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CHARACTERIZATION OF FUCHSIAN GROUPS AS LAMINAR GROUPS & THE STRUCTURE THEOREM OF HYPERBOLIC 2-ORBIFOLDS

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ABSTRACT. Thurston and Calegari-Dunfield showed that the fundamental group of a tautly foliated hyperbolic 3-manifold acts on the circle in a distinctive way that the action preserves some structure of S^1 , so-called a circle lamination. Indeed, a large class of Kleinian groups acts on the circle preserving a circle lamination. In this paper, we are concerned with the converse problem that a group acting on the circle with at least two invariant circle laminations is Kleinian. We prove that a subgroup of the orientation preserving circle homeomorphism group is a Fuchsian group whose quotient orbifold is not a geometric pair of pants (or turnover) if and only if it preserves three circle laminations with a certain transversality. This is the complete generalization of the previous result of the first author which is proven under the assumption that the subgroup is discrete and torsion-free. On the way to our main result, we also show a structure theorem (generalized pants decomposition) for complete 2-dimensional hyperbolic orbifolds, including the case of infinite type, which is of independent interest.

1. INTRODUCTION

1.1. *Background and Motivation.* The classification of a group action on a manifold is a fundamental question in mathematics. Along this line, one of the outstanding studies is the Zimmer's program. One part of Zimmer's program is to understand the actions of a lattice G of a semi-simple Lie group Γ on a compact manifold M . See [18] for the survey of Zimmer's program.

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The most accessible case is that M is dimension one, namely, $M = S^1$. When Γ is of real rank at least two, Ghys [21] gave a complete characterization for homomorphisms from G to the orientation preserving circle homeomorphism group $\text{Homeo}^+(S^1)$. This result says that the circle action of a higher rank lattice is rigid.

On the contrary, in the rank one case, the circle actions of G are more flexible. For instance, when $\Gamma = \text{PSL}_2(\mathbb{R})$ and G is a closed hyperbolic surface group, the space of G -actions is studied by Mann [25] and Mann-Wolff [26] in terms of the Euler number. They showed the equivalency of the rigidity and the geometricity of the G -actions. This implies that non-geometric actions have no rigidity in some reasonable sense. Hence, in general, given any subgroup H of $\text{Homeo}^+(S^1)$, it is difficult to determine whether H comes from a lattice of a semi-simple Lie group of real rank one.

Nonetheless, in the case of $\text{PSL}_2(\mathbb{C})$, we can observe that a large class of lattices act on the circle in a distinctive way that the actions preserve some structure of S^1 , so-called a *circle lamination*. This feature is first observed by Thurston along the proof of the hyperbolization theorem for surface bundles over S^1 . More generally, in [32], Thurston showed that the fundamental group of a tautly foliated hyperbolic 3-manifold faithfully acts on S^1 , preserving a pair of circle laminations. In various circumstance, Thurston's theorem has been generalized by many authors, e.g. [12], [15], [13], [17], [19] and so on. Also, we can see that almost all torsion-free lattices in $\text{PSL}_2(\mathbb{R})$ which are the fundamental groups of hyperbolic surfaces of finite type act on S^1 preserving infinitely many distinct circle laminations.

Hence, it is natural to ask whether a subgroup of $\text{Homeo}^+(S^1)$ preserving several circle laminations, so-called a *laminar group*, is a discrete subgroup of $\text{PSL}_2(\mathbb{C})$, so-called a *Kleinian group*. Also, we may ask that a laminar group with infinite many invariant circle laminations is a *Fuchsian group* which is a discrete subgroup of $\text{PSL}_2(\mathbb{R})$. In this paper, we are mainly concerned with the second question. We will show that three is the enough number of invariant circle laminations to characterize Fuchsian groups.

1.2. Main results. To present the precise statements for the questions and our main theorems, we roughly introduce basic notions about laminations. Let $\mathbb{H}^2$ be the hyperbolic plane. For now, we use the Poincaré disk for $\mathbb{H}^2$ which is the open unit disk in the complex plane $\mathbb{C}$. The Gromov boundary of $\mathbb{H}^2$ is the unit circle S^1 in $\mathbb{C}$. Hence, we can think of the orientation preserving isometry group $\text{PSL}_2(\mathbb{R})$ for $\mathbb{H}^2$ as a subgroup of $\text{Homeo}^+(S^1)$ since each element of $\text{PSL}_2(\mathbb{R})$ is continuously and uniquely extended to S^1 .

A given geodesic in $\mathbb{H}^2$ joins two points in S^1 . Conversely, given two distinct points in S^1 , there is a unique geodesic in $\mathbb{H}^2$ connecting them. Hence, the set $g(\mathbb{H}^2)$ of geodesics in $\mathbb{H}^2$ is parametrized by the set $\mathcal{M}$ of subsets of

S^1 with cardinality of 2. We denote the parametrization from $g(\mathbb{H}^2)$ to $\mathcal{M}$ by ϵ .

A *geodesic lamination* Λ in $\mathbb{H}^2$ is a non-empty subcollection of $g(\mathbb{H}^2)$ such that $\bigcup \Lambda$ is a closed subset in $\mathbb{H}^2$ and each pair of elements of Λ are disjoint. We say that for each component of $\mathbb{H}^2 \setminus \Lambda$, the closure of the component in $\overline{\mathbb{H}^2}$ is a *non-leaf gap* of Λ . A geodesic lamination is *quite full* if every non-leaf gap is either an ideal polygon or a crown. Here, a *crown* $\mathcal{G}$ is a non-leaf gap having a sequence $\{t_i\}_{i \in \mathbb{Z}}$ in S^1 such that $\epsilon^{-1}(\{t_i, t_{i+1}\})$ is a boundary geodesic of $\mathcal{G}$ and $\{t_i\}_{i \in \mathbb{Z}}$ accumulates to a unique point p . The point p is called the *pivot* of the crown. In particular, a quite full geodesic lamination without crowns is said to be *very full*.

A *circle lamination* $\mathcal{L}$ is a subcollection of $\mathcal{M}$ if $\mathcal{L} = \epsilon(\Lambda)$ for some geodesic lamination Λ in $\mathbb{H}^2$. For convenience, we think of circle laminations as geodesic laminations in $\mathbb{H}^2$. Given a collection $\mathcal{C} = \{\mathcal{L}_\alpha\}_{\alpha \in \Gamma}$ of circle laminations, we denote by $\text{Aut}(\mathcal{C})$ the set of all elements in $\text{Homeo}^+(S^1)$ preserving $\mathcal{C}$, namely, $\text{Aut}(\mathcal{C}) := \{g \in \text{Homeo}^+(S^1) : g(\mathcal{L}_\alpha) = \mathcal{L}_\alpha \text{ for all } \alpha \in \Gamma\}$. We also call $\text{Aut}(\mathcal{C})$ the *laminar automorphism group* of $\mathcal{C}$. A subgroup of $\text{Homeo}^+(S^1)$ is *laminar* if it is a subgroup of the laminar automorphism group of some circle lamination. The laminar groups were first introduced in [11].

Now, we are ready to give the precise statements for the questions explained in the previous section.

QUESTION 1.1. *Let $\mathcal{C} = \{\mathcal{L}_\alpha\}_{\alpha \in \Gamma}$ be a collection of quite full circle laminations whose endpoint sets are pairwise disjoint, namely, $(\bigcup \mathcal{L}_\alpha) \cap (\bigcup \mathcal{L}_\beta) = \emptyset$ for any $\alpha \neq \beta$. Is every subgroup of $\text{Aut}(\mathcal{C})$ isomorphic to a discrete subgroup of $\text{PSL}_2(\mathbb{C})$?*

QUESTION 1.2. *Are the Fuchsian groups the only case to act faithfully on S^1 preserving an infinite collection of quite full circle laminations whose endpoint sets are pairwise disjoint? In other words, if $\mathcal{C}$ is an infinite collection of quite full circle laminations whose endpoint sets are pairwise disjoint, is every subgroup of $\text{Aut}(\mathcal{C})$ conjugated to a Fuchsian group as subgroups in $\text{Homeo}^+(S^1)$?*

Indeed, Question 1.1 was proposed in [3] and [1] in terms of pants-like COL_n groups. See Section 3 for the definition of a pants-like COL_n group. According to [4], given a finite collection $\mathcal{C} = \{\mathcal{L}_\alpha\}_{\alpha \in \Gamma}$ of quite full circle laminations whose endpoint sets are pairwise disjoint, for any subgroup G of $\text{Aut}(\mathcal{C})$, we can construct a pants-like collection $\mathcal{C}' = \{\mathcal{L}'_\alpha\}_{\alpha \in \Gamma}$ of circle laminations preserved by G . Hence, even though the definition of a pants-like COL_n group looks artificial, this is a weaker condition than preserving n quite full circle laminations with the disjoint endpoint set condition. Therefore, Question 1.1 assumes conditions which are stronger than in [3] and weaker than in [1].

In this paper, we prove the following corollary, which gives an affirmative answer to Question 1.2. In particular, three invariant laminations are sufficient to characterize circle actions of Fuchsian groups.

COROLLARY 1.3. *If $\mathcal{C} = \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$ is a collection of quite full circle laminations whose endpoint sets are pairwise disjoint, then every subgroup of $\text{Aut}(\mathcal{C})$ is conjugate to a Fuchsian group as a subgroup of $\text{Homeo}^+(S^1)$. Conversely, given a Fuchsian group G , we can construct three quite full circle laminations satisfying the disjoint endpoint condition and preserved by G , provided that $\mathbb{H}^2/G$ is not a geometric pair of pants.*

In fact, we prove a stronger statement than the first assertion of Corollary 1.3.

THEOREM A. Every pants-like COL_3 group is a convergence group.

Together with the following well-known theorem, Theorem A implies the desired Corollary 1.5.

THEOREM 1.4 (Convergence group theorem ([33],[20],[16], cf. [23] for the indiscrete case)). *A subgroup of $\text{Homeo}^+(S^1)$ is a convergence group if and only if it is conjugate to a Fuchsian group in $\text{Homeo}^+(S^1)$.*

COROLLARY 1.5. *Every pants-like COL_3 group G is a discrete Möbius group; namely, there is an element ϕ in $\text{Homeo}^+(S^1)$ such that $\phi G \phi^{-1}$ is a Fuchsian group.*

For the converse direction, we prove the following theorem.

THEOREM B. Let G be a Fuchsian group. Suppose that $\mathbb{H}/G$ is not a geometric pair of pants. Then G is a pants-like COL_3 group.

With a minor modification, it implies the second assertion of Corollary 1.3. See the paragraphs following the proofs of Theorem B and Corollary 6.11 for details of this modification.

The combination of Theorem A and Theorem B is the complete generalization of the following theorem shown in [3].

THEOREM 1.6. *Let G be a torsion-free discrete subgroup of $\text{Homeo}^+(S^1)$. Then G is a pants-like COL_3 group if and only if G is a Fuchsian group whose quotient is not the thrice-punctured sphere.*

Note that Theorem A and Theorem B are not immediately implied by Theorem 1.6. First, in [3], the proof of the only-if-direction depends on the fact that every non-trivial element has a fixed point which is implied by the torsion-free condition. Also, along the proof, discreteness was used in a lemma which is corresponding to Lemma 7.3 in this paper. Hence, we have to overcome these difficulties to prove Theorem A. For the if-direction, the construction of three different geodesic laminations contained some technical issues.

Also, the Selberg's Lemma does not make it easy to construct the geodesic laminations on hyperbolic 2-orbifolds since we also consider the infinite type orbifolds. Instead, we have to use the structure theorem (generalized pants decomposition) for hyperbolic 2-orbifolds including the case of infinite type as follows.

THEOREM 6.1 (The structure theorem for complete hyperbolic 2-dimensional orbifolds). *Let G be a non-elementary Fuchsian group and S be the complete 2-dimensional hyperbolic orbifold $\mathbb{H}/G$. Suppose that there is a pants decomposition $\mathcal{C}$ of S . Let Λ be the union of all geodesic realizations in S of elements of $\mathcal{C}$. Then $\Lambda \cup (\partial CH(L(G))/G)$ is a geometric pants decomposition of $CC(S)$.*

This theorem is analogous of the structure theorem for complete hyperbolic surfaces of Álvarez and Rodríguez [2] and Basmajian and Šarić [6]. The pants decomposition theorem for hyperbolic orbifolds of finite type is well known. For instance, see Thurston [31, Chapter 13, Orbifolds]. However, we could not find the structure theorem of hyperbolic 2-orbifold including the case of infinite type in the literature. Hence, we collect and reorganize basic facts about the hyperbolic geometry of orbifolds, and prove the structure theorem for hyperbolic orbifolds for the sake of completeness. This is of the independent interest and could serve as a useful reference to the community (for instance, see [8]).

Now, by Corollary 1.3, the remaining case of Question 1.1 is when $\mathcal{C}$ consists of at most two circle laminations. The case where $\mathcal{C}$ consists of two circle laminations was studied in [4]. In [4], Jung and the author show that if $\mathcal{C} = \{\mathcal{L}_1, \mathcal{L}_2\}$ is a *veering pair* satisfying the disjoint endpoints condition and some extra conditions, then every subgroup in $\text{Aut}(\mathcal{C})$ is the fundamental group of an irreducible 3-orbifold. Moreover, if the 3-orbifold is finite in some sense, then the orbifold admits a complete hyperbolic metric, and hence the fundamental group is Kleinian. We note that a key step in the proof in [4] is Lemma 4.9, involving the concept of “pre-approximation sequences”, which is used to prove the classification theorem for gap stabilizers.

Throughout the paper, we use basic notations and statements of [5] and [3]. Especially, instead of the circle laminations, we use the notion of laminations systems which are more proper to keep track of the dynamic of the circle actions.

1.3. Organization. We give a brief review of necessary preliminaries in Section 2. In Section 3, we show that Möbius-like laminar groups with two invariant laminations satisfying some special conditions are discrete in $\text{Homeo}^+(S^1)$. This implies that every pants-like COL_3 group is discrete. In Section 4, we prove Theorem A. In Section 5, we give a summary of elementary facts about the geometry and topology of 2-dimensional hyperbolic orbifolds and for completeness, we prove some basic facts which are well-known among experts. In

Section 6.1, we show a structure theorem of complete 2-dimensional hyperbolic orbifolds, Theorem 6.1, following the proof for hyperbolic surfaces in [6]. Note that the structure theorem for complete hyperbolic surfaces was firstly proven by Álvarez and Rodríguez [2], and after that, a geometric proof was suggested by Basmajian and Šarić [6]. Using Theorem 6.1, in Section 6.2 and Section 6.3, we construct three different laminations in S^1 and prove Theorem B.

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2. BASIC NOTIONS AND NOTATIONS TO STUDY LAMINAR GROUPS

In first three sections, we review basic notions about Riemann surfaces and the hyperbolic geometry on orbifolds. In this part, to prove Theorem 6.1, we adopt a proper collection of definitions. This is based on [27], [7], and [24]. Then, we recall circle laminations and lamination systems. Also, we summarize related notions and technical lemmas. This part is a summary of [5].

2.1. The Riemann sphere and the hyperbolic plane. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the extended real line. We denote the upper half plane by $\mathbb{H}$ and the lower half plane by $\mathbb{H}^*$. The upper half plane $\mathbb{H}$ has a standard hyperbolic metric ρ , so-called the Poincaré metric. With this metric, $\hat{\mathbb{R}}$ is the Gromov boundary $\partial\mathbb{H}$. On the other hand, a Cayley transformation p expressed as

$$p(z) = i \frac{1+z}{1-z}$$

gives an identification between the Poincaré disk $\mathbb{D}$ and the upper half plane $\mathbb{H}$ where $\mathbb{D}$ is the unit open disk in $\mathbb{C}$ with the standard hyperbolic metric.

Recall that the automorphism groups of $\hat{\mathbb{C}}$ and $\mathbb{H}$ are $\mathrm{PSL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{R})$, respectively. The group $\mathrm{PSL}_2(\mathbb{C})$ acts on $\hat{\mathbb{C}}$ by linear fractional transformations and especially the subgroup $\mathrm{PSL}_2(\mathbb{R})$ acts on $\hat{\mathbb{C}}$ preserving $\hat{\mathbb{R}}$. Recall that the orientation preserving isometry group of $\mathbb{H}$ is exactly $\mathrm{PSL}_2(\mathbb{R})$. Note that the automorphism group of $\mathbb{D}$ can be identified with $\mathrm{PSL}_2(\mathbb{R})$, via p , and so we think of the automorphism group of $\mathbb{D}$ as $\mathrm{PSL}_2(\mathbb{R})$.

A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Especially, a Kleinian group in $\mathrm{PSL}_2(\mathbb{R})$ is called a *Fuchsian group*. We denote by $\Omega(G)$ the domain

of discontinuity of a Fuchsian group G on $\hat{\mathbb{C}}$ and by $L(G)$ the limit set of G . Note that $L(G)$ is a closed subset of $\hat{\mathbb{R}}$, and $\mathbb{H} \cup \mathbb{H}^* \subset \Omega(G)$. The Fuchsian group G is *of the first kind* if $L(G) = \hat{\mathbb{R}}$, and *of the second kind*, otherwise. In the first kind case, $\Omega(G)$ has exactly two connected components $\mathbb{H}$ and $\mathbb{H}^*$. If $L(G)$ is finite, then $L(G)$ contains at most two points. In this case, we say that G is *elementary*. Recall that every elementary Fuchsian group is virtually abelian and the converse is also true. When $L(G) \neq \hat{\mathbb{R}}$ is infinite, $L(G)$ is a closed, perfect, nowhere dense subset of $\hat{\mathbb{R}}$, hence a Cantor set. Hence, in the second kind case, $\Omega(G)$ has the only one connected component which is conformally equivalent to the Riemann sphere without a cantor set.

2.2. Riemann surfaces with branched coverings. Let G be a subgroup of $\mathrm{PSL}_2(\mathbb{C})$, and let Ω be a non-empty G -invariant open subset of $\hat{\mathbb{C}}$. Suppose that G acts properly discontinuously on Ω . If Ω has more than two boundary points, then, for each $z_0 \in \Omega$, the stabilizer

$$G_{z_0} = \{g \in G : g(z_0) = z_0\}$$

is a finite cyclic group generated by an elliptic element fixing z_0 . Moreover, we can take a complex structure on the orbit space Ω/G so that the quotient map q from Ω to Ω/G is holomorphic and a branched covering.

In other words, for each $z_0 \in \Omega$, there are charts $\varphi_1 : U_1 \rightarrow V_1$ on Ω and $\varphi_2 : U_2 \rightarrow V_2$ on Ω/G , with $q(U_1) \subset U_2$, such that $\varphi_1(z_0) = 0$ and

$$\varphi_2 \circ q \circ \varphi_1^{-1}(z) = z^m,$$

where $m = |G_{z_0}|$. We call m the *ramification index* of q at z_0 , and z_0 is called a *ramification point* if its ramification index is greater than 1. See [24] for a more detailed discussion.

Let G be a non-elementary Fuchsian group. We write π_G^d for the quotient map from $\Omega(G)$ to $\Omega(G)/G$, and π_G for the quotient map from $\mathbb{H}$ to $\mathbb{H}/G$. With the conformal structures defined above, the quotient spaces $\Omega(G)/G$ and $\mathbb{H}/G$ are Riemann surfaces.

If G is of the first kind, then $\Omega(G)/G$ has exactly two connected components; otherwise, it is connected. If G is of the second kind, there is a natural anti-conformal involution J on $\Omega(G)/G$ induced by complex conjugation on $\hat{\mathbb{C}}$. The map J interchanges $\mathbb{H}/G$ and $\mathbb{H}^*/G$ and fixes $(\hat{\mathbb{R}} - L(G))/G$ pointwise.

2.3. 2-dimensional hyperbolic orbifolds. Let S be a connected metric space. The space S is called a *2-dimensional hyperbolic orbifold* if there is a subgroup $G(S)$ of $\mathrm{PSL}_2(\mathbb{R})$ and a non-empty $G(S)$ -invariant simply connected open set $C(S)$ of $\mathbb{H}$ such that $G(S)$ acts properly discontinuously on $C(S)$ and $C(S)/G(S)$ with the metric induced by ρ is isometric to S . We say that S is *complete* if $C(S) = \mathbb{H}$.

Similarly, the space S is a *2-dimensional hyperbolic orbifold with geodesic boundary* if there is a subgroup $G(S)$ of $\mathrm{PSL}_2(\mathbb{R})$ and a $G(S)$ -invariant closed

convex subset $C(S)$ of $\mathbb{H}$ with non-empty interior whose boundary is a disjoint union of bi-infinite geodesics such that $G(S)$ acts properly discontinuously on $C(S)$ and $C(S)/G(S)$ with the metric induced by ρ is isometric to S . Here, $C(S)$ is allowed to be $\mathbb{H}$. By definition, every 2-dimensional hyperbolic orbifold with geodesic boundary is complete as a metric space. Note that $G(S)$ is uniquely determined up to conjugation by $\mathrm{PSL}_2(\mathbb{R})$.

Let π_S be the quotient map from $C(S)$ to $C(S)/G(S)$. As discussed above, when S is a 2-dimensional hyperbolic orbifold, π_S is a branched covering. Even though S is a 2-dimensional hyperbolic orbifold with geodesic boundary, the restriction $\pi_S|_{\mathrm{int}(C(S))}$ to the interior $\mathrm{int}(C(S))$ of $C(S)$ is a branched covering onto $\mathrm{int}(C(S))/G(S)$ of S . This map can be also extended continuously onto the geodesic boundary $\partial C(S)$. Therefore, π_S is a branched covering.

Let S be a 2-dimensional hyperbolic orbifold with or without geodesic boundary. The set $C(S)$ is a universal orbifold covering space and π_S is the orbifold covering map. We say that a point $x \in S$ is a *cone point of order n* if x is an image of a ramification point with ramification index n and we denote the set of all cone points in S by $\Sigma(S)$. Note that by definition, a ramification point is not in $\partial C(S)$ when S is a 2-dimensional hyperbolic orbifold with geodesic boundary.

We define the *geometric interior* S° of S by

$$S^\circ = \mathrm{int}(C(S))/G(S) - \Sigma(S),$$

and call points of S° *regular points*. Points of $S - \mathrm{int}(C(S))/G(S)$ are called *boundary points*, and the set of boundary points of S is denoted by ∂S . Note that if S is a 2-dimensional hyperbolic orbifold with geodesic boundary, then ∂S is a disjoint union of geodesics.

Recall that if a bordered Riemann surface X has an open subset which is conformally equivalent to a unit disk without center, then we say that X has a *puncture*. Moreover, when X has a puncture, we can uniquely extend S to a bigger Riemann surface $\bar{X}$ by adding the center. In this sense, we think of a puncture as the added center of the disk. If a point x is in $\partial S \cup \Sigma(S)$ or a puncture of S , x is called a *geometric boundary point* and ΔS denotes the set of all geometric boundary points of S .

Let S be a hyperbolic surface with geodesic boundary. Under the above definition, S is a 2-dimensional hyperbolic orbifold with geodesic boundary and without cone points. As usual, a *geodesic lamination* on S is a nonempty closed subset of S which is a disjoint union of simple geodesics, and the geodesics composing the lamination are called *leaves*. For hyperbolic 2-orbifolds with geodesic boundary, we define geodesic laminations as follows:

DEFINITION 2.1. *Let S be a 2-dimensional hyperbolic orbifold with geodesic boundary. Fix the corresponding group $G(S)$. Then a closed subset Λ of S is called a geodesic lamination on S if $\pi_S^{-1}(\Lambda)$ is a geodesic lamination of $C(S)$, namely $\pi_S^{-1}(\Lambda)$ is a closed subset of $C(S)$ which consists of pairwise*

disjoint bi-infinite geodesics. In particular, if $\pi_S^{-1}(\Lambda) = G(S) \cdot \ell$ for some bi-infinite geodesic ℓ in $\mathbb{H}$, then we call Λ a simple geodesic in S .

When S is a complete 2-dimensional hyperbolic orbifold and Λ is a geodesic lamination of S , we think of Λ as a closed subset of S which is a disjoint union of simple geodesics. Hence, the geodesics composing the lamination are called *leaves*. As $\pi_S^{-1}(\Lambda)$ is a geodesic lamination of $C(S)$, $\pi_S^{-1}(\Lambda)$ is a geodesic lamination on $\mathbb{H}$. Since given two points in $\partial\mathbb{H}$, there is a unique bi-infinite geodesic which has the points as end points, and vice versa, we may think of $\pi_S^{-1}(\Lambda)$ as a collection of two-point sets of S^1 corresponding to leaves of $\pi_S^{-1}(\Lambda)$. Moreover, since $G(S)$ acts on $\hat{\mathbb{R}}$ and $G(S)$ permutes leaves of $\pi_S^{-1}(\Lambda)$, $G(S)$ acts on the collection of two-point sets of S^1 and permutes two-point sets. In the following sections, we will formulate precisely the collection of two-point sets corresponding to $\pi_S^{-1}(\Lambda)$ in terms of the topology of S^1 and group actions on such collections.

2.4. The topology of S^1 . Let S^1 be the unit circle on $\mathbb{C}$ which is the boundary of the Poincaré disk $\mathbb{D}$. Now we introduce a circular order to S^1 which is helpful to discuss the topology of S^1 .

We start with a set X . For each $n \in \mathbb{N}$ with $n \geq 2$, we write $\Delta_n(X)$ for the set

$$\{(x_1, \dots, x_n) \in X^n : x_i = x_j \text{ for some } i \neq j\}.$$

A *circular order* of X means a map φ from X^3 to $\{-1, 0, 1\}$ satisfying following properties :

(DV) $\varphi^{-1}(0) = \Delta_3(X)$, and

(C) $\varphi(x_1, x_2, x_3) - \varphi(x_0, x_2, x_3) + \varphi(x_0, x_1, x_3) - \varphi(x_0, x_1, x_2) = 0$ for all $x_0, x_1, x_2, x_3 \in X$.

Let $n \in \mathbb{N}$ with $n \geq 3$. A n -tuple $(x_1, \dots, x_n)$ in $(S^1)^n - \Delta_n(S^1)$ is called *positively oriented* in S^1 if for each $i \in \{2, \dots, n-1\}$, $p(x_1^{-1}x_i) < p(x_1^{-1}x_{i+1})$ where p is the Cayley transformation. Similarly, a n -tuple $(x_1, \dots, x_n)$ in $(S^1)^n - \Delta_n(S^1)$ is called *negatively oriented* in S^1 if for each $i \in \{2, \dots, n-1\}$, $p(x_1^{-1}x_{i+1}) < p(x_1^{-1}x_i)$. Let φ be a map from $(S^1)^3$ to $\{-1, 0, 1\}$ defined by

$$\varphi(x) = \begin{cases} -1 & \text{if } x \text{ is negatively oriented,} \\ 0 & \text{if } x \in \Delta_3(S^1), \\ 1 & \text{if } x \text{ is positively oriented.} \end{cases}$$

We can see that φ is a circular order of S^1 . From now on, the circular order of S^1 means the map φ .

