t},\log h_v\rangle_{H_{0,v}} = \langle h_v^{-1}\frac{\partial h_v}{\partial t},\log h_v\rangle_{H_{0,v}} = \frac{1}{2}\frac{\partial}{\partial t}\Big(\sum_{v\in Q_0}|\log h_v|^2_{H_{0,v}}\Big)$$

and

$$\sum_{v\in Q_0}\langle 4tr_g\overline{\partial}_{E_v}(h_v^{-1}\partial_{H_{0,v}}h_v),\log h_v\rangle_{H_{0,v}} \ge -\frac{1}{2}\widetilde{\Delta}\Big(\sum_{v\in Q_0}|\log h_v|^2_{H_{0,v}}\Big).$$

Then

$$\begin{aligned}
&\frac{1}{2}(\frac{\partial}{\partial t}-\widetilde{\Delta})(\sum_{v\in Q_0}|\log h_v|^2_{H_{0,v}})\\
&\le -\varepsilon\sum_{v\in Q_0}\sigma_v|\log h_v|^2_{H_{0,v}}+\sum_{v\in Q_0}\frac{4}{\sigma_v}|\Phi(H_{0,v})|_{H_{0,v}}|\log h_v|_{H_{0,v}}\\
&\le -\varepsilon C_2\sum_{v\in Q_0}|\log h_v|^2_{H_{0,v}}+C_3\sum_{v\in Q_0}|\Phi(H_{0,v})|_{H_{0,v}}|\log h_v|_{H_{0,v}},
\end{aligned}$$

which together with the maximum principle implies (4.12).

□

For future reference, we recall the following lemma.

LEMMA 4.6 ([39, Lemma 6.7]). *Let $u(x,t)$ be a function on $M_\rho \times [0,T]$ satisfying*

$$\left(\frac{\partial}{\partial t} - \widetilde{\Delta}\right) u \leq 0, \quad u|_{t=0} = 0,$$

and $\sup_{M_\rho} u \leq C_4$. Then, we have

$$u(x,t) \leq \frac{C_4}{\rho}(\varphi(x) + C_5 t),$$

where C_5 is the bound of $\widetilde{\Delta}\varphi$ as in Condition 2.

We postulate that for all $v \in Q_0$, the norm $|\Phi_v(H_{0,v})|_{H_{0,v}}$ is bounded on the affine Gauduchon manifold (M, D, g, ν). Given any compact subset $\Omega \subset M$, there exists a constant ρ_0 such that $\Omega \subseteq M_{\rho_0}$. Consider $\mathbf{H}_\rho(t) = \{H_{\rho,v}(t)\}_{v\in Q_0}$ and $\mathbf{H}_{\rho_1}(t) = \{H_{\rho_1,v}(t)\}_{v\in Q_0}$ as long-term solutions to the perturbed heat flow (3.2) satisfying the Dirichlet boundary condition (4.11) for $\rho_0 < \rho_1 < \rho$. Define $u = \sigma(\mathbf{H}_\rho(t), \mathbf{H}_{\rho_1}(t))$. By Proposition 4.5, u is uniformly bounded and serves as a subsolution for the heat operator with $u(0) = 0$. Applying Lemma 4.6, we obtain

$$\sigma(\mathbf{H}_\rho(t), \mathbf{H}_{\rho_1}(t)) \leq C_4 \frac{(\rho_0 + C_5 T)}{\rho}$$

on $M_{\rho_0} \times [0,T]$. Thus, $\mathbf{H}_\rho$ forms a Cauchy sequence on $M_{\rho_0} \times [0,T]$ as $\rho \to \infty$. For each $v \in Q_0$, Proposition 4.5 guarantees the uniform C^0 bound of $\mathbf{H}_\rho(t)$, and local C^1 estimates can be derived similarly to [44, Proposition 3.5]. Using the standard Schauder estimate for parabolic equations, we obtain local uniform and smooth estimates for $H_{\rho,v}(t)$ for each $v \in Q_0$. Note that the parabolic Schauder estimate only yields a uniform and smooth estimate for $h_v(t)$ on $M_{\rho_0} \times [\iota, T]$ with $\iota > 0$, depending on ι^{-1}. To address this, we apply the maximum principle to obtain a local uniform bound on the curvature $|F_{H_{\rho,v}}|_{H_{\rho,v}}$ for each $v \in Q_0$, followed by standard elliptic estimates to obtain locally uniform and smooth estimates. This step is omitted due to its similarity to [29, Lemma 2.5]. By taking a subsequence with $\rho \to \infty$, the metric $\mathbf{H}_\rho(t)$ converges in C^∞_{loc}-topology on the twisted quiver bundle $\mathcal{R}$ to a long-term solution $\mathbf{H}(t)$ of the perturbed heat flow (3.2) on $M \times [0, \infty)$. In summary, we have the following proposition.

PROPOSITION 4.7. *Let $\mathcal{R}$ denote the twisted quiver bundle, endowed with a fixed Hermitian metric $\mathbf{H}_0$, over the non-compact affine Gauduchon manifold (M, D, g, ν) fulfilling Condition 2. Assuming $\sup_M |\Phi_v(H_{0,v})|_{H_{0,v}}$ is finite, it can be demonstrated that the perturbed heat flow (3.2) admits a long-term solution $\mathbf{H}(t)$ satisfying the following bound on the whole M:*

$$\sup_{(x,t)\in M\times[0,+\infty)} |\log h_v|_{H_0}(x,t) \leq \frac{C_1}{\varepsilon} \sum_{v\in Q_0} \sup_M |\Phi_v(H_{0,v})|_{H_{0,v}}.$$

5. Solution to the perturbed equation

5.1. *Dirichlet problem on compact affine Gauduchon manifold.* We first tackle the Dirichlet problem related to the perturbed equation, leading to the following proposition.

Theorem 5.1. *Consider $\mathcal{R}$, the twisted quiver bundle, endowed with a fixed Hermitian metric $\mathbf{H}_0 = \{H_{0,v}\}_{v\in Q_0}$, over a compact affine Gauduchon manifold (M, D, g, ν) with a non-empty boundary ∂M. There exists a unique Hermitian metric $\mathbf{H} = \{H_v\}_{v\in Q_0}$ on $\mathcal{R}$ fulfilling the conditions*

$$\Phi_{\varepsilon,v}(H_v) = 0 \quad \textit{and} \quad H_v|_{\partial M} = H_{0,v}, \quad \forall \varepsilon \geq 0. \tag{5.13}$$

For $\varepsilon > 0$, it holds that

$$\sup_{x\in M} |s_v|_{H_{0,v}}(x) \leq \frac{C_1}{\varepsilon} \sum_{v\in Q_0} \sup_M |\Phi_v(H_{0,v})|_{H_{0,v}}. \tag{5.14}$$

Furthermore,

$$\sum_{v\in Q_0} \|\overline{\partial}_{E_v} s_v\|_{L^2(M)} \leq C(\varepsilon^{-1}, \Phi_v(H_{0,v}), \mathrm{Vol}(M)), \tag{5.15}$$

where $s_v = \log(H_{0,v}^{-1}H_v)$. If the initial metric $\mathbf{H}_0$ on $\mathcal{R}$ satisfies

$$tr_{E_v}(\Phi_v(H_{0,v})) = 0, \tag{5.16}$$

then $\sum_{v\in Q_0} \sigma_v tr_{E_v}(s_v) = 0$, and $\mathbf{H}$ on $\mathcal{R}$ also meets condition (5.16).

