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PAIR CORRELATION OF ZEROS OF DIRICHLET L-FUNCTIONS AND PROPORTION OF SIMPLE ZEROS

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Abstract. Baluyot, Goldston, Suriajaya, and Turnage-Butterbaugh obtained an unconditional form of Montgomery's theorem concerning pair correlation of zeros of the Riemann zeta function. They used it to prove that under certain assumptions at least 61.7% of the zeros are simple. In this paper, we obtain an analogous theorem for Dirichlet L-functions and apply it to prove a similar result regarding simple zeros of Dirichlet L-functions.

1. Introduction

Let $\rho = \beta + i\gamma$ be a nontrivial zero of the Riemann zeta function $\zeta(s)$ and $W(u) = \frac{4}{4-u^2}$. Following Montgomery [8], Baluyot et al. [1] defined

$$F(x,T) = \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T}} x^{\rho-\rho'} W(\rho-\rho'),$$

and

$$F(\alpha) := \left(\frac{T}{2\pi}\log T\right)^{-1} F(T^\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T}} T^{\alpha(\rho-\rho')} W(\rho-\rho').$$

They generalised Montgomery's pair correlation method to prove unconditionally that (we state corrected version, see Baluyot et al. [2])

$$F(\alpha) = T^{-2\alpha}\left(\log T + O(1)\right) + \alpha + O\left(\frac{1}{\sqrt{\log T}}\right) \tag{1.1}$$

uniformly for $0 \le \alpha \le 1,\ T \to \infty$.

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They used this to prove that if all the zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ with $T^{\frac{3}{8}}<\gamma\le T$ satisfy $\frac{1}{2}-\frac{1}{2\log T}<\beta<\frac{1}{2}+\frac{1}{2\log T}$ then at least 61.7% of the nontrivial zeros are simple.

Baluyot et al. [1] also proved that assuming the weaker hypothesis

$$N(\sigma,T)=o\left(T^{2(1-\sigma)}(\log T)^{-1}\right)\quad \text{for}\quad \frac{1}{2}+\frac{1}{2\log T}\le\sigma\le 1,$$

at least 61.7% of the nontrivial zeros are simple.

Here, we are interested in pair correlation of zeros of Dirichlet L-functions. Let γ_χ denote an imaginary part of the zero of the Dirichlet L-function $L(s,\chi)$. Fujii [4, Theorem 13] assumed Generalised Riemann Hypothesis (GRH) and showed that if χ, ψ are primitive characters mod q,k respectively, and

$$F_{\chi,\psi}(\alpha):=\left(\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right)\right)^{-1}\operatorname{Re}\left(\sum_{0<\gamma_\chi,\gamma_\psi\le T}\left(\frac{T}{2\pi}\right)^{i\alpha(\gamma_\chi-\gamma_\psi)}W(\gamma_\chi-\gamma_\psi)\right),$$

then for $0\le\alpha\le1$, $\delta_{\chi,\psi}=\begin{cases}1 & \text{if } \chi=\psi\\ 0 & \text{if } \chi\ne\psi\end{cases}$, we have

$$F_{\chi,\psi}(\alpha)=\delta_{\chi,\psi}\alpha+\left(\frac{T}{2\pi}\right)^{-2\alpha}\log T+O\left(T^{-2\alpha}\right)+O\left(\frac{1}{\sqrt{\log T}}\right).$$

Karabulut and Yıldırım [6, Theorem 3] proved that if χ and ψ are primitive characters mod $q_\chi,q_\psi>1$, then, assuming GRH,

$$\begin{aligned}F_{\chi,\psi}(x,T):=&\sum_{0<\gamma_\chi,\gamma_\psi\le T}x^{i(\gamma_\chi-\gamma_\psi)}W(\gamma_\chi-\gamma_\psi)\\ =&\delta_{\chi,\psi}\left(\frac{T\log x}{2\pi}+O\left(\frac{T\log x\log q_\chi}{x}\right)\right)+\frac{T\log^2 T}{2\pi x^2}\left(1+O_{q_\chi,q_\psi}\left(\frac{1}{\log T}\right)\right)\\ &+O(Tx^{-\frac{1}{2}}\log^3 2x)+O(x\log 2x\log\log 3x)+O(x\log T).\end{aligned}$$

In this paper, we will obtain results analogous to [1] concerning pair correlation for Dirichlet L-functions and the proportion of simple zeros of Dirichlet L-functions. Let χ be a primitive Dirichlet character mod $q>1$, and from now on, let $\rho=\beta+i\gamma$ denote a nontrivial zero of $L(s,\chi)$. Define, for $x>0$ and $T\ge2$,

$$F(x,T,\chi)=\sum_{\substack{\rho,\rho'\\ 0<\gamma,\gamma'\le T}}x^{\rho-\rho'}W(\rho-\rho'),\quad\text{where}\quad W(u)=\frac{4}{4-u^2},$$

and for $\alpha \in \mathbb{R}$,

$$F(\alpha,\chi) := \left(\frac{T}{2\pi}\log(qT)\right)^{-1} F((qT)^{\alpha},T,\chi) = \left(\frac{T}{2\pi}\log(qT)\right)^{-1} \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T}} (qT)^{\alpha(\rho-\rho')}W(\rho-\rho'). \tag{1.2}$$

We will prove the following unconditional theorem.

THEOREM 1.1. *Let χ be a primitive Dirichlet character mod $q>1$. Then $F(\alpha,\chi)$ is real, even, and nonnegative. Moreover, as $T\to\infty$, we have*

$$F(\alpha,\chi) = (qT)^{-2\alpha}\left(\log(qT)+O(1)\right)+\alpha+O\left(\frac{q(\log q)^2\log\log(qT)}{\log(qT)}\right) \tag{1.3}$$

uniformly for $0\le\alpha\le 1$, $1<q\le\log T$.

In the proof of Theorem 1.1, we will use ideas from the proof of Theorem 1 in [1] and from Montgomery and Vaughan [10]. Note that in (1.3), the last error term is smaller in the T aspect compared to the corresponding term in (1.1). This is because the Dirichlet L-function for a primitive Dirichlet character mod $q>1$ is an entire function, whereas the Riemann zeta function has a pole at $s=1$. This difference is reflected in the corresponding explicit formulas, compare formula (2.6) below with formula (2.3) in [1].

There are many results about the proportion of simple trivial zeros of Dirichlet L-functions averaged over characters. For example, Sono [12], by examining a specific averaged zero pair correlation function, proved that, assuming GRH, at least 93.22% of low-lying zeros of Dirichlet L-functions, averaged over primitive characters, are simple.