We call a nonempty proper connected open subset of S^1 an *open interval* in S^1 . Let u, v be elements of S^1 . Then, we define the subset $(u, v)_{S^1}$ of S^1 as follows:

1. If $u \neq v$, $(u, v)_{S^1}$ is the set $(u, v)_{S^1} = \{p \in S^1 : \varphi(u, p, v) = 1\}$ and is called a *nondegenerate open interval* in S^1 .

2. If $u = v$, $(u, v)_{S^1}$ is the set $S^1 - \{u\}$ and is called a *degenerate open interval* in S^1 .

In particular, if $(u, v)_{S^1}$ is a nondegenerate open interval, then we denote $(v, u)_{S^1}$ by $(u, v)_{S^1}^*$, and call it the *dual interval* of $(u, v)_{S^1}$. We can see that the set of all nondegenerate open intervals of S^1 is a basis for the topology of S^1 which is induced from the standard topology of $\mathbb{C}$. For convenience, when $(u, v)_{S^1}$ is a nondegenerate open interval, we use the following notations: $[u, v)_{S^1} = \{u\} \cup (u, v)_{S^1}$; $(u, v]_{S^1} = (u, v)_{S^1} \cup \{v\}$; and $[u, v]_{S^1} = (u, v)_{S^1} \cup \{u, v\}$.

REMARK 2.2. In [5], the map p is the stereographic projection

$$p(z) = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z) - 1}.$$

Although we use the Cayley transform to define the circular order φ , the induced circular order is the same. Hence this choice does not affect the results of [5].

2.5. *Laminations of S^1* . Let $\mathcal{M}$ be the set of all two-point subsets of S^1 . More precisely,

$$\mathcal{M} = ((S^1)^2 - \Delta_2(S^1)) / (x, y) \sim (y, x).$$

For each $\{a, b\}$ and $\{c, d\}$ of $\mathcal{M}$, they are said to be *linked* if each connected component of $S^1 - \{a, b\}$ contains precisely one of c or d . They are called *unlinked* if they are not linked. We can see easily that $\mathcal{M}$ has a hausdorff distance induced from the Euclidean metric of $\mathbb{C}$.

DEFINITION 2.3. A lamination of S^1 is a nonempty closed subset Λ of $\mathcal{M}$ whose elements are pairwise unlinked. We call each element $\{a, b\}$ of Λ a leaf of Λ and the points a and b are called the end points of the leaf.

REMARK 2.4. Let A be a geodesic laminations of $\mathbb{H}$. For each leaf in $\mathbb{H}$, the end point set of the leaf is a two elements subset of S^1 . In this sense, there is a subset of $\mathcal{M}$ corresponding to A . Then, this subset is obviously a lamination of S^1 . For more detailed discussion, consult [14, Chapter 2].

2.6. *Lamination systems on S^1* . From now on, we recall the notion of lamination systems on S^1 which is first introduced in [5]. This makes it easier to do general topological arguments for laminations of S^1 . It turns out that a lamination system on S^1 is a subset of the topology of S^1 .

The idea of making a lamination system on S^1 from a lamination of S^1 is to take all connected components of complements of each leaf of the lamination of S^1 . So, we will define lamination systems as a collection of nondegenerate open intervals in S^1 with some conditions corresponding to unlinkedness and closedness.

Let I and J be two nondegenerate open intervals. If $I \subseteq J$ or $I^* \subseteq J$, then we say that the pair of points $\{I, I^*\}$ lies on J (See Figure 1). If $\bar{I} \subseteq J$

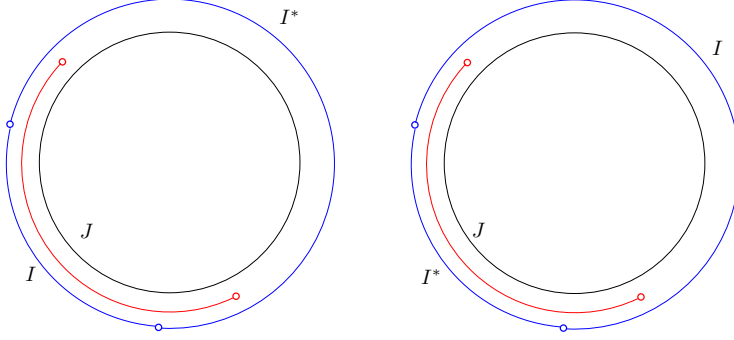


FIGURE 1. The red segment represents the nondegenerate open interval J and the blue parts represent I and I^* . Two figures show all possible cases where $\{I, I^*\}$ lies on J .

or $\overline{I^*} \subseteq J$, then we say that the pair of points $\{I, I^*\}$ *properly lies on* J . Now, we can define lamination systems as follows.

DEFINITION 2.5 ([5]). Let $\mathcal{L}$ be a nonempty family of nondegenerate open intervals of S^1 . $\mathcal{L}$ is called a lamination system on S^1 if it satisfies the following properties :

1. If $I \in \mathcal{L}$, then $I^* \in \mathcal{L}$.
2. For any $I, J \in \mathcal{L}$, $\{I, I^*\}$ lies on J or J^* .
3. If there is a sequence $\{I_n\}_{n=1}^\infty$ on $\mathcal{L}$ such that for any $n \in \mathbb{N}$, $I_n \subseteq I_{n+1}$, and $\bigcup_{n=1}^\infty I_n$ is a nondegenerate open interval in S^1 , then $\bigcup_{n=1}^\infty I_n \in \mathcal{L}$.

The third condition will imply the closedness of lamination systems on S^1 .

We define leaves and gaps on a lamination system $\mathcal{L}$ as follows. A subset $\mathcal{G}$ of $\mathcal{L}$ is a *leaf* of $\mathcal{L}$ if $\mathcal{G} = \{I, I^*\}$ for some $I \in \mathcal{L}$. We denote such a leaf $\mathcal{G}$ by $\ell(I)$. Likewise, a subset $\mathcal{G}$ of $\mathcal{L}$ is a *gap* of $\mathcal{L}$ if $\mathcal{G}$ satisfies following conditions:

1. Elements of $\mathcal{G}$ are pairwise disjoint , and
2. for each $I \in \mathcal{L}$, there is a J in $\mathcal{G}$ on which $\ell(I)$ lies.

By the second condition on gaps, every gap is nonempty. Obviously, a leaf is also a gap with two elements. Hence, we say that a gap is a *non-leaf gap* if it is not a leaf. We denote $S^1 - \bigcup_{I \in \mathcal{G}} I$ by $v(\mathcal{G})$ and call it a *vertex set* of $\mathcal{G}$ or an *end points set* of $\mathcal{G}$. Each element of a vertex set is called a *vertex* or an *end point*. Note that in general, a vertex set need not be a discrete subset of S^1 .

Let us talk about the closedness of lamination systems. First, we must have the concept of convergence in lamination systems. The third condition of lamination systems allows us define the notion of the limit of a sequence of leaves.

DEFINITION 2.6 ([5]). *Let $\mathcal{L}$ be a lamination system, and $\{\ell_n\}_{n=1}^\infty$ be a sequence of leaves in $\mathcal{L}$. Let J be a nondegenerate open interval. We say that $\{\ell_n\}_{n=1}^\infty$ converges to J if there is a sequence $\{I_n\}_{n=1}^\infty$ in $\mathcal{L}$ such that for each $n \in \mathbb{N}$, $\ell_n = \ell(I_n)$, and*

$$J \subseteq \liminf I_n \subseteq \limsup I_n \subseteq \overline{J}.$$

We denote this by $\ell_n \rightarrow J$.

This definition is symmetric in the following sense.

PROPOSITION 2.7 ([5]). *Let $\mathcal{L}$ be a lamination system and $\{\ell_n\}_{n=1}^\infty$ be a sequence of leaves in $\mathcal{L}$. Let J be a nondegenerate open interval. Suppose that there is a sequence $\{I_n\}_{n=1}^\infty$ in $\mathcal{L}$ such that for each $n \in \mathbb{N}$, $\ell_n = \ell(I_n)$ and*

$$J \subseteq \liminf I_n \subseteq \limsup I_n \subseteq \overline{J}.$$

Then

$$J^* \subseteq \liminf I_n^* \subseteq \limsup I_n^* \subseteq \overline{J^*}.$$

Since the third condition on lamination systems guarantees that the limit of an ascending sequence on a lamination system is in the lamination system, we need to consider descending sequences to say about limits of arbitrary sequences on lamination systems. The following lemma implies the closedness of descending sequences in a lamination system $\mathcal{L}$.

LEMMA 2.8 ([5]). *Let $\{I_n\}_{n=1}^\infty$ be a sequence on a lamination system $\mathcal{L}$ such that $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$, and $\bigcup_{n=1}^\infty I_n^* = J \in \mathcal{L}$. Then $\text{Int}\left(\bigcap_{n=1}^\infty I_n\right) = J^* \in \mathcal{L}$.*

With this lemma, the following proposition shows the closedness of lamination systems.

PROPOSITION 2.9 ([5]). *If a sequence $\{\ell_n\}_{n=1}^\infty$ of leaves of a lamination system $\mathcal{L}$ converges to a nondegenerate open interval J , then $J \in \mathcal{L}$.*

Moreover, by Proposition 2.7 we can prove that if $\ell_n \rightarrow J$, then $\ell_n \rightarrow J^*$, and so $J^* \in \mathcal{L}$. Due to that, we can make a concept of the limit of a sequence of leaves.

DEFINITION 2.10 ([5]). *Let $\mathcal{L}$ be a lamination system, and $\{\ell_n\}_{n=1}^\infty$ be a sequence of leaves on $\mathcal{L}$. Let ℓ be a leaf of $\mathcal{L}$. Then, we say that $\{\ell_n\}_{n=1}^\infty$ converges to ℓ if $\ell_n \rightarrow I$ for some $I \in \ell$ and we denote it by $\ell_n \rightarrow \ell$.*

REMARK 2.11. Let Λ be a lamination of S^1 . Then the set

$$\mathcal{L} = \{(a, b)_{S^1} : \{a, b\} \in \Lambda\}$$

is a lamination system. Conversely, for a given lamination system $\mathcal{L}$, the set

$$\Lambda = \{v(\ell(I)) : I \in \mathcal{L}\}.$$

Note that if a sequence $\{\ell_n\}_{n=1}^\infty$ of leaves of $\mathcal{L}$ converges to ℓ , then $\{v(\ell_n)\}_{n=1}^\infty$ is also a sequence of leaves of Λ and converges to $v(\ell)$.

2.7. *Lamination systems and laminar groups.* A homeomorphism f on S^1 is *orientation preserving* if for each positively oriented triple $(x, y, z) \in (S^1)^3 - \Delta_3(S^1)$, $(f(x), f(y), f(z))$ is positively oriented. $\text{Homeo}^+(S^1)$ is the set of orientation preserving homeomorphisms on S^1 . For each $f \in \text{Homeo}^+(S^1)$, we denote the set of fixed points of f by Fix_f .

Let G be a subgroup of $\text{Homeo}^+(S^1)$. The group G is called a *laminar group* if there is a lamination system $\mathcal{L}$ such that for any $g \in G$ and $I \in \mathcal{L}$, $g(I) \in \mathcal{L}$. We say a lamination system $\mathcal{L}$ to be *G-invariant* if for each $g \in G$ and $I \in \mathcal{L}$, $g(I) \in \mathcal{L}$. Note that on a G -invariant lamination system, every gap is mapped to a gap and the converging property is preserved under the given G -action.

Let $\mathcal{L}$ be a lamination system on S^1 . On a lamination system $\mathcal{L}$ on S^1 , a gap $\mathcal{G}$ with $|v(\mathcal{G})| < \infty$ is called an *ideal polygon*. In particular, an ideal polygon is called a *non-leaf ideal polygon* if it is not a non-leaf gap. We say that a lamination system $\mathcal{L}$ is *very full* if every gap of $\mathcal{L}$ is an ideal polygon. Note that for an ideal polygon $\mathcal{G}$, since $v(\mathcal{G})$ is a finite set, we can write $v(\mathcal{G}) = \{x_1, x_2, \dots, x_n\}$ where $|v(\mathcal{G})| = n$, and $(x_1, x_2, \dots, x_n)$ is a positively oriented n -tuple. Moreover, we can represent $\mathcal{G} = \{(x_1, x_2)_{S^1}, (x_2, x_3)_{S^1}, \dots, (x_{n-1}, x_n)_{S^1}, (x_n, x_1)_{S^1}\}$.

Let $\{I, J\}$ be a subset of $\mathcal{L}$. Then, $\{I, J\}$ is called a *distinct pair* if $I \cap J = \emptyset$, and $\{I, J\}$ is not a leaf. A distinct pair $\{I, J\}$ is *separated* if there is a non-leaf gap $\mathcal{G}$ such that $I \subseteq K$ and $J \subseteq L$ for some $K, L \in \mathcal{G}$, not necessarily $K \neq L$. The lamination system $\mathcal{L}$ is *totally disconnected* if every distinct pair is separated.

Let $I \in \mathcal{L}$ and $\{\ell_n\}_{n=1}^\infty$ be a sequence of leaves of $\mathcal{L}$. Then we call $\{\ell_n\}_{n=1}^\infty$ an *I-side sequence* if for all $n \in \mathbb{N}$, $I \notin \ell_n$, and ℓ_n lies on I , and $\ell_n \rightarrow I$. And we say that I is *isolated* if there is no I -side sequence on $\mathcal{L}$. Moreover, a leaf ℓ is *isolated* if each element of ℓ is isolated.

For a lamination system $\mathcal{L}$, let $E(\mathcal{L}) = \bigcup_{I \in \mathcal{L}} v(\ell(I))$ and we call it the *end points set* of $\mathcal{L}$. When we consider several lamination systems at the same time, considering transversality is useful.

DEFINITION 2.12. Two lamination systems $\mathcal{L}_1$ and $\mathcal{L}_2$ on S^1 are *transverse* if $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$. They are said to be *strongly transverse* if $E(\mathcal{L}_1) \cap E(\mathcal{L}_2) = \emptyset$.

2.8. *Some technical lemmas.* The following structure is useful to deal with the configuration of leaves and to argue the existence of a gap.

DEFINITION 2.13 ([5]). Let $\mathcal{L}$ be a lamination system on S^1 and I be a nondegenerate open interval. Then, for $p \in I$, we define C_p^I as the set $C_p^I = \{J \in \mathcal{L} : p \in J \subseteq I\}$.

We can see that C_p^I is totally ordered by the set inclusion [5]. The following lemma gives good descending or ascending sequences of C_p^I to make sure the existence of the maximal or minimal interval in C_p^I .

LEMMA 2.14 ([5]). Let $\mathcal{L}$ be a lamination system on S^1 , and I be a nondegenerate open interval. Let x be an element of I . Assume that C_x^I is nonempty. Then there is a sequence $\{J_n\}_{n=1}^\infty$ on C_x^I such that for all $n \in \mathbb{N}$, $J_n \subseteq J_{n+1}$, and $\bigcup_{n=1}^\infty J_n = \bigcup_{K \in C_x^I} K$. Also, there is a sequence $\{K_n\}_{n=1}^\infty$ on C_x^I

such that for all $n \in \mathbb{N}$, $K_{n+1} \subseteq K_n$, and $\bigcap_{n=1}^\infty K_n = \bigcap_{K \in C_x^I} K$.

The following lemma shows that the one-sided isolation condition is a sufficient condition for a gap to exist.

LEMMA 2.15 ([5]). Let $\mathcal{L}$ be a lamination system on S^1 and $I \in \mathcal{L}$. Suppose that I^* is isolated. Then there is a non-leaf gap $\mathcal{G}$ such that $I \in \mathcal{G}$.

Like leaves, there is a kind of unlinkedness for gaps.

LEMMA 2.16 ([5]). Let $\mathcal{L}$ be a lamination system on S^1 , and $\mathcal{G}, \mathcal{G}'$ be two gaps with $|\mathcal{G}|, |\mathcal{G}'| \geq 2$. Then, $\mathcal{G} = \mathcal{G}'$ or there are I in $\mathcal{G}$, and I' in $\mathcal{G}'$ such that $I^* \subseteq I'$, and for all $J \in \mathcal{G}$, $\ell(J)$ lies on I' , and for all $J' \in \mathcal{G}'$, $\ell(J')$ lies on I .

Let $\mathcal{L}$ be a lamination system on S^1 . $\mathcal{L}$ is called *dense* if $E(\mathcal{L})$ is a dense subset of S^1 . Let $p \in S^1$ and $\mathcal{L}$ be a dense lamination system. Suppose that there is a sequence $\{I_n\}_{n=1}^\infty$ on $\mathcal{L}$ such that for all $n \in \mathbb{N}$, $I_{n+1} \subseteq I_n$, and $\bigcap_{n \in \mathbb{N}} I_n = \{p\}$. We call such a sequence a *rainbow* at p . In [3], it is observed that very full laminations have abundant rainbows.

THEOREM 2.17 ([3],[5]). Let $\mathcal{L}$ be a very full lamination system of S^1 . Then, for each $p \in S^1$, either $p \in E(\mathcal{L})$ or p has a rainbow, and these two possibilities are mutually exclusive. Moreover, $E(\mathcal{L})$ is dense in S^1 .

Totally disconnectedness is an important condition since it guarantees the existence of a gap. The transversality condition makes lamination systems be totally disconnected.

LEMMA 2.18 ([1],[5]). If two dense lamination systems are strongly transverse, then each of the lamination systems is totally disconnected.

3. LAMINAR GROUPS WITH TWO LAMINATION SYSTEMS AND FIXED POINT SETS

Analyzing fixed point sets of elements of laminar groups is useful to study laminar groups. Let g be an element of $\text{Homeo}^+(S^1)$. If g is not trivial, the fixed point set Fix_g is a proper closed subset of S^1 . Hence, the complement of the fixed set Fix_g is an at most countable disjoint union of open intervals. Therefore, we can see that the action of g is just translations on the intervals of the complement of Fix_g . Fortunately, unlike laminar groups with one invariant lamination system, in the case where laminar groups with more than two lamination systems, the fixed point set of each element is a finite set [5].

DEFINITION 3.1 ([3]). *A nontrivial element g in $\text{Homeo}^+(S^1)$ is said to be :*

1. elliptic if $|\text{Fix}_g| = 0$,
2. parabolic if $|\text{Fix}_g| = 1$, and
3. hyperbolic if $|\text{Fix}_g| = 2$

A point p of S^1 is called a *cuspid point* of G if there is a parabolic element g in G fixing p .

REMARK 3.2 ([3]). *Let G be a laminar group and $\mathcal{L}$ a G -invariant lamination system. Suppose that $\mathcal{L}$ is very full. Given a cuspid point, there are infinitely many leaves of $\mathcal{L}$ having the cuspid point as an end point.*

Now we introduce some classes of laminar groups to study laminar groups having more than two lamination systems.

DEFINITION 3.3 ([3]). *Let G be a subgroup of $\text{Homeo}^+(S^1)$ and let $\mathcal{C} = \{\mathcal{L}_\alpha\}_{\alpha \in J}$ be a collection of G -invariant very full lamination systems, where J is an index set. Then $\mathcal{C}$ is called pants-like if the lamination systems in $\mathcal{C}$ are pairwise transverse and for any $\alpha \neq \beta \in J$, every element of $E(\mathcal{L}_\alpha) \cap E(\mathcal{L}_\beta)$ is a cuspid point of G . A subgroup G of $\text{Homeo}^+(S^1)$ is said to be pants-like COL_n for some $n \in \mathbb{N}$ if it admits a pants-like collection with n elements.*

A subgroup $G < \text{Homeo}^+(S^1)$ is *Möbius-like* if each element of G is conjugate in $\text{Homeo}^+(S^1)$ into $\text{PSL}_2(\mathbb{R})$. It is *Möbius* if G itself is conjugate in $\text{Homeo}^+(S^1)$ into $\text{PSL}_2(\mathbb{R})$.

We recall the following theorem.

THEOREM 3.4 ([3]). *Let G be a pants-like COL_3 group. Then every non-trivial element of G is conjugate in $\text{Homeo}^+(S^1)$ to an element of $\text{PSL}_2(\mathbb{R})$. Moreover, this conjugacy preserves type: elliptic elements are conjugate to finite-order elliptics in $\text{PSL}_2(\mathbb{R})$, parabolic elements to parabolics, and hyperbolic elements to hyperbolics. Furthermore, G is discrete in $\text{Homeo}^+(S^1)$ under the compact-open topology*

Thus every pants-like COL_3 group is Möbius-like and discrete. In Section 4, we promote this Möbius-like property to the Möbius property by proving Theorem A.

In Lemma 2.18, transversality implies total disconnectedness. A similar statement holds for pants-like COL_2 groups:

PROPOSITION 3.5. *Let G be a pants-like COL_2 group and $\mathcal{C} = \{\mathcal{L}_1, \mathcal{L}_2\}$ be a pants-like collection for G . Then both $\mathcal{L}_1$ and $\mathcal{L}_2$ are totally disconnected.*

PROOF. First we show that $\mathcal{L}_1$ is totally disconnected. Assume that there is a $\{I, J\}$ be a distinct pair of $\mathcal{L}_1$. We claim that if there is a point p in $I^* \cap J^*$ such that $C_p^{I^*} \cap C_p^{J^*}$ is nonempty in $\mathcal{L}_1$, then $\{I, J\}$ is separated. Let K be the union of elements of $C_p^{I^*} \cap C_p^{J^*}$. Then K is a nondegenerate open interval such that $C_p^K = C_p^{I^*} \cap C_p^{J^*}$. Moreover, by Lemma 2.14 and the closedness of $\mathcal{L}_1$, we have that $K \in \mathcal{L}_1$. Now we show that K^* is isolated. Assume that there is a K^* -side sequence $\{\ell_n\}_{n=1}^\infty$ of $\mathcal{L}_1$. Then we can take a sequence $\{I_n\}_{n=1}^\infty$ in $\mathcal{L}_1$ so that $\ell_n = \ell(I_n)$ for all $n \in \mathbb{N}$ and

$$K^* \subseteq \liminf I_n \subseteq \limsup I_n \subseteq \overline{K^*}.$$

Choose a point p_I in I and a point p_J in J . Since $\{p_I, p_J\} \subseteq K^*$ and $K^* \subseteq \liminf I_n$, there is a number N in $\mathbb{N}$ such that $\{p_I, p_J\} \subseteq \bigcap_{k=N}^\infty I_k$, that is for all n with $N \leq n$, $\{p_I, p_J\} \subseteq I_n$. Similarly choose a point p_K in K . Since $\limsup I_n \subseteq \overline{K^*}$ and $p_K \notin K^c = \overline{K^*}$, there is a number M in $\mathbb{N}$ such that $p_K \notin \bigcup_{k=M}^\infty I_k$, that is for all $n \in \mathbb{N}$ with $M \leq n$, $p_K \notin I_n$. Fix n_0 with $\max\{N, M\} < n_0$. Now we consider the unlinkedness of I and I_{n_0} . Since $p_I \in I \cap I_{n_0}$ and $I \cap I_{n_0}$ is nonempty, there are three possible cases: $I \subseteq I_{n_0}$; $I_{n_0} \subseteq I$; $I_{n_0}^* \subseteq I$. If $I_{n_0} \subseteq I$, then $p_J \notin I_{n_0}$ as $p_J \notin I$. This is a contradiction. If $I_{n_0}^* \subseteq I$, then $p_K \in \overline{I_{n_0}^*} \subseteq \overline{I}$ but $p_K \in K \subseteq (I^* \cap J^*) \subseteq I^* = (\overline{I})^c$. This is also a contradiction. Hence $I \subseteq I_{n_0}$ is the only possible case. Likewise $J \subseteq I_{n_0}$. Moreover since $\ell(I_{n_0})$ lies in K^* , $I_{n_0} \subseteq K^*$ or $I_{n_0}^* \subseteq K^*$. However, if $I_{n_0}^* \subseteq K^*$, then $p_K \in K \subseteq I_{n_0}$ so this is in contradiction to that $p_K \notin I_{n_0}$. So $I_{n_0} \subseteq K^*$ is the only possible case and then $p \in I_{n_0}^*$. Therefore $I_{n_0}^* \subseteq C_p^{I^*} \cap C_p^{J^*}$ and $K \subseteq I_{n_0}^*$. However since $\{\ell_n\}_{n=1}^\infty$ is a K^* -side sequence, $\ell(I_{n_0}) \neq \ell(K)$ and so $K \subsetneq I_{n_0}^*$. This contradicts the maximality of K in C_p^K . Therefore K^* is isolated. By Lemma 2.15, there is a non-leaf gap $\mathcal{G}$ of $\mathcal{L}_1$ such that $K \in \mathcal{G}$. Then since $\mathcal{G}$ is a gap, there are two elements U_I and U_J in $\mathcal{G}$ such that $\ell(I)$ lies on U_I and $\ell(J)$ lies on U_J . If $I^* \subseteq U_I$, then $K \subseteq I^* \subseteq U_I$ so $K = I^* = U_I$. However $p_J \in I^*$ and $p_J \notin K$. Hence this is a contradiction. Therefore $I \subseteq U_I$. Similarly we can get that $J \subseteq U_J$. Thus the distinct pair $\{I, J\}$ is separated so the claim is proved.