Proof. Based on Proposition 4.4, we establish the existence of a long-term solution $\mathbf{H}(t)$ for the perturbed heat equation (4.10). By applying Proposition 3.1 and the inequality $|\nabla\zeta|^2 \geq |\nabla|\zeta||^2$, we obtain

$$\left(\frac{\partial}{\partial t} - \widetilde{\Delta}\right)\left[\sum_{v\in Q_0} \frac{1}{\sigma_v}|\Phi_{\varepsilon,v}(H_v)|_{H_v}\right] \leq 0. \tag{5.17}$$

When the initial metric $\mathbf{H}_0$ satisfies (5.16), combining (3.4) with the maximum principle yields

$$\sum_{v\in Q_0} tr_{E_v}(\Phi_{\varepsilon,v}(H_v)) = 0.$$

As a result,

$$\sum_{v\in Q_0} \sigma_v tr_{E_v}(\log(H_{0,v}^{-1}H_v(t))) = 0$$

holds true, ensuring that $\mathbf{H}(t)$ meets the condition (5.16) for all $t \geq 0$.

Referring to [40, Chapter 5, Proposition 1.8], our objective is to address the Dirichlet problem on M defined as:

$$\widetilde{\Delta} f = -|\Phi_v(H_{0,v})|_{H_{0,v}}, \quad f|_{\partial M} = 0. \tag{5.18}$$

We introduce $w(x,t) = \int_0^t |\Phi_{\varepsilon,v}(H_v)|_{H_v}(x,\varrho)\mathrm{d}\varrho - f(x)$. By considering (5.17), (5.18), and the boundary conditions of H_v, it becomes clear that for $t > 0$, $|\Phi_{\varepsilon,v}(H_v)|_{H_v}(x,t)$ diminishes on ∂M. Hence,

$$\left(\frac{\partial}{\partial t} - \widetilde{\Delta}\right) w(x,t) \leq 0, \quad w(x,0) = -f(x), \quad w(x,t)|_{\partial M} = 0.$$

Applying the maximum principle, for all $x \in M$ and $t \in (0,+\infty)$, we derive

$$\int_0^t |\Phi_{\varepsilon,v}(H_v)|_{H_v}(x,\varrho)\mathrm{d}\varrho \leq \sup_{y\in M} f(y). \tag{5.19}$$

Given the assumptions $t_1 \leq t \leq t_2$ and defining $\bar{h}_v(x,t) = H_v^{-1}(x,t_1)H_v(x,t)$, one can easily deduce that

$$\frac{\partial}{\partial t} \log tr_{E_v}(\bar{h}_v) \leq 2|\Phi_{\varepsilon,v}(H_v)|_{H_v}.$$

Through integration, we find

$$tr_{E_v}(H_v^{-1}(x,t_1)H_v(x,t)) \leq r \exp\left(2\int_{t_1}^t |\Phi_{\varepsilon,v}(H_v)|_{H_v}\mathrm{d}\varrho\right).$$

Similarly, an equivalent bound holds for $tr_{E_v}(H_v^{-1}(x,t)H_v(x,t_1))$. Consequently,

$$\sigma(H_v(x,t), H_v(x,t_1)) \leq 2r\left(\exp\left(2\int_{t_1}^t |\Phi_{\varepsilon,v}(H_v)|_{H_v}\mathrm{d}\varrho\right) - 1\right). \tag{5.20}$$

Using (5.19) and (5.20), we deduce that as $t \to \infty$, the metric $\mathbf{H}(t)$ on the twisted quiver bundle $\mathcal{R}$ converges to a continuous metric $\mathbf{H}_\infty$ in the C^0 topology. By Lemma 4.3, for each vertex $v \in Q_0$, $H_v(t)$ is uniformly bounded in both C^1_{loc} and $L^p_{2,loc}$ $(1 < p < +\infty)$. Additionally, $|H_v^{-1}\frac{\partial H_v}{\partial t}|$ is uniformly bounded for each $v \in Q_0$. Employing elliptic regularity, we conclude the existence of a subsequence $H_v(t)$ converging to $H_{v,\infty}$ in C^∞_{loc}-topology. From (5.19), $H_{v,\infty}$ meets the boundary condition. Uniqueness follows from the maximum principle and Proposition 3.4.

If $\varepsilon > 0$, the implication in Proposition 4.5, as noted in (4.12), implies (5.14). By definition, it is evident that

$$|\overline{\partial}_{E_v}s_v|^2_{H_{0,v}} \leq C_6\langle\Psi(s)(\overline{\partial}_{E_v}s_v), \overline{\partial}_{E_v}s_v\rangle_{H_{0,v}},$$

where C_6 is a constant dependent only on the L^∞-bound of s_v.

Applying inequality (3.6) from Proposition 3.5 and equation (5.13), we derive

$$\begin{aligned}\sum_{v\in Q_0}\int_M |\overline{\partial}_{E_v}s_v|^2_{H_{0,v}}\frac{\omega^n}{\nu} &\leq C_6\sum_{v\in Q_0}\int_M \langle \Psi(s_v)(\overline{\partial}_{E_v}s_v),\overline{\partial}_{E_v}s_v\rangle_{H_{0,v}}\frac{\omega^n}{\nu}\\ &= C_6\sum_{v\in Q_0}\int_M(-tr_{E_v}(\Phi_v(H_{0,v})s_v)-\varepsilon\sigma_v|s_v|^2_{H_{0,v}})\frac{\omega^n}{\nu}\\ &\leq \frac{C_7}{\varepsilon}\sum_{v\in Q_0}\sup_M|\Phi_v(H_{0,v})|^2_{H_{0,v}}\cdot \mathrm{Vol}(M),\end{aligned}$$

which directly results in the conclusion of (5.15).

□

5.2. *Solution on non-compact affine Gauduchon manifold.* Let (M, D, g, ν) denote a non-compact affine Gauduchon manifold, with $\{M_\rho\}$ constituting an exhaustive sequence of its compact subdomains. Consider a twisted quiver bundle $\mathcal{R}$ over the base M_ρ, equipped with a set of Hermitian metrics $\mathbf{H}_0$ on $\mathcal{R}$. According to Theorem 5.1, the Dirichlet problem on M_ρ is solvable, producing a Hermitian metric $\mathbf{H}_\rho(x) = \{H_{\rho,v}\}_{v\in Q_0}$ on $\mathcal{R}$ that fulfills:

$$\begin{cases}\Phi_{\varepsilon,v}(H_{\rho,v}) = 0,\\ H_{\rho,v}(x)|_{\partial M_\rho} = H_{0,v}(x).\end{cases}$$

To extend the solution across the entire manifold M, we depend on a priori estimates, notably the C^0-estimate. Define by

$$h_{\rho,v} = H_{0,v}^{-1}H_{\rho,v}$$

for each $v \in Q_0$. By Proposition 5.1, we have for all $v \in Q_0$:

$$\sup_{x\in M_\rho}|\log h_{\rho,v}|_{H_{0,v}}(x) \leq \frac{C_1}{\varepsilon}\sum_{v\in Q_0}\sup_{M_\rho}|\Phi_v(H_{0,v})|_{H_{0,v}}.$$