We will demonstrate the following two statements concerning the proportion of simple trivial zeros of individual Dirichlet L-functions.

THEOREM 1.2. *Suppose that all the zeros $\rho=\beta+i\gamma$ of $L(s,\chi)$, where χ is a primitive Dirichlet character mod $q>1$, satisfy $\frac{13}{64}<\beta<\frac{51}{64}$, and also that all the zeros with $(qT)^{\frac{3}{8}}<\gamma\le T$, $1<q\le\sqrt{\log T}$, satisfy $\frac{1}{2}-\frac{1}{2\log(qT)}<\beta<\frac{1}{2}+\frac{1}{2\log(qT)}$. Then as $T\to\infty$, $1<q\le\sqrt{\log T}$, at least 61.7% of the nontrivial zeros of $L(s,\chi)$ are simple.*

THEOREM 1.3. *Let χ be a primitive Dirichlet character mod $q>1$. Assume that*

$$N(\sigma,T,\chi) = o\left(q^{-1}(qT)^{2(1-\sigma)}(\log(qT))^{-1}\right) \tag{1.4}$$

uniformly for $\frac{1}{2}+\frac{1}{2\log(qT)}\le\sigma\le 1$, $1<q\le\sqrt{\log T}$. Then as $T\to\infty$, $1<q\le\sqrt{\log T}$, at least 61.7% of the nontrivial zeros of $L(s,\chi)$ are simple.

Most of the results on zero density estimates for Dirichlet L-functions are of the form

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} N(\sigma,T,\chi) \ll (Q^2T^a)^{A(\sigma)(1-\sigma)+\epsilon},$$

where the asterisk means that the sum is over primitive characters only. It is conjectured that this holds with $a=1$, $A(\sigma)=2$ for the interval $\frac{1}{2}\le\sigma\le 1$. Heath-Brown [5] proved this for $\frac{11}{14}\le\sigma\le 1$.

The next section is devoted to the proof of Theorem 1.1. In Section 3, we consider certain Fourier transforms and prove some results about Tsang kernels, which will be useful in Section 4, where we prove Theorems 1.2 and 1.3.

2. Proof of Theorem 1.1

Write $\beta=\frac{1}{2}+\delta$, where $-\frac{1}{2}<\delta<\frac{1}{2}$. For Dirichlet L-functions we have the following explicit formula.

Lemma 2.1. *Let χ be a primitive character mod $q>1$. Then for $x\ge 1$ and $t\in\mathbb{R}$ we have*

(2.5)
$$2\sum_{\rho}\frac{x^{\delta+i(\gamma-t)}}{1+(t-\gamma+i\delta)^2} = -\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)}{n^{\frac{1}{2}+it}}\min\left\{\frac{n}{x},\frac{x}{n}\right\} + x^{-1}\left(\log q\tau + O(1)\right) + O(x^{-\frac{1}{2}-\mathfrak{a}}\tau^{-1})$$

uniformly in q, where $\tau=|t|+2$, and $\mathfrak{a}=\begin{cases}0 & \text{if } \chi(-1)=1,\\ 1 & \text{if } \chi(-1)=-1.\end{cases}$

Proof. If $x>1$, $x\ne p^m$, $s\ne\rho$, $s\ne -(2n+\mathfrak{a})$, then (Yıldırım [14, formula (6)])

$$\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n^s} = -\frac{L'}{L}(s,\chi) - \sum_{\rho}\frac{x^{\rho-s}}{\rho-s} + \sum_{n=0}^{\infty}\frac{x^{-(2n+\mathfrak{a})-s}}{2n+\mathfrak{a}+s}. \tag{2.6}$$

Denoting $\rho=\frac{1}{2}+\delta+i\gamma$ we have

$$\sum_{\rho}\frac{x^{\delta+i(\gamma-t)}}{\sigma-\frac{1}{2}-\delta+i(t-\gamma)} = x^{-\frac{1}{2}}\left(x^{\sigma}\left(\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n^s}+\frac{L'}{L}(s,\chi)\right)-\sum_{n=0}^{\infty}\frac{x^{-(2n+\mathfrak{a})-it}}{2n+\mathfrak{a}+s}\right). \tag{2.7}$$

If $\sigma>1$, then

$$\frac{L'}{L}(s,\chi) = -\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)}{n^s}.$$

Substituting this into (2.7) we get, for $\sigma > 1$,
(2.8)
$$\sum_{\rho} \frac{x^{\delta+i(\gamma-t)}}{\sigma-\frac{1}{2}-\delta+i(t-\gamma)} = -x^{-\frac{1}{2}}\left(x^{\sigma}\sum_{n>x}\frac{\Lambda(n)\chi(n)}{n^{s}} + \sum_{n=0}^{\infty}\frac{x^{-(2n+\mathfrak{a})-it}}{2n+\mathfrak{a}+s}\right).$$

Replace σ by $1-\sigma$, so that s is replaced by $1-\bar{s}$. Then we also need $s \neq 1-\bar{\rho}$ and $s \neq 1+2n+\mathfrak{a}$. We get

(2.9)
$$\sum_{\rho}\frac{x^{\delta+i(\gamma-t)}}{\frac{1}{2}-\sigma-\delta+i(t-\gamma)}$$
$$= x^{-\frac{1}{2}}\left(\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n^{it}}\left(\frac{x}{n}\right)^{1-\sigma} + x^{1-\sigma}\frac{L'}{L}(1-\sigma+it,\chi) - \sum_{n=0}^{\infty}\frac{x^{-(2n+\mathfrak{a})-it}}{2n+\mathfrak{a}+1-\sigma+it}\right).$$

Subtract (2.9) from (2.8) to get, for $1 < \sigma < 2$,

(2.10)
$$\begin{aligned}(2\sigma-1)\sum_{\rho}&\frac{x^{\delta+i(\gamma-t)}}{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma+i\delta)^2}\\ &= - x^{-\frac{1}{2}}\left(\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n^{it}}\left(\frac{x}{n}\right)^{1-\sigma} + \sum_{n>x}\frac{\Lambda(n)\chi(n)}{n^{it}}\left(\frac{x}{n}\right)^{\sigma}\right)\\ &\quad - x^{\frac{1}{2}-\sigma}\frac{L'}{L}(1-\sigma+it,\chi)\\ &\quad + (2\sigma-1)x^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{x^{-(2n+\mathfrak{a})-it}}{(2n+\mathfrak{a}+\sigma+it)(2n+\mathfrak{a}+1-\sigma+it)}.\end{aligned}$$