Now observe that the intersection $I^* \cap J^*$ is a nonempty open set as $\{I, J\}$ is not a leaf of $\mathcal{L}_1$. Then since $E(\mathcal{L}_2)$ is dense by Lemma 2.17, there is a point p in $I^* \cap J^* \cap E(\mathcal{L}_2)$. If $p \notin E(\mathcal{L}_1)$, then by Lemma 2.17, there is a rainbow $\{J_n\}_{n=1}^\infty$ at p in $\mathcal{L}_1$. Therefore we can show that $C_p^{I^*} \cap C_p^{J^*}$ is nonempty in this case, proving that $J_n \in C_p^{I^*} \cap C_p^{J^*}$ for some $n \in \mathbb{N}$. Hence by the claim, $\{I, J\}$ is separated. If not, p is in $E(\mathcal{L}_1) \cap E(\mathcal{L}_2)$ and is a cusp point of G since $\{\mathcal{L}_1, \mathcal{L}_2\}$ is a pants-like collection. Then there is a parabolic element g of G fixing p and there is a leaf ℓ of $\mathcal{L}_1$ having p as an end point. Now we take a nondegenerate open interval U which is a neighborhood of p so that $U \cap (I \cup J) = \emptyset$. Since g is parabolic, $g^L(\ell)$ lies on U for some $L \in \mathbb{N}$. Hence there is an element V in $g^L(\ell)$ such that $V \subseteq U$ and we pick a point q in V . Then $C_q^{I^*} \cap C_q^{J^*}$ is nonempty as $V \in C_q^{I^*} \cap C_q^{J^*}$. By the claim, $\{I, J\}$ is separated. Thus $\mathcal{L}_1$ is totally disconnected. Likewise $\mathcal{L}_2$ is also totally disconnected. $\square$

COROLLARY 3.6. *Let G be a pants-like COL_2 group and $\mathcal{C} = \{\mathcal{L}_1, \mathcal{L}_2\}$ be a pants-like collection for G . Then each of $\mathcal{L}_1$ and $\mathcal{L}_2$ has a non-leaf ideal polygon.*

PROOF. Let $\mathcal{L}$ be an element of $\mathcal{C}$. Since $\mathcal{L}$ is nonempty, there is an element I of $\mathcal{L}$ and because $E(\mathcal{L})$ is dense in S^1 , there is a point p in $I \cap E(\mathcal{L})$. Hence there is a leaf ℓ having p as an end point and then since ℓ lies on I , there is an element J of ℓ such that $J \subseteq I$. Note that $I \notin \ell$ since $p \notin v(\ell(I))$. Therefore $\{J, I^*\}$ is a distinct pair and so by Proposition 3.5, there is a non-leaf gap $\mathcal{G}$ of $\mathcal{L}$ which makes $\{J, I^*\}$ be separated. Thus $\mathcal{G}$ is a non-leaf ideal polygon since $\mathcal{L}$ is very full. $\square$

COROLLARY 3.7. *Let G be a pants-like COL_2 group and $\mathcal{C} = \{\mathcal{L}_1, \mathcal{L}_2\}$ be a pants-like collection for G . If G is Möbius-like, then G is a discrete subgroup of $\text{Homeo}^+(S^1)$ with the compact-open topology.*

PROOF. Let $\mathcal{L}$ be an element of $\mathcal{C}$. Then by Corollary 3.6, there is a non-leaf ideal polygon $\mathcal{G}$ of $\mathcal{L}$. Then we write

$$\mathcal{G} = \{(x_1, x_2)_{S^1}, (x_2, x_3)_{S^1}, \dots, (x_m, x_1)_{S^1}\}$$

for some $m \in \mathbb{N}$ with $2 < m$. We choose a number ϵ in $\mathbb{R}$ so that $0 < \epsilon < d_{\mathbb{C}}(x_r, x_s)/3$ for all $r, s \in \mathbb{Z}_m$.

Now we assume that G is not discrete, that is there is a sequence $\{g_n\}_{n=1}^\infty$ of nontrivial distinct elements of G converging to the identity map id_{S^1} under the compact-open topology. Then for each $r \in \mathbb{Z}_m$, let U_r be the ϵ -neighborhood of x_r in S^1 , i.e. $U_r = \{z \in S^1 : d_{\mathbb{C}}(x_r, z) < \epsilon\}$. Fix $r \in \mathbb{Z}_m$. Since $\{x_r\}$ is compact and $\{g_n\}_{n=1}^\infty$ converges to id_{S^1} , there is a number N_r in $\mathbb{N}$ such that $g_n(x_r) \subseteq U_r$ whenever $N_r < n$.

Then we choose a number N in $\mathbb{N}$ which is greater than $\max\{N_1, N_2, \dots, N_m\}$. Then for each $r \in \mathbb{Z}_m$, $g_N(x_r) \in U_r$. On the other hand, by Lemma 2.16,

$g_N(\mathcal{G}) = \mathcal{G}$ or $g_N(\{x_1, x_2, \dots, x_m\}) \subseteq [x_i, x_{i+1}]_{S^1}$ for some $i \in \mathbb{Z}_m$. If $g_N(\{x_1, x_2, \dots, x_m\}) \subseteq [x_i, x_{i+1}]_{S^1}$ for some $i \in \mathbb{Z}_m$, then $\epsilon < d_{\mathbb{C}}(x_{i+2}, g_N(x_{i+2}))$ so this is in contradiction to that $g_N(x_{i+2}) \in U_{i+2}$. If $g_N(\mathcal{G}) = \mathcal{G}$, then by the choice of N , g_N must fix the end points of $\mathcal{G}$ and so g_N has at least three fixed points since $\mathcal{G}$ is not a leaf. However because $\mathcal{G}$ is Möbius-like, g_N must be the identity map but this is a contradiction since g_N was taken to be nontrivial. Thus G is a discrete subgroup of $\text{Homeo}^+(S^1)$. $\square$

COROLLARY 3.8. *Every pants-like COL_3 group is a discrete subgroup of $\text{Homeo}^+(S^1)$ with the compact-open topology.*

PROOF. It follows from Theorem 3.4 and Corollary 3.7. $\square$

Hence we could expect the following lemma.

LEMMA 3.9. *Let G be a pants-like COL_2 group and $\mathcal{C} = \{\mathcal{L}_1, \mathcal{L}_2\}$ be a pants-like collection for G . If G is Möbius-like, then hyperbolic elements can not share fixed points with parabolic elements, that is for any hyperbolic element g and parabolic element h of G , $\text{Fix}_g \cap \text{Fix}_h = \emptyset$.*

PROOF. Let $\mathcal{C} = \{\mathcal{L}_1, \mathcal{L}_2\}$ be a pants-like collection for G . Assume that there is a hyperbolic element g and a parabolic element h of G such that $\text{Fix}_g \cap \text{Fix}_h = \{p\}$. We denote the other fixed point of g by q . For each $i \in \mathbb{Z}_2$, there is a leaf ℓ^i of $\mathcal{L}_i$ since p is a cusp point.

Fix $i \in \mathbb{Z}_2$. We claim that there is a leaf ℓ_∞^i of $\mathcal{L}_i$ such that $v(\ell_\infty^i) = \text{Fix}_g$. If $q \in v(\ell^i)$, then $\ell_\infty^i = \ell^i$. If not, there is an element I^i in ℓ^i containing the point q . Consider the sequences $\{g^n(I^i)\}_{n=1}^\infty$ and $\{g^{-n}(I^i)\}_{n=1}^\infty$. When $g(I^i) \subsetneq I^i$, $I_\infty^i = \text{Int}(\bigcap_{n=1}^\infty g^n(I^i))$ is in $\mathcal{L}_i$ by Lemma 2.8 and $v(\ell(I_\infty^i)) = \{p, q\}$.

Hence $\ell_\infty^i = \ell(I_\infty^i)$. Similarly if $I^i \subsetneq g(I^i)$, then $I_\infty^i = \text{Int}(\bigcup_{n=1}^\infty g^{-n}(I^i))$ is in $\mathcal{L}_i$ and $v(\ell(I_\infty^i)) = \{p, q\}$. Therefore $\ell_\infty^i = \ell(I_\infty^i)$. However this implies that $\mathcal{L}_1 \cap \mathcal{L}_2 \neq \emptyset$ and so it is a contradiction since $\mathcal{L}_1$ and $\mathcal{L}_2$ are transverse. Thus we are done. $\square$

4. PANTS-LIKE COL_3 GROUPS ARE CONVERGENCE GROUPS

In this section, we show that every pants-like COL_3 group is a convergence group. Let G be a subgroup of $\text{Homeo}^+(S^1)$. We say a sequence of distinct elements of G to have the *convergence property* if there exist two points a and b in S^1 and a subsequence $\{g_n\}_{n=1}^\infty$ such that the sequence $\{g_n|_{S^1 - \{b\}}\}_{n=1}^\infty$ of maps restricted to $S^1 - \{b\}$ converges uniformly to the constant map from $S^1 - \{b\}$ to $\{a\}$ on every compact subset of $S^1 - \{b\}$. The group G is called a *convergence group* if every sequence of distinct elements of G has the convergence property.

REMARK 4.1. Let $\{g_n\}_{n=1}^\infty$ be a sequence of distinct elements of a subgroup $G < \text{Homeo}^+(S^1)$. Then $\{g_n\}_{n=1}^\infty$ has the convergence property if and only if $\{g_n^{-1}\}_{n=1}^\infty$ has the convergence property.

In the proofs of several theorems related to the convergence property, we frequently take subsequences. Hence for clarity we sometimes think of a sequence $\{s_n\}_{n=1}^\infty$ in a set S as a function s from $\mathbb{N}$ to S so that $s(n) = s_n$ for all $n \in \mathbb{N}$. Also, in this sense, a subsequence $\{s_{n_k}\}_{k=1}^\infty$ of the sequence has a function α on $\mathbb{N}$ increasing strictly so that $\alpha(k) = n_k$ for all $k \in \mathbb{N}$. Let α be a strictly increasing function on $\mathbb{N}$. A strictly increasing function γ on $\mathbb{N}$ is called a *subindex* of α if there is a strictly increasing function β on $\mathbb{N}$ such that $\gamma = \alpha \circ \beta$.

In the pants-like COL_3 case, it is not so easy to directly show that a given sequence of distinct elements has the convergence property. Theorem 4.2 gives another equivalent condition for convergence groups which makes the situation be slightly easier. Let $\Theta(S^1)$ be the set of all subsets of S^1 of cardinality 3 or equivalently $\Theta(S^1)$ is the quotient space of $(S^1)^3 - \Delta_3(S^1)$ by the symmetric group S_3 permuting coordinates (following the notation in [9]). We say that a sequence $\{h_n\}_{n=1}^\infty$ of distinct elements of G is *properly discontinuous on triples* if for any two compact subsets K and L of $\Theta(S^1)$, the set $\{m \in \mathbb{N} : h_m(K) \cap L \neq \emptyset\}$ is finite.

THEOREM 4.2 ([9]). Let G be a subgroup of $\text{Homeo}^+(S^1)$ and $\{g_n\}_{n=1}^\infty$ be a sequence of distinct elements of G . If $\{g_n\}_{n=1}^\infty$ is properly discontinuous on triples, then every subsequence of $\{g_n\}_{n=1}^\infty$ has the convergence property. The converse also holds.

The strategy for proving that pants-like COL_3 groups are convergence groups is first to assume that there is a sequence $\{g_n\}_{n=1}^\infty$ not having the convergence property. By Theorem 4.2, the sequence is not properly discontinuous on triples. Then there are two compact subsets K and L of $\Theta(S^1)$ such that the set $\{m \in \mathbb{N} : g_m(K) \cap L \neq \emptyset\}$ is infinite. Namely, there is a strictly increasing function α on $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $g_{\alpha(n)}(K) \cap L \neq \emptyset$.

LEMMA 4.3. Let X be a topological space and $\{g_n\}_{n=1}^\infty$ a sequence of distinct homeomorphisms on X . Suppose that there are two sequentially compact subsets K and L of X such that $g_n(K) \cap L \neq \emptyset$ for infinitely many $n \in \mathbb{N}$. Then there is a sequence $\{x_k\}_{k=1}^\infty$ in K and a subsequence $\{g_{n_k}\}_{k=1}^\infty$ such that $\{x_k\}_{k=1}^\infty$ converges to a point x in K and $\{g_{n_k}(x_k)\}_{k=1}^\infty$ converges to a point x' in L .

Now by Lemma 4.3, there is a sequence $\{x_n\}_{n=1}^\infty$ in K and a subindex β of α such that the sequence $\{x_n\}_{n=1}^\infty$ converges to a point x_∞ in K and the sequence $\{g_{\beta(n)}(x_n)\}_{n=1}^\infty$ converges to a point y_∞ in L . Set $x_n = (x_n^1, x_n^2, x_n^3)$, $x_\infty = (x_\infty^1, x_\infty^2, x_\infty^3)$ and $y_\infty = (y_\infty^1, y_\infty^2, y_\infty^3)$. In other words we have that for each $i \in \mathbb{Z}_3$, the sequence $\{x_n^i\}_{n=1}^\infty$ converges to x_∞^i and the sequence

$\{g_{\beta(n)}(x_n^i)\}_{n=1}^\infty$ converges to y_∞^i . Then, using lamination systems, we show that the subsequence $\{g_{\beta(n)}\}_{n=1}^\infty$ has the convergence property. This is in contradiction to the assumption.

Therefore we will focus on analyzing the sequences $\{x_n^i\}_{n=1}^\infty$ and $\{g_{\beta(n)}\}_{n=1}^\infty$. To control the behavior of sequences of homeomorphisms, we need to consider a slightly generalized concept of rainbows.

DEFINITION 4.4. *Let p be a point in S^1 and $\mathcal{L}$ a lamination system. Then a sequence $\{I_n\}_{n=1}^\infty$ in $\mathcal{L}$ is called a quasi-rainbow at the point p if for each $n \in \mathbb{N}$, $I_{n+1} \subseteq I_n$ and $\bigcap_{n=1}^\infty \overline{I_n} = \{p\}$.*

Every rainbow at p is also a quasi-rainbow at p . Let $\mathcal{L}$ be a lamination system and $\{I_n\}_{n=1}^\infty$ a rainbow at $p \in S^1$. Then $\{p\} = \bigcap_{n=1}^\infty I_n \subseteq \bigcap_{n=1}^\infty \overline{I_n}$.

Assume that there is a element q such that $q \in \bigcap_{n=1}^\infty \overline{I_n} - \{p\}$. Then $q \in \overline{I_n}$ for

all $n \in \mathbb{N}$. On the other hand, since $\{p\} = \bigcap_{n=1}^\infty I_n$ and $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$,

there is N such that $q \notin I_n$ for all $n > N$. Therefore, for each $n > N$, $q \in \partial I_n$ so one of $(p, q)_{S^1}$ or $(q, p)_{S^1}$ is contained in I_n . If there is M and M' such that $M, M' > N$, $(p, q)_{S^1} \subseteq I_M$, and $(q, p)_{S^1} \subseteq I_{M'}$, then one of I_M and $I_{M'}$ must be $(q, q)_{S^1}$ since it is either $I_M \subseteq I_{M'}$ or $I_{M'} \subseteq I_M$. It is a contradiction since $(q, q)_{S^1} \notin \mathcal{L}$. Therefore, if $(p, q)_{S^1} \subseteq I_{N'}$ for some $N' > N$, then $(p, q)_{S^1} \subseteq I_n$

for all $n > N'$ but $\{p\} \subsetneq [p, q)_{S^1} \subseteq \bigcap_{n=1}^\infty I_n$. It is a contradiction. Similarly,

if $(q, p)_{S^1} \subseteq I_{N''}$ for some $N'' > N$, then $(q, p)_{S^1} \subseteq I_n$ for all $n > N''$ but $\{p\} \subsetneq (q, p]_{S^1} \subseteq \bigcap_{n=1}^\infty I_n$. It is also a contradiction. Thus the rainbow $\{I_n\}_{n=1}^\infty$

is a quasi-rainbow at p .

DEFINITION 4.5. *Let p be a point in S^1 and $\{p_n\}_{n=1}^\infty$ a sequence in S^1 converging to the point p . If $\varphi(p, p_1, p_2) \neq 0$ and $\varphi(p, p_1, p_2) = \varphi(p, p_n, p_{n+1})$ for all $n \in \mathbb{N}$, then we say that the sequence $\{p_n\}_{n=1}^\infty$ converges monotonically to the point p .*

Let $\{p_n\}_{n=1}^\infty$ be a sequence in S^1 converging monotonically to a point p in S^1 . Observe that for each $n \in \mathbb{N}$, $\varphi(p, p_n, p_m) = \varphi(p, p_1, p_2)$ for all $m \in \mathbb{N}$ with $n < m$. Hence every subsequence also converges monotonically. If $\varphi(p, p_1, p_2) = -1$ and $\varphi(p, p_2, p_1) = 1$, then $p_n \in (p, p_1)_{S^1}$ for all $n \in \mathbb{N}$ with $n > 1$ as $\varphi(p, p_n, p_1) = 1$ for all $n \in \mathbb{N}$ with $n > 1$. Moreover, for each $n \in \mathbb{N}$, $p_m \in (p, p_n)_{S^1}$ for all $m \in \mathbb{N}$ with $n < m$. Then for any n and m in $\mathbb{N}$ with $n < m$, $(p, p_m)_{S^1} \subsetneq (p, p_n)_{S^1}$ and since $\{p_n\}_{n=1}^\infty$ converges to p ,

$\bigcap_{n=1}^{\infty} (p, p_n)_{S^1} = \emptyset$ and $\bigcap_{n=1}^{\infty} [p, p_n]_{S^1} = \{p\}$. This looks like a quasi-rainbow at p . Likewise, if $\varphi(p, p_1, p_2) = 1$, then we can derive similar results.

LEMMA 4.6. *Let G be a pants-like COL_2 group and $\mathcal{C} = \{\mathcal{L}_1, \mathcal{L}_2\}$ be a pants-like collection for G . Let p be a point in S^1 and $\{p_n\}_{n=1}^{\infty}$ be a sequence in S^1 converging monotonically to p . Then there exists a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ and a quasi-rainbow $\{I_k\}_{k=1}^{\infty}$ in $\mathcal{L}_i$ for some $i \in \mathbb{Z}_2$ such that $p_{n_k} \in I_k$ for all $k \in \mathbb{N}$.*

PROOF. First, we consider the case where one of $\mathcal{L}_1$ or $\mathcal{L}_2$ has a rainbow $\{I_n\}_{n=1}^{\infty}$ at p . For each $n \in \mathbb{N}$, there is a number $m(n)$ in $\mathbb{N}$ such that for all m with $m(n) < m$, $p_m \in I_n$ because I_n is a neighborhood of p . Then a subindex α of the identity $\text{id}_{\mathbb{N}}$ on $\mathbb{N}$ can be given by $\alpha(n) = \sum_{k=1}^n m(k)$ so that $p_{\alpha(n)} \in I_n$ for all $n \in \mathbb{N}$.

Then assume that there is no rainbow at p in both $\mathcal{L}_1$ and $\mathcal{L}_2$. By Theorem 2.17, $p \in E(\mathcal{L}_1) \cap E(\mathcal{L}_2)$, so p is a cusp point of G since $\mathcal{C}$ is pants-like. As observed previously, there is a sequence $\{J_n\}_{n=1}^{\infty}$ of nondegenerate open intervals such that $J_{n+1} \subseteq J_n$. Write $\partial J_n = \{p, p_n\}$. Choose a leaf ℓ of $\mathcal{L}_1$ having the point p as an end point. Let q denote the other end point of ℓ .

Because $\bigcap_{n=1}^{\infty} \overline{J_n} = \{p\}$, there is a number N such that $q \notin \overline{J_N}$. Since $S^1 - \{p, q\} = (p, q)_{S^1} \cup (q, p)_{S^1}$, $J_N \subseteq S^1 - \{p, q\}$, and J_N is connected, there is an element I in ℓ such that $\overline{J_N} \cap \overline{I^*} = \{p\}$. Therefore, $p_m \in I$ for all m with $N \leq m$. Since p is a cusp point of G , we can take a parabolic element g in G fixing p so that $g(q) \in I$ and $g(I) \subset I$.

Now we claim that for each $n \in \mathbb{N}$, there is a number $o(n)$ in $\mathbb{N}$ such that $p_m \in g^{n-1}(I)$ for all $m \in \mathbb{N}$ with $o(n) \leq m$. When $n = 1$, $o(1) = N$. Fix a number n . Since $g^n(q) \in \overline{g^{n-1}(I)} \subset S^1 - \{p\}$, there is a number M with $N < M$ such that $g^n(q) \notin \overline{J_M}$. Then one of $g^n(I)$ or $g^n(I^*)$ contains J_M . If $J_M \subseteq g^n(I^*)$, then $J_M \subseteq g^n(I^*) \cap I$, and so $p_m \in g^n(I^*) \cap I$ for all m with $M < m$. Hence the open neighborhood $g^n(I) \cup \{p\} \cup I^*$ of p contains only finitely many points of the sequence $\{p_n\}_{n=1}^{\infty}$ since $g^n(I) \cup \{p\} \cup I^*$ and $g^n(I^*) \cap I$ are disjoint. This contradicts the fact that $\{p_n\}_{n=1}^{\infty}$ converges to p . Therefore $J_M \subseteq g^n(I)$ and $\overline{J_M} \cap \overline{g^n(I^*)} = \{p\}$. Hence $p_m \in g^n(I)$ for all m with $M \leq m$. Thus if we set $o(n+1) = M$, then the claim is proved.

Finally, define a subindex β of $\text{id}_{\mathbb{N}}$ by $\beta(n) = \sum_{k=1}^n o(k)$ so that $p_{\beta(n)} \in g^{n-1}(I)$ for all $n \in \mathbb{N}$. The sequence $\{g^{n-1}(I)\}_{n=1}^{\infty}$ is contained in $\mathcal{L}_1$, and hence it is a quasi-rainbow at p in $\mathcal{L}_1$. Thus we are done. $\square$

DEFINITION 4.7. Let G be a laminar group and $\mathcal{L}$ a G -invariant lamination system. Let $p \in S^1$. Assume that there is a quasi-rainbow $\{I_n\}_{n=1}^\infty$ at p . A sequence $\{(g_n, I_n)\}_{n=1}^\infty$ of elements of $G \times \mathcal{L}$ is called a pre-approximation sequence at p if there is a point x in S^1 such that $g_n(x) \in I_n$ for all $n \in \mathbb{N}$.

REMARK 4.8. Any subsequence of a pre-approximation sequence at p is also a pre-approximation sequence at p .

The following is a key lemma to prove the main result of this chapter. This lemma allows us to properly pass to subsequences.

LEMMA 4.9. Let G be a pants-like COL_2 group and $\mathcal{C} = \{\mathcal{L}_1, \mathcal{L}_2\}$ be a pants-like collection for G . Suppose that we have a sequence $\{x_n\}_{n=1}^\infty$ of elements of S^1 converging to $x \in S^1$ and a sequence $\{g_n\}_{n=1}^\infty$ of distinct elements of G such that $\{g_n(x_n)\}_{n=1}^\infty$ converges to $x' \in S^1$. Then we can have one of the following cases:

1. there is a subsequence $\{g_{n_k}\}_{k=1}^\infty$ such that $g_{n_k}(x) = x'$ for all $k \in \mathbb{N}$;
2. there is a subsequence $\{g_{n_k}\}_{k=1}^\infty$ and a quasi-rainbow $\{I_k\}_{k=1}^\infty$ at x' in $\mathcal{L}_i$ for some $i \in \mathbb{Z}_2$ such that the sequence $\{(g_{n_k}, I_k)\}_{k=1}^\infty$ is a pre-approximation sequence at x' ;
3. there is a subsequence $\{g_{n_k}\}_{k=1}^\infty$ and a quasi-rainbow $\{I_k\}_{k=1}^\infty$ at x in $\mathcal{L}_i$ for some $i \in \mathbb{Z}_2$ such that the sequence $\{(g_{n_k}^{-1}, I_k)\}_{k=1}^\infty$ is a pre-approximation sequence at x .

PROOF. Observe that every convergence sequence in S^1 has a constant subsequence or a subsequence converging monotonically. First we will take a subsequence of $\{x_n\}_{n=1}^\infty$. Then we need to consider two possible cases by the observation. Assume that there is a subindex α of the identity $\text{id}_{\mathbb{N}}$ such that the subsequence $\{x_{\alpha(n)}\}_{n=1}^\infty$ is constant i.e. $x_{\alpha(n)} = x$ for all $n \in \mathbb{N}$. The sequence $\{g_{\alpha(n)}(x_{\alpha(n)})\}_{n=1}^\infty$ still converges to x' . We will take again a subindex of α for $\{g_{\alpha(n)}(x_{\alpha(n)})\}_{n=1}^\infty$. If there is a subindex β of α such that $\{g_{\beta(n)}(x_{\beta(n)})\}_{n=1}^\infty$ is constant, then $g_{\beta(n)}(x) = g_{\beta(n)}(x_{\beta(n)}) = x'$ for all $n \in \mathbb{N}$ and this is the first case of the statement. If there is a subindex β of α such that the sequence $\{g_{\beta(n)}(x_{\beta(n)})\}_{n=1}^\infty$ converges monotonically to x' , then by Lemma 4.6, we can take a subindex γ of β and a quasi-rainbow $\{I_n\}_{n=1}^\infty$ at x' in $\mathcal{L}_i$ for some $i \in \mathbb{Z}_2$ so that $g_{\gamma(n)}(x_{\gamma(n)}) \in I_n$ for all $n \in \mathbb{N}$. Since the sequence $\{x_{\gamma(n)}\}_{n=1}^\infty$ is constant, the sequence $\{(g_{\gamma(n)}, I_n)\}_{n=1}^\infty$ is a pre-approximation sequence at x' and this is the second case of the statement.

Next we consider the case where there is a subindex α of the identity $\text{id}_{\mathbb{N}}$ such that the subsequence $\{x_{\alpha(n)}\}_{n=1}^\infty$ converges monotonically to x . Similarly we will take a subsequence of the sequence $\{g_{\alpha(n)}(x_{\alpha(n)})\}_{n=1}^\infty$. If there is a subindex β of α such that the sequence $\{g_{\beta(n)}(x_{\beta(n)})\}_{n=1}^\infty$ is constant, then $x_{\beta(n)} = g_{\beta(n)}^{-1}(x')$ for all $n \in \mathbb{N}$. Because the sequence $\{x_{\beta(n)}\}_{n=1}^\infty$ still converges monotonically to x , Lemma 4.6 can apply so there is a subindex γ of β and a quasi-rainbow $\{I_n\}_{n=1}^\infty$ at x in $\mathcal{L}_i$ for some $i \in \mathbb{Z}_2$ such that

$g_{\gamma(n)}^{-1}(x') = x_{\gamma(n)} \in I_n$ for all $n \in \mathbb{N}$. Then the sequence $\{(g_{\gamma(n)}^{-1}, I_n)\}_{n=1}^\infty$ is a pre-approximation sequence at x and this is the third case.