For any compact $\Omega \subset M$, there is a ρ_0 such that $\Omega \subseteq M_{\rho_0}$. Employing arguments akin to those in [38, Proposition 4.1], we secure local uniform C^1-estimates. Specifically, for $\rho > \rho_0$,

$$\sup_{x\in\Omega}|h_{\rho,v}^{-1}\partial_{H_{0,v}}h_{\rho,v}|_{H_{0,v}} \leq C_8, \tag{5.21}$$

where C_8 is a constant uniform across ρ. Utilizing the perturbed equation $\Phi_{\varepsilon,v}(H_v) = 0$ and standard elliptic theory, we infer uniform local higher-order estimates. By extracting a subsequence, for each $v \in Q_0$, $H_{\rho,v}$ converges in C^∞_{loc}-topology to $H_{\infty,v}$, which satisfies $\Phi_{\varepsilon,v}(H_v) = 0$ on M. This establishes the following proposition:

Proposition 5.2. *Consider a twisted quiver bundle $\mathcal{R}$ equipped with a fixed Hermitian metric $\mathbf{H}_0$ over a non-compact affine Gauduchon manifold (M, D, g, ν). Provided that $\sup_M |\Phi_v(H_0)|_{H_{0,v}}$ is bounded, for any $\varepsilon > 0$, there exists a metric $\mathbf{H} = \{H_v\}_{v\in Q_0}$ on $\mathcal{R}$ satisfying*

$$\Phi_{\varepsilon,v}(H_v) = 0,$$

$$\sup_{x\in M} |\log(H_{0,v}^{-1}H_v)|_{H_{0,v}}(x) \leq \frac{C_1}{\varepsilon} \sum_{v\in Q_0} \sup_M |\Phi_v(H_{0,v})|_{H_{0,v}},$$

and

$$\|\overline{\partial}_{E_v}(\log(H_{0,v}^{-1}H_v))\|_{L^2} \leq C(\varepsilon^{-1}, \Phi_v(H_{0,v}), \mathrm{Vol}(M)).$$

Furthermore, if $\mathbf{H}_0$ meets the criterion (5.16), then

$$\sum_{v\in Q_0} \sigma_v tr_{E_v} \log(H_{0,v}^{-1}H_v) = 0,$$

ensuring $\mathbf{H}$ also satisfies (5.16).

6. Stability implies the existence of the Hermite–Einstein metric

Let (M, D, g, ν) denote the non-compact affine Gauduchon manifold as specified in Theorem 1.1, and let $\mathcal{R}$ signify a twisted quiver bundle over M. Provided a suitable background metric $\mathbf{K} = \{K_v\}_{v\in Q_0}$ on $\mathcal{R}$ satisfying $tr_g F_{K_v} \leq 0$, $\sup_M |tr_g F_{K_v}|_{K_v} < +\infty$, and $\sup_M |\phi|_{K_v} < +\infty$, we invoke [44, Proposition 4.3] to resolve the Poisson equation on M:

$$tr_g \bar{\partial}\partial f = -\frac{1}{\sum_{v\in Q_0} \sigma_v \cdot \mathrm{rk}(E_v)} \sum_{v\in Q_0} tr_{E_v}(\Phi_v(K_v)). \tag{6.22}$$

Applying the conformal transformation $\overline{K}_v = e^f K_v$, direct calculation yields

$$tr_{E_v} \Phi_v(\overline{K}_v) = tr_{E_v} \Phi_v(K_v) + tr_{E_v}\Big(\sigma_v tr_g(\bar{\partial}\partial f)\mathrm{Id}_{E_v}\Big). \tag{6.23}$$

Using (6.22) and (6.23), we find that $\overline{K}_v$ meets the requirement:

$$\sum_{v\in Q_0} tr_{E_v}(\Phi_v(\overline{K}_v)) = 0. \tag{6.24}$$

Analyzing the function f, we observe that if $\mathcal{R}$ displays analytic (σ, τ)-stability relative to the Hermitian metric $\mathbf{K}$, it retains this stability relative to the transformed metric $\overline{\mathbf{K}} = \{\overline{K}_v\}_{v\in Q_0}$. Hence, we may assume without loss of generality that the initial metric $\mathbf{K}$ on $\mathcal{R}$ already satisfies Equation (6.24).

By Proposition 5.2, for every vertex $v \in Q_0$ and any $\varepsilon > 0$, the following perturbed equation is solvable:

$$\Phi_{\varepsilon,v}(H_{\varepsilon,v}) := \Phi_v(H_{\varepsilon,v}) + \varepsilon\sigma_v(\log h_{\varepsilon,v}) = 0, \tag{6.25}$$

where $h_{\varepsilon,v}$ is given by $K_v^{-1}H_{\varepsilon,v} = e^{s_{\varepsilon,v}}$ and

$$\Phi_v(H_{\varepsilon,v}) = \sigma_v tr_g F_{H_{\varepsilon,v}} + \sum_{a\in\mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_{\varepsilon,v}} - \sum_{a\in\mathfrak{t}^{-1}(v)} \phi_a^{*H_{\varepsilon,v}} \circ \phi_a - \tau_v \cdot \mathrm{Id}_{E_v}.$$

Considering that the initial Hermitian metric $\mathbf{K}$ on $\mathcal{R}$ satisfies (6.24), it follows that:

$$\sum_{v\in Q_0} \sigma_v tr_{E_v}(\log h_{\varepsilon,v}) = 0.$$

We denote by

$$\mathrm{Herm}(E_v, K_v) = \{\eta \in \mathrm{End}(E_v) : \eta^{*K_v} = \eta\}$$

and

$$\mathrm{Herm}^+(E_v, K_v) = \{\rho \in \mathrm{Herm}(E_v, K_v) : \rho > 0\},$$

where ρ means all eigenvalues of ρ are positive.

Using analogous arguments as in [31, Corollary 19] and [32, Lemma 3.3.4], we can easily derive the following lemma.

Lemma 6.1. *For $h_{\varepsilon,v} \in \mathrm{Herm}^+(E_v, K_v)$ fulfilling $\Phi_{\varepsilon,v}(H_{\varepsilon,v}) = 0$ with some $\varepsilon > 0$, it follows that*

$$\sigma_v \sup_M |\log h_{\varepsilon,v}|_{K_v} \leq C_9 \left(\sum_{v\in Q_0} \sigma_v \|\log h_{\varepsilon,v}\|_{L^2(M)} + C_{10} \right),$$

where C_9 and C_{10} depend solely on ω and K_v.

When $\mathcal{R}$ displays analytic (σ,τ)-stability with respect to the Hermitian metric $\mathbf{K}$, our objective is to show that, by choosing a subsequence, $\mathbf{H}_\varepsilon$ converges to an affine (σ,τ)-Hermite–Einstein metric $\mathbf{H}$ in the C^∞_{loc}-topology as $\varepsilon \to 0$. Utilizing the local C^1-estimates from (5.21) combined with standard elliptic theory, our primary goal becomes obtaining a uniform C^0-estimate. By virtue of Lemma 6.1, this task reduces to proving a uniform bound on $\sum_{v\in Q_0} \sigma_v \|\log h_{\varepsilon,v}\|_{L^2(M)}$.