Both sides of (2.10) are continuous for all $x \ge 1$, so we no longer exclude the values $x = 1, p^m$. Using the functional equation of $L(s,\chi)$ and the fact that $\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(|s|^{-1})$ in $-\pi+\delta < \arg s < \pi+\delta$ for any fixed $\delta > 0$ we get, for $\sigma > 1$,
$$-\frac{L'}{L}(1-\sigma+it,\chi) = \log q\tau + O_{\sigma}(1).$$
The last term of (2.10) is $O(x^{-\frac{1}{2}-\mathfrak{a}}\tau^{-1})$. Take $\sigma = \frac{3}{2}$ to get (2.5). □

Write (2.5) as $l(x,t,\chi) = r(x,t,\chi)$ and define
$$L(x,T,\chi) := \int_0^T |l(x,t,\chi)|^2\,dt = \int_0^T |r(x,t,\chi)|^2\,dt =: R(x,T,\chi).$$

In the same way as Lemma 3 in [1] it can be shown that

$$F(x,T,\chi)=\frac{2}{\pi}\int\limits_{-\infty}^{\infty}\left|\sum_{\substack{\rho\\0<\gamma\le T}}\frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2}\right|^2dt. \tag{2.11}$$

Lemma 2.2. *For $x\ge 1$, $1<q\le\log T$, we have as $T\to\infty$,*

$$L(x,T,\chi)=2\pi F(x,T,\chi)+O((qT)^{\frac12})+O(x(\log q)^2). \tag{2.12}$$

Proof. We have

$$L(x,T,\chi)=4\int\limits_0^T\left|\sum_{\rho}\frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2}\right|^2dt. \tag{2.13}$$

Using the estimate $N(T+1,\chi)-N(T,\chi)\ll\log(qT)$ we obtain, for $T\ge 2$, $|t|\le T$,

$$\sum_{|\gamma|>T}\frac{1}{1+(t-\gamma)^2}\ll\frac{\log(qT)}{T-|t|+1}, \tag{2.14}$$

and, for $|t|\ge T$,

$$\sum_{|\gamma|\le T}\frac{1}{1+(t-\gamma)^2}\ll\frac{\log(qT)}{|t|-T+1}. \tag{2.15}$$

By (2.14) and (2.15) we have, for $t\in\mathbb{R}$,

$$\sum_{\gamma}\frac{1}{1+(t-\gamma)^2}\ll\log(q\tau). \tag{2.16}$$

By (2.16),

$$\left|\sum_{\rho}\frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2}\right|\ll x^{\frac12}\sum_{\gamma}\frac{1}{1+(t-\gamma)^2}\ll x^{\frac12}\log(q\tau). \tag{2.17}$$

For $|t|\le T$ and $Z\ge 2T$ the bound (2.14) leads to

$$\left|\sum_{\substack{\rho\\|\gamma|>Z}}\frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2}\right|\ll x^{\frac12}\sum_{|\gamma|>Z}\frac{1}{1+\gamma^2}\ll\frac{x^{\frac12}\log(qZ)}{Z}. \tag{2.18}$$

Hence by (2.17) and (2.18), taking $Z = T\log^2(qT)$ we get
(2.19)

$$\int_0^T \left|\sum_{\substack{\rho \\ |\gamma|>Z}} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2}\right| \left|\sum_{\rho} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2}\right| dt \ll \frac{xT\log^2(qZ)}{Z} \ll x.$$

Also

$$\int_0^T \left|\sum_{\substack{\rho \\ |\gamma|>Z}} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2}\right|^2 dt \ll \frac{xT\log^2(qZ)}{Z^2} \ll x. \tag{2.20}$$

From (2.13), (2.19) and (2.20) it follows that

$$L(x,T,\chi) = 4\int_0^T \left|\sum_{\substack{\rho \\ |\gamma|\le Z}} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2}\right|^2 dt + O(x). \tag{2.21}$$

The function $L(s,\chi)$ has a zero-free region

$$\sigma > 1-\eta(t,\chi), \quad |t| \ge 10,$$

where $0 < \eta(t,\chi) < \frac{1}{2}$ and $\eta(t,\chi)$ is decreasing in t. Thus, for $x \ge 1$, $|t| \le T$, by (2.14) we have

$$\sum_{T<\gamma\le Z} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2} \ll x^{\frac{1}{2}-\eta(Z,\chi)} \frac{\log(qT)}{T-|t|+1},$$

and by (2.16),

$$\sum_{|\gamma|\le Z} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2} \ll x^{\frac{1}{2}} \sum_{|\gamma|\le Z} \frac{1}{1+(t-\gamma)^2} \le x^{\frac{1}{2}} \sum_{\gamma} \frac{1}{1+(t-\gamma)^2}$$
$$\ll x^{\frac{1}{2}}\log(qT).$$

Hence
(2.22)

$$\int_0^T \left|\sum_{T<\gamma\le Z} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2}\right| \left|\sum_{|\gamma|\le Z} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2}\right| dt \ll x^{1-\eta(Z,\chi)}\log^3(qT).$$

If $0 \le t \le T$ then

$$\sum_{\gamma\le 0} \frac{1}{1+(t-\gamma)^2} \ll \frac{\log(qT)}{t+1},$$

so for $0 \le t \le T$

$$\sum_{-Z\le\gamma\le 0} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2} \ll x^{\frac{1}{2}-\eta(Z,\chi)} \frac{\log(qT)}{t+1}.$$

Therefore
(2.23)

$$\int_0^T \left| \sum_{-Z\le\gamma\le 0} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2} \right| \left| \sum_{|\gamma|\le Z} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2} \right| dt \ll x^{1-\eta(Z,\chi)} \log^3(qT).$$

By (2.21), (2.22) and (2.23) we get
(2.24)

$$L(x,T,\chi) = 4\int_0^T \left| \sum_{\substack{\rho \\ 0<\gamma\le T}} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2} \right|^2 dt + O(x^{1-\eta(Z,\chi)} \log^3(qT)) + O(x).$$

If $t \ge T$ then

$$\sum_{0<\gamma\le 10} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2} \ll x^{\frac12} \sum_{0<\gamma\le 10} \frac{1}{1+(t-\gamma)^2} \ll x^{\frac12} \sum_{0<\gamma\le 10} \frac{1}{t^2}$$
$$= \frac{x^{\frac12} N(10,\chi)}{t^2} \ll \frac{x^{\frac12}\log q}{t^2},$$

and by (2.15),

$$\sum_{10<\gamma\le T} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2} \ll x^{\frac12-\eta(T,\chi)} \sum_{|\gamma|\le T} \frac{1}{1+(t-\gamma)^2} \ll x^{\frac12-\eta(T,\chi)} \frac{\log(qT)}{t-T+1}.$$