Finally we assume that there is a subindex β of α such that the sequence $\{g_{\beta(n)}(x_{\beta(n)})\}_{n=1}^\infty$ converges monotonically to x' . By Lemma 4.6, there is a subindex γ of β and a quasi-rainbow $\{J_n\}_{n=1}^\infty$ at x' in $\mathcal{L}_i$ for some $i \in \mathbb{Z}_2$ such that $g_{\gamma(n)}(x_{\gamma(n)}) \in J_n$ for all $n \in \mathbb{N}$. For brevity, we write $h_n = g_{\gamma(n)}$ and $y_n = x_{\gamma(n)}$. Let us consider the sequence $\{h_n^{-1}(J_n)\}_{n=1}^\infty$.

First, assume that $x \in \overline{h_n^{-1}(J_n)}$ for infinitely many $n \in \mathbb{N}$. Then there is a subindex δ of the identity $\text{id}_{\mathbb{N}}$ such that $x \in \overline{h_{\delta(n)}^{-1}(J_{\delta(n)})} = h_{\delta(n)}^{-1}(\overline{J_{\delta(n)}})$ for all $n \in \mathbb{N}$. Then $h_{\delta(n)}(x) \in \overline{J_{\delta(n)}}$ for all $n \in \mathbb{N}$ and since $\{J_{\delta(n)}\}_{n=1}^\infty$ is a quasi-rainbow at x' , the sequence $\{h_{\delta(n)}(x)\}_{n=1}^\infty$ converges to x' . Then there are two possible cases. First, if there is a subindex ϵ of δ such that $\{h_{\epsilon(n)}(x)\}_{n=1}^\infty$ is a constant sequence, then $\{g_{\gamma \circ \epsilon(n)}\}_{n=1}^\infty$ is in the first case. Second, if we can take a subindex ϵ of δ so that the sequence $\{h_{\epsilon(n)}(x)\}_{n=1}^\infty$ converges monotonically to x' , then by Lemma 4.6, there are a subindex ζ of ϵ and a quasi-rainbow $\{I_n\}_{n=1}^\infty$ at x' in $\mathcal{L}_i$ for some $i \in \mathbb{Z}_2$ such that $h_{\zeta(n)}(x) \in I_n$ for all $n \in \mathbb{N}$. Then the sequence $\{(g_{\gamma \circ \zeta(n)}, I_n)\}_{n=1}^\infty$ is a pre-approximation sequence at x' and this is the second case.

Next, assume that $x \in \overline{h_n^{-1}(J_n)}$ for only finitely many $n \in \mathbb{N}$. Then we can take a subindex δ of $\text{id}_{\mathbb{N}}$ such that $x \notin \overline{h_{\delta(n)}^{-1}(J_{\delta(n)})}$ for all $n \in \mathbb{N}$. Now, we consider the case where there is n_0 in $\mathbb{N}$ such that $y_{\delta(n_0)} \in h_{\delta(n_0)}^{-1}(J_{\delta(n_0)})$ for infinitely many n in $\mathbb{N}$. Then we can take a subindex ζ of δ so that $\zeta(1) = \delta(n_0)$ and $y_{\zeta(1)} \in h_{\zeta(1)}^{-1}(J_{\zeta(1)})$ for all $n \in \mathbb{N}$. Observe that if $y_{\zeta(m)} \in h_{\zeta(m)}^{-1}(J_{\zeta(m)})$ for infinitely many $m \in \mathbb{N}$, then $x \in \overline{h_{\zeta(n)}^{-1}(J_{\zeta(n)})}$, so, by assumption, for each $n \in \mathbb{N}$, $y_{\zeta(m)} \in h_{\zeta(m)}^{-1}(J_{\zeta(m)})$ for only finitely many $m \in \mathbb{N}$. Then there is a number M in $\mathbb{N}$ such that $y_{\zeta(m)} \notin h_{\zeta(1)}^{-1}(J_{\zeta(1)})$ for all $m \in \mathbb{N}$ with $M < m$. Choose a number m with $M < m$. Now there are four cases: $h_{\zeta(m)}^{-1}(J_{\zeta(m)}) \subseteq h_{\zeta(1)}^{-1}(J_{\zeta(1)})$; $h_{\zeta(m)}^{-1}(J_{\zeta(m)})^* \subseteq h_{\zeta(1)}^{-1}(J_{\zeta(1)})$; $h_{\zeta(1)}^{-1}(J_{\zeta(1)}) \subseteq h_{\zeta(m)}^{-1}(J_{\zeta(m)})^*$; $h_{\zeta(1)}^{-1}(J_{\zeta(1)}) \subseteq h_{\zeta(m)}^{-1}(J_{\zeta(m)})$. Since $y_{\zeta(m)} \in h_{\zeta(m)}^{-1}(J_{\zeta(m)}) - h_{\zeta(1)}^{-1}(J_{\zeta(1)})$, the first case that $h_{\zeta(m)}^{-1}(J_{\zeta(m)}) \subseteq h_{\zeta(1)}^{-1}(J_{\zeta(1)})$ is not possible. Also, since $x \in h_{\zeta(m)}^{-1}(J_{\zeta(m)})^* - h_{\zeta(1)}^{-1}(J_{\zeta(1)})$, the second case is not possible. If $h_{\zeta(1)}^{-1}(J_{\zeta(1)}) \subseteq h_{\zeta(m)}^{-1}(J_{\zeta(m)})^*$, then $h_{\zeta(1)}^{-1}(J_{\zeta(1)}) \cap h_{\zeta(m)}^{-1}(J_{\zeta(m)}) = \emptyset$ but this is in contradiction with that $y_{\zeta(1)} \in h_{\zeta(1)}^{-1}(J_{\zeta(1)}) \cap h_{\zeta(m)}^{-1}(J_{\zeta(m)})$. Therefore, $h_{\zeta(1)}^{-1}(J_{\zeta(1)}) \subseteq h_{\zeta(m)}^{-1}(J_{\zeta(m)})$, and $y_{\zeta(1)} \in h_{\zeta(m)}^{-1}(J_{\zeta(m)})$ for all $m \in \mathbb{N}$ with $M < m$. So we have that $h_{\zeta(m)}(y_{\zeta(1)}) \in J_{\zeta(m)}$ for all $m \in \mathbb{N}$ with $M < m$ and we define a subindex η of ζ as $\eta(n) = \zeta(n + M)$ for all $n \in \mathbb{N}$. Then, $\{(g_{\gamma \circ \eta(n)}, J_{\eta(n)})\}_{n=1}^\infty$ is a pre-approximation sequence at x' since $g_{\gamma \circ \eta(n)}(y_{\zeta(1)}) = h_{\zeta(n+M)}(y_{\zeta(1)}) \in J_{\eta(n)}$ for all $n \in \mathbb{N}$. Thus, this is the second case.

Now we also assume that there is no such a n_0 . Then for each $n \in \mathbb{N}$, we define $L_n = \{h_{\delta(m)}^{-1}(J_{\delta(m)}) : m \in \mathbb{N} \text{ and } y_{\delta(n)} \in h_{\delta(m)}^{-1}(J_{\delta(m)})\}$. Obviously, any L_n is not empty since $y_{\delta(n)} \in h_{\delta(n)}^{-1}(J_{\delta(n)})$ and $h_{\delta(n)}^{-1}(J_{\delta(n)}) \in L_n$, and is finite by the assumption. Fix a number $n \in \mathbb{N}$. First we show that L_n is totally ordered by the set inclusion $\subseteq$. Let M and N be elements of L_n . Then since each element belong to the same lamination system, $M \subseteq N$, $M^* \subseteq N$, $M \subseteq N^*$ or $M^* \subseteq N^*$ by unlinkedness. If $M^* \subseteq N$, then $x \in M^* \subseteq N$ as $x \notin \overline{M} = (M^*)^c$. It is in contradiction to that $x \notin N$. The case where $M \subseteq N^*$ is also impossible since $y_{\delta(n)} \in M$ but $y_{\delta(n)} \notin N^*$. Therefore $M \subseteq N$ or $M^* \subseteq N^*$, or equivalently $M \subseteq N$ or $N \subseteq M$. Thus the claim is proved.

Then we want to show that for any m and n in $\mathbb{N}$, $\bigcup L_m$ and $\bigcup L_n$ are same or disjoint. Note that $\bigcup L_n$ and $\bigcup L_m$ are the maximal elements of L_m and L_n respectively. By unlinkedness, $\bigcup L_m \subseteq \bigcup L_n, [\bigcup L_m]^* \subseteq \bigcup L_n, \bigcup L_m \subseteq [\bigcup L_n]^*$ or $[\bigcup L_m]^* \subseteq [\bigcup L_n]^*$. If $\bigcup L_m \subseteq \bigcup L_n$ or $[\bigcup L_m]^* \subseteq [\bigcup L_n]^*$, then $\bigcup L_m$ and $\bigcup L_n$ are same by the maximality. If $[\bigcup L_m]^* \subseteq \bigcup L_n$, then $x \in [\bigcup L_m]^* \subseteq \bigcup L_n$ but $x \notin \bigcup L_n$. So this is not a possible case. If $\bigcup L_m \subseteq [\bigcup L_n]^*$, then $\bigcup L_m$ and $\bigcup L_n$ are disjoint. Therefore the claim can be attained.

Next we claim that for any n and m in $\mathbb{N}$ with $n \leq m$, if $\bigcup L_n = \bigcup L_m$, then $\bigcup L_n = \bigcup L_k$ for all k with $n \leq k \leq m$. Assume that for some n and m with $n < m$, there is a number k in $\mathbb{N}$ with $n < k < m$ such that $\bigcup L_n$ and $\bigcup L_k$ are disjoint. There is a nondegenerate open interval I such that $\partial I = \{y_{\delta(n)}, y_{\delta(m)}\}$ and $x \in I^*$. Then $[\bigcup L_n]^* \subseteq I^*$ or equivalently $I \subseteq \bigcup L_n$ since $x \in [\bigcup L_n]^*$, $[\bigcup L_n]^* \subseteq S^1 - \{y_{\delta(n)}, y_{\delta(m)}\}$ and $[\bigcup L_n]^*$ is connected. Hence I and $\bigcup L_k$ are disjoint as $\bigcup L_n$ and $\bigcup L_k$ are disjoint. This implies that $y_{\delta(k)} \in I^c = I^* \cup \partial I$. Because $y_{\delta(k)} \neq y_{\delta(n)}$ and $y_{\delta(k)} \neq y_{\delta(m)}$, we can get that $y_{\delta(k)} \in I^*$ and so $\varphi(y_{\delta(n)}, y_{\delta(k)}, y_{\delta(m)}) = \varphi(y_{\delta(n)}, x, y_{\delta(m)})$. On the other hand $\varphi(x, y_{\delta(1)}, y_{\delta(2)}) = \varphi(x, y_{\delta(n)}, y_{\delta(m)}) = \varphi(x, y_{\delta(n)}, y_{\delta(k)}) = \varphi(x, y_{\delta(k)}, y_{\delta(m)})$ since the sequence $\{y_{\delta(n)}\}_{n=1}^\infty$ converges monotonically to x . By the cocycle condition for the 4-tuple $(x, y_{\delta(n)}, y_{\delta(k)}, y_{\delta(m)})$,

$$\begin{aligned} 0 &= \varphi(y_{\delta(n)}, y_{\delta(k)}, y_{\delta(m)}) - \varphi(x, y_{\delta(k)}, y_{\delta(m)}) + \varphi(x, y_{\delta(n)}, y_{\delta(m)}) - \varphi(x, y_{\delta(n)}, y_{\delta(k)}) \\ &= \varphi(y_{\delta(n)}, y_{\delta(k)}, y_{\delta(m)}) - \varphi(x, y_{\delta(1)}, y_{\delta(2)}) + \varphi(x, y_{\delta(1)}, y_{\delta(2)}) - \varphi(x, y_{\delta(1)}, y_{\delta(2)}) \\ &= \varphi(y_{\delta(n)}, y_{\delta(k)}, y_{\delta(m)}) - \varphi(x, y_{\delta(1)}, y_{\delta(2)}). \end{aligned}$$

Therefore,

$$\varphi(y_{\delta(n)}, y_{\delta(k)}, y_{\delta(m)}) = \varphi(x, y_{\delta(1)}, y_{\delta(2)}) = \varphi(x, y_{\delta(n)}, y_{\delta(m)}) = -\varphi(y_{\delta(n)}, x, y_{\delta(m)}).$$

However this is in contradiction to that $\varphi(y_{\delta(n)}, y_{\delta(k)}, y_{\delta(m)}) = \varphi(y_{\delta(n)}, x, y_{\delta(m)})$ since $\varphi(y_{\delta(n)}, x, y_{\delta(m)}) \neq 0$. Thus we can get the result which we wanted.

Then we define $C = \{\bigcup L_n : n \in \mathbb{N}\}$ and can define a partial order $\leq$ on C as follows. For any pair of two elements U and V in C , $U \leq V$ if and only if there are numbers n and m in $\mathbb{N}$ such that $n \leq m$, $y_{\delta(n)} \in U$, and

$y_{\delta(m)} \in V$. This order is well-defined by the previous claims and is a total order. Recall that as mentioned before, for each $n \in \mathbb{N}$, $y_{\delta(m)} \in h_{\delta(n)}^{-1}(J_{\delta(n)})$ for only finitely many $m \in \mathbb{N}$. Then there is the order preserving bijection μ from $(C, \leq)$ to $\mathbb{N}$ with the standard total order. Also we can define a function ν from $\{y_{\delta(n)} : n \in \mathbb{N}\}$ to C so that $\nu(y_{\delta(n)}) = \bigcup L_n$. Then the composition map $\mu \circ \nu \circ \xi$ is a monotonically increasing surjection on $\mathbb{N}$, where the function ξ from $\mathbb{N}$ to $\{y_{\delta(n)} : n \in \mathbb{N}\}$ is defined by $n \mapsto y_{\delta(n)}$. Hence the map $\mu \circ \nu \circ \xi$ has a right inverse π . It follows at once that the map π on $\mathbb{N}$ is an order preserving injection, that is the map π is a strictly increasing function on $\mathbb{N}$. Let ϵ be a subindex of δ defined by $\delta \circ \pi$.

Now we consider the sequence $\{h_{\epsilon(n)}^{-1}(J_{\epsilon(n)})\}_{n=1}^{\infty}$. Since the sequence $\{y_{\epsilon(n)}\}_{n=1}^{\infty}$ converges monotonically to x , as discussed previously, there is the sequence $\{K_n\}_{n=1}^{\infty}$ of nondegenerate open intervals such that for each $n \in \mathbb{N}$, $\partial K_n = \{x, y_{\epsilon(n)}\}$ and $K_{n+1} \subsetneq K_n$. We claim that for each $n \in \mathbb{N}$, $h_{\epsilon(m)}^{-1}(J_{\epsilon(m)}) \subseteq K_n$ for all $m \in \mathbb{N}$ with $n < m$. Choose a pair of two numbers n and m in $\mathbb{N}$ with $n < m$. Note that $\nu \circ \xi \circ \pi$ is a bijection since $\nu \circ \xi \circ \pi = \mu^{-1} \circ \text{id}_{\mathbb{N}}$. It follows that U_n and U_m are disjoint where we write $U_n = \nu \circ \xi \circ \pi(n)$ and $U_m = \nu \circ \xi \circ \pi(m)$. Because $U_m \subset S^1 - \{x, y_{\epsilon(n)}\}$, $y_{\epsilon(m)} \in K_n$ and U_m is connected, we can conclude that $U_m \subset K_n$. Therefore $h_{\epsilon(m)}^{-1}(J_{\epsilon(m)}) \subseteq U_m \subset K_n$. Thus the claim is proved.

Finally we consider the sequence $\{h_{\epsilon(n)}^{-1}(x')\}_{n=1}^{\infty}$. Since the sequence $\{J_{\epsilon(n)}\}_{n=1}^{\infty}$ is a quasi-rainbow at x' , by the previous claim, for each $n \in \mathbb{N}$, $h_{\epsilon(m)}^{-1}(x') \in h_{\epsilon(m)}^{-1}(\overline{J_{\epsilon(m)}}) = \overline{h_{\epsilon(m)}^{-1}(J_{\epsilon(m)})} \subset \overline{K_n}$ for all $m \in \mathbb{N}$ with $n < m$. Hence the sequence $\{h_{\epsilon(n)}^{-1}(x')\}_{n=1}^{\infty}$ converges to x as $\bigcap_{n=1}^{\infty} \overline{K_n} = \{x\}$. Note that for each $n \in \mathbb{N}$, $h_{\epsilon(n)}^{-1}(x') \neq x$ since $x \notin \overline{h_{\epsilon(n)}^{-1}(J_{\epsilon(n)})}$. Then we can take a subindex ζ of $\gamma \circ \epsilon$ so that the sequence $\{g_{\zeta(n)}^{-1}(x')\}_{n=1}^{\infty}$ converges monotonically to x . By Lemma 4.6, there is a subindex η of ζ and a quasi-rainbow $\{I_n\}_{n=1}^{\infty}$ at x in $\mathcal{L}_i$ for some $i \in \mathbb{Z}_2$ such that $g_{\eta(n)}^{-1}(x') \in I_n$ for all $n \in \mathbb{N}$. Then the sequence $\{(g_{\eta(n)}^{-1}, I_n)\}_{n=1}^{\infty}$ is a pre-approximation sequence at x . Thus this is the third case. $\square$

Now we will apply this lemma in turn to $\{x_n^1\}_{n=1}^{\infty}$, $\{x_n^2\}_{n=1}^{\infty}$ and $\{x_n^3\}_{n=1}^{\infty}$ with $\{g_{\beta(n)}\}_{n=1}^{\infty}$. Then each subsequencing step has three possible cases as Lemma 4.9. Then it turns out that after passing to a subsequence of $\{g_{\beta(n)}\}_{n=1}^{\infty}$, the subsequence has the following property.

DEFINITION 4.10. *Let G be a Möbius-like subgroup of $\text{Homeo}^+(S^1)$ and $\{g_n\}_{n=1}^{\infty}$ be a sequence of distinct elements of G . Let p and q be two distinct points in S^1 . Then the sequence $\{g_n\}_{n=1}^{\infty}$ is called an approximation sequence to $\{p, q\}$ if for any pair of two numbers m and k of $\mathbb{N}$ with $m < k$, $g_k \circ g_m^{-1}$*

is hyperbolic and the sequence $\{\text{Fix}_{g_{\alpha(n+1)} \circ g_{\alpha(n)}^{-1}}\}_{n=1}^{\infty}$ in $\mathcal{M}$ converges to $\{p, q\}$ for any subindex α of $\text{id}_{\mathbb{N}}$.

REMARK 4.11. Every subsequence of an approximation sequence to $\{p, q\}$ is an approximation sequence to $\{p, q\}$.

First we consider the following lemmas.

LEMMA 4.12. Let G be a Möbius-like subgroup of $\text{Homeo}^+(S^1)$ and $\{g_n\}_{n=1}^{\infty}$ be a sequence of distinct elements of G . Let x' and y' be two distinct points in S^1 . Assume that there are two distinct points x and y in S^1 such that for any $n \in \mathbb{N}$, $g_n(x) = x'$ and $g_n(y) = y'$. Then $\{g_n\}_{n=1}^{\infty}$ is an approximation sequence to $\{x', y'\}$.

PROOF. Choose two distinct numbers n and m in $\mathbb{N}$ with $n < m$. Then $g_m \circ g_n^{-1}(x') = x'$ and $g_m \circ g_n^{-1}(y') = y'$ and since $\{g_n\}_{n=1}^{\infty}$ is a sequence of distinct elements so $g_n \neq g_m$, the element $g_m \circ g_n^{-1}$ is a nontrivial element having a fixed point. Hence $g_m \circ g_n^{-1}$ must be a hyperbolic element since G is Möbius-like, $\{x', y'\} \subseteq \text{Fix}_{g_m \circ g_n^{-1}}$ and $x' \neq y'$. Moreover for any subindex α of $\text{id}_{\mathbb{N}}$, $g_{\alpha(n+1)} \circ g_{\alpha(n)}^{-1}$ are hyperbolic and $\text{Fix}_{g_{\alpha(n+1)} \circ g_{\alpha(n)}^{-1}} = \{x', y'\}$. Thus $\{g_n\}_{n=1}^{\infty}$ is an approximation sequence to $\{x', y'\}$. $\square$

LEMMA 4.13. Let G be a Möbius-like laminar group and $\mathcal{L}$ be a G -invariant lamination system. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of distinct elements of G , and x' and y' be two distinct points in S^1 . Suppose that there is a point x in S^1 such that $g_n(x) = x'$ for all $n \in \mathbb{N}$. Assume that we can take a quasi-rainbow $\{I_n\}_{n=1}^{\infty}$ at y' in $\mathcal{L}$ so that the sequence $\{(g_n, I_n)\}_{n=1}^{\infty}$ is a pre-approximation sequence at y' . Further assume that $x' \notin \overline{I_1}$. Then $\{g_n\}_{n=1}^{\infty}$ is an approximation sequence to $\{x', y'\}$.

PROOF. Since $\{(g_n, I_n)\}_{n=1}^{\infty}$ is a pre-approximation sequence at y' , there is a point y in S^1 such that $g_n(y) \in I_n$ for all $n \in \mathbb{N}$. Choose two distinct numbers n and m in $\mathbb{N}$ with $n < m$. Then $g_m \circ g_n^{-1}(x') = x'$ and $x' \in \text{Fix}_{g_m \circ g_n^{-1}}$. Since $\{g_n\}_{n=1}^{\infty}$ is a sequence of distinct elements so $g_n \neq g_m$, the element $g_m \circ g_n^{-1}$ is a nontrivial element having a fixed point.

Now we consider I_n . The element $g_m(y)$ belongs to $(g_m \circ g_n^{-1}(I_n)) \cap I_n$ as $y \in g_n^{-1}(I_n)$ and $g_m(y) \in I_m \subset I_n$. Hence $g_m \circ g_n^{-1}(I_n)$ and I_n are not disjoint so by the unlinkedness of them, there are three cases: $g_m \circ g_n^{-1}(I_n) \subset I_n$; $(g_m \circ g_n^{-1}(I_n))^* \subset I_n$; $(g_m \circ g_n^{-1}(I_n))^* \subset I_n^*$. If $(g_m \circ g_n^{-1}(I_n))^* \subset I_n$ or equivalently $g_m \circ g_n^{-1}(I_n^*) \subset I_n$, then $x' = g_m \circ g_n^{-1}(x') \in g_m \circ g_n^{-1}(I_n^*) \subset I_n$ as $x' \in I_n^*$. This is a contradiction since I and I^* are disjoint.

Therefore $g_m \circ g_n^{-1}(I_n) \subseteq I_n$ or $(g_m \circ g_n^{-1}(I_n))^* \subset I_n^*$, namely $g_m \circ g_n^{-1}(\overline{I_n}) \subseteq \overline{I_n}$ or $\overline{I_n} \subset g_m \circ g_n^{-1}(\overline{I_n})$. Then the Brouwer's fixed point theorem can apply and implies that there must be a fixed point of $g_m \circ g_n^{-1}$ in $\overline{I_n}$. Since $x' \notin \overline{I_1}$, $\overline{I_n} \subset \overline{I_1}$, and G is Möbius-like, the element $g_m \circ g_n^{-1}$ is a hyperbolic element and there is a unique fixed point in $\overline{I_n}$.

Finally let α be a subindex of $\text{id}_{\mathbb{N}}$. Note that for all $n \in \mathbb{N}$, the element $g_{\alpha(n+1)} \circ g_{\alpha(n)}^{-1}$ is hyperbolic and $x' \in \text{Fix}_{g_{\alpha(n+1)} \circ g_{\alpha(n)}^{-1}}$. We can write $\text{Fix}_{g_{\alpha(n+1)} \circ g_{\alpha(n)}^{-1}} = \{x', p_n\}$ for all $n \in \mathbb{N}$. As we saw, $p_n \in \overline{I_{\alpha(n)}}$. Moreover the sequence $\{p_n\}_{n=1}^{\infty}$ converges to y' as $\{I_{\alpha(n)}\}_{n=1}^{\infty}$ is a quasi-rainbow at y' . Therefore the sequence $\{x', p_n\}_{n=1}^{\infty}$ in $\mathcal{M}$ converges to $\{x', y'\}$. Thus we are done. $\square$

LEMMA 4.14. *Let G be a Möbius-like laminar group. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of distinct elements of G , and let x'_1 and x'_2 be two distinct points in S^1 . Suppose that, for each $i \in \mathbb{Z}_2$, there exists a G -invariant lamination system $\mathcal{L}_i$, not necessarily the same for $i = 1, 2$, containing a quasi-rainbow $\{I_n^i\}_{n=1}^{\infty}$ at x'_i such that $\{(g_n, I_n^i)\}_{n=1}^{\infty}$ is a pre-approximation sequence at x'_i . Assume that $\overline{I_1^1}$ and $\overline{I_1^2}$ are disjoint. Then $\{g_n\}_{n=1}^{\infty}$ is an approximation sequence to $\{x'_1, x'_2\}$.*

PROOF. Choose two distinct numbers n and m in $\mathbb{N}$ with $n < m$. For each $i \in \mathbb{Z}_2$, there is a point x_i in S^1 such that $g_n(x_i) \in I_n^i$ for all $n \in \mathbb{N}$ since $\{(g_n, I_n^i)\}_{n=1}^{\infty}$ is a pre-approximation sequence at x'_i . Hence for each $i \in \mathbb{Z}_2$, $g_m(x_i) \in g_m \circ g_n^{-1}(I_n^i)$ so $g_m \circ g_n^{-1}(I_n^i) \cap I_n^i \neq \emptyset$ since $g_m(x_i) \in I_m^i \subset I_n^i$. Fix $i \in \mathbb{Z}_2$. By the unlinkness of I_n^i and $g_m \circ g_n^{-1}(I_n^i)$, there are three cases: $g_m \circ g_n^{-1}(I_n^i) \subset I_n^i$; $g_m \circ g_n^{-1}(I_n^i)^* \subset I_n^i$; $g_m \circ g_n^{-1}(I_n^i)^* \subset (I_n^i)^*$. If $g_m \circ g_n^{-1}(I_n^i)^* \subset I_n^i$, then $g_m \circ g_n^{-1}(I_n^{i+1}) \subset g_m \circ g_n^{-1}((I_n^i)^*) \subset I_n^i$ since $\overline{I_1^1}$ and $\overline{I_1^2}$ are disjoint and $I_n^{i+1} \subset (I_n^i)^*$. Hence $g_m(x_{i+1}) \in g_m \circ g_n^{-1}(I_n^{i+1}) \subset I_n^i$ as $x_{i+1} \in g_n^{-1}(I_n^{i+1})$, and so $g_m(x_{i+1}) \in (I_m^{i+1} \cap I_n^i) \subset (I_n^{i+1} \cap I_n^i)$. However this is a contradiction since $\overline{I_1^1}$ and $\overline{I_1^2}$ are disjoint. Therefore $g_m \circ g_n^{-1}(I_n^i) \subset I_n^i$ or $g_m \circ g_n^{-1}(I_n^i)^* \subset (I_n^i)^*$, namely $g_m \circ g_n^{-1}(\overline{I_n^i}) \subset \overline{I_n^i}$ or $\overline{I_n^i} \subset g_m \circ g_n^{-1}(\overline{I_n^i})$. Then the Brouwer's fixed point theorem can apply and implies that for each $i \in \mathbb{Z}_2$, I_n^i has a fixed point of the element $g_m \circ g_n^{-1}$. Therefore the element $g_m \circ g_n^{-1}$ has at least two fixed points since $\overline{I_1^1}$ and $\overline{I_1^2}$ are disjoint. Note that $g_m \circ g_n^{-1}$ is nontrivial since $\{g_n\}_{n=1}^{\infty}$ is a sequence of distinct elements so $g_n \neq g_m$. Since G is Möbius-like, $g_m \circ g_n^{-1}$ must be hyperbolic and so for each $i \in \mathbb{Z}_2$, I_n^i has exactly one fixed point of the element $g_m \circ g_n^{-1}$.