We proceed by contradiction. If our assertion fails, there must exist a positive constant δ and a subsequence $\varepsilon_i \to 0$ as $i \to \infty$ such that

$$\sum_{v\in Q_0} \sigma_v \|\log h_{\varepsilon_i,v}\|_{L^2(M)} \to +\infty.$$

Let us define

$$s_{\varepsilon_i,v} = \log h_{\varepsilon_i,v}, \quad l_{i,v} = \|s_{\varepsilon_i,v}\|_{L^2(M)}, \quad \text{and} \quad u_{\varepsilon_i,v} = \frac{s_{\varepsilon_i,v}}{l_{i,v}}.$$

From these definitions, it follows that $\sum_{v\in Q_0}\operatorname{tr}(\sigma_v u_{\varepsilon_i,v})=0$ and $\|u_{\varepsilon_i,v}\|_{L^2}=1$. By applying Lemma 6.1, we derive:

$$\sup_M |u_{\varepsilon_i,v}| \leq \frac{C_9}{l_{i,v}}\left(\sum_{v\in Q_0}\sigma_v l_{i,v}+C_{10}\right) < C_{11} < +\infty. \tag{6.26}$$

Step 1 We will show that for each $v\in Q_0$, the L_1^2 norms of $u_{\varepsilon_i,v}$ remain uniformly bounded. Given that the L^2 norms of $u_{\varepsilon_i,v}$ are normalized to 1, our primary objective is to establish uniform boundedness for the L^2 norms of $\nabla u_{\varepsilon_i,v}$.

Utilizing Proposition 3.5 and the perturbed equation (6.25), we infer that for all $u_{\varepsilon_i,v}$, the following inequality is satisfied:

$$\begin{aligned}&\sum_{v\in Q_0}\Bigg(\int_M \operatorname{tr}_{E_v}(\Phi_v(K_v)u_{\varepsilon_i,v})\frac{\omega_g^n}{\nu}\\&\qquad+\sigma_v\int_M l_{i,v}\left\langle\Psi(l_{i,v}u_{\varepsilon_i,v})(\overline{\partial}_{E_v}u_{\varepsilon_i,v}),\overline{\partial}_{E_v}u_{\varepsilon_i,v}\right\rangle_{K_v}\frac{\omega_g^n}{\nu}\Bigg)\\&\leq -\varepsilon_i\sum_{v\in Q_0}\sigma_v l_{i,v},\end{aligned}$$

Next, consider the function defined by:

$$\gamma\Psi(\gamma x,\gamma y)=\begin{cases}\gamma, & \text{if } x=y,\\ \frac{e^{\gamma(y-x)}-1}{y-x}, & \text{if } x\neq y.\end{cases}$$

From (6.26), we infer that (x,y) lies in the domain $[-C_{12},C_{12}]\times[-C_{12},C_{12}]$. A simple verification yields:

$$\gamma\Psi(\gamma x,\gamma y)\to\begin{cases}(x-y)^{-1}, & \text{if } x>y,\\ +\infty, & \text{if } x\leq y,\end{cases} \tag{6.27}$$

which increases monotonically as $\gamma\to+\infty$. We introduce ζ, a smooth function mapping $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}^+$ such that $\zeta(x,y)<(x-y)^{-1}$ for $x>y$. By applying (6.27) and adopting reasoning from [39, Lemma 5.4], we derive:

$$\begin{aligned}&\sum_{v\in Q_0}\Bigg(\int_M \operatorname{tr}_{E_v}(\Phi_v(K_v)u_{\varepsilon_i,v})\frac{\omega_g^n}{\nu}\\&\qquad+\sigma_v\int_M\left\langle\zeta(u_{\varepsilon_i,v})(\overline{\partial}_{E_v}u_{\varepsilon_i,v}),\overline{\partial}_{E_v}u_{\varepsilon_i,v}\right\rangle_{K_v}\frac{\omega_g^n}{\nu}\Bigg)\\&\leq 0,\end{aligned}\tag{6.28}$$

for i large enough.

Specifically, we opt for $\zeta(x,y) = \frac{1}{3C_{12}}$. Consequently, within the domain $(x,y) \in [-C_{12}, C_{12}] \times [-C_{12}, C_{12}]$ and under the condition $x > y$, it holds that $\frac{1}{3C_{12}} < \frac{1}{x-y}$. Hence,

$$\sum_{v \in Q_0} \left(\int_M tr_{E_v}(\Phi_v(K_v) u_{\varepsilon_i,v}) \frac{\omega_g^n}{\nu} + \frac{\sigma_v}{3C_{12}} \int_M |\overline{\partial}_{E_v} u_{\varepsilon_i,v}|^2_{K_v} \frac{\omega_g^n}{\nu} \right) \leq 0$$

for sufficiently large i. This implies

$$\sum_{v \in Q_0} \int_M |\overline{\partial}_{E_v} u_{\varepsilon_i,v}|^2_{K_v} \frac{\omega_g^n}{\nu} \leq C_{13} \sum_{v \in Q_0} \sup_M |\Phi_v(K_v)|_{K_v} \cdot \mathrm{Vol}(M).$$

Hence, for every $v \in Q_0$, the sequence $u_{\varepsilon_i,v}$ remains bounded in the L_1^2 norm, enabling us to extract a weakly convergent subsequence in L_1^2, denoted $\{u_{\varepsilon_{i_j},v}\}$, which converges to $u_{\infty,v}$. For brevity, we continue to use $\{u_{\varepsilon_i,v}\}$ to represent this subsequence. Considering the embedding of L_1^2 into L^2, it follows that

$$1 = \lim_{i \to \infty} \int_M |u_{\varepsilon_i,v}|^2_{H_{0,v}} \frac{\omega_g^n}{\nu} = \int_M |u_{\infty,v}|^2_{H_{0,v}} \frac{\omega_g^n}{\nu},$$

indicating that $u_{\infty,v}$ has an L^2 norm of 1 and is therefore non-trivial.

Utilizing equation (6.28) and paralleling the argument in [39, Lemma 5.4], we obtain the inequality

$$\begin{aligned} &\sum_{v \in Q_0} \left(\int_M tr_{E_v}(\Phi_v(K_v) u_{\infty,v}) \frac{\omega_g^n}{\nu} \right. \\ &\qquad \left. + \sigma_v \int_M \left\langle \zeta(u_{\infty,v})(\overline{\partial}_{E_v} u_{\infty,v}), \overline{\partial}_{E_v} u_{\infty,v} \right\rangle_{K_v} \frac{\omega_g^n}{\nu} \right) \\ &\leq 0. \end{aligned} \tag{6.29}$$

Step 2 Utilizing the reasoning presented by Uhlenbeck and Yau in [41], we construct a quiver subsheaf that contradicts the (σ, τ)-analytic stability of $\mathcal{R}$.

By leveraging equation (6.29) and the technique described in [39, Lemma 5.5], we infer that for all $v \in Q_0$, the eigenvalues of $u_{\infty,v}$ are constant over almost every point. Let $\mu_{1,v} < \mu_{2,v} < \cdots < \mu_{l,v}$ denote the unique eigenvalues of $u_{\infty,v}$. Given the constraints $\sum_{v \in Q_0} tr_{E_v}(\sigma_v u_{\infty,v}) = 0$ and $\|u_{\infty,v}\|_{L^2(M)} = 1$, it follows that $2 \leq l \leq r$. For each eigenvalue $\mu_{j,v}$ with $1 \leq j \leq l-1$, we define a function

$$\Upsilon_{j,v}(x) : \mathbb{R} \to \mathbb{R}$$

by

$$\Upsilon_{j,v}(x) = \begin{cases} 1, & \text{if } x \leq \mu_{j,v}, \\ 0, & \text{if } x \geq \mu_{j+1,v}. \end{cases}$$

We define $\pi_{j,v}$ as $\Upsilon_{j,v}(u_{\infty,v})$ and denote $E_{j,v}$ by $\pi_{j,v}(E_v)$. Based on [39, p. 887], we ascertain the following properties:

1. $\pi_{j,v}$ belongs to L_1^2;
2. $\pi_{j,v}$ is idempotent and self-adjoint with respect to $H_{0,v}$;
3. $\pi_{j,v}$ commutes with $\overline{\partial}_{E_{j,v}}$ under the projection $\mathrm{Id}_{E_{j,v}} - \pi_{j,v}$;
4. For every $a \in Q_1$, the composition $(\mathrm{Id}_{E_{j,\mathfrak{h}a}} - \pi_{j,\mathfrak{h}a}) \circ \phi_a \circ (\pi_{j,\mathfrak{t}a} \otimes \mathrm{Id}_{\widetilde{E}_a})$ vanishes.