Hence, for $t \ge T$,

$$\sum_{0<\gamma\le T} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2} \ll x^{\frac12-\eta(T,\chi)} \frac{\log(qT)}{t-T+1} + \frac{x^{\frac12}\log q}{t^2}.$$

It follows that

$$\int_T^\infty \left| \sum_{\substack{\rho \\ 0<\gamma\le T}} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2} \right|^2 dt \ll x^{1-\eta(T,\chi)} \log^2(qT) + x(\log q)^2. \tag{2.25}$$

If $t \le 0$ then

$$\sum_{0<\gamma\le T} \frac{1}{1+(t-\gamma)^2} \ll \frac{\log qT}{1-t},$$

so

$$\sum_{0<\gamma\le T} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2} \ll x^{\frac12-\eta(T,\chi)} \frac{\log qT}{1-t} + \frac{x^{\frac12}\log q}{1-t}.$$

Hence

$$\int_{-\infty}^0 \left| \sum_{\substack{\rho \\ 0<\gamma\le T}} \frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2} \right|^2 dt \ll x^{1-\eta(T,\chi)} \log^2(qT) + x(\log q)^2. \tag{2.26}$$

By (2.24), (2.25), (2.26) and (2.11), we obtain

$$\begin{aligned}L(x,T,\chi) =& 4\int_{-\infty}^{\infty}\left|\sum_{\substack{\rho\\0<\gamma\le T}}\frac{x^{\delta+i\gamma}}{1+(t-\gamma+i\delta)^2}\right|^2 dt\\ &+O(x^{1-\eta(Z,\chi)}\log^3(qT))+O(x(\log q)^2)\\ =&2\pi F(x,T,\chi)+O(x^{1-\eta(T\log^2(qT),\chi)}\log^3(qT))+O(x(\log q)^2).\end{aligned}$$

By Khale [7] we can take,

$$\eta(t,\chi)=\frac{c}{\log q+(\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}}\quad \text{for } q\ge 3,\ |t|\ge 10$$

for some constant $c>0$. Therefore for primitive χ, we choose

$$\eta(t,\chi)=\frac{c}{(\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}}\quad \text{for } 1<q\le\log|t|,\ |t|\ge 10 \tag{2.27}$$

for some constant $c>0$.
For $1<q\le\log T$, $(qT)^{\frac{1}{2}}\le x$, we have

$$x^{1-\eta(T\log^2(qT),\chi)}\log^3(qT)\ll x\exp\left(-c\frac{\log x}{(\log x)^{2/3}(\log\log x)^{1/3}}\right)\log^3(qT)\ll x,$$

while for $1<q\le\log T$, $1\le x\le (qT)^{\frac{1}{2}}$,

$$x^{1-\eta(T\log^2(qT),\chi)}\log^3(qT)\ll(qT)^{\frac{1}{2}}.$$

It follows that

$$L(x,T,\chi)=2\pi F(x,T,\chi)+O((qT)^{\frac{1}{2}})+O(x(\log q)^2)$$

uniformly for $x\ge 1$, $1<q\le\log T$. □

To obtain Lemma 2.5, we will use the following two lemmas, which are proved in Montgomery and Vaughan [10, Corollary 26.6 and Theorem 26.7].

LEMMA 2.3. *Suppose that $a_1,a_2,\dots$ are real or complex numbers such that $\sum_{n=1}^{\infty}|a_n|<\infty$, and let d_n be an integer such that $d_n\le\frac{n}{2}$ and with the property that $a_m=0$ whenever $0<|m-n|<d_n$. Then for any $T>0$,*

$$\int_0^T\left|\sum_{n=1}^{\infty}a_n n^{-it}\right|^2 dt=T\sum_{n=1}^{\infty}|a_n|^2+O\left(\sum_{n=1}^{\infty}\frac{n|a_n|^2}{d_n}\right).$$

LEMMA 2.4. *Let Q denote the set of primepowers. For $n \in Q$, let d_n be the minimum of $|m-n|$ for $m \in Q$, $m \neq n$. Then for $U \geq 4$,*

$$\sum_{\substack{n\in Q\\ U\leq n\leq 2U}} \frac{1}{d_n} \ll \frac{U\log\log U}{(\log U)^2}.$$

LEMMA 2.5. *For $1 \leq x \leq qT$ and any $q > 1$ we have as $T \to \infty$*

$$\begin{aligned} R(x,T,\chi) =& x^{-2}T\log^2(qT)\left(1+O\left(\frac{1}{\log qT}\right)\right)+T\log x\\ &+O(qT(\log q)^2\log\log(qT)). \end{aligned}$$

PROOF. We see that $|r(x,t,\chi)| = |l(x,t,\chi)| \ll \sqrt{x}\log q\tau$. Therefore

$$R(x,1,\chi) = \int_0^1 |r(x,t,\chi)|^2 dt \ll x(\log q)^2.$$

By Lemma 2.3 we obtain

$$\begin{aligned} \int_1^T \left|\sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^{\frac{1}{2}+it}} \min\left\{\frac{n}{x},\frac{x}{n}\right\}\right|^2 dt =& (T-1)\sum_{n=1}^{\infty}\frac{\Lambda(n)^2|\chi(n)|^2}{n}\min\left\{\frac{n}{x},\frac{x}{n}\right\}^2\\ &+O\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)^2}{d_n}\min\left\{\frac{n}{x},\frac{x}{n}\right\}^2\right), \end{aligned}$$

where d_n is the distance from n to the nearest other prime power.
By the prime number theorem $\theta(x) = x + O\left(\frac{x}{\log x}\right)$. Hence

$$\begin{aligned} \sum_{n\leq x}\Lambda(n)^2|\chi(n)|^2 &= \sum_{p\leq x}|\chi(p)|\log^2 p + \sum_{p\leq\sqrt{x}}\left(\left[\frac{\log x}{\log p}\right]-1\right)|\chi(p)|\log^2 p\\ &= \sum_{p\leq x}|\chi(p)|\log^2 p + O\left(\theta(\sqrt{x})\log x\right)\\ &= \sum_{p\leq x}|\chi(p)|\log^2 p + O(\sqrt{x}\log x)\\ &= \sum_{p\leq x}\log^2 p - \sum_{\substack{p\leq x\\ p|q}}\log^2 p + O(\sqrt{x}\log x)\\ &= \sum_{p\leq x}\log^2 p + O\left(\log x\log q\right) + O(\sqrt{x}\log x)\\ &= x\log x + O(x\log q). \end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)^2|\chi(n)|^2}{n} \min\left\{\frac{n}{x}, \frac{x}{n}\right\}^2 = \log x + O(\log q).$$