Finally let α be a subindex of $\text{id}_{\mathbb{N}}$. Note that for each $n \in \mathbb{N}$, the element $g_{\alpha(n+1)} \circ g_{\alpha(n)}^{-1}$ is hyperbolic and for each $i \in \mathbb{Z}_2$, $\overline{I_{\alpha(n)}^i}$ has exactly one fixed point p_n^i of $g_{\alpha(n+1)} \circ g_{\alpha(n)}^{-1}$. Hence we can write $\text{Fix}_{g_{\alpha(n+1)} \circ g_{\alpha(n)}^{-1}} = \{p_n^1, p_n^2\}$ for all $n \in \mathbb{N}$. Therefore the sequence $\{p_n^1, p_n^2\}_{n=1}^{\infty}$ in $\mathcal{M}$ converges to $\{x'_1, x'_2\}$ as $\{I_{\alpha(n)}^1\}_{n=1}^{\infty}$ and $\{I_{\alpha(n)}^2\}_{n=1}^{\infty}$ are quasi-rainbows at x'_1 and x'_2 respectively. Thus we are done. $\square$

We now prove the first main theorem. We shall use Lemma 7.3; since its proof requires some additional notions which are not used elsewhere in this section, we defer it to Appendix 7.

Theorem A Every pants-like COL_3 group is a convergence group.

PROOF. Let G be a pants-like COL_3 group and $\mathcal{C} = \{\mathcal{L}_i\}_{i \in \mathbb{Z}_3}$ be a pants-like collection for G . We want to show that every sequence of distinct elements of G has the convergence property. Now we assume that there is a sequence $\{g_n\}_{n=1}^\infty$ of distinct elements of G that does not have the convergence property. Then by Proposition 4.2, the sequence is not properly discontinuous on triples, that is there are two compact subsets K and L of $\Theta(S^1)$ such that $\{m \in \mathbb{N} : g_m(K) \cap L \neq \emptyset\}$ is infinite. Hence there is a sequence $\{x_n\}_{n=1}^\infty$ in K converging to a point x_∞ in K and a subindex α of $\text{id}_{\mathbb{N}}$ such that the sequence $\{g_{\alpha(n)}(x_n)\}_{n=1}^\infty$ converges to a point y_∞ in L by Lemma 4.3. Then we write $x_n = (x_n^1, x_n^2, x_n^3)$ for all $n \in \mathbb{N} \cup \{\infty\}$ and $y_\infty = (y_\infty^1, y_\infty^2, y_\infty^3)$ so that for each $i \in \mathbb{Z}_3$, the sequence $\{x_n^i\}_{n=1}^\infty$ in S^1 converges to the point x_∞^i of S^1 and the sequence $\{g_{\alpha(n)}(x_n^i)\}_{n=1}^\infty$ converges to the point y_∞^i of S^1 . Choose a number ϵ in $\mathbb{R}$ so that $0 < \epsilon < d_{\mathbb{C}}(x_\infty^i, x_\infty^{i+1})/3$ and $0 < \epsilon < d_{\mathbb{C}}(y_\infty^i, y_\infty^{i+1})/3$ for all $i \in \mathbb{Z}_3$. For each $p \in x_\infty \cup y_\infty$, we write $N_\epsilon(p) = \{z \in S^1 : d_{\mathbb{C}}(p, z) < \epsilon\}$.

First if there is a subindex β of α such that for some $i_0 \in \mathbb{Z}_3$, $g_{\beta(n)}(x_\infty^{i_0}) = y_\infty^{i_0}$ and $g_{\beta(n)}(x_\infty^{i_0+1}) = y_\infty^{i_0+1}$ for all $n \in \mathbb{N}$. Then by Lemma 4.12, the sequence $\{g_{\beta(n)}\}_{n=1}^\infty$ is an approximation sequence to $\{y_\infty^{i_0}, y_\infty^{i_0+1}\}$ and also $\{g_{\beta(n)}^{-1}\}_{n=1}^\infty$ is an approximation sequence to $\{x_\infty^{i_0}, x_\infty^{i_0+1}\}$. Applying Lemma 4.9 to the index i_0+2 , we can take a subindex γ of β so that $\{g_{\gamma(n)}\}_{n=1}^\infty$ is one of the three cases in Lemma 4.9. In the case where $g_{\gamma(n)}(x_\infty^{i_0+2}) = y_\infty^{i_0+2}$ for all $n \in \mathbb{N}$, we get that $y_\infty \subset \text{Fix}_{g_{\gamma(n+1)} \circ g_{\gamma(n)}^{-1}}$ for all $n \in \mathbb{N}$. This implies that $g_{\gamma(n+1)} \circ g_{\gamma(n)}^{-1} = \text{id}_{S^1}$ for all $n \in \mathbb{N}$ since every pants-like COL_3 group is Möbius-like. This is in contradiction to that the sequence $\{g_n\}_{n=1}^\infty$ is a sequence of distinct elements of G . The first case is not possible. Hence there is a quasi-rainbow $\{I_n\}_{n=1}^\infty$ in some lamination system in $\mathcal{C}$ such that either $\{(g_{\gamma(n)}, I_n)\}_{n=1}^\infty$ is a pre-approximation sequence at $y_\infty^{i_0+2}$ or $\{(g_{\gamma(n)}^{-1}, I_n)\}_{n=1}^\infty$ is a pre-approximation sequence at $x_\infty^{i_0+2}$. When $\{(g_{\gamma(n)}, I_n)\}_{n=1}^\infty$ is a pre-approximation sequence at $y_\infty^{i_0+2}$, there is a number N in $\mathbb{N}$ such that $\overline{I_N} \subset N_\epsilon(y_\infty^{i_0+2})$ since $\{I_n\}_{n=1}^\infty$ is a quasi-rainbow at $y_\infty^{i_0+2}$. Then we can take a subindex δ of $\text{id}_{\mathbb{N}}$ so that $\overline{I_{\delta(1)}} \subset N_\epsilon(y_\infty^{i_0+2})$. Note that the sequence $\{(g_{\gamma \circ \delta(n)}, I_{\delta(n)})\}_{n=1}^\infty$ is still a pre-approximation sequence at $y_\infty^{i_0+2}$. Since $y_\infty^{i_0} \notin N_\epsilon(y_\infty^{i_0+2})$ and $y_\infty^{i_0} \notin \overline{I_{\delta(1)}}$, the sequence $\{g_{\gamma \circ \delta(n)}\}_{n=1}^\infty$ is an approximation sequence to $\{y_\infty^{i_0}, y_\infty^{i_0+2}\}$ by Lemma 4.13. Then there is a number n_0 in $\mathbb{N}$ such that the hyperbolic element $g_{\gamma \circ \delta(n_0+1)} \circ g_{\gamma \circ \delta(n_0)}^{-1}$ has exactly one fixed point in each of $N_\epsilon(y_\infty^{i_0})$ and $N_\epsilon(y_\infty^{i_0+2})$. Since $y_\infty^{i_0+1} \in \text{Fix}_{g_{\gamma \circ \delta(n_0+1)} \circ g_{\gamma \circ \delta(n_0)}^{-1}}$, the element $g_{\gamma \circ \delta(n_0+1)} \circ g_{\gamma \circ \delta(n_0)}^{-1}$ has at least three fixed points by the choice of ϵ . Hence $g_{\gamma \circ \delta(n_0+1)} \circ g_{\gamma \circ \delta(n_0)}^{-1}$ must be id_{S^1} since G is Möbius-like. This is a contradiction since $\{g_n\}_{n=1}^\infty$ is a sequence of distinct elements of G . Likewise the case where $\{(g_{\gamma(n)}^{-1}, I_n)\}_{n=1}^\infty$

is a pre-approximation sequence at $x_\infty^{i_0+2}$ is also impossible. Therefore there is no such a subindex β .

Then we consider the case where there is a subindex β of α such that for some $i_0 \in \mathbb{Z}_3$, $g_{\beta(n)}(x_\infty^{i_0}) = y_\infty^{i_0}$ for all $n \in \mathbb{N}$. Applying Lemma 4.9 to the index i_0+1 , we can take a subindex γ of $\text{id}_\mathbb{N}$ so that $\{g_{\beta \circ \gamma(n)}\}_{n=1}^\infty$ is one of the second and third cases in Lemma 4.9 since the first case is ruled out by the previous case. Namely there is a quasi-rainbow $\{J_n^{i_0+1}\}_{n=1}^\infty$ in some lamination system of $\mathcal{C}$ such that either $\{(g_{\beta \circ \gamma(n)}, J_n^{i_0+1})\}_{n=1}^\infty$ is a pre-approximation sequence at $y_\infty^{i_0+1}$ or $\{(g_{\beta \circ \gamma(n)}^{-1}, J_n^{i_0+1})\}_{n=1}^\infty$ is a pre-approximation sequence at $x_\infty^{i_0+1}$. Applying Lemma 4.9 again to the index i_0+2 , we can take a subindex δ of $\text{id}_\mathbb{N}$ so that $\{g_{\beta \circ \gamma \circ \delta(n)}\}_{n=1}^\infty$ is one of the second and third cases in Lemma 4.9, that is there is a quasi-rainbow $\{I_n^{i_0+2}\}_{n=1}^\infty$ in some lamination system of $\mathcal{C}$ such that either $\{(g_{\beta \circ \gamma \circ \delta(n)}, I_n^{i_0+2})\}_{n=1}^\infty$ is a pre-approximation sequence at $y_\infty^{i_0+2}$ or $\{(g_{\beta \circ \gamma \circ \delta(n)}^{-1}, I_n^{i_0+2})\}_{n=1}^\infty$ is a pre-approximation sequence at $x_\infty^{i_0+2}$. Now for brevity we write $h_n = g_{\beta \circ \gamma \circ \delta(n)}$ and $I_n^{i_0+1} = J_{\delta(n)}^{i_0+1}$ for all $n \in \mathbb{N}$.

First we consider the case where $\{(h_n, I_n^{i_0+1})\}_{n=1}^\infty$ and $\{(h_n, I_n^{i_0+2})\}_{n=1}^\infty$ are pre-approximation sequences at $y_\infty^{i_0+1}$ and $y_\infty^{i_0+2}$, respectively. Since $\{I_n^{i_0+1}\}_{n=1}^\infty$ and $\{I_n^{i_0+2}\}_{n=1}^\infty$ are quasi-rainbows, there is a number N in $\mathbb{N}$ such that $\overline{I_N^{i_0+1}} \subset N_\epsilon(y_\infty^{i_0+1})$ and $\overline{I_N^{i_0+2}} \subset N_\epsilon(y_\infty^{i_0+2})$. Hence we can take a subindex ζ of $\text{id}_\mathbb{N}$ so that $\overline{I_{\zeta(1)}^{i_0+1}} \subset N_\epsilon(y_\infty^{i_0+1})$ and $\overline{I_{\zeta(1)}^{i_0+2}} \subset N_\epsilon(y_\infty^{i_0+2})$. Since $\overline{I_{\zeta(1)}^{i_0+1}} \cap \overline{I_{\zeta(1)}^{i_0+2}} = \emptyset$, the sequence $\{h_{\zeta(n)}\}_{n=1}^\infty$ is an approximation sequence to $\{y_\infty^{i_0+1}, y_\infty^{i_0+2}\}$ by Lemma 4.14. Hence we can choose a number M in $\mathbb{N}$ such that the element $h_{\zeta(M+1)} \circ h_{\zeta(M)}^{-1}$ has exactly one fixed point in each of $N_\epsilon(y_\infty^{i_0+1})$ and $N_\epsilon(y_\infty^{i_0+2})$. On the other hand, the element $y_\infty^{i_0} \in \text{Fix}_{h_{\zeta(M+1)} \circ h_{\zeta(M)}^{-1}}$. Then the element $h_{\zeta(M+1)} \circ h_{\zeta(M)}^{-1}$ has at least three fixed points by the choice of ϵ and this is a contradiction since $h_{\zeta(M+1)} \circ h_{\zeta(M)}^{-1}$ is hyperbolic. Likewise the case where $\{(h_n^{-1}, I_n^{i_0+1})\}_{n=1}^\infty$ and $\{(h_n^{-1}, I_n^{i_0+2})\}_{n=1}^\infty$ are pre-approximation sequences at $x_\infty^{i_0+1}$ and $x_\infty^{i_0+2}$ respectively is not possible.

Next we consider the case where $\{(h_n^{-1}, I_n^{i_0+1})\}_{n=1}^\infty$ and $\{(h_n, I_n^{i_0+2})\}_{n=1}^\infty$ are pre-approximation sequences at $x_\infty^{i_0+1}$ and $y_\infty^{i_0+2}$, respectively. As $\{I_n^{i_0+1}\}_{n=1}^\infty$ and $\{I_n^{i_0+2}\}_{n=1}^\infty$ are quasi-rainbows at $x_\infty^{i_0+1}$ and $y_\infty^{i_0+2}$, respectively, there is a subindex ζ of $\text{id}_\mathbb{N}$ such that $\overline{I_{\zeta(1)}^{i_0+1}} \subset N_\epsilon(x_\infty^{i_0+1})$ and $\overline{I_{\zeta(1)}^{i_0+2}} \subset N_\epsilon(y_\infty^{i_0+2})$. Then since $x_\infty^{i_0} \notin \overline{I_{\zeta(1)}^{i_0+1}}$ and $y_\infty^{i_0} \notin \overline{I_{\zeta(1)}^{i_0+2}}$, by Lemma 4.13, the sequence $\{h_{\zeta(n)}^{-1}\}_{n=1}^\infty$ is an approximation sequence to $\{x_\infty^{i_0}, x_\infty^{i_0+1}\}$ and the sequence $\{h_{\zeta(n)}\}_{n=1}^\infty$ is an approximation sequence to $\{y_\infty^{i_0}, y_\infty^{i_0+2}\}$. Now we write $\text{Fix}_{h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)}} = \{x_\infty^{i_0}, a_n\}$ and $\text{Fix}_{h_{\zeta(n+1)} \circ h_{\zeta(n)}^{-1}} = \{y_\infty^{i_0}, b_n\}$. Then the two sequences $\{a_n\}_{n=1}^\infty$

and $\{b_n\}_{n=1}^\infty$ converge to $x_\infty^{i_0+1}$ and $y_\infty^{i_0+2}$. Note that for each $n \in \mathbb{N}$,

$$\begin{aligned} h_{\zeta(n)}(\text{Fix}_{h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)}}) &= h_{\zeta(n)}(\text{Fix}_{h_{\zeta(n)}^{-1} \circ h_{\zeta(n+1)}}) \\ &= \text{Fix}_{h_{\zeta(n)} \circ (h_{\zeta(n)}^{-1} \circ h_{\zeta(n+1)}) \circ h_{\zeta(n)}^{-1}} \\ &= \text{Fix}_{h_{\zeta(n+1)} \circ h_{\zeta(n)}^{-1}}. \end{aligned}$$

Hence $h_{\zeta(n)}(a_n) = b_n$ for all $n \in \mathbb{N}$. Now we have that $\{x_{\beta' \circ \gamma \circ \delta \circ \zeta(n)}^{i_0+1}\}_{n=1}^\infty$ and $\{x_{\beta' \circ \gamma \circ \delta \circ \zeta(n)}^{i_0+2}\}_{n=1}^\infty$ converge to $x_\infty^{i_0+1}$ and $x_\infty^{i_0+2}$, respectively and that $\{h_{\zeta(n)}(x_{\beta' \circ \gamma \circ \delta \circ \zeta(n)}^{i_0+1})\}_{n=1}^\infty$ and $\{h_{\zeta(n)}(x_{\beta' \circ \gamma \circ \delta \circ \zeta(n)}^{i_0+2})\}_{n=1}^\infty$ converge to $y_\infty^{i_0+1}$ and $y_\infty^{i_0+2}$, respectively, where β' is the subindex of $\text{id}_\mathbb{N}$ satisfying that $\beta = \alpha \circ \beta'$. Moreover for each $n \in \mathbb{N}$, a_n is a fixed point of the hyperbolic element $h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)}$, and $\{a_n\}_{n=1}^\infty$ and $\{h_{\zeta(n)}(a_n)\}_{n=1}^\infty$ converge to $x_\infty^{i_0+1}$ and $y_\infty^{i_0+2}$, respectively. Also the sequence $\{h_{\zeta(n)}\}_{n=1}^\infty$ is an approximation sequence. Therefore by Lemma 7.3, the sequence $\{h_{\zeta(n)}\}_{n=1}^\infty$ has the convergence property and so $\{g_n\}_{n=1}^\infty$ has the convergence property. However this is in contradiction to the assumption that $\{g_n\}_{n=1}^\infty$ does not have the convergence property. Likewise we can see that the case where $\{(h_n, I_n^{i_0+1})\}_{n=1}^\infty$ and $\{(h_n^{-1}, I_n^{i_0+2})\}_{n=1}^\infty$ are pre-approximation sequences at $y_\infty^{i_0+1}$ and $x_\infty^{i_0+2}$, respectively, is not possible. Therefore, there is no such subindex β .

Then we consider the case where there is a subindex β of α such that for each $i \in \mathbb{Z}_3$, we can take a quasi-rainbow $\{I_n^i\}_{n=1}^\infty$ in some lamination system of $\mathcal{C}$ at y_∞^i so that $\{(g_{\beta(n)}, I_n^i)\}_{n=1}^\infty$ is a pre-approximation sequence at y_∞^i . Then there is a subindex γ of β such that for each $i \in \mathbb{Z}_3$, $\overline{I_{\gamma(1)}^i} \subset N_\epsilon(y_\infty^i)$. By the choice of ϵ , for each $i \in \mathbb{Z}_3$, $\overline{I_{\gamma(1)}^i} \cap \overline{I_{\gamma(1)}^{i+1}} = \emptyset$. Hence by Lemma 4.14, for each $i \in \mathbb{Z}_3$, the sequence $\{g_{\gamma(n)}\}_{n=1}^\infty$ is an approximation sequence to $\{y_\infty^i, y_\infty^{i+1}\}$. Then there is a number $n_0 \in \mathbb{N}$ such that for each $i \in \mathbb{Z}_3$, $g_{\gamma(n_0+1)} \circ g_{\gamma(n_0)}^{-1}$ has a fixed point in $N_\epsilon(y_\infty^i)$. This implies that the element $g_{\gamma(n_0+1)} \circ g_{\gamma(n_0)}^{-1}$ has at least three fixed points. This is a contradiction since $g_{\gamma(n_0+1)} \circ g_{\gamma(n_0)}^{-1}$ is hyperbolic. Therefore, there is no such β . Likewise, there is no subindex β of α such that for each $i \in \mathbb{Z}_3$, we can take a quasi-rainbow $\{I_n^i\}_{n=1}^\infty$ in some lamination system of $\mathcal{C}$ at x_∞^i so that $\{(g_{\beta(n)}^{-1}, I_n^i)\}_{n=1}^\infty$ is a pre-approximation sequence at x_∞^i .

Now we apply Lemma 4.9 to the index 1. Then by the previous discussion, the first case of Lemma 4.9 can not occur so there is a subindex β of $\text{id}_\mathbb{N}$ and a quasi-rainbow $\{I_n^1\}_{n=1}^\infty$ in some lamination system of $\mathcal{C}$ such that

- $\{(g_{\alpha \circ \beta(n)}, I_n^1)\}_{n=1}^\infty$ is a pre-approximation sequence at y_∞^1 or
- $\{(g_{\alpha \circ \beta(n)}^{-1}, I_n^1)\}_{n=1}^\infty$ is a pre-approximation sequence at x_∞^1 .

Then we apply Lemma 4.9 to the index 2. Like the previous case, there is a subindex γ of $\text{id}_{\mathbb{N}}$ and a quasi-rainbow $\{I_n^2\}_{n=1}^{\infty}$ in some lamination system of $\mathcal{C}$ such that either $\{(g_{\alpha\circ\beta\circ\gamma(n)}, I_n^2)\}_{n=1}^{\infty}$ is a pre-approximation sequence at y_{∞}^2 or $\{(g_{\alpha\circ\beta\circ\gamma(n)}^{-1}, I_n^2)\}_{n=1}^{\infty}$ is a pre-approximation sequence at x_{∞}^2 . Again we apply Lemma 4.9 to the index 3. Then there is a subindex δ of $\text{id}_{\mathbb{N}}$ and a quasi-rainbow $\{I_n^3\}_{n=1}^{\infty}$ such that either $\{(g_{\alpha\circ\beta\circ\gamma\circ\delta(n)}, I_n^3)\}_{n=1}^{\infty}$ is a pre-approximation sequence at y_{∞}^3 or $\{(g_{\alpha\circ\beta\circ\gamma\circ\delta(n)}^{-1}, I_n^3)\}_{n=1}^{\infty}$ is a pre-approximation sequence at x_{∞}^3 . For brevity, we write $h_n = g_{\alpha\circ\beta\circ\gamma\circ\delta(n)}$, $J_n^1 = I_{\gamma\circ\delta(n)}^1$, $J_n^2 = I_{\delta(n)}^2$, and $J_n^3 = I_n^3$ for all $n \in \mathbb{N}$. Also we write $u_n^i = x_{\beta\circ\gamma\circ\delta(n)}^i$ and $v_n^i = y_{\beta\circ\gamma\circ\delta(n)}^i$ for all $n \in \mathbb{N}$ and for all $i \in \mathbb{Z}_3$.

Then by the previous discussion, there are only two cases which we need to deal with. The first case is that there is an index $i_0 \in \mathbb{Z}_3$ such that the two sequences $\{(h_n, J_n^{i_0})\}_{n=1}^{\infty}$ and $\{(h_n, J_n^{i_0+1})\}_{n=1}^{\infty}$ are pre-approximation sequences at $y_{\infty}^{i_0}$ and $y_{\infty}^{i_0+1}$, respectively, and the sequence $\{(h_n^{-1}, J_n^{i_0+2})\}_{n=1}^{\infty}$ is a pre-approximation sequence at $x_{\infty}^{i_0+2}$. The second case is that there is an index $i_0 \in \mathbb{Z}_3$ such that the two sequences $\{(h_n^{-1}, J_n^{i_0})\}_{n=1}^{\infty}$ and $\{(h_n^{-1}, J_n^{i_0+1})\}_{n=1}^{\infty}$ are pre-approximation sequences at $x_{\infty}^{i_0}$ and $x_{\infty}^{i_0+1}$, respectively, and the sequence $\{(h_n, J_n^{i_0+2})\}_{n=1}^{\infty}$ is a pre-approximation sequence at $y_{\infty}^{i_0+2}$.