Invoking Uhlenbeck and Yau's regularity theorem for L_1^2-subbundles from [41], the collection $\{\pi_{j,v}\}_{j=1}^{l-1}$ determines $l-1$ coherent sub-sheaves of E_v for each $v \in Q_0$. By applying the arguments in [48, p. 288], which extend [15, Theorem 0.2], we can obtain a sequence of desirable weakly quiver sub-bundles $\mathcal{R}_j$ of $\mathcal{R}$.

Given that

$$\sum_{v\in Q_0} tr_{E_v}(\sigma_v u_{\infty,v}) = 0$$

and

$$u_{\infty,v} = \mu_{l,v} \cdot \mathrm{Id}_{E_v} - \sum_{j=1}^{l-1}(\mu_{j+1,v} - \mu_{j,v}) \cdot \pi_{j,v},$$

it follows that

$$\sum_{v\in Q_0}\left(\sigma_v \mu_{l,v} \cdot \mathrm{rk}(E_v) - \sum_{j=1}^{l-1}(\mu_{j+1,v} - \mu_{j,v})\sigma_v \cdot \mathrm{rk}(E_{j,v})\right) = 0. \tag{6.30}$$

Let

$$\mu_{l,\widetilde{v}} = \max_{v\in Q_0} \mu_{l,v}, \quad \sum_{j=1}^{l-1}(\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}}) = \min_{v\in Q_0} \sum_{j=1}^{l-1}(\mu_{j+1,v} - \mu_{j,v}).$$

Then, from (6.30), we deduce

$$\sum_{v\in Q_0} \sigma_v \cdot \mu_{l,\widetilde{v}} \cdot \mathrm{rk}(E_v) \geq \sum_{v\in Q_0}\sum_{j=1}^{l-1}(\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}})\sigma_v \cdot \mathrm{rk}(E_{j,v}). \tag{6.31}$$

Define the quantity χ as follows:

$$\chi = n\left(\mu_{l,\widetilde{v}} \deg_{\sigma,\tau}(\mathcal{R}, \mathbf{K}) - \sum_{j=1}^{l-1}(\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}}) \deg_{\sigma,\tau}(\mathcal{R}_j, \mathbf{K})\right). \tag{6.32}$$

By substituting (6.31) into χ, we obtain:

$$\chi \geq n\sum_{j=1}^{l-1}(\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}}) \sum_{v\in Q_0} \sigma_v \mathrm{rk}(E_{j,v}) \left(\mathcal{S}_{\sigma,\tau}(\mathcal{R}, \mathbf{K}) - \mathcal{S}_{\sigma,\tau}(\mathcal{R}_j, \mathbf{K})\right). \tag{6.33}$$

Furthermore, according to [39, Lemma 3.2], the Chern–Weil formula with respect to the metric $\mathbf{K}$ on the twisted quiver bundle $\mathcal{R}$ is given by:
(6.34)
$$\deg(E_{j,v}, K_v) = \frac{1}{n} \sum_{v \in Q_0} \left(\int_M \langle tr_g F_{H_{0,v}}, \pi_{j,v} \rangle_{K_v} \frac{\omega_g^n}{\nu} - \int_M |\overline{\partial}_{E_v} \pi_{j,v}|^2_{K_v} \frac{\omega_g^n}{\nu} \right).$$

Substituting (6.34) into (6.32), we have

(6.35)

$$\begin{aligned}
\chi = & \sum_{v \in Q_0} \int_M \Big\langle \sigma_v \operatorname{tr}_g F_{K_v}, \mu_{l,\widetilde{v}} \operatorname{Id}_{E_v} - \sum_{j=1}^{l-1} (\mu_{j+1,\widehat{v}} - \mu_{i,\widehat{v}}) \pi_{j,v} \Big\rangle_{K_v} \frac{\omega_g^n}{\nu} \\
& + \sum_{v \in Q_0} \sigma_v \sum_{j=1}^{l-1} (\mu_{j+1,\hat{v}} - \mu_{j,\hat{v}}) \|\overline{\partial}_{E_v} \pi_{j,v}\|^2_{L^2} \\
& - \sum_{v \in Q_0} \tau_v \cdot \Big(\mu_{l,\widetilde{v}} \operatorname{rk}(E_v) - \sum_{j=1}^{l-1} (\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}}) \operatorname{rk}(E_{j,v}) \Big) \\
= & \sum_{v \in Q_0} \int_M \Big\langle \sigma_v \operatorname{tr}_g F_{K_v}, \mu_{l,v} \cdot \operatorname{Id}_{E_v} - \sum_{j=1}^{l-1} (\mu_{j+1,v} - \mu_{j,v}) \pi_{j,v} \Big\rangle_{K_v} \frac{\omega_g^n}{\nu} \\
& + \sum_{v \in Q_0} \sigma_v \sum_{j=1}^{l-1} (\mu_{j+1,v} - \mu_{j,v}) \|\overline{\partial}_{E_v} \pi_{j,v}\|^2_{L^2} \\
& - \sum_{v \in Q_0} \tau_v \cdot \Big(\mu_{l,v} \operatorname{rk}(E_v) - \sum_{j=1}^{l-1} (\mu_{j+1,v} - \mu_{j,v}) \operatorname{rk}(E_{j,v}) \Big) \\
& + \sum_{v \in Q_0} \int_M \Big\langle \sigma_v \operatorname{tr}_g F_{K_v}, (\mu_{l,\widetilde{v}} - \mu_{l,v}) \cdot \operatorname{Id}_{E_v} \\
& \quad + \Big(\sum_{j=1}^{l-1} (\mu_{j+1,v} - \mu_{j,v}) - \sum_{j=1}^{l-1} (\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}}) \Big) \pi_{j,v} \Big\rangle_{K_v} \frac{\omega_g^n}{\nu} \\
& + \sum_{v \in Q_0} \Big(\sigma_v \Big(\sum_{j=1}^{l-1} (\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}}) - \sum_{j=1}^{l-1} (\mu_{j+1,v} - \mu_{j,v}) \Big) \\
& \quad \times \|\overline{\partial}_{E_v} \pi_{i,v}\|^2_{L^2} \Big) \\
& + \sum_{v \in Q_0} \tau_v \cdot \Big((\mu_{l,v} - \mu_{l,\widetilde{v}}) \cdot \operatorname{rk}(E_v)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \Big(\sum_{j=1}^{l-1} (\mu_{i+1,\widehat{v}} - \mu_{j,\widehat{v}}) - \sum_{j=1}^{l-1} (\mu_{j+1,v} - \mu_{j,v}) \Big) \operatorname{rk}(E_{j,v}) \Big) \\
&\leq \sum_{v \in Q_0} \int_M \Big(\langle \Phi_v(K_v), u_{\infty,v} \rangle_{K_v} \\
&\quad + \Big\langle \sigma_v \sum_{j=1}^{l-1} (\mu_{j+1,v} - \mu_{j,v}) (\mathrm{d}\,\Upsilon_{j,v})^2 (u_{\infty,v}) (\overline{\partial}_{E_v} u_{\infty,v}), \overline{\partial}_{E_v} u_{\infty,v} \Big\rangle_{K_v} \Big) \frac{\omega_g^n}{\nu} \\
&\leq 0,
\end{aligned}
$$