Lemma 2.4 leads to

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)^2}{d_n} \min\left\{\frac{n}{x}, \frac{x}{n}\right\}^2 = O(x \log\log 4x).$$

Hence

(2.28)

$$\begin{aligned}\int_1^T \left|\sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^{\frac{1}{2}+it}} \min\left\{\frac{n}{x}, \frac{x}{n}\right\}\right|^2 dt &= (T-1)\left(\log x + O(\log q)\right) + O(x\log\log 4x)\\ &= T\log x + O(T\log q) + O(x\log\log 4x).\end{aligned}$$

We have

(2.29)

$$x^{-2}\int_1^T (\log(q\tau) + O(1))^2\, dt = x^{-2} T\log^2(qT)\left(1 + O\left(\frac{1}{\log qT}\right)\right) + O(x^{-2}(\log q)^2).$$

By integration by parts we get

$$\int_1^T n^{-it}\log(q\tau) dt \ll \frac{\log(qT)}{\log n} \quad \text{for} \quad n \geq 2.$$

Hence, the correlation between the two main terms of (2.5) is

(2.30)

$$x^{-1}\int_1^T \left(\sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^{\frac{1}{2}+it}} \min\left\{\frac{n}{x}, \frac{x}{n}\right\}\right)(\log q\tau + O(1)) dt \ll \frac{\log qT}{\sqrt{x}\log(2x)} \ll T\log q.$$

From (2.28), (2.29) and (2.30) it follows that

$$\begin{aligned}&\int_1^T \left|-\sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^{\frac{1}{2}+it}} \min\left\{\frac{n}{x}, \frac{x}{n}\right\} + x^{-1}\left(\log q\tau + O(1)\right)\right|^2 dt\\ &= x^{-2} T\log^2(qT)\left(1 + O\left(\frac{1}{\log qT}\right)\right) + T\log x + O(T\log q)\\ &\quad + O(x\log\log 4x) + O(x^{-2}(\log q)^2).\end{aligned}$$

The integral of the square of the last error term of (2.5) is

$$\int_1^T x^{-1-2\mathfrak{a}}\tau^{-2}dt \ll 1.$$

If $M_i = \int_a^b |f_i(t)|^2 dt$ for $i = 1, 2$, and $M_1 \geq M_2$, then $\int_a^b |f_1(t) + f_2(t)|^2 dt = M_1 + O(\sqrt{M_1 M_2})$.
Thus, for $1 \leq x \leq qT$,

$$\begin{aligned} R(x,T,\chi) =& x^{-2}T\log^2(qT)\left(1+O\left(\frac{1}{\log qT}\right)\right) + T\log x + O(T\log q) \\ &+ O(x\log\log 4x) + O(x(\log q)^2) \\ =& x^{-2}T\log^2(qT)\left(1+O\left(\frac{1}{\log qT}\right)\right) + T\log x \\ &+ O(qT(\log q)^2\log\log(qT)). \end{aligned}$$

□

Proof of Theorem 1.1. $F(\alpha)$ is even because we may interchange ρ and ρ' in (1.2). $F(\alpha)$ is real and nonnegative by (2.11). By Lemma 2.2, for $1 \leq x \leq qT$ and $1 < q \leq \log T$,

$$L(x,T,\chi) = 2\pi F(x,T,\chi) + O((qT)^{\frac{1}{2}}) + O(x(\log q)^2) = 2\pi F(x,T,\chi) + O(qT(\log q)^2).$$

By Lemma 2.5,

$$R(x,T,\chi) = \frac{T\log^2(qT)}{x^2}\left(1+O\left(\frac{1}{\log qT}\right)\right) + T\log x + O(qT(\log q)^2\log\log(qT))$$

for $1 \leq x \leq qT$.
Since $L(x,T,\chi) = R(x,T,\chi)$, it follows that, for $1 \leq x \leq qT,\ 1 < q \leq \log T$, we have

$$F(x,T,\chi) = \frac{T\log^2(qT)}{2\pi x^2}\left(1+O\left(\frac{1}{\log(qT)}\right)\right) + \frac{T\log x}{2\pi} + O(qT(\log q)^2\log\log(qT)).$$

From (1.2) we deduce that, for $0 \leq \alpha \leq 1$, $1 < q \leq \log T$,

$$F(\alpha,\chi) = \frac{\log(qT)}{(qT)^{2\alpha}}\left(1+O\left(\frac{1}{\log(qT)}\right)\right) + \alpha + O\left(\frac{q(\log q)^2\log\log(qT)}{\log(qT)}\right).$$

□

3. Tsang's Kernel

Define the Fourier transform $\widehat{g}(z)$ of $g(\alpha)$ by $\widehat{g}(z) = \int_{-\infty}^{\infty} g(\alpha)e^{-2\pi i z\alpha}d\alpha$.
Then

$$\widehat{g}\left(i(\rho-\rho')\frac{\log(qT)}{2\pi}\right) = \int_{-\infty}^{\infty} g(\alpha)(qT)^{\alpha(\rho-\rho')}d\alpha,$$

and so

(3.31)
$$\sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T}} \widehat{g}\left(i(\rho-\rho')\frac{\log(qT)}{2\pi}\right) W(\rho-\rho') = \frac{T}{2\pi}\log(qT)\int_{-\infty}^{\infty} g(\alpha)F(\alpha,\chi)d\alpha.$$

Lemma 3.1. *Let $r(\alpha)$ be a real-valued even function in $L^1(\mathbb{R})$ with support in $[-1,1]$ and Lipschitz continuous at $\alpha = 0$. Then $\widehat{r}(z)$ is an even analytic function,*

$$\widehat{r}(z) = 2\int_0^1 r(\alpha)\cos(2\pi z\alpha)\,d\alpha,$$

and we have

$$\sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T}} \widehat{r}\left(i(\rho-\rho')\frac{\log(qT)}{2\pi}\right) W(\rho-\rho')$$
$$= \frac{T}{2\pi}\log(qT)\left(r(0) + 2\int_0^1 \alpha r(\alpha)\,d\alpha + O\left(\frac{q(\log q)^2\log\log(qT)}{\log(qT)}\right)\right)$$

uniformly as $T\to\infty$, $1 < q \le \log T$.