First we consider the first case. There is a subindex ζ of $\text{id}_{\mathbb{N}}$ such that $\overline{J_{\zeta(1)}^{i_0}} \subset N_{\epsilon}(y_{\infty}^{i_0})$ and $\overline{J_{\zeta(1)}^{i_0+1}} \subset N_{\epsilon}(y_{\infty}^{i_0+1})$. By Lemma 4.14, $\{h_{\zeta(n)}\}_{n=1}^{\infty}$ is an approximation sequence to $\{y_{\infty}^{i_0}, y_{\infty}^{i_0+1}\}$. Note that for each $n \in \mathbb{N}$, $h_{\zeta(n+1)} \circ h_{\zeta(n)}^{-1}$ is hyperbolic so $h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)}$ is also hyperbolic since

$$h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)} = h_{\zeta(n)}^{-1} \circ (h_{\zeta(n)} \circ h_{\zeta(n+1)}^{-1}) \circ h_{\zeta(n)} = h_{\zeta(n)}^{-1} \circ (h_{\zeta(n+1)} \circ h_{\zeta(n)}^{-1})^{-1} \circ h_{\zeta(n)}.$$

Moreover, since $\{(h_{\zeta(n)}^{-1}, J_{\zeta(n)}^{i_0+2})\}_{n=1}^{\infty}$ is a pre-approximation sequence at $x_{\infty}^{i_0+2}$, there is a point $\tilde{x}$ in S^1 such that $h_{\zeta(n)}^{-1}(\tilde{x}) \in J_{\zeta(n)}^{i_0+2}$ for all $n \in \mathbb{N}$ and so

$$\begin{aligned} h_{\zeta(n+1)}^{-1}(\tilde{x}) &= (h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)})(h_{\zeta(n)}^{-1}(\tilde{x})) \\ &\in (h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)})(J_{\zeta(n)}^{i_0+2}) \cap J_{\zeta(n+1)}^{i_0+2} \\ &\subset (h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)})(J_{\zeta(n)}^{i_0+2}) \cap J_{\zeta(n)}^{i_0+2}. \end{aligned}$$

Hence we can claim that for each $n \in \mathbb{N}$, $\overline{J_{\zeta(n)}^{i_0+2}}$ has a fixed point of the hyperbolic element $h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)}$. For each $n \in \mathbb{N}$, since $(h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)})(J_{\zeta(n)}^{i_0+2}) \cap J_{\zeta(n)}^{i_0+2} \neq \emptyset$, there are three possible cases: $(h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)})(J_{\zeta(n)}^{i_0+2}) \subset J_{\zeta(n)}^{i_0+2}$; $(h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)})(J_{\zeta(n)}^{i_0+2})^* \subset J_{\zeta(n)}^{i_0+2}$; $J_{\zeta(n)}^{i_0+2} \subset (h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)})(J_{\zeta(n)}^{i_0+2})$. In the first and third cases, the claim follows at once so we only need to consider the second case. Fix $n_0 \in \mathbb{N}$. Assume that $(h_{\zeta(n_0+1)}^{-1} \circ h_{\zeta(n_0)})(J_{\zeta(n_0)}^{i_0+2})^* \subset J_{\zeta(n_0)}^{i_0+2}$. If

there is a point a in $\text{Fix}_{h_{\zeta(n_0+1)}^{-1} \circ h_{\zeta(n_0)}} \overline{J_{\zeta(n_0)}^{i_0+2}}$, then

$$a \in (J_{\zeta(n_0)}^{i_0+2})^* \subset (h_{\zeta(n_0+1)}^{-1} \circ h_{\zeta(n_0)})(J_{\zeta(n_0)}^{i_0+2}).$$

This implies that $a = (h_{\zeta(n_0+1)}^{-1} \circ h_{\zeta(n_0)})^{-1}(a) \in J_{\zeta(n_0)}^{i_0+2}$ as $\text{Fix}_{h_{\zeta(n_0+1)}^{-1} \circ h_{\zeta(n_0)}} = \text{Fix}_{(h_{\zeta(n_0+1)}^{-1} \circ h_{\zeta(n_0)})^{-1}}$, so this is a contradiction. Therefore $\text{Fix}_{h_{\zeta(n_0+1)}^{-1} \circ h_{\zeta(n_0)}} \subset \overline{J_{\zeta(n_0)}^{i_0+2}}$ and the claim is proved. Therefore we can take a sequence $\{a_n\}_{n=1}^\infty$ so that for each $n \in \mathbb{N}$, $a_n \in \overline{J_{\zeta(n)}^{i_0+2}} \cap \text{Fix}_{h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)}}$. Then as $\{J_{\zeta(n)}^{i_0+2}\}_{n=1}^\infty$ is a quasi-rainbow at $x_\infty^{i_0+2}$, the sequence $\{a_n\}_{n=1}^\infty$ converges to $x_\infty^{i_0+2}$. Note again that for each $n \in \mathbb{N}$,

$$\begin{aligned} h_{\zeta(n)}(\text{Fix}_{h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)}}) &= h_{\zeta(n)}(\text{Fix}_{h_{\zeta(n)}^{-1} \circ h_{\zeta(n+1)}}) \\ &= \text{Fix}_{h_{\zeta(n)} \circ (h_{\zeta(n)}^{-1} \circ h_{\zeta(n+1)}) \circ h_{\zeta(n)}^{-1}} \\ &= \text{Fix}_{h_{\zeta(n+1)} \circ h_{\zeta(n)}^{-1}}. \end{aligned}$$

Hence the sequence $\{h_{\zeta(n)}(a_n)\}_{n=1}^\infty$ has at most two subsequential limits belonging to $\{y_\infty^{i_0}, y_\infty^{i_0+1}\}$ as the sequence $\{\text{Fix}_{h_{\zeta(n+1)}^{-1} \circ h_{\zeta(n)}}\}_{n=1}^\infty$ converges to $\{y_\infty^{i_0}, y_\infty^{i_0+1}\}$. Therefore, there is a subindex η of $\text{id}_\mathbb{N}$ such that the sequence $\{h_{\zeta \circ \eta(n)}(a_{\eta(n)})\}_{n=1}^\infty$ converges to $y_\infty^{j_0}$ for some $j_0 \in \{i_0, i_0+1\}$. Now we have that the two sequences $\{u_{\zeta \circ \eta(n)}^{i_0+2}\}_{n=1}^\infty$ and $\{u_{\zeta \circ \eta(n)}^{j_0}\}_{n=1}^\infty$ converge to $x_\infty^{i_0+2}$ and $x_\infty^{j_0}$, respectively, and two sequences $\{h_{\zeta \circ \eta(n)}(u_{\zeta \circ \eta(n)}^{i_0+2})\}_{n=1}^\infty$ and $\{h_{\zeta \circ \eta(n)}(u_{\zeta \circ \eta(n)}^{j_0})\}_{n=1}^\infty$ converge to $y_\infty^{i_0+2}$ and $y_\infty^{j_0}$, respectively. Moreover, for each $n \in \mathbb{N}$, $a_{\eta(n)}$ is a fixed point of the hyperbolic element $h_{\zeta(\eta(n)+1)}^{-1} \circ h_{\zeta(\eta(n))}$, and the two sequences $\{a_{\eta(n)}\}_{n=1}^\infty$ and $\{h_{\zeta \circ \eta(n)}(a_{\eta(n)})\}_{n=1}^\infty$ converge to $x_\infty^{i_0+2}$ and $y_\infty^{j_0}$, respectively. Also the sequence $\{h_{\zeta \circ \eta(n)}\}_{n=1}^\infty$ is an approximation sequence as the sequence $\{h_{\zeta(n)}\}_{n=1}^\infty$ is an approximation sequence. Therefore by Lemma 7.3, the sequence $\{h_{\zeta \circ \eta(n)}\}_{n=1}^\infty$ has the convergence property so the sequence $\{g_n\}_{n=1}^\infty$ has the convergence property. This is in contradiction to the assumption that the sequence $\{g_n\}_{n=1}^\infty$ does not have the convergence property. Hence, the first case is not possible. Likewise, in the second case, we can show that the sequence $\{g_n^{-1}\}_{n=1}^\infty$ has the convergence property and this implies that the sequence $\{g_n\}_{n=1}^\infty$ has the convergence property. This is also a contradiction. Therefore, we can conclude that there is no sequence $\{g_n\}_{n=1}^\infty$ of distinct elements of G that does not have the convergence property. Thus, G is a convergence group. $\square$

5. THE ELEMENTARY GEOMETRY AND TOPOLOGY OF 2-DIMENSIONAL HYPERBOLIC ORBIFOLDS

5.1. *Simple closed curves and simple closed geodesics in 2-dimensional hyperbolic orbifolds.* Let S be a complete hyperbolic 2-orbifold, and let $\alpha : I =$

$[0, 1] \rightarrow S$ be a closed curve. Fix a presentation $S = \mathbb{H}/G(S)$. A continuous map $\bar{\alpha} : \mathbb{R} \rightarrow \mathbb{H}$ is called a *full lift* of α if $\pi_S \circ \bar{\alpha}$ is the 1-periodic extension of α ; equivalently, $\pi_S \circ \bar{\alpha}$ is 1-periodic and the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\bar{\alpha}} & \mathbb{H} \\ i \uparrow & & \downarrow \pi_S \\ [0, 1] & \xrightarrow{\alpha} & S. \end{array}$$

A curve α is *regular* if it does not pass through a cone point, that is, if $\alpha(I) \cap \Sigma(S) = \emptyset$. Equivalently, a regular curve is a curve in $S - \Sigma(S)$.

Assume that α is regular and simple. Since π_S is a branched covering, the restriction $\pi_S : \mathbb{H} - \pi_S^{-1}(\Sigma(S)) \rightarrow S^\circ$ is a covering map. Hence, once a lift of $\alpha(0)$ is chosen, there is a unique lift $\tilde{\alpha} : [0, 1] \rightarrow \mathbb{H}$ of α . Since α is closed, there is a unique element $g \in G(S)$ such that $\tilde{\alpha}(1) = g\tilde{\alpha}(0)$. Fix a lift of $\alpha(0)$, and define $\beta : \mathbb{R} \rightarrow \mathbb{H}$ by $\beta(t) = \tilde{\alpha}(t)$ for $t \in [0, 1]$ and $\beta(t+1) = g\beta(t)$ for all $t \in \mathbb{R}$. Then β is a full lift of α .

We record the following elementary properties of full lifts.

PROPOSITION 5.1. *Let S be a complete hyperbolic 2-orbifold, and let α be a regular simple closed curve in S . Fix a presentation $S = \mathbb{H}/G(S)$. Then the following hold.*

1. *If $\tilde{p}$ is a lift of $\alpha(0)$, then there is a unique full lift $\bar{\alpha} : \mathbb{R} \rightarrow \mathbb{H}$ of α such that $\bar{\alpha}(0) = \tilde{p}$.*
2. *If $\bar{\alpha}$ is a full lift of α , then there is a unique element $g \in G(S)$ such that $\bar{\alpha}(t+1) = g\bar{\alpha}(t)$ for all $t \in \mathbb{R}$. We call this element the holonomy of $\bar{\alpha}$ and denote it by $\text{hol}(\bar{\alpha})$.*
3. *If β_1 and β_2 are full lifts of α , then there is a unique element $g \in G(S)$ such that $\beta_2 = g\beta_1$. In this case, $\text{hol}(\beta_2) = g\text{hol}(\beta_1)g^{-1}$.*

We now classify regular simple closed curves according to their holonomies. Let S be a complete hyperbolic 2-orbifold, and let α be a regular simple closed curve in S . Fix a presentation $S = \mathbb{H}/G(S)$. We call α *trivial*, *elliptic*, *parabolic*, or *hyperbolic* according as the holonomy of any full lift of α is trivial, nontrivial elliptic, parabolic, or hyperbolic. This is well-defined, since the holonomies of full lifts of α lie in the same conjugacy class in $G(S)$.

LEMMA 5.2. *Let S be a complete 2-dimensional hyperbolic orbifold and α be a regular simple closed curve from $I = [0, 1]$ to S . Then the following holds.*

1. *If α is trivial, then α is freely homotopic to a point in S° .*
2. *If α is elliptic, then α bounds a cone point.*
3. *If α is parabolic, then α bounds a puncture.*

PROOF. Fix a presentation $S = \mathbb{H}/G(S)$. First, we assume that α is trivial. Choose a full lift $\bar{\alpha}$ and denote $\text{hol}(\bar{\alpha})$ by g . As α is trivial, g is trivial. Then by definition, $\bar{\alpha}|_{[0, 1]}$ is a simple closed curve in $\mathbb{H}$. Hence there is a

closed disk B whose boundary is $\bar{\alpha}([0, 1])$. Now we want to show that there is no ramification point in B . Obviously, the boundary ∂B has no ramification point as α is regular. Assume that there is a ramification point x in the interior $\text{int } B$ and h is a generator of the stabilizer G_x . Since α is a regular simple closed curve, the pre-image $\pi_S^{-1}(\alpha(I))$ is a 1-dimensional manifold. Hence $\pi_S^{-1}(\alpha(I)) = \cup \{f(\partial B) : f \in G(S)\}$ and $\pi_S^{-1}(\alpha(I))$ is a disjoint union of simple closed curves. Therefore, if $h(\partial B) \neq \partial B$, then there are two possible cases: $h(B) \subsetneq B$ or $B \subsetneq h(B)$. Assume that $h(B) \neq B$ and choose a shortest geodesic segment ℓ joining x and ∂B . Now we say that the other end point of ℓ is y . As h is an isometry, $h(\ell)$ is also a shortest geodesic segment joining x and $h(\partial B)$. If $h(y)$ is in the interior $\text{int } B$ of B , then $h(B) \subset \text{int } B$ but this implies that the length of shortest geodesic segments joining x and $h(\partial B)$ is less than the length of ℓ . It is a contradiction. Similarly, $h(y)$ cannot be in the exterior $\text{ext } B$. Therefore, $h(B) = B$ and so $h(\partial B) = \partial B$. However, this is also a contradiction since α is simple and $\bar{\alpha}([0, 1])$ is a simple closed curve in $\mathbb{H}$. Thus, there is no ramification point in B so $\pi_S(B)$ is a disk with $\partial\pi_S(B) = \alpha(I)$.

Next, we assume that α is elliptic. Choose a full lift $\bar{\alpha}$ and denote $\text{hol}(\bar{\alpha})$ by g . As α is elliptic, g is an elliptic element of order n for some $n \in \mathbb{N}$ with $2 \leq n$. Then the full lift is periodic with period n and so $\bar{\alpha}([0, n])$ is a simple closed curve in $\mathbb{H}$. Moreover, $\bar{\alpha}([0, n])$ bounds a disk with a ramification point x which is the fixed point of g . Let B be the closed disk with $\partial B = \bar{\alpha}([0, n])$. As α is regular, ∂B has no ramification point. Now we claim that there is no ramification point except x in $\text{int } B$. Assume that there is a ramification point y in $\text{int } B - \{x\}$ and h is a generator of the stabilizer G_y . Since α is a regular simple closed curve, the pre-image $\pi_S^{-1}(\alpha(I))$ is a 1-dimensional manifold. Hence $\pi_S^{-1}(\alpha(I)) = \cup \{f(\partial B) : f \in G(S)\}$ and $\pi_S^{-1}(\alpha(I))$ is a disjoint union of simple closed curves. Assume that $h(\partial B) \neq \partial B$ and choose a shortest geodesic segment ℓ joining y and ∂B . Then $h(\ell)$ is also a shortest geodesic segment joining y and $h(\partial B)$ as h is an isometry. Now let z be the other end point of ℓ . If $h(z) \in \text{int } B$, then $h(B) \subset \text{int } B$ and this implies that the length of shortest geodesic segments joining y and $h(\partial B)$ is less than the length of ℓ . This is a contradiction. Similarly, the case where $h(z) \in \text{ext } B$ is not possible, so $h(\partial B) = \partial B$. Note that $A = \{g^i \bar{\alpha}(0) : i \in \{0, 1, \dots, n-1\}\}$ equals $\pi_S^{-1}(\alpha(0)) \cap \partial B$. Hence h preserves A so $h\bar{\alpha}(0) = g^m \bar{\alpha}(0)$ for some $m \in \{1, \dots, n-1\}$. Then $g^{-m}h$ fixes $\bar{\alpha}(0)$. If $g^{-m}h$ is not trivial and is elliptic, this is in contradiction with that α is regular. Therefore, $g^{-m}h$ is trivial but this is also a contradiction since $x \neq y$. Thus, there is no such a y so we can conclude that $\pi_S(B)$ is a closed disk with one cone point and $\partial\pi_S(B) = \alpha(I)$.

Finally, we consider the case where α is parabolic. Choose a full lift $\bar{\alpha}$ and denote $\text{hol } \bar{\alpha}$ by g . As α is parabolic, g is parabolic. Then by definition,

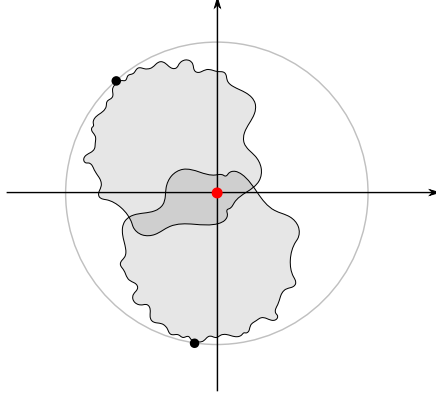


FIGURE 2. This is a schematic picture on the Poincaré disk when B contains a ramification point. The black marked points correspond to x_∞ and $h(x_\infty)$, and the shaded regions are B and $h(B)$. The red point at the origin is the ramification point and h is the rational rotation fixing the red point.

$\bar{\alpha}$ is an embedding and there is a point x_∞ in $\partial_\infty \mathbb{H}$ which is the fixed point of g such that $\lim_{t \rightarrow \infty} \bar{\alpha}(t) = \lim_{t \rightarrow -\infty} \bar{\alpha}(t) = x_\infty$. Hence in $\hat{\mathbb{C}}$, there is a closed disk B such that $\text{int } B \subset \mathbb{H}$ and $\partial B = \bar{\alpha}(\mathbb{R}) \cup \{x_\infty\}$. Now we want to show that B has no ramification point. Obviously, there is no ramification point in ∂B as α is regular. Assume that there is a ramification point y in $\text{int } B$ which is the fixed point of a non-trivial elliptic element $h \in G(S)$. Then $h(B) \cap B \neq \emptyset$ and since $h(x_\infty) \neq x_\infty$, $h(\partial B) \neq \partial B$ and $h(\partial B) \cap \partial B \neq \emptyset$ (See Figure 2). This is a contradiction since α is simple. Therefore, there is no ramification point in B . Thus, $\pi_S(B)$ is a disk with one puncture and $\partial \pi_S(B) = \alpha(I)$. $\square$

REMARK 5.3. *The converse of each assertion in Lemma 5.2 also holds.*

A regular simple closed curve is said to be *essential* if it does not bound a disk, a disk with one cone point, or a disk with one puncture.

PROPOSITION 5.4. *Let S be a complete 2-dimensional hyperbolic orbifold and α be an essential regular simple closed curve in S . Fix a presentation $S = \mathbb{H}/G(S)$. Then α is hyperbolic and every full lift of α is a quasi-geodesic in $\mathbb{H}$.*

When α is a hyperbolic simple closed curve, each full lift is a quasi-geodesic so it has a unique geodesic connecting two end points of the full lift. So $\pi_S^{-1}(\alpha)$ is a disjoint union of quasi-geodesics and is $G(S)$ -invariant. Now let Q be the set of images of full lifts of α and Q' be the set of geodesic

representatives of elements of Q , namely Q' is the set of hyperbolic axes of holonomies of all full lifts of α . Then Q' is also a disjoint union of bi-infinite geodesics and is $G(S)$ -invariant. By construction, for any geodesic ℓ in Q' , $Q' = \bigcup G(S) \cdot \ell$. Now we say that $\pi_S(Q')$ is the *geodesic realization* of α . Note that geodesic realizations are simple geodesics.

The following lemma was proved in [29] in a slightly different form.

LEMMA 5.5 ([29]). *Let S be a complete 2-dimensional hyperbolic orbifold and α be an essential regular simple closed curve in S . The following holds.*

1. *If α does not bound a disk with two cone points of order two, then the geodesic realization of α is in the free homotopy class of α in S^o .*
2. *If α bounds a disk with two cone points of order two, then a geodesic segment connecting the cone points is the geodesic realization of α .*

PROOF. Fix a presentation $S = \mathbb{H}/G(S)$. First assume that α is a hyperbolic simple closed curve. Choose a full lift $\bar{\alpha}$ of α and denote $\text{hol}(\bar{\alpha})$ by g . Since $\bar{\alpha}$ is a quasi-geodesic and $J = \bar{\alpha}(\mathbb{R})$ is preserved by g , the hyperbolic axis ℓ of g is the geodesic representative of $\bar{\alpha}$. Now we say that x and y in $\partial_\infty \mathbb{H}$ are two end points of ℓ and write $\bar{\ell}$ and $\bar{J}$ for $\ell \cup \{x, y\}$ and $J \cup \{x, y\}$, respectively. Now we think of $\bar{\ell}$ and $\bar{J}$ as images of two proper embedding δ_ℓ and δ_J , respectively, from $[0, 1]$ to $\bar{\mathbb{H}}$ such that $\delta_\ell(0) = \delta_J(0) = x$ and $\delta_\ell(1) = \delta_J(1) = y$.

Choose a ramification point r in $\mathbb{H} - (\ell \cup J)$ and a generator h of the stabilizer G_r . Note that since α is regular, there is no ramification point in J . Now we write π_{G_r} for the quotient map from $\mathbb{H}$ to $\mathbb{H}/G_r$. If h preserves ℓ , then the order of h must be two and so r must be in ℓ . This is a contradiction. Hence, h does not preserve ℓ . Therefore, $\pi_{G_r}(\delta_\ell)$ and $\pi_{G_r}(\delta_J)$ are paths in $\bar{\mathbb{H}}/G_r$ which do not pass through the cone point in $\mathbb{H}/G_r$. Moreover, as α is simple, $\pi_{G_r}(\delta_\ell)$ and $\pi_{G_r}(\delta_J)$ are simple. Now we consider the case where ℓ has no ramification point. Then the stabilizer $\text{Stab}_{G(S)}(\ell)$ of ℓ is generated by the hyperbolic element g so $\pi_S(\ell)$ is a regular simple closed geodesic. On the other hand, the choice of r is arbitrary and so for any ramification point r , we get that $\pi_{G_r}(\delta_\ell)$ and $\pi_{G_r}(\delta_J)$ are simple. This implies that the regular simple closed geodesic $\pi_S(\ell)$ and $\alpha(I)$ are in same free homotopy class in S^o (see Figure 3 in [29]). Therefore, we can take a free homotopy between $\pi_S(\ell)$ and $\alpha(I)$ in S^o . Thus, the first statement is proven.

Then we consider the case where there is a ramification point r' in ℓ . Then the ramification index of r' must be two and the stabilizer $G_{r'}$ preserves ℓ . Now we focus on the stabilizer group $\text{Stab}_{G(S)}(\ell)$ of ℓ . Since we may think of $\text{Stab}_{G(S)}(\ell)$ as a discrete subgroup of the isometry group of the real line, $\text{Stab}_{G(S)}(\ell)$ is isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}$, or the infinite dihedral group Dih_∞ . Obviously, $G_{r'} \subset \text{Stab}_{G(S)}(\ell)$ and $g \in \text{Stab}_{G(S)}(\ell)$. Let s be a geodesic segment joining r' and $g^{-1}(r')$ and f be the non-trivial element of $G_{r'}$. Then the end

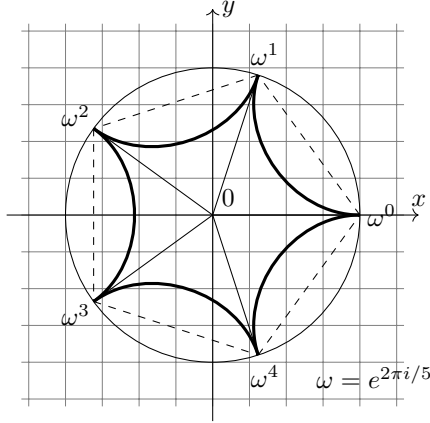


FIGURE 3. Thick arcs are hyperbolic geodesics in the Poincaré disk $\mathbb{D}$ joining consecutively the points $\{1, \omega, \omega^2, \omega^3, \omega^4\}$ in the infinite circle $\partial\mathbb{D}$. Hence, the geodesics bound an ideal 5-gon. Moreover, it is preserved by an elliptic isometry defined as $z \mapsto \omega z$. Therefore, the ideal 5-gon is regular and 0 is the center.

points of $g(s)$ are r' and $g(r')$. Therefore, $(f \circ g)(g^{-1}(r')) = f(r') = r'$, $(f \circ g)(r') = g^{-1}(r')$, and $(f \circ g)(s) = s$ since f is an isometry in $G_{r'}$. Hence, fg fixes the midpoint r'' between r' and $g^{-1}(r')$ and preserves ℓ . Then we can see that f and fg are elliptic elements of order two and generate $\text{Stab}_{G(S)}(\ell)$. Therefore, $\text{Stab}_{G(S)}(\ell)$ is isomorphic to Dih_∞ , namely $\text{Stab}_{G(S)}(\ell) = \langle f, fg | f^2 = (fg)^2 = \text{id} \rangle$.

Now we write $\pi_{\text{Stab}_{G(S)}(\ell)}$ for the projection from $\mathbb{H}$ to $\mathbb{H}/\text{Stab}_{G(S)}(\ell)$. Then

$$\Sigma(\mathbb{H}/\text{Stab}_{G(S)}(\ell)) = \{\pi_{\text{Stab}_{G(S)}(\ell)}(r'), \pi_{\text{Stab}_{G(S)}(\ell)}(r'')\}$$

and $\pi_{\text{Stab}_{G(S)}(\ell)}(\ell)$ is the geodesic segment joining two cone points. Also, $\pi_{\text{Stab}_{G(S)}(\ell)}(\bar{\alpha})[0, 1]$ is a simple closed curve bounding a disk with the cone points. This implies the second statement. $\square$

The proof of the following proposition is similar to the corresponding argument for complete hyperbolic surfaces.

PROPOSITION 5.6. *Let S be a complete 2-dimensional hyperbolic orbifold, and α and β essential regular simple closed curves in S . If α and β are disjoint and are not freely homotopic in S^o , then the geodesic realizations are disjoint.*

5.2. Elementary 2-dimensional hyperbolic orbifolds.

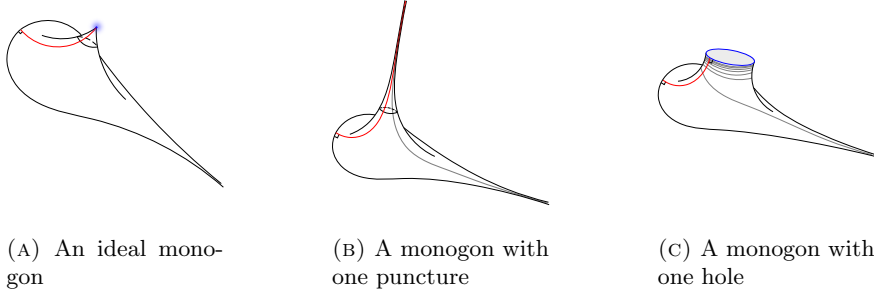


FIGURE 4. In Figure 4a, the blue point is the cone point. In Figure 4b and Figure 4c, the gray lines are the images of s_1 and s_2 under the identification by f_{12} . Hence, the gray lines are geodesics in each 2-dimensional hyperbolic orbifold with geodesic boundary. In particular, the blue circle in Figure 4c is the simple closed geodesic added to be complete. And the red arcs are the geodesics ℓ_S .

5.2.1. *Ideal monogons.* Let n be a natural number which is greater than 2. Recall that an *ideal n -gon* is a hyperbolic surface S with geodesic boundary if the interior of S is homeomorphic to the disk, the boundary of S is a disjoint union of n bi-infinite geodesics, and the area of S is finite (see Figure 3). Each boundary component of S is called a *side* and each end point of a side is called a *vertex*. We think of an ideal n -gon S as a closed convex subset of $\mathbb{H}$. The ideal n -gon S is called *regular* if there is an elliptic element r_S in $\mathrm{PSL}_2(\mathbb{R})$ which is a rotation through angle $2\pi/n$ about some point c_S in S and preserves S . We call c_S the *center* of S . Then the quotient space $S/\langle r_S \rangle$ of S by the group generated by r_S is called an *ideal monogon* with a cone point of order n (see Figure 4a). The quotient space $S/\langle r_S \rangle$ is a 2-dimensional hyperbolic orbifold with one geodesic boundary and the cone point is the image of the center c_S . Hence, the order of the cone point is n and there is a unique shortest geodesic segment ℓ_S joining the cone point with the geodesic boundary. Note that the segment ℓ_S intersects perpendicularly with the geodesic boundary.