where the differential $\mathrm{d}\Upsilon_{j,v}(x,y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$
\mathrm{d}\Upsilon_{j,v}(x,y) = \begin{cases} \dfrac{\Upsilon_{j,v}(x) - \Upsilon_{j,v}(y)}{x-y}, & \text{if} \quad x \neq y; \\ \Upsilon'_{j,v}(x), & \text{if} \quad x = y. \end{cases}
$$

Combining (6.33) and (6.35) leads to a contradiction with the analytic (σ,τ)-stability of the bundle $\mathcal{R}$. □

7. Semi-stability implies the existence of the approximate Hermite–Einstein structure

The proof for Theorem 1.3 bears resemblance to that of Theorem 1.1. To facilitate readers, we will provide a detailed proof here.

Let (M, D, g, ν) denote the non-compact affine Gauduchon manifold as described in Theorem 1.3, and let $\mathcal{R}$ represent a twisted quiver bundle over M. Given a suitable background metric $\mathbf{K} = \{K_v\}_{v \in Q_0}$ on $\mathcal{R}$ that satisfies $tr_g F_{K_v} \leq 0$, $\sup_M |tr_g F_{K_v}|_{K_v} < +\infty$, and $\sup_M |\phi|_{K_v} < +\infty$. By applying the conformal transformation $\overline{K}_v = e^f K_v$, we also observe that $\overline{K}_v$ fulfills the condition:

$$
\sum_{v \in Q_0} tr_{E_v}(\Phi_v(\overline{K}_v)) = 0. \tag{7.36}
$$

Upon analyzing the function f, we notice that if $\mathcal{R}$ exhibits analytic (σ,τ)-semi-stability with respect to the Hermitian metric $\mathbf{K}$, it maintains this semi-stability with respect to the transformed metric $\overline{\mathbf{K}} = \{\overline{K}_v\}_{v \in Q_0}$. Therefore, it suffices to consider the initial metric $\mathbf{K}$ on $\mathcal{R}$ that already fulfills Equation (7.36), without loss of generality.

According to Proposition 5.2, for any vertex $v \in Q_0$ and any positive ε, the following perturbed equation admits a solution:

$$
\Phi_{\varepsilon,v}(H_{\varepsilon,v}) := \Phi_v(H_{\varepsilon,v}) + \varepsilon \sigma_v (\log h_{\varepsilon,v}) = 0, \tag{7.37}
$$

where $h_{\varepsilon,v}$ is defined by $K_v^{-1} H_{\varepsilon,v} = e^{s_{\varepsilon,v}}$ and

$$
\Phi_v(H_{\varepsilon,v}) = \sigma_v tr_g F_{H_{\varepsilon,v}} + \sum_{a \in \mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_{\varepsilon,v}} - \sum_{a \in \mathfrak{t}^{-1}(v)} \phi_a^{*H_{\varepsilon,v}} \circ \phi_a - \tau_v \cdot \mathrm{Id}_{E_v}.
$$

Given that the initial Hermitian metric $\mathbf{K}$ on $\mathcal{R}$ fulfills (7.36), it consequently holds that:

$$\sum_{v\in Q_0} \sigma_v tr_{E_v}(\log h_{\varepsilon,v}) = 0.$$

We will demonstrate that if the quiver bundle $\mathcal{R}$ is analytic (σ,τ)-semi-stable, then as $\varepsilon \to 0$, it follows that

$$\sup_M \Bigg| \sigma_v \operatorname{tr}_g F_{H_{\varepsilon,v}} + \sum_{a\in\mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_{\varepsilon,v}} - \sum_{a\in\mathfrak{t}^{-1}(v)} \phi_a^{*H_{\varepsilon,v}} \circ \phi_a - \tau_v \cdot \mathrm{Id}_{E_v} \Bigg|_{H_{\varepsilon,v}} \to 0.$$

We will use the techniques developed by Nie–Zhang [35] and Simpson [39].

Case1 Suppose there exists a uniform constant C_{14} such that

$$\|\log h_{\varepsilon,v}\|_{L^2(M)} \le C_{14} < +\infty.$$

Then by Lemma 6.1, we have

$$\begin{aligned}
&\sup_M |\sigma_v tr_g F_{H_{\varepsilon,v}} + \sum_{a\in\mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_{\varepsilon,v}} - \sum_{a\in\mathfrak{t}^{-1}(v)} \phi_a^{*H_v} \circ \phi_a - \tau_v \cdot \mathrm{Id}_{E_v}|_{H_{\varepsilon,v}} \\
&= \varepsilon\sigma_v \sup_M |\log h_{\varepsilon_i,v}|_{H_{\varepsilon,v}} \\
&< \varepsilon C_9(C_{15}C_{14} + C_{10}).
\end{aligned}$$

Hence when $\varepsilon \to 0$, we have

$$\sup_M |\sigma_v tr_g F_{H_{\varepsilon,v}} + \sum_{a\in\mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_{\varepsilon,v}} - \sum_{a\in\mathfrak{t}^{-1}(v)} \phi_a^{*H_{\varepsilon,v}} \circ \phi_a - \tau_v \cdot \mathrm{Id}_{E_v}|_{H_{\varepsilon,v}} \to 0.$$

Case2 $\overline{\lim}_{\varepsilon_i\to 0} \|\log h_{\varepsilon_i,v}\|_{L^2(M)} \to \infty$.

Claim If $\mathcal{R}$ is analytic (σ,τ)-semi-stable with respect to the metric $\mathbf{K}$, then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sup_M |\sigma_v tr_g F_{H_{\varepsilon,v}} + \sum_{a\in\mathfrak{h}^{-1}(\varepsilon,v)} \phi_a \circ \phi_a^{*H_{\varepsilon,v}} - \sum_{a\in\mathfrak{t}^{-1}(\varepsilon,v)} \phi_a^{*H_{\varepsilon,v}} \circ \phi_a - \tau_v \cdot \mathrm{Id}_{E_v}|_{H_{\varepsilon,v}} \\
&= \varepsilon\sigma_v \sup_M |\log h_{\varepsilon,v}|_{H_{\varepsilon,v}} \\
&= 0.
\end{aligned}$$

Should the claim fail to hold, there would exist a $\delta > 0$ and a subsequence $\{\varepsilon_i\} \to 0$ as $i \to +\infty$, such that

$$\|\log h_{\varepsilon_i,v}\|_{L^2} \to +\infty,$$

and

$$\begin{aligned}&\sup_M |\sigma_v tr_g F_{H_{\varepsilon_i,v}} + \sum_{a\in\mathfrak{h}^{-1}(\varepsilon,v)} \phi_a\circ\phi_a^{*H_{\varepsilon_i,v}} - \sum_{a\in\mathfrak{t}^{-1}(\varepsilon,v)} \phi_a^{*H_{\varepsilon_i,v}}\circ\phi_a - \tau_v\cdot \mathrm{Id}_{E_v}|_{H_{\varepsilon_i,v}}\\&=\varepsilon_i\sigma_v\sup_M|\log h_{\varepsilon,v}|_{H_{\varepsilon_i,v}}\\&\geq\delta.\end{aligned}$$