Proof. Theorem 1.1 implies

$$\int_{-\infty}^{\infty} F(\alpha,\chi)r(\alpha)d\alpha = 2\int_0^1 F(\alpha,\chi)r(\alpha)d\alpha$$
$$=2\left(1+O\left(\frac{1}{\log(qT)}\right)\right)\int_0^1 (qT)^{-2\alpha}\log(qT)r(\alpha)d\alpha$$
$$+ 2\int_0^1 \alpha r(\alpha)d\alpha + O\left(\frac{q(\log q)^2\log\log(qT)}{\log(qT)}\right).$$

Since $r(\alpha)$ is Lipschitz continuous at $\alpha = 0$, we get $r(\alpha) = r(0) + O\left(\frac{\log\log(qT)}{\log(qT)}\right)$ as $T \to \infty$ and $0 \le \alpha \le \frac{\log\log(qT)}{\log(qT)}$. Hence as $T \to \infty$, we obtain

$$\begin{aligned}
2\int\limits_0^1 (qT)^{-2\alpha}\log(qT)r(\alpha)d\alpha &= 2\int\limits_0^{\frac{\log\log(qT)}{\log(qT)}} (qT)^{-2\alpha}\log(qT)r(\alpha)d\alpha \\
&\quad + O\left(\int\limits_{\frac{\log\log(qT)}{\log(qT)}}^1 (qT)^{-2\alpha}\log(qT)|r(\alpha)|d\alpha\right) \\
= 2\left(r(0) + O\left(\frac{\log\log(qT)}{\log(qT)}\right)\right) &\int\limits_0^{\frac{\log\log(qT)}{\log(qT)}} (qT)^{-2\alpha}\log(qT)d\alpha \\
&\quad + O\left(\frac{1}{\log(qT)}\int\limits_0^1 |r(\alpha)|d\alpha\right) \\
= 2\left(r(0) + O\left(\frac{\log\log(qT)}{\log(qT)}\right)\right)&\left(\frac{1}{2} - \frac{1}{2\log^2(qT)}\right) + O\left(\frac{1}{\log(qT)}\right) \\
= r(0) + O\left(\frac{\log\log(qT)}{\log(qT)}\right)&.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int\limits_{-\infty}^{\infty} F(\alpha,\chi)r(\alpha)d\alpha &= \left(1 + O\left(\frac{1}{\log(qT)}\right)\right)\left(r(0) + O\left(\frac{\log\log(qT)}{\log(qT)}\right)\right) \\
&\quad + 2\int\limits_0^1 \alpha r(\alpha)d\alpha + O\left(\frac{q(\log q)^2\log\log(qT)}{\log(qT)}\right) \\
&= \left(r(0) + 2\int\limits_0^1 \alpha r(\alpha)d\alpha + O\left(\frac{q(\log q)^2\log\log(qT)}{\log(qT)}\right)\right).
\end{aligned}$$

Then the lemma follows in view of (3.31). ∎

Define the Tsang kernel $K(z)$ by

$$\widehat{K}(t) := j(2\pi t)\operatorname{sech}(2\pi t),$$

where $j(\alpha)$ is even, nonnegative, bounded, twice differentiable on $[0,1]$, with $\mathrm{supp}(j) \subset [-1,1]$, and $0 \le \widehat{j}(w) \ll \frac{1}{1+w^2}$ for all $w \in \mathbb{R}$. Then

$$K(z) = \frac{1}{\pi} \int_0^1 j(\alpha)\,\mathrm{sech}(\alpha)\cos(z\alpha) d\alpha.$$

The Fejér kernel is

$$j_F(\alpha) = \max\{0, 1-|\alpha|\}, \qquad \widehat{j}_F(w) = \left(\frac{\sin \pi w}{\pi w}\right)^2.$$

The Montgomery-Taylor kernel is (Montgomery [9], Cheer and Goldston [3])

$$j_M(\alpha) = \frac{1}{1-\cos\sqrt{2}}\left(\frac{1}{2\sqrt{2}}\sin\left(\sqrt{2}j_F(\alpha)\right) + \frac{1}{2}j_F(\alpha)\cos\left(\sqrt{2}\alpha\right)\right),$$

$$\widehat{j}_M(w) = \frac{1}{1-\cos\sqrt{2}}\left(\frac{\sin(\frac{1}{2}(\sqrt{2}-2\pi w))}{\sqrt{2}-2\pi w} + \frac{\sin(\frac{1}{2}(\sqrt{2}+2\pi w))}{\sqrt{2}+2\pi w}\right)^2.$$

LEMMA 3.2 (K.-M. Tsang). *The kernel $K(z)$ is an even entire function such that:*

(a) $K(x) > 0$ for all $x \in \mathbb{R}$,
(b) For $z \in \mathbb{C} - \{0\}$, $K(z) \ll \frac{e^{|\mathrm{Im}(z)|}}{|z|^2}$,
(c) For $z = x+iy$, $x, y \in \mathbb{R}$, when $|y| < 1$, we have $\mathrm{Re}\, K(x+iy) > 0$.

For the proof of this lemma, see Tsang [13, Lemma 1] and Baluyot et al. [1, Lemma 6].

LEMMA 3.3. *For a Tsang kernel $K(z)$, we have*

(3.32)
$$2\pi \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|<\frac{1}{\log(qT)}}} \mathrm{Re}\, K\left(i(\rho-\rho')\log(qT)\right) + \mathcal{S}(T)$$
$$= \frac{T}{2\pi}\log(qT)\left(\widehat{K}(0) + 2\int_0^1 \alpha \widehat{K}\left(\frac{\alpha}{2\pi}\right) d\alpha + O\left(\frac{q(\log q)^2 \log\log(qT)}{\log(qT)}\right)\right)$$

uniformly as $T \to \infty$, $1 < q \le \log T$, where

(3.33)
$$\mathcal{S}(T) := 2\pi \mathrm{Re} \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|\ge\frac{1}{\log(qT)}}} K\left(i(\rho-\rho')\log(qT)\right) W(\rho-\rho').$$

Moreover, $\mathrm{Re}\, K > 0$ *for every term in the sum in* (3.32)*, and* $\widehat{K} \ge 0$*.*