5.2.2. *Geometric annuli.* Let P_3 be an ideal triangle in $\mathbb{H}$. Note that every ideal triangle is regular. We say that s_1 , s_2 , and s_3 are the sides of P_3 . For each $i \in \{1, 2, 3\}$, there is the unique perpendicular from the center c_{P_3} to the geodesic s_i and so we say that p_i is the foot of the perpendicular. Now we glue s_1 and s_2 by an element f_{12} in $\mathrm{PSL}_2(\mathbb{R})$ such that $f_{12}(s_1) = s_2$. Then f_{12} fixes the end point q in $\partial_\infty \mathbb{H}$ at which s_1 and s_2 intersect.

If f_{12} is a parabolic element and $f_{12}(p_1) = p_2$, the resulting surface S is Cauchy complete under the metric induced by the hyperbolic metric of P_3 , and

has one bi-infinite geodesic boundary. Hence, we call this resulting orbifold a *monogon with one puncture* (see Figure 4b). In this case, there is a unique geodesic ℓ_S joining the puncture with s_3 which intersect perpendicularly with s_3 . Moreover, $\ell_S \cap s_3 = \{p_3\}$.

If f_{12} is hyperbolic and so $f_{12}(p_1) \neq p_2$, then the resulting space is incomplete. By adding a simple closed geodesic, we can make the resulting space be complete, and in this case, the resulting space S is a hyperbolic surface which has one bi-infinite geodesic boundary and one closed geodesic boundary. Hence, we call this space a *monogon with one hole* (see Figure 4c). The length of the closed geodesic boundary is determined by the translation length of f_{12} . Hence, for any given positive number l , there is a monogon with one hole whose closed geodesic boundary is of length l . Note that there is a unique shortest geodesic segment ℓ_S joining two geodesic boundaries which intersect perpendicularly with both geodesic boundaries. Moreover, it turns out that $\ell_S \cap s_3 = \{p_3\}$.

From the construction of a monogon S with one hole or with one puncture, there is a bi-infinite geodesic decomposing S into an ideal triangle which is the gray geodesics in Figure 4. Now we call a 2-dimensional hyperbolic orbifold S with geodesic boundary a *geometric annulus* if S is isometric to an ideal monogon with one cone point, a monogon with one puncture, or a monogon with one hole (see Figure 4). Note that a geometric interior of a geometric annulus is homeomorphic to the annulus. Also, we call the intersection point of ℓ_S and the bi-infinite geodesic boundary the *shearing point* of S .

5.2.3. Geometric pair of pants.

DEFINITION 5.7. A geometric pair of pants is a 2-dimensional hyperbolic orbifold with geodesic boundary satisfying the following:

1. The geometric interior is homeomorphic to the thrice-punctured sphere;
2. The geometric boundary is compact and has exactly three connected components;
3. The hyperbolic area is finite.

All geometric pairs of pants fall into four types according to the number of closed geodesic boundary components (See 5). For brevity, until the end of this section, we think of a puncture as a cone point of order ∞ . Then in Figure 5, each marked point is a cone point of order n for some $n \in \mathbb{N} \cup \{\infty\}$ with $n \neq 1$. More precisely, in Figure 5c, two marked points can not be cone points of order two simultaneously. In Figure 5d, if three marked points are of order n_1 , n_2 and n_3 , respectively, then n_1 , n_2 and n_3 satisfy

$$2 - \left\{ \left(1 - \frac{1}{n_1}\right) + \left(1 - \frac{1}{n_2}\right) + \left(1 - \frac{1}{n_3}\right) \right\} < 0$$

,or

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} < 1$$

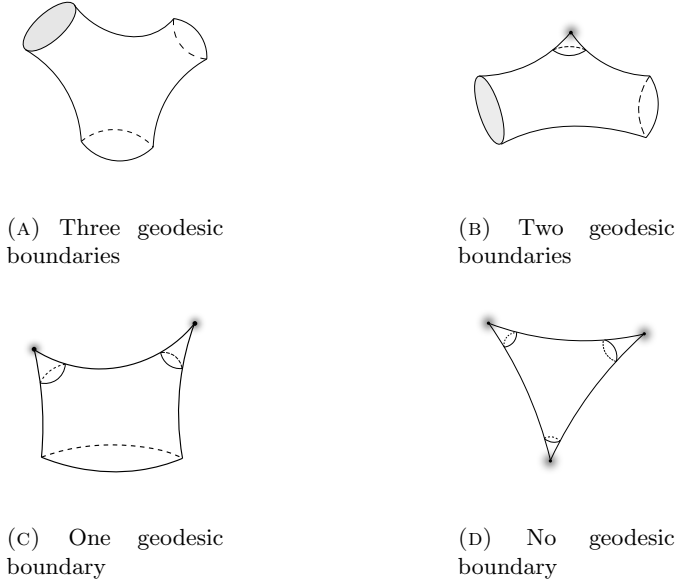


FIGURE 5. There are four types of geometric pairs of pants according to the number of closed geodesic boundary components. Each marked point is a cone point or a puncture. However, in Figure 5c, two marked points can not be cone points of order two simultaneously.

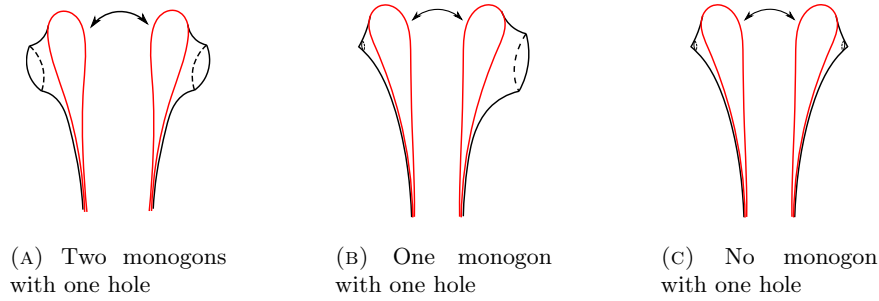


FIGURE 6. Each pointy point represents a puncture or a cone of order $n > 2$. Then there are three cases according to the number of used monogons with one hole in the glueing process. The red lines are the bi-infinite geodesics b_1 and b_2 along which we glue two monogons.

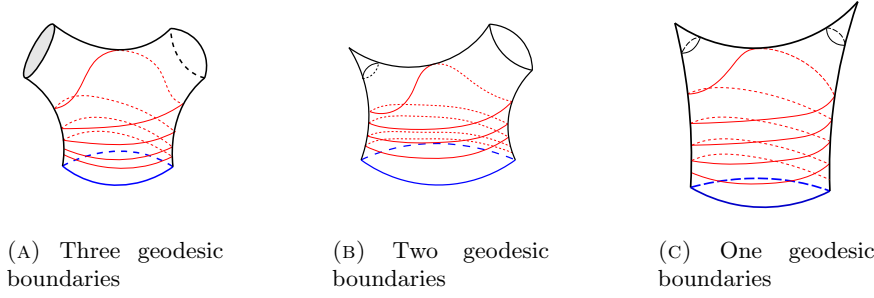


FIGURE 7. Each pointy point represents a puncture or a cone of order $n > 2$. These pairs of pants are corresponding to the first three types in Figure 5. In Figure 6, after gluing along red lines, the resulting metric spaces are not usually complete. Therefore, we need to add a simple closed geodesic to be complete. The blue curves represent the additional simple closed geodesics. Then, after gluing and completion, we get the geometric pairs of pants. Precisely, Figure 7a, Figure 7b or Figure 7c are obtained from Figure 6a, Figure 6b or Figure 6c, respectively.

where $1/\infty = 0$. In other words, the Euler characteristic must be negative.

Let A_1 and A_2 be geometric annuli and say that the bi-infinite geodesic boundaries of A_1 and A_2 are b_1 and b_2 , respectively. Each boundary b_i has the shearing point sh_i . Now we glue A_1 and A_2 along b_1 and b_2 by an isometry f_{12} of real line such that $f_{12}(b_1) = b_2$ (See Figure 6). Then we get a space S which may be incomplete. If S is complete, then S is a geometric pair of pants. If S is not complete, by adding a simple closed geodesic as a boundary, we can make S be complete. Then S is also a geometric pair of pants (see Figure 7). The length of the additional closed geodesic is determined by the distance between $f_{12}(sh_1)$ and sh_2 . Note that for any given positive real number l , we can find a map f_{12} which makes the additional closed geodesic be of length l . Using this construction, we may get any given geometric pair of pants without cone point of order two that is one of the first three types in Figure 5. Also, observe that in the geometric pairs of pants, the red geodesics in Figure 7 and the gray geodesics in Figure 4a with the boundary geodesics decompose the geodesic pairs of pants into ideal monogons and ideal triangles.

Similarly, we consider the case where a geometric pair of pants has a cone point of order 2. Let f_{11} be an isometry of b_1 such that f_{11} is a reflection on b_1 and so there is a fixed point c_1 (see Figure 8). Now a quotient space S is

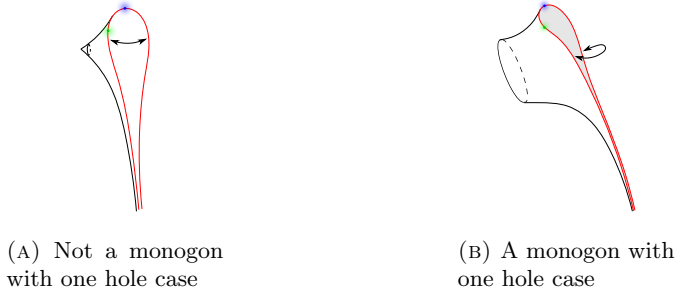


FIGURE 8. The pointy point in Figure 8a represents a puncture or a cone point of order $n > 2$. The red lines are the bi-infinite geodesics b_1 . The blue marked points are the fixed points c_1 of the isometries f_{11} . The green marked points are the shearing points sh_1 of the geometric annuli.

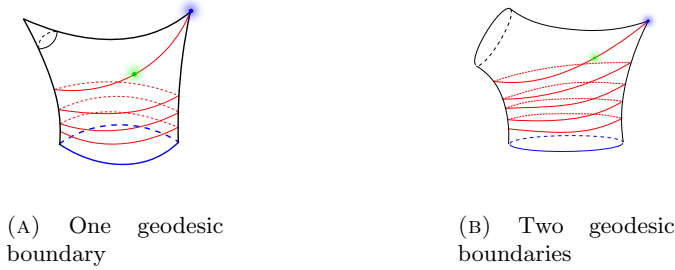


FIGURE 9. The blue marked points are the images of b_1 and the green ones are the images of sh_1 . Moreover, the blue points are cone points of order two. The blue curves are the additional simple closed geodesics.

obtained from A_1 by identifying $x \in b_1$ with $f_{11}(x)$. Then if $c_1 = sh_1$, then S is complete and is a geometric pair of pants with one cone point of order two. If $c_1 \neq sh_1$, then S is not complete and so by adding a closed geodesic as a boundary, we make S be complete (see 7). Hence, S is a geometric pair of pants with one cone point of order two. The length of the additional closed geodesic boundary component is determined by the distance between c_1 and sh_1 . Like a previous case, using this construction, we can get any given geometric pair of pants with a cone point of order two that is second or third type in Figure 5. Also, observe that in the geometric pairs of pants, the red geodesics in Figure 7 and the gray geodesics in Figure 4a with the boundary

geodesics decompose the geodesic pair of pants into ideal monogons and ideal triangles.

From observations, we may conclude that any geometric pair of pants, which does not have three cone points, can be decomposed into two or one geometric annuli. Moreover, the geometric pairs of pants are decomposed into ideal monogons and ideal triangles. Now we summarize the above in the following lemma.

LEMMA 5.8. *Let S be a geometric pair of pants. Suppose that S does not have three cone points of finite order. (i.e. S is not the type of Figure 5d) There is a geodesic lamination such that the metric completion of each connected component of the complement of the lamination is isometric to an ideal triangle or an ideal monogon with one cone point.*

6. GEODESIC LAMINATIONS IN COMPLETE 2-DIMENSIONAL HYPERBOLIC ORBIFOLDS

In this section, we prove the structure theorem for complete 2-dimensional hyperbolic orbifolds following [6]. Based on this, we show that any Fuchsian group G such that $\mathbb{H}/G$ is not a geometric pair of pants is pants-like COL_3 ; see Theorem B.

6.1. A structure theorem for complete 2-dimensional hyperbolic orbifolds.

6.1.1. *Half-planes and funnels in a complete 2-dimensional hyperbolic orbifold.* Let S be a hyperbolic surface with geodesic boundary. We define $H = \{z \in \mathbb{H} : 0 \leq \text{Re}(z)\}$. Then if S is isometric to H with the metric induced from ρ , then S is called a *half-plane*. If S is isometric to H/G where G is generated by $z \mapsto e^h z$ for some $h \in \mathbb{R}_{>0}$, then S is called a *funnel* with boundary length h .

Let G be a Fuchsian group and S the complete 2-dimensional hyperbolic orbifold $\mathbb{H}/G$. For a closed subset A of $\partial\mathbb{H}$, there is a unique smallest convex subset $CH(A)$ in $\mathbb{H}$ such that $\overline{CH(A)} \cap \partial\mathbb{H} = A$. Then $CH(A) = \overline{CH(A)} \cap \mathbb{H}$ is also a convex subset. Hence, we call $CH(A)$ the *convex hull* of A . When G is not elementary, the *convex core* $CC(S)$ of S is the quotient of the convex hull of $L(G)$ by G , $CH(L(G))/G$. In general, the convex core of a complete 2-dimensional hyperbolic orbifold is a 2-dimensional hyperbolic orbifold with geodesic boundary. In particular, when G is a Fuchsian group of the first kind, S itself is the convex core of S .

Assume that G is non-elementary and of the second kind. Then its limit set $L(G)$ is a Cantor set in $\partial\mathbb{H}$, and $\Omega(G) \cap \partial\mathbb{H}$ is a countable disjoint union of nondegenerate open intervals. For each component I of $\Omega(G) \cap \partial\mathbb{H}$, the stabilizer $\text{Stab}_G(I)$ is either trivial or infinite cyclic.

Let $\bar{I}$ be the closure of I in $\partial\mathbb{H}$, and consider $CH(\bar{I})$. Then $CH(\bar{I})$ is a hyperbolic half-plane, and $\text{Stab}_G(CH(\bar{I})) = \text{Stab}_G(I)$. Thus $CH(\bar{I})/\text{Stab}_G(CH(\bar{I}))$

isometrically embeds in S . Moreover, this quotient is a half-plane if $\text{Stab}_G(CH(\bar{I}))$ is trivial, and is a funnel if $\text{Stab}_G(CH(\bar{I}))$ is infinite cyclic.

In the latter case, $\text{Stab}_G(CH(\bar{I}))$ is generated by a hyperbolic element $g \in \text{PSL}_2(\mathbb{R})$, whose axis is the geodesic boundary of $CH(\bar{I})$. Hence the boundary length of the funnel $CH(\bar{I})/\text{Stab}_G(CH(\bar{I}))$ is equal to the translation length of g . Finally, note that $\pi_1(CC(S)) = \pi_1(S)$, where $S = \mathbb{H}/G$.

6.1.2. Pants decompositions. Let S be a 2-dimensional hyperbolic orbifold with geodesic boundary. Then a collection $\mathcal{C}$ of non-trivial simple closed curves in S° which are homotopically distinct and disjoint is called a *pants decomposition* of S if each connected component of $S^\circ - \bigcup \mathcal{C}$ is homeomorphic to the sphere without three points and $\mathcal{C}$ is locally finite. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two pants decompositions of S . We say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *distinct* if for each $(c_1, c_2) \in \mathcal{C}_1 \times \mathcal{C}_2$, c_1 and c_2 are not freely homotopic in S° .

A geodesic lamination Λ of S is called a *geometric pants decomposition* if the Cauchy completion of each connected component of $S - \Lambda$ is a geometric pair of pants. Two geodesic laminations Λ_1 and Λ_2 in S are called *transverse* if they share no leaf.

Let G be a Fuchsian group that is not a finite group and S the complete 2-dimensional hyperbolic orbifold $\mathbb{H}/G$. If G is torsion free, then S is a complete hyperbolic surface. If not, there is a ramification point p in $\mathbb{H}$ and any point in the orbit $G \cdot p$ is also a ramification point. Then, since G is not finite and the set of ramification points is discrete in $\mathbb{H}$, there are countably many ramification points. Hence, the complement $\mathbb{H} - \pi_G^{-1}(\Sigma(S))$ of ramification points in $\mathbb{H}$ is a Riemann surface whose fundamental group is a free group of infinite rank. Note that $(\pi_G)_*(\pi_1(\mathbb{H} - \pi_G^{-1}(\Sigma(S))))$ is a normal subgroup of $\pi_1(S^\circ)$. Then the universal covering of $\mathbb{H} - \pi_G^{-1}(\Sigma(S))$ is $\mathbb{H}$ with a holomorphic covering map $\tilde{\pi}_G$ such that the deck transformation group of $\tilde{\pi}_G$ is isomorphic to $(\pi_G)_*(\pi_1(\mathbb{H} - \pi_G^{-1}(\Sigma(S))))$. We write K_G for the deck transformation group of $\tilde{\pi}_G$. Then we define

$$H_G = \{h \in \text{Aut}(\mathbb{H}) : \tilde{\pi}_G \circ h = g \circ \tilde{\pi}_G \text{ for some } g \in G\}.$$

Then H_G is isomorphic to $\pi_1(S^\circ)$ and is a Fuchsian group. Hence, $\pi_G \circ \tilde{\pi}_G$ is a universal covering map from $\mathbb{H}$ to S° whose deck transformation group is H_G .

Now we define a homomorphism $\tilde{\pi}_G^*$ from H_G to G so that for each $h \in H_G$, $\tilde{\pi}_G \circ h = \tilde{\pi}_G^*(h) \circ \tilde{\pi}_G$. Then we have an exact sequence

$$\{1\} \longrightarrow K_G \xrightarrow{i} H_G \xrightarrow{\tilde{\pi}_G^*} G \longrightarrow \{1\}$$

where i is the inclusion map. Hence, H_G/K_G is isomorphic to G . For more detail, consult the Kra's book [24].

Now we prove the following lemma which is analogous to Proposition 3.1 in [6].

THEOREM 6.1 (The structure theorem for complete hyperbolic 2-dimensional orbifolds). *Let G be a non-elementary Fuchsian group and S be the complete 2-dimensional hyperbolic orbifold $\mathbb{H}/G$. Suppose that there is a pants decomposition $\mathcal{C}$ of S . Let Λ be the union of all geodesic realizations in S of elements of $\mathcal{C}$. Then $\Lambda \cup (\partial CH(L(G))/G)$ is a geometric pants decomposition of $CC(S)$.*

PROOF. Note that any pants decomposition induces an exhaustion by finite type subsurfaces. Here being finite type means that the fundamental group is finitely generated. Hence there is a sequence $\{S_n\}_{n=1}^\infty$ of subsurfaces of S^o such that for each $n \in \mathbb{N}$, $S_n \subset S_{n+1}$ and the manifold boundary ∂S_n consists of curves of $\mathcal{C}$ and $\bigcup_{n=1}^\infty \text{int}(S_n) = S^o$. We may assume that for each $n \in \mathbb{N}$, every boundary curve of S_n does not bound a disk with two cone points of order two in S . Therefore, there is a sequence $\{H_n\}_{n=1}^\infty$ of subgroups of H_G such that for all $n \in \mathbb{N}$, $H_n < H_{n+1}$ and H_n is a corresponding subgroup of H_G to $\pi_1(S_n)$. Note that $H_G = \bigcup_{n=1}^\infty H_n$ as $S^o = \bigcup_{n=1}^\infty S_n$. Moreover, for each $n \in \mathbb{N}$, there is a unique connected component $\tilde{S}_n$ of $(\pi_G \circ \tilde{\pi}_G)^{-1}(S_n)$ such that the stabilizer $\text{Stab}_G(\tilde{S}_n)$ is H_n and so $\tilde{S}_n \subseteq \tilde{S}_{n+1}$. Note that for each $n \in \mathbb{N}$, the manifold boundary $\partial \tilde{S}_n$ consists of full lifts of the boundary curves of S_n .

Now we consider the sequence $\{G_n\}_{n=1}^\infty$ of subgroups of G where $G_n = \tilde{\pi}_G^*(H_n)$. Obviously, $G = \bigcup_{n=1}^\infty G_n$. Also, for each $n \in \mathbb{N}$, the stabilizer of $\tilde{\pi}_G(\tilde{S}_n)$ is G_n and the manifold boundary of $\tilde{\pi}_G(\tilde{S}_n)$ consists of full lifts of the boundary curves of S_n . Fix $n \in \mathbb{N}$. By assumption, each boundary component of $\tilde{\pi}_G(\tilde{S}_n)$ is a quasi-geodesic under the Poincaré metric in $\mathbb{H}$ and $\tilde{\pi}_G(\tilde{S}_n)$ is a connected component of $\pi_G^{-1}(S_n)$. Let B_n be the set of all hyperbolic axes of holonomies of boundary components of $\tilde{\pi}_G(\tilde{S}_n)$. Then there is a closed convex subset C_n of $\mathbb{H}$ such that $\partial C_n = B_n$. Then by the choice of C_n , G_n preserves C_n and the stabilizer of C_n is G_n under the G -action. Therefore, $\pi_G(C_n)$ is isometric to the 2-dimensional hyperbolic orbifold C_n/G_n with geodesic boundary. Moreover, $\pi_G(C_n)$ is the 2-dimensional hyperbolic orbifold which is bounded by the geodesic realizations of the boundary curves of S_n .

Since C_n is a closed convex subset of $\mathbb{H}$ preserved by G_n , $CH(L(G_n)) \subseteq C_n$. Since each boundary component of C_n is a hyperbolic axis for some hyperbolic element of G_n , the end points of each boundary component of C_n are contained in $L(G_n)$. Hence, $\partial C_n \subset \partial CH(L(G_n))$. Since G_n is finitely generated, for each boundary component ℓ in $\partial CH(L(G_n)) - \partial C_n$, $\pi_G(\ell)$ bounds a funnel in $\pi_G(C_n)$. (See Section 10.4. in [7])

Now we claim that $CH(L(G)) = \overline{\bigcup_{n=1}^{\infty} CH(L(G_n))}$. As $L(G_n) \subseteq L(G_{n+1})$, we have that $CH(L(G_n)) \subseteq CH(L(G_{n+1}))$ and so $\bigcup_{n=1}^{\infty} CH(L(G_n))$ is convex. Then G preserves the closed convex subset $\overline{\bigcup_{n=1}^{\infty} CH(L(G_n))}$ of $\mathbb{H}$. Hence, $CH(L(G)) \subseteq \overline{\bigcup_{n=1}^{\infty} CH(L(G_n))}$. On the other hand, for any $n \in \mathbb{N}$, $L(G_n) \subseteq L(G)$ and $CH(L(G_n)) \subseteq CH(L(G))$. Therefore,

$$\bigcup_{n=1}^{\infty} CH(L(G_n)) \subseteq CH(L(G)) \subseteq \overline{\bigcup_{n=1}^{\infty} CH(L(G_n))}.$$

The claim is proved since $CH(L(G))$ is closed in $\mathbb{H}$.

Let ℓ be a bi-infinite geodesic in $\partial CH(L(G))$. If ℓ is a manifold boundary component of $\bigcup_{n=1}^{\infty} CH(L(G_n))$, there is a natural number n_0 such that $\ell \subset \partial CH(L(G_{n_0}))$. Then $\pi_G(\ell)$ bounds a funnel in S . If ℓ is in $CH(L(G)) - \bigcup_{n=1}^{\infty} CH(L(G_n))$, then there is a sequence $\{\ell_n\}_{n=1}^{\infty}$ of geodesics such that for each $n \in \mathbb{N}$, $\ell_n \subset \partial CH(L(G_n))$ and $\{\ell_n\}_{n=1}^{\infty}$ converges to ℓ . Then $\{\pi_G(\ell_n)\}_{n=1}^{\infty}$ converges to $\pi_G(\ell)$. If $\pi_G(\ell)$ is a regular simple closed geodesic, then $\pi_G(\ell) \subset \text{int}(S_{n_1})$ for some $n_1 \in \mathbb{N}$ since $\bigcup_{n=1}^{\infty} \text{int}(S_n) = S^\circ$. Hence, $\pi_G(\ell) \subset \pi_G(C_{n_1})$ and $\pi_G(\ell) \subset \pi_G(CH(L(G_{n_1})))$, so $\ell \subset CH(L(G_{n_1}))$. This is a contradiction. Therefore, $\pi_G(\ell)$ is a bi-infinite geodesic and bounds a half-plane in S .

Now we have that $\{\pi_G(CH(L(G_n)))\}_{n=1}^{\infty}$ is an increasing sequence of 2-dimensional hyperbolic orbifolds with geodesic boundary whose orbifold fundamental group are finitely generated and such that $\bigcup_{n=1}^{\infty} \pi_G(CH(L(G_n)))$ is $CC(S)$ without bi-infinite geodesic boundaries. Finally, by the definition of a pants decomposition, each element in $\mathcal{C}$ has a geodesic realization and the geodesic realization is contained in $\pi_G(CH(L(G_{n_2})))$ for some $n_2 \in \mathbb{N}$. Therefore, for $n \in \mathbb{N}$, $\Lambda \cap \pi_G(CH(L(G_n)))$ is a geometric pants decomposition of $\pi_G(CH(L(G_n)))$. Thus, we are done. $\square$

REMARK 6.2. *If G is of the first kind, then Λ itself is a geometric pants decomposition of S .*

6.2. *Fuchsian groups of the first kind are pants-like COL_3 .*

LEMMA 6.3. *Let G be a Fuchsian group of the first kind and S be the complete 2-dimensional hyperbolic orbifold $\mathbb{H}/G$. Suppose that S is not a geometric pants. Then there are three pairwise transverse geometric pants decompositions.*

PROOF. If S° is a Riemann surface of finite type, then there is a nonempty pants decomposition since S is not a geometric pants. Otherwise, by Richards classification result [28], S° has a topological exhaustion by finite type surfaces. Hence there is a pants decomposition $\mathcal{C}_1$. Note that Lemma 4.2 and Proposition 4.3 in [3] does not depend on the metric of surfaces. Hence by applying the argument of Lemma 4.2 and Proposition 4.3 in [3] to S° and $\mathcal{C}_1$, we can get pants decompositions $\mathcal{C}_2$ and $\mathcal{C}_3$ such that $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are pairwise distinct. By Theorem 6.1, there are geometric pants decompositions Λ_1 , Λ_2 and Λ_3 which are pairwise transverse. $\square$

THEOREM 6.4. *Let G be a Fuchsian group of the first kind. Suppose that $\mathbb{H}/G$ is not a geometric pair of pants. Then G is a pants-like COL_3 group.*

PROOF. By Lemma 6.3, there are three geometric pants decompositions Λ_1 , Λ_2 , and Λ_3 of S which are pairwise transverse. By Lemma 5.8, we can add geodesics in each geometric pair of pants and so we can get three geodesic laminations Λ'_1 , Λ'_2 , and Λ'_3 such that the completion of any connected component of the complement of each geodesic lamination is an ideal triangle or a monogon with one cone point. Therefore, $\pi_G^{-1}(\Lambda'_1)$, $\pi_G^{-1}(\Lambda'_2)$, and $\pi_G^{-1}(\Lambda'_3)$ are geodesic laminations whose complements are ideal polygons. Therefore, they induce three laminations of S^1 and so they also induce three very full lamination systems. Moreover, we can see that the very full lamination systems comprise a pants-like collection for G . Thus, G is a pants-like COL_3 group. $\square$

6.3. *Proof: Fuchsian groups of the second kind are pants-like COL_3 .* We now consider the case of the second kind to complete the proof of Theorem B. Let G be a non-elementary Fuchsian group of the second kind. We begin by outlining the strategy.

We first take the double of $\mathbb{H}/G$. This double is a hyperbolic surface whose fundamental group is a Fuchsian group of the first kind. Thus Theorem 6.1 gives three pairwise transverse geometric pants decompositions on the double. We then restrict these laminations to the original surface. The resulting three geodesic laminations on $\mathbb{H}/G$ need not be very full, so we add further geodesics to make them very full.

First, we construct geodesic laminations for elementary pieces.

LEMMA 6.5. *Let H be the half plane and $\overline{H}$ be the closure of H in $\overline{\mathbb{H}} = \mathbb{H} \cup \hat{\mathbb{R}}$. Suppose that Q is a countable dense subset of the set $\overline{H} - H$ of non-negative real numbers, containing $\{0, \infty\}$. There is a geodesic lamination Λ*

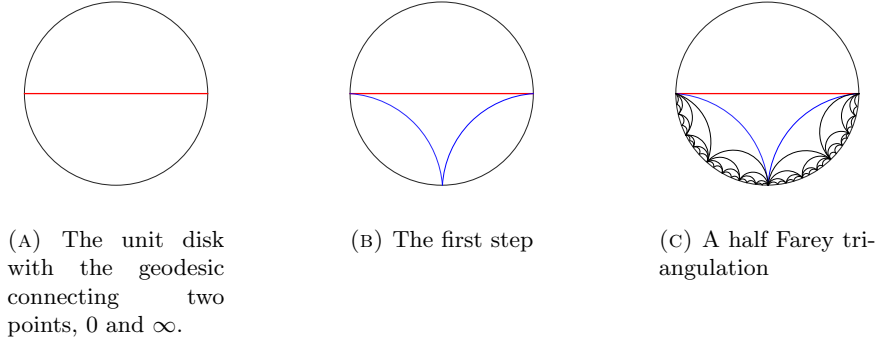


FIGURE 10. These pictures show the steps to draw a half Farey triangulation.

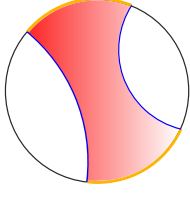
in H such that the Cauchy completion of each connected component of $H - \Lambda$ is the ideal 3-gon and such that in the corresponding lamination system $\mathcal{L}$, every leaf is isolated and $E(\mathcal{L}) = Q$.

PROOF. First, we say that Q is a sequence $\{q_n\}_{n=1}^\infty$ such that $q_1 = 0$ and $q_2 = \infty$. Then there is a geodesic joining q_1 and q_2 . See Figure 10a. The red line represents the geodesic and the geodesic is the boundary of the half plane H . Note that $H = CH([0, \infty])$.

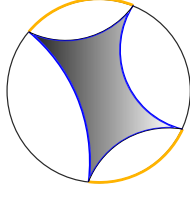
Now, we consider q_3 . We can draw two geodesics. One is the geodesic joining two points q_1 and q_3 and the other is the geodesic joining q_2 and q_3 . The blue lines in Figure 10b represent the geodesics. Then, q_4 is contained in (q_1, q_3) or (q_3, q_2) and both $CH([q_1, q_3])$ and $CH([q_3, q_2])$ are half planes. So we can do similarly in $CH([q_1, q_3])$ or $CH([q_3, q_2])$. Obviously, we can repeat this process. Finally, we can get Figure 10c. These geodesics give a geodesic lamination in H that we want. $\square$

LEMMA 6.6. Let I and J be two nondegenerate open intervals in $\partial\mathbb{H}$. Suppose that $\bar{I}$ and $\bar{J}$ are disjoint. We say that S is the hyperbolic surface $CH(\bar{I} \cup \bar{J})$. (See Figure 11a) Assume that there is a countable set Q such that Q is dense in S and $\partial I \cup \partial J \subset Q$. Then there is a geodesic lamination Λ in S such that the Cauchy completion of each connected component of $S - \Lambda$ is the ideal 3-gon and the geodesic boundaries of S are contained in Λ and such that in the corresponding lamination system $\mathcal{L}$, every leaf is isolated and $E(\mathcal{L}) = Q$.

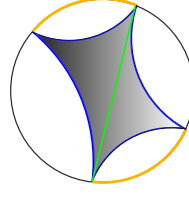
PROOF. S is a complete hyperbolic surface with two geodesic boundaries which are the blue lines in Figure 11a. Now, we say that S' is the convex hull



(A) The red region is the convex hull of orange arcs which represent $\bar{I}$ and $\bar{J}$



(B) The shaded region is the convex hull of ∂I and ∂J . This is a kind of convex core of S .



(C) The green line is one of the diagonal geodesics of the shaded region.

FIGURE 11. These pictures show the steps to draw a triangulation.

of $\partial I \cup \partial J$ which is an ideal 4-gon. (See Figure 11b) Then S is the union of S' , $CH(\bar{I})$ and $CH(\bar{J})$.

Now, we say that $Q_I = Q \cap \bar{I}$ and $Q_J = Q \cap \bar{J}$. Then Q_I and Q_J are countable dense subsets of $\bar{I}$ and $\bar{J}$, respectively. Since $CH(\bar{I})$ and $CH(\bar{J})$ are half planes, by Lemma 6.5, there are two lamination systems $\mathcal{L}_I$ and $\mathcal{L}_J$ in $CH(\bar{I})$ and $CH(\bar{J})$, respectively, such that $E(\mathcal{L}_I) = Q_I$ and $E(\mathcal{L}_J) = Q_J$.

Finally, we consider S' . In S' , we just add one diagonal geodesic like in Figure 11c. The boundary geodesics of S' with the diagonal geodesic give a lamination system $\mathcal{L}_{S'}$ in S' such that every gap of $\mathcal{L}_{S'}$ is an ideal triangle, every leaf is isolated, and $E(\mathcal{L}_{S'}) = \partial I \cup \partial J$. Therefore, $\mathcal{L}_{S'} \cup \mathcal{L}_I \cup \mathcal{L}_J$ is a lamination system which we want. $\square$

Let G be an Fuchsian group of the second kind. Then $\partial\mathbb{H} - L(G)$ is non-empty. Now, we consider the surface $(\mathbb{H} - L(G))/G$ with boundary. Then $(\mathbb{H} - L(G))/G - \mathbb{H}/G$ is exactly $(\partial\mathbb{H} - L(G))/G$ since G acts properly discontinuously on $\partial\mathbb{H} - L(G)$. Therefore, we call each connected component of $(\partial\mathbb{H} - L(G))/G$ an **ideal boundary** of $\mathbb{H}/G$ and each point of ideal boundaries an **ideal point** of $\mathbb{H}/G$. We denote the quotient map from $\mathbb{H} - L(G)$ to $(\mathbb{H} - L(G))/G$ by π_G . Note that $\pi_G|_{\mathbb{H}} = \pi_G$.

DEFINITION 6.7. Let G be a Fuchsian group. Suppose that there is a cone point c_1 in $\mathbb{H}/G$. A subset ℓ of $\mathbb{H}/G$ is called a **regular simple geodesic ray** emanating from c_1 if there is a geodesic ray α_ℓ from $[0, \infty)$ to $\mathbb{H}$ such that $\pi_G \circ \alpha_\ell$ is injective, $(\pi_G \circ \alpha_\ell)([0, \infty)) = \ell$, $\pi_G \circ \alpha_\ell(0) = c_1$, and for each $s \in (0, \infty)$, $\pi_G \circ \alpha_\ell(s)$ is a regular point in $\mathbb{H}/G$. In particular, when $\lim_{s \rightarrow \infty} \alpha_\ell(s)$ is a point p_∞ in $\partial\mathbb{H} - L(G)$, the point $\pi_G(p_\infty)$ in an ideal boundary is called

the *ideal end point* of the geodesic ray ℓ . Also, when $\lim_{s \rightarrow \infty} \alpha_\ell(s)$ is a cusp point of G , we say that ℓ is *emanating* to a puncture which corresponds to the cusp point.

Let G be a Fuchsian group of the second kind. Suppose that there is a regular simple geodesic ray ℓ emanating from a cone point c having p as the ideal end point. Then there is an injective continuous map β_ℓ from $[0, 1]$ to $(\mathbb{H} - L(G))/G$ such that $\beta_\ell(0) = c$, $\beta_\ell(1) = p$, and $\beta_\ell([0, 1)) = \ell$.

PROPOSITION 6.8. *Let G be a Fuchsian group of the second kind and p be an ideal point of $\mathbb{H}/G$. Suppose that there is a cone point q in $\mathbb{H}/G$. If there is a continuous map α from $[0, 1]$ to $(\mathbb{H} - L(G))/G$ such that $\alpha(0) = q$, $\alpha(1) = p$ and $\alpha([0, 1)) \cap \Sigma(\mathbb{H}/G) = \{q\}$, and $\alpha|_{(0, 1)}$ is an embedding in the geometric interior $(\mathbb{H}/G)^o$, then there is an isotopy H on $(\mathbb{H} - L(G))/G$ satisfying the following.*

1. $H(s, 0) = \alpha(s)$,
2. $H(0, t) = q$, $H(1, t) = p$,
3. $H((0, 1) \times [0, 1]) \cap \Sigma(\mathbb{H}/G) = \emptyset$ and
4. $H([0, 1], 1)$ is a regular simple geodesic ray emanating from q with the ideal end point p .

PROOF. First observe that there is a regular simple geodesic ray γ in $\mathbb{H}/G$ starting at q and with ideal endpoint p .

Suppose that q has order n , and let $\tilde{q}$ be a lift of q . Then there is an elliptic element $g \in G$ rotating about $\tilde{q}$ through angle $2\pi/n$. Moreover, the component A of $\pi_G^{-1}(\alpha([0, 1]))$ containing $\tilde{q}$ is a star graph with n leaves and central vertex $\tilde{q}$. Each leaf of A projects to α . Let $\tilde{\alpha}_0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_{n-1}$ denote the leaves of A , ordered counterclockwise, such that $\pi_G \circ \tilde{\alpha}_i = \alpha$. Moreover, every component of $\pi_G^{-1}(\alpha([0, 1]))$ is a non-trivial translate of A under the G -action.

Note that g preserves A , with $g(\tilde{\alpha}_i) = \tilde{\alpha}_{i+1}$ and $g(\tilde{q}) = \tilde{q}$. Write $\tilde{p}_i = \tilde{\alpha}_i(1)$ for the corresponding lifts of p . For each i , there is a unique geodesic ray $\tilde{\gamma}_i : [0, \infty) \rightarrow \mathbb{H}$ starting at $\tilde{q}$ and landing at $\tilde{p}_i$. Again, $g(\tilde{\gamma}_i) = \tilde{\gamma}_{i+1}$. Set $A = \bigcup_{i=0}^{n-1} \text{im } \tilde{\gamma}_i$, which is the geodesic representative of A .

We claim that $\pi_G(\tilde{\gamma}_i)$ is the desired ray. To this end, it suffices to show that $\tilde{q}$ is the only ramification point lying on A .

Assume, to the contrary, that $x \neq \tilde{q}$ is a ramification point of index $m > 1$ lying on $\tilde{\gamma}_i$ for some i . Then there is an elliptic element $h \in G$ rotating about x through angle $2\pi/m$.

If $m > 2$, then A and $h(A)$ intersect transversely at x . Since h rotates $\tilde{\gamma}_i$ about the interior point x through an angle different from π , we have $h(\tilde{p}_i) \neq \tilde{p}_i$. The possible configurations of the points $\tilde{p}_j$ and $h(\tilde{p}_j)$ show that $h(A)$ is a non-trivial translate of A . Thus A intersects its non-trivial translate $h(A)$, contradicting the choice of A .

Likewise, the case $n > 2$ is ruled out by the same argument, interchanging the roles of h and g . Hence it remains only to consider the case $(n, m) = (2, 2)$.

In this case, gh is a hyperbolic element whose axis is the geodesic A . Hence the two endpoints $\tilde{p}_0$ and $\tilde{p}_1$ are fixed points of gh , and therefore belong to $L(G)$. This contradicts the assumption that p is an ideal point of $\mathbb{H}/G$. This completes the claim.

We now isotope α to γ , avoiding the other cone points. First isotope α , rel endpoints, to a simple arc $\alpha_1 : [0, 1] \rightarrow (\mathbb{H} - L(G))/G$ whose initial and terminal subarcs agree with those of γ .

We describe the construction near the initial cone point q . Choose a small closed ball $B \subset \mathbb{H}/G$, centered at q , such that $B \cap \Sigma(S) = \{q\}$. Let $\alpha(t_0)$ be the first point at which α , run from q , meets ∂B . There is a unique geodesic segment a_0 from q to $\alpha(t_0)$. Since B contains no other cone points, we may isotope $\alpha([0, t_0])$ to a_0 by an isotopy supported in B and fixing q .

The intersection $B \cap \gamma$ is the geodesic initial segment of γ , joining q to a unique point in ∂B . Choose a larger closed ball B_{out} , also centered at q , with $B_{\text{out}} \cap \Sigma(S) = \{q\}$. A twist in the annulus between ∂B and ∂B_{out} , rotating B , carries a_0 to $B \cap \gamma$. Hence, after an isotopy supported in B_{out} , the initial segment of α agrees with the initial segment $B \cap \gamma$ of γ .

It remains to treat the terminal end. Since p is an ideal point of $\mathbb{H}/G$, Theorem 6.1 implies that the topological end of γ is contained in a funnel or a half-plane $\mathcal{E}$. In either case, choose an embedded half-plane $H \subset \mathcal{E}$ whose ideal boundary is a closed interval containing p in its interior. Since α limits to p , the terminal tail of α is eventually contained in $\text{int}(H)$. Thus, if t_1 is the last time at which α meets ∂H , then $\alpha(t_1) \in \partial H$ and $\alpha((t_1, 1)) \subset \text{int}(H)$.

There is a unique geodesic ray a_1 in H , starting at $\alpha(t_1)$ with ideal endpoint p . Since H is a half-plane and contains no cone point, we may isotope $\alpha([t_1, 1))$ to a_1 , fixing $\alpha(t_1)$ and p , by an isotopy supported in H . Now choose a slightly larger half-plane H_{out} , with $H \subset H_{\text{out}} \subset \mathcal{E}$, whose ideal boundary still contains p in its interior. Then we may take an isotopy supported in H_{out} , preserving H , which carries a_1 to the terminal geodesic ray $H \cap \gamma$ of γ . Indeed, such an isotopy is obtained by modifying an earthquake isotopy on the rectangular region bounded by the geodesic boundaries of H and H_{out} . Thus, after an isotopy supported in H_{out} , the terminal subarc of α agrees with the terminal ray $H \cap \gamma$ of γ .

Combining the initial and terminal isotopies, whose supports are disjoint after choosing B_{out} and H_{out} suitably, we obtain a simple arc, still denoted by α , whose initial and terminal subarcs agree with those of γ , and the whole isotopy avoids all cone points other than the fixed endpoint q .

It remains to isotope the compact middle subarc. Choose a smaller closed ball $B_{\text{in}} \subset B$, centered at q , and a smaller half-plane $H_{\text{in}} \subset H$, whose ideal boundary is a closed interval containing p in its interior, so that $\alpha([0, 1]) \cap B_{\text{in}} = \gamma \cap B_{\text{in}}$ and $\alpha([0, 1]) \cap H_{\text{in}} = \gamma \cap H_{\text{in}}$. Let $x \in \partial B_{\text{in}}$ and $y \in \partial H_{\text{in}}$ be the

endpoints of the complementary compact subarcs, and denote these subarcs by $\alpha_{xy} \subset \alpha$ and $\gamma_{xy} \subset \gamma$. Thus α_{xy} and γ_{xy} both join x to y , while γ_{xy} is geodesic. Moreover, α_{xy} is a simple compact arc in the geometric interior of $\mathbb{H}/G$, meeting $\partial B_{\text{in}} \cup \partial H_{\text{in}}$ only at its endpoints.

After an arbitrarily small perturbation rel endpoints, we may assume that α_{xy} is piecewise geodesic and still has these properties. We remark that a geodesic segment joining two regular points cannot pass through a cone point. Indeed, otherwise one could shorten the segment by replacing a sufficiently small subarc passing through the cone point with a geodesic segment lying in the geometric interior near the cone point (recall the proof of Lemma 5.5). Also, around each regular point, one can take a sufficiently small disk contained in the geometric interior and hence disjoint from $\Sigma(S)$.

With these facts in hand, we may modify the usual polygonal shortening argument, following the arguments in Buser's book [10, Appendix A.19 and A.20]; for the orbifold case, see also [30, Theorem 4.1]. The same conclusion can also be obtained by adapting the disk-flow straightening argument of Hass–Scott [22]. We sketch the argument below; see [10, Appendix A.19 and A.20] for the detailed algorithm.

Since any piecewise geodesic simple arc contained in the geometric interior of $\mathbb{H}/G$ is compact, it can be covered by finitely many sufficiently small disks, each disjoint from $\Sigma(\mathbb{H}/G)$. Modifying the polygonal shortening argument with respect to this finite cover, we obtain a sequence of piecewise geodesic simple arcs $\{\beta_n\}_{n \in \mathbb{N}}$ such that $\beta_1 = \alpha_{xy}$, each β_n meets ∂B_{in} and ∂H_{in} only at its endpoints x and y , and the lengths of the β_n are nonincreasing as $n \rightarrow \infty$. Moreover, each β_n is isotopic to β_{n+1} by an isotopy supported outside $B_{\text{in}} \cup H_{\text{in}}$ and avoiding $\Sigma(\mathbb{H}/G)$. By the Arzelà–Ascoli theorem [10, Appendix A.19], after passing to a subsequence if necessary, we may assume that $\{\beta_n\}$ converges to a simple geodesic arc β_∞ .

The polygonal shortening process therefore yields an isotopy, relative to endpoints, from α_{xy} to the geodesic representative in its relative homotopy class, through simple arcs in the geometric interior. By uniqueness, $\gamma_{xy} = \beta_\infty$. Hence α_{xy} is isotopic to γ_{xy} , relative to endpoints, by an isotopy supported away from $\Sigma(\mathbb{H}/G)$. Combining this isotopy with the initial and terminal isotopies constructed above gives an isotopy from α to γ , fixing the endpoints and avoiding every cone point other than the fixed initial endpoint q . This completes the proof. $\square$

By Proposition 6.8, any simple embedded arc in $\overline{\mathbb{H}}/G$ connecting a cone point and an ideal point has a geodesic realization. Now, we are ready to prove the elementary cases.

LEMMA 6.9. *Let G be an elementary Fuchsian group. Then G is a pants-like COL_3 group.*

PROOF. There are following four cases.

1. G is the trivial group.
2. G is a finite cyclic group which is generated by some elliptic element.
3. G is an infinite cyclic group which is generated by some non-elliptic element.
4. G is an infinite dihedral group which is generated by two elliptic elements of order two.

First, assume that G is the trivial group. Assume that P is a countable dense subset of $\partial\mathbb{H}$. Choose two distinct points p_1 and p_2 in P . Set $I = [p_1, p_2]_{S^1}$ and $J = [p_2, p_1]_{S^1}$, and set $P_I = I \cap P$ and $P_J = J \cap P$. Then P_I and P_J are countable dense subsets of I and J , respectively, and $CH(I)$ and $CH(J)$, containing p_1 and p_2 , are half planes. Therefore, by Lemma 6.5, there are two lamination systems $\mathcal{L}_I$ and $\mathcal{L}_J$ such that $E(\mathcal{L}_I) = P_I$ and $E(\mathcal{L}_J) = P_J$. Now, we write $\mathcal{L}_P$ for $\mathcal{L}_I \cup \mathcal{L}_J$. Then $\mathcal{L}_P$ is a very full lamination system such that $E(\mathcal{L}_I \cup \mathcal{L}_J) = P$. Observe that we can choose three countable dense subsets P_1 , P_2 and P_3 in $\partial\mathbb{H}$ that are pairwise disjoint. Therefore, $\{\mathcal{L}_{P_1}, \mathcal{L}_{P_2}, \mathcal{L}_{P_3}\}$ is a pants-like collection. Therefore, we can conclude that G is a pants-like COL_3 group with $\{\mathcal{L}_{P_1}, \mathcal{L}_{P_2}, \mathcal{L}_{P_3}\}$.

Next, we consider the case where G is a non-trivial finite cyclic group which is generated by some elliptic element. Let n be the order of G . We may assume that in $\mathbb{D}$, the elliptic element ϕ defined by $z \mapsto e^{\frac{2\pi i}{n}} z$ generates G . Let Q^0 be the set $\{e^{2\pi i q} : q \in \mathbb{Q}\}$. Then, Q^0 is a countable dense subset of $\partial\mathbb{D}$. Then $\mathcal{G} = \{(1, \phi(1))_{S^1}, \dots, (\phi^{n-1}(1), \phi^n(1))_{S^1}\}$ is an ideal polygon with $v(\mathcal{G}) \subset Q^0$. For each $k \in \{0, \dots, n-1\}$, $Q_k^0 = [\phi^k(1), \phi^{k+1}(1)]_{S^1} \cap Q^0$ is a countable dense subset of $[\phi^k(1), \phi^{k+1}(1)]_{S^1}$ and $CH([\phi^k(1), \phi^{k+1}(1)]_{S^1})$ is a half plane. Therefore, by Lemma 6.5, for each $k \in \{0, \dots, n-1\}$, there is a lamination system $\mathcal{L}_k^0$ such that $E(\mathcal{L}_k^0) = Q_k^0$. Then $\mathcal{L}^0 = \bigcup_{k=0}^{n-1} \mathcal{L}_k^0$ is a very full lamination system such that $E(\mathcal{L}^0) = Q^0$ and $\mathcal{G}$ is a gap of $\mathcal{L}^0$. Moreover, by construction, G preserves $\mathcal{L}^0$.

Now, we set $Q^{\sqrt{2}} = \{e^{2\pi i(q+\sqrt{2})} : q \in \mathbb{Q}\}$ and $Q^{\sqrt{3}} = \{e^{2\pi i(q+\sqrt{3})} : q \in \mathbb{Q}\}$. We can define $\mathcal{L}^{\sqrt{2}}$ by the set $\{(e^{2\pi i\sqrt{2}}a, e^{2\pi i\sqrt{2}}b)_{S^1} : (a, b)_{S^1} \in \mathcal{L}^0\}$ and $\mathcal{L}^{\sqrt{3}}$ by the set $\{(e^{2\pi i\sqrt{3}}a, e^{2\pi i\sqrt{3}}b)_{S^1} : (a, b)_{S^1} \in \mathcal{L}^0\}$. Then $E(\mathcal{L}^{\sqrt{2}}) = Q^{\sqrt{2}}$ and $E(\mathcal{L}^{\sqrt{3}}) = Q^{\sqrt{3}}$. Therefore, $\{\mathcal{L}^0, \mathcal{L}^{\sqrt{2}}, \mathcal{L}^{\sqrt{3}}\}$ is a pants-like COL_3 collection. Moreover, these lamination systems are preserved by G by construction. Therefore, G is a pants-like COL_3 group.

Then we consider the case where G is an infinite cyclic group which is generated by some non-elliptic element. First, we assume that in $\mathbb{H}$, G is generated by the parabolic element ϕ defined by $z \mapsto z + 1$. We set $Q^0 = \mathbb{Q} \cup \infty$. Since $Q^0 \cap [0, 1]$ is a countable dense subset of $[0, 1]$, and $CH([0, 1])$ is a half plane, by Lemma 6.5, there is a lamination system $\mathcal{L}_0^0$ such that

$E(\mathcal{L}_0^0) = p^{-1}(Q^0 \cap [0, 1])$. Now, we define

$$\mathcal{L}^0 = \left[\bigcup_{k \in \mathbb{Z}} (p^{-1} \phi^k p)(\mathcal{L}_0^0) \right] \cup \left[\bigcup_{k \in \mathbb{Z}} \{ \phi^k(p^{-1}((0, \infty))), \phi^k(p^{-1}((0, \infty))^*) \} \right].$$

Obviously, $\mathcal{L}^0$ is a very full G -invariant lamination system such that $E(\mathcal{L}^0) = p^{-1}(Q^0)$.

Now we set $Q^{\sqrt{2}} = \sqrt{2} + \mathbb{Q}$ and $Q^{\sqrt{3}} = \sqrt{3} + \mathbb{Q}$. Then, we define

$$\mathcal{L}^{\sqrt{2}} = \{(a, b)_{S^1} : p^{-1}((-\sqrt{2}) + p((a, b)_{S^1})) \in \mathcal{L}^0\}$$

and

$$\mathcal{L}^{\sqrt{3}} = \{(a, b)_{S^1} : p^{-1}((-\sqrt{3}) + p((a, b)_{S^1})) \in \mathcal{L}^0\}.$$

Then $E(\mathcal{L}^{\sqrt{2}}) = p^{-1}(Q^{\sqrt{2}})$ and $E(\mathcal{L}^{\sqrt{3}}) = p^{-1}(Q^{\sqrt{3}})$, and both $\mathcal{L}^{\sqrt{2}}$ and $\mathcal{L}^{\sqrt{3}}$ are very full G -invariant lamination systems. Therefore, $\{\mathcal{L}^0, \mathcal{L}^{\sqrt{2}}, \mathcal{L}^{\sqrt{3}}\}$ is a pants-like COL_3 collection since for two distinct indices i and j in $\{0, \sqrt{2}, \sqrt{3}\}$, $Q^i \cap Q^j = \{\infty\}$ and ∞