Similar to the previous section, we define

$$s_{\varepsilon_i,v}=\log h_{\varepsilon_i,v},\quad l_{i,v}=\|s_{\varepsilon_i,v}\|_{L^2(M)},\quad \text{and}\quad u_{\varepsilon_i,v}=\frac{s_{\varepsilon_i,v}}{l_{i,v}}.$$

From these definitions, we deduce that $\sum_{v\in Q_0} tr_{E_v}(\sigma_v u_{\varepsilon_i,v})=0$ and $\|u_{\varepsilon_i,v}\|_{L^2} = 1$. Utilizing Lemma 6.1, we obtain

$$l_i\geq\frac{\delta}{\varepsilon_i C_9}-\frac{C_{10}}{C_9},$$

and

$$\sup_M|u_{\varepsilon_i,v}|\leq\frac{C_9}{l_{i,v}}\left(\sum_{v\in Q_0}\sigma_v l_{i,v}+C_{10}\right)<C_{15}<+\infty. \tag{7.38}$$

Step 1 We will demonstrate that for each $v\in Q_0$, the L^2_1 norms of $u_{\varepsilon_i,v}$ remain uniformly bounded. Since the L^2 norms of $u_{\varepsilon_i,v}$ are normalized to 1, our main goal is to establish a uniform bound for the L^2 norms of $\nabla u_{\varepsilon_i,v}$.

By leveraging Proposition 3.5 and the perturbed equation (7.37), we deduce that the following inequality holds for all $u_{\varepsilon_i,v}$:

$$\begin{aligned}&\sum_{v\in Q_0}\Bigg(\int_M \mathrm{tr}_{E_v}(\Phi_v(K_v)u_{\varepsilon_i,v})\frac{\omega_g^n}{\nu}\\&\qquad+\sigma_v\int_M l_{i,v}\left\langle\Psi(l_{i,v}u_{\varepsilon_i,v})(\overline{\partial}_{E_v}u_{\varepsilon_i,v}),\overline{\partial}_{E_v}u_{\varepsilon_i,v}\right\rangle_{K_v}\frac{\omega_g^n}{\nu}\Bigg)\\&\leq-\varepsilon_i\sum_{v\in Q_0}\sigma_v l_{i,v},\end{aligned}$$

Next, consider the function defined by:

$$\gamma\Psi(\gamma x,\gamma y)=\begin{cases}\gamma, & \text{if } x=y,\\ \frac{e^{\gamma(y-x)}-1}{y-x}, & \text{if } x\neq y.\end{cases}$$

From (7.38), we conclude that (x,y) lies within the domain $[-C_{16},C_{16}]\times[-C_{16},C_{16}]$. A straightforward verification reveals:

$$\gamma\Psi(\gamma x,\gamma y)\to\begin{cases}(x-y)^{-1}, & \text{if } x>y,\\ +\infty, & \text{if } x\leq y,\end{cases}$$

which increases monotonically as $\gamma \to +\infty$. We introduce ζ, a smooth function mapping $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}^+$ such that $\zeta(x,y) < (x-y)^{-1}$ for $x > y$. Utilizing (6.27) and following reasoning similar to that in [39, Lemma 5.4], for sufficiently large i, we obtain:

$$\begin{aligned}&\delta C_{17} + \sum_{v \in Q_0} \left(\int_M \mathrm{tr}_{E_v}(\Phi_v(K_v) u_{\varepsilon_i,v}) \frac{\omega_g^n}{\nu} \right. \\ &\qquad \left. + \sigma_v \int_M \left\langle \zeta(u_{\varepsilon_i,v})(\overline{\partial}_{E_v} u_{\varepsilon_i,v}), \overline{\partial}_{E_v} u_{\varepsilon_i,v} \right\rangle_{K_v} \frac{\omega_g^n}{\nu} \right) \\ &\leq \varepsilon_i C_{18},\end{aligned} \tag{7.39}$$

where C_{17} and C_{18} are uniformly positive constants.

Specifically, we choose $\zeta(x,y) = \frac{1}{3C_{16}}$. Consequently, within the domain $(x,y) \in [-C_{16}, C_{16}] \times [-C_{16}, C_{16}]$ and for $x > y$, it holds that $\frac{1}{3C_{16}} < \frac{1}{x-y}$. Hence, for sufficiently large i, we have:

$$\delta C_{17} + \sum_{v \in Q_0} \left(\int_M tr_{E_v}(\Phi_v(K_v) u_{\varepsilon_i,v}) \frac{\omega_g^n}{\nu} + \frac{\sigma_v}{3C_{16}} \int_M |\overline{\partial}_{E_v} u_{\varepsilon_i,v}|^2_{K_v} \frac{\omega_g^n}{\nu} \right) \leq 0.$$

This implies:

$$\sum_{v \in Q_0} \int_M |\overline{\partial}_{E_v} u_{\varepsilon_i,v}|^2_{K_v} \frac{\omega_g^n}{\nu} \leq C_{19} \sum_{v \in Q_0} \max_M |\Phi_v(K_v)|_{K_v} \cdot \mathrm{Vol}(M).$$

Hence, for each $v \in Q_0$, the sequence $u_{\varepsilon_i,v}$ remains bounded in the L_1^2 norm, allowing us to extract a weakly convergent subsequence, denoted $\{u_{\varepsilon_{i_k},v}\}$, which converges to $u_{\infty,v}$ in L_1^2. For simplicity, we will continue to use $\{u_{\varepsilon_i,v}\}$ to represent this subsequence. Given the embedding of L_1^2 into L^2, it follows that:

$$1 = \lim_{i \to \infty} \int_M |u_{\varepsilon_i,v}|^2_{H_{0,v}} \frac{\omega_g^n}{\nu} = \int_M |u_{\infty,v}|^2_{H_{0,v}} \frac{\omega_g^n}{\nu},$$

indicating that $u_{\infty,v}$ has an L^2 norm of 1 and is thus non-trivial.

By utilizing equation (7.39) and following a similar argument as in [39, Lemma 5.4], we derive the inequality:

$$\begin{aligned}&\delta C_{17} + \sum_{v \in Q_0} \left(\int_M \mathrm{tr}_{E_v}(\Phi_v(K_v) u_{\infty,v}) \frac{\omega_g^n}{\nu} \right. \\ &\qquad \left. + \sigma_v \int_M \left\langle \zeta(u_{\infty,v})(\overline{\partial}_{E_v} u_{\infty,v}), \overline{\partial}_{E_v} u_{\infty,v} \right\rangle_{K_v} \frac{\omega_g^n}{\nu} \right) \\ &\leq 0.\end{aligned} \tag{7.40}$$

Step 2 Drawing on the reasoning presented by Uhlenbeck and Yau in [41], we construct a quiver subsheaf that contradicts the (σ, τ)-semi-stability of $\mathcal{R}$.