Proof. In Lemma 3.1 take $r(\alpha) = \widehat{K}(\frac{\alpha}{2\pi}) = j(\alpha)\operatorname{sech}(\alpha)$. Then $\widehat{r}(z) = 2\pi K(2\pi z)$. By Lemma 3.1,

$$2\pi \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T}} K\left(i(\rho-\rho')\log(qT)\right) W(\rho-\rho')$$
$$= \frac{T}{2\pi}\log(qT)\left(\widehat{K}(0) + 2\int_0^1 \alpha\widehat{K}\left(\frac{\alpha}{2\pi}\right)\,d\alpha + O\left(\frac{q(\log 2q)^2\log\log(qT)}{\log(qT)}\right)\right).$$

We have

$$\sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T}} |W(\rho-\rho')| \ll \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T}} \frac{1}{1+(\gamma-\gamma')^2} \ll T\log^2(qT).$$

Hence, since $W(0)=1$, using Lemma 3.2 (b) we get

$$\sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|<\frac{1}{\log(qT)}}} K\left(i(\rho-\rho')\log(qT)\right)\left(W(\rho-\rho')-1\right)$$
$$= \sum_{\substack{\rho\ne\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|<\frac{1}{\log(qT)}}} K\left(i(\rho-\rho')\log(qT)\right)\frac{(\rho-\rho')^2}{4-(\rho-\rho')^2}$$
$$\ll \sum_{\substack{\rho\ne\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|<\frac{1}{\log(qT)}}} K\left(i(\rho-\rho')\log(qT)\right)\frac{(\rho-\rho')^2}{4-(\rho-\rho')^2}$$
$$\ll \sum_{\substack{\rho\ne\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|<\frac{1}{\log(qT)}}} \frac{(qT)^{|\beta-\beta'|}}{|\rho-\rho|^2\log^2(qT)}\frac{|\rho-\rho'|^2}{|4-(\rho-\rho')^2|}$$
$$= \frac{1}{4\log^2(qT)} \sum_{\substack{\rho\ne\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|<\frac{1}{\log(qT)}}} (qT)^{|\beta-\beta'|}|W(\rho-\rho')| \ll T$$

We get (3.32) by taking real parts.
If $|\beta-\beta'| < \frac{1}{\log(qT)}$ then $\operatorname{Re} K\left(i(\rho-\rho')\log(qT)\right) > 0$ by Lemma 3.2 (c). ◻

Lemma 3.4. *Assume that*

$$N(\sigma,T,\chi) = o\left(q^{-1}(qT)^{2(1-\sigma)}(\log(qT))^{-1}\right) \tag{3.34}$$

uniformly for $\frac{1}{2}+\frac{1}{2\log(qT)}\le\sigma\le 1,\ 1<q\le\sqrt{\log T}$. *Then*

$$\mathcal{S}(T)=o(T\log(qT))\quad \text{as}\quad T\to\infty,\ 1<q\le\sqrt{\log T}.$$

Proof. Lemma 3.2 (b) leads to

$$K\left(i(\rho-\rho')\log(qT)\right)\ll\frac{(qT)^{|\beta-\beta'|}}{|\rho-\rho|^2\log^2(qT)}.$$

Further,

$$\begin{aligned}|W(\rho-\rho')|&=\frac{4}{|4-(\rho-\rho')^2|}\le\frac{4}{\mathrm{Re}(4-(\rho-\rho')^2)}\\&=\frac{4}{4-(\beta-\beta')^2+(\gamma-\gamma')^2}\le\frac{4}{3+(\gamma-\gamma')^2}\ll 1.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{S}(T)&=2\pi\mathrm{Re}\sum_{\substack{0<\gamma,\gamma'\le T\\|\beta-\beta'|\ge\frac{1}{\log(qT)}}}K\left(i(\rho-\rho')\log(qT)\right)W(\rho-\rho')\\&\ll\sum_{\substack{0<\gamma,\gamma'\le T\\|\beta-\beta'|\ge\frac{1}{\log(qT)}}}\frac{(qT)^{|\beta-\beta'|}}{((\beta-\beta')\log(qT))^2+((\gamma-\gamma')\log(qT))^2}\\&\ll\sum_{\substack{0<\gamma,\gamma'\le T\\|\beta-\beta'|\ge\frac{1}{\log(qT)}}}\frac{(qT)^{|\beta-\beta'|}}{1+(\gamma-\gamma')^2}=\sum_{\substack{0<\gamma,\gamma'\le T\\|\delta-\delta'|\ge\frac{1}{\log(qT)}}}\frac{(qT)^{|\delta-\delta'|}}{1+(\gamma-\gamma')^2}\\&\ll\sum_{\substack{0<\gamma,\gamma'\le T\\|\delta-\delta'|\ge\frac{1}{\log(qT)}}}\frac{(qT)^{2|\delta|}+(qT)^{2|\delta'|}}{1+(\gamma-\gamma')^2}.\end{aligned}$$

In the sum above we have $|\delta|+|\delta'|\ge|\delta-\delta'|\ge\frac{1}{\log(qT)}$, so either $|\delta|\ge\frac{1}{2\log(qT)}$ or $|\delta'|\ge\frac{1}{2\log(qT)}$. Therefore

$$\begin{aligned}\mathcal{S}(T)&\ll\sum_{\substack{0<\gamma,\gamma'\le T\\|\delta|\ge\frac{1}{2\log(qT)}}}\frac{(qT)^{2|\delta|}+(qT)^{2|\delta'|}}{1+(\gamma-\gamma')^2}+\sum_{\substack{0<\gamma,\gamma'\le T\\|\delta'|\ge\frac{1}{2\log(qT)}}}\frac{(qT)^{2|\delta|}+(qT)^{2|\delta'|}}{1+(\gamma-\gamma')^2}\\&\ll\sum_{\substack{0<\gamma,\gamma'\le T\\|\delta|\ge\frac{1}{2\log(qT)}}}\frac{(qT)^{2|\delta|}+(qT)^{2|\delta'|}}{1+(\gamma-\gamma')^2}\end{aligned}$$

$$\ll \sum_{\substack{0<\gamma,\gamma'\le T\\ |\delta|\ge \frac{1}{2\log(qT)}\\ |\delta|>|\delta'|}} \frac{(qT)^{2|\delta|}}{1+(\gamma-\gamma')^2} + \sum_{\substack{0<\gamma,\gamma'\le T\\ |\delta'|\ge \frac{1}{2\log(qT)}\\ |\delta|\le|\delta'|}} \frac{(qT)^{2|\delta'|}}{1+(\gamma-\gamma')^2}$$

$$\ll \sum_{\substack{0<\gamma,\gamma'\le T\\ |\delta|\ge \frac{1}{2\log(qT)}}} \frac{(qT)^{2|\delta|}}{1+(\gamma-\gamma')^2} = \sum_{\substack{0<\gamma\le T\\ |\delta|\ge \frac{1}{2\log(qT)}}} (qT)^{2|\delta|} \sum_{0<\gamma'\le T} \frac{1}{1+(\gamma-\gamma')^2}$$

$$\ll \sum_{\substack{0<\gamma\le T\\ |\delta|\ge \frac{1}{2\log(qT)}}} (qT)^{2|\delta|}\log(q|\gamma|+2) \ll \log(qT) \sum_{\substack{0<\gamma\le T\\ |\delta|\ge \frac{1}{2\log(qT)}}} (qT)^{2|\delta|}.$$