By employing equation (7.40) and the technique outlined in [39, Lemma 5.5], we deduce that for all $v \in Q_0$, the eigenvalues of $u_{\infty,v}$ are constant almost everywhere. Let $\mu_{1,v} < \mu_{2,v} < \cdots < \mu_{l,v}$ denote the distinct eigenvalues of $u_{\infty,v}$. Given the constraints $\sum_{v\in Q_0} tr_{E_v}(\sigma_v u_{\infty,v}) = 0$ and $\|u_{\infty,v}\|_{L^2(M)} = 1$, it follows that $2 \leq l \leq r$. For each eigenvalue $\mu_{j,v}$ with $1 \leq j \leq l-1$, we define a function

$$\Upsilon_{j,v}(x) : \mathbb{R} \to \mathbb{R}$$

by

$$\Upsilon_{j,v}(x) = \begin{cases} 1, & \text{if } x \leq \mu_{j,v}, \\ 0, & \text{if } x \geq \mu_{j+1,v}. \end{cases}$$

We define $\pi_{j,v}$ as $\Upsilon_{j,v}(u_{\infty,v})$ and denote $E_{j,v}$ by $\pi_{j,v}(E_v)$. Based on the argument from [39, p. 887], we ascertain the following properties:

1. $\pi_{j,v}$ belongs to L_1^2;
2. $\pi_{j,v}$ is idempotent and self-adjoint with respect to $H_{0,v}$;
3. $\pi_{j,v}$ commutes with $\overline{\partial}_{E_{j,v}}$ under the projection $\mathrm{Id}_{E_{j,v}} - \pi_{j,v}$;
4. For every $a \in Q_1$, the composition $(\mathrm{Id}_{E_{j,\mathfrak{h}a}} - \pi_{j,\mathfrak{h}a}) \circ \phi_a \circ (\pi_{j,\mathfrak{t}a} \otimes \mathrm{Id}_{\widetilde{E}_a})$ vanishes.

By invoking Uhlenbeck and Yau's regularity theorem for L_1^2-subbundles from [41], the set $\{\pi_{j,v}\}_{j=1}^{l-1}$ determines $l-1$ coherent sub-sheaves of E_v for each $v \in Q_0$. By utilizing the arguments presented in [48, p. 288], which build upon [15, Theorem 0.2], we can construct a sequence of desired weakly quiver sub-bundles $\mathcal{R}_j$ of $\mathcal{R}$.

Given the equations

$$\sum_{v\in Q_0} tr_{E_v}(\sigma_v u_{\infty,v}) = 0,$$

$$u_{\infty,v} = \mu_{l,v} \cdot \mathrm{Id}_{E_v} - \sum_{j=1}^{l-1} (\mu_{j+1,v} - \mu_{j,v}) \cdot \pi_{j,v},$$

it follows that

$$\sum_{v\in Q_0} \left(\sigma_v \mu_{l,v} \cdot \mathrm{rk}(E_v) - \sum_{j=1}^{l-1} (\mu_{j+1,v} - \mu_{j,v}) \sigma_v \cdot \mathrm{rk}(E_{j,v}) \right) = 0. \tag{7.41}$$

To move forward, let us introduce the following definitions:

$$\mu_{l,\widetilde{v}} := \max_{v\in Q_0} \mu_{l,v}, \quad \sum_{j=1}^{l-1} (\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}}) := \min_{v\in Q_0} \sum_{j=1}^{l-1} (\mu_{j+1,v} - \mu_{j,v}).$$

Then, from (7.41), we deduce

$$\sum_{v\in Q_0} \sigma_v \cdot \mu_{l,\widetilde{v}} \cdot \mathrm{rk}(E_v) \geq \sum_{v\in Q_0} \sum_{j=1}^{l-1} (\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}}) \sigma_v \cdot \mathrm{rk}(E_{j,v}). \tag{7.42}$$

Define the quantity χ as follows:

$$\chi = n\left(\mu_{l,\widehat{v}}\deg_{\sigma,\tau}(\mathcal{R},\mathbf{K}) - \sum_{j=1}^{l-1}(\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}})\deg_{\sigma,\tau}(\mathcal{R}_j,\mathbf{K})\right). \tag{7.43}$$

By substituting (7.42) into χ, we obtain:

$$\chi \geq n\sum_{i=1}^{l-1}(\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}})\sum_{v\in Q_0}\sigma_v \mathrm{rk}(E_{j,v})\left(\mathcal{S}_{\sigma,\tau}(\mathcal{R},\mathbf{K}) - \mathcal{S}_{\sigma,\tau}(\mathcal{R}_j,\mathbf{K})\right). \tag{7.44}$$

Furthermore, as stated in [39, Lemma 3.2], the Chern–Weil formula for the twisted quiver bundle $\mathcal{R}$ with respect to the metric $\mathbf{K}$ is expressed as

$$\deg(E_{j,v},K_v) = \frac{1}{n}\sum_{v\in Q_0}\left(\int_M \langle tr_g F_{H_{0,v}},\pi_{j,v}\rangle_{K_v}\frac{\omega_g^n}{\nu} - \int_M |\overline{\partial}_{E_v}\pi_{j,v}|^2_{K_v}\frac{\omega_g^n}{\nu}\right). \tag{7.45}$$

Substituting (7.45) into (7.43), and using the same argument in [28, Pages 793-794], we have

$$\begin{aligned}
\chi &= \sum_{v\in Q_0}\int_M \left\langle \sigma_v tr_g F_{K_v}, \mu_{l,\widetilde{v}}\mathrm{Id}_{E_v} - \sum_{j=1}^{l-1}(\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}})\pi_{j,v}\right\rangle_{K_v}\frac{\omega_g^n}{\nu} \\
&\quad + \sum_{v\in Q_0}\sigma_v\sum_{j=1}^{l-1}(\mu_{j+1,\hat{v}} - \mu_{j,\hat{v}})\|\overline{\partial}_{E_v}\pi_{j,v}\|^2_{L^2} \\
&\quad - \sum_{v\in Q_0}\tau_v\cdot\Big(\mu_{l,\widehat{v}}\mathrm{rk}(E_v) - \sum_{j=1}^{l-1}(\mu_{j+1,\widehat{v}} - \mu_{j,\widehat{v}})\mathrm{rk}(E_{j,v})\Big) \\
&\leq \sum_{v\in Q_0}\int_M \Big(\langle \Phi_v(K_v), u_{\infty,v}\rangle_{K_v} \\
&\quad + \langle \sigma_v\sum_{j=1}^{l-1}(\mu_{j+1,v} - \mu_{j,v})(\mathrm{d}\Upsilon_{j,v})^2(u_{\infty,v})(\overline{\partial}_{E_v}u_{\infty,v}), \overline{\partial}_{E_v}u_{\infty,v}\rangle_{K_v}\Big)\frac{\omega_g^n}{\nu} \\
&\leq -\delta C_{17} \\
&< 0,
\end{aligned} \tag{7.46}$$

where the differential $\mathrm{d}\Upsilon_{j,v}(x,y) : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is given by

$$\mathrm{d}\Upsilon_{j,v}(x,y) = \begin{cases} \dfrac{\Upsilon_{j,v}(x) - \Upsilon_{j,v}(y)}{x-y}, & \text{if } \quad x \neq y; \\ \Upsilon'_{j,v}(x), & \text{if } \quad x = y. \end{cases}$$

Combining equations (7.44) and (7.46) results in a contradiction to the analytic (σ,τ)-semi-stability of the bundle $\mathcal{R}$. □

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Declarations

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