Thus, in view of the symmetry of the zeros with respect to the critical line,

$$\mathcal{S}(T) \ll \log(qT) \sum_{\substack{0<\gamma\le T\\ 1/2+1/(2\log(qT))\le\beta<1}} (qT)^{2\beta-1}.$$

By the hypothesis (3.34),

$$\begin{aligned}\mathcal{S}(T) \ll \log(qT) &\int_{1/2+1/(2\log(qT))}^{1} (qT)^{2u-1} d(-N(u,T,\chi))\\ =&e\log(qT)N\left(\frac{1}{2}+\frac{1}{2\log(qT)},T,\chi\right)\\ &+2\log^2(qT)\int_{1/2+1/(2\log(qT))}^{1} N(u,T,\chi)(qT)^{2u-1}du\\ =&o(T\log(qT)).\end{aligned}$$

□

4. Proof of Theorems 1.2 and 1.3

We will show that the assumption of Theorem 1.2 implies the assumption of Theorem 1.3 and then we will prove Theorem 1.3.

Lemma 4.1. *Suppose that all the zeros* $\rho=\beta+i\gamma$ *of* $L(s,\chi)$ *with* $(qT)^{\frac{3}{8}} < \gamma \le T$, $1<q\le\sqrt{\log T}$ *satisfy* $\frac{1}{2}-\frac{1}{2\log(qT)} < \beta < \frac{1}{2}+\frac{1}{2\log(qT)}$. *Then*

$$N(\sigma,T,\chi) = o\left(q^{-1}(qT)^{2(1-\sigma)}(\log(qT))^{-1}\right)$$

uniformly for $\frac{1}{2}+\frac{1}{2\log(qT)} \le \sigma \le \frac{51}{64}$, $1<q\le\sqrt{\log T}$.

Proof. If $\sigma \ge \frac{1}{2}+\frac{1}{2\log(qT)}$ then by our hypothesis we have

$$N(\sigma,T,\chi) = N(\sigma,(qT)^{\frac{3}{8}},\chi) \le N((qT)^{\frac{3}{8}},\chi) \ll (qT)^{\frac{3}{8}}\log(qT).$$

If $\sigma \le \frac{51}{64}$ and $1 < q \le \sqrt{\log T}$ then as $T \to \infty$,

$$\frac{(qT)^{\frac{3}{8}} \log(qT)}{q^{-1}(qT)^{2(1-\sigma)}(\log(qT))^{-1}} \le \frac{q^{\frac{31}{32}} \log^2(qT)}{T^{\frac{1}{32}}} \le \frac{(\log T) \log^2(T \log T)}{T^{\frac{1}{32}}} \to 0.$$

Hence, for $\frac{1}{2} + \frac{1}{2\log(qT)} \le \sigma \le \frac{51}{64},\ 1 < q \le \sqrt{\log T}$, we obtain

$$N(\sigma, T, \chi) = o\left(q^{-1}(qT)^{2(1-\sigma)}(\log(qT))^{-1}\right).$$

□

Proof of Theorem 1.3. Let m_ρ denote the multiplicity of a zero ρ of $L(s,\chi)$.
We have
(4.35)

$$\frac{q(\log q)^2 \log\log(qT)}{\log(qT)} = o(1) \quad \text{uniformly for} \quad T \to \infty, 1 < q \le \sqrt{\log T}.$$

Using Lemma 3.3, Lemma 3.4 and (4.35), we derive, for $1 < q \le \sqrt{\log T}$,

$$\begin{aligned}
\sum_{\substack{\rho \\ 0<\gamma\le T}} m_\rho &= \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ \rho=\rho'}} 1 = \frac{1}{K(0)} \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ \rho=\rho'}} \mathrm{Re} K(0) \\
&= \frac{1}{K(0)} \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ \rho=\rho'}} \mathrm{Re}\, K\left(i(\rho-\rho')\log(qT)\right) \\
&\le \frac{1}{K(0)} \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|<\frac{1}{\log(qT)}}} \mathrm{Re}\, K\left(i(\rho-\rho')\log(qT)\right) \\
&= \frac{1}{2\pi K(0)} \frac{T}{2\pi} \log(qT) \left(\widehat{K}(0) + 2\int_0^1 \alpha \widehat{K}\left(\frac{\alpha}{2\pi}\right) d\alpha + o(1)\right).
\end{aligned}$$

Let $N(T,\chi)$ denote the number of nontrivial zeros of $L(s,\chi)$ with $0 < \gamma \le T$. Then

$$N(T,\chi) = \frac{T}{2\pi} \log\left(\frac{qT}{2\pi}\right) - \frac{T}{2\pi} + O(\log qT)$$

uniformly for all q (see Corollary 14.7 of [11]). The number of nontrivial zeros of $L(s,\chi)$ with $-T \le \gamma < 0$ is $N(T,\overline{\chi})$, and the number of nontrivial zeros with $\gamma = 0$ is $O(\log q)$.

For $1 < q \le \sqrt{\log T}$, the proportion of nontrivial zeros of $L(s,\chi)$ in the upper half-plane that are simple is

$$\frac{1}{N(T,\chi)} \sum_{\substack{\rho:\text{simple} \\ 0<\gamma\le T}} 1 \ge 2 - \frac{1}{N(T,\chi)} \sum_{\substack{\rho \\ 0<\gamma\le T}} m_\rho$$
$$\ge 2 - \frac{1}{2\pi K(0)} \left(\widehat{K}(0) + 2\int_0^1 \alpha \widehat{K}\left(\frac{\alpha}{2\pi}\right) d\alpha + o(1) \right).$$

Then the Montgomery-Taylor kernel $j(\alpha) = j_M(\alpha)$ leads to (see Baluyot et al. [1, Section 7])

$$\frac{1}{N(T,\chi)} \sum_{\substack{\rho:\text{simple} \\ 0<\gamma\le T}} 1 \ge 0.617483786... + o(1).$$

□

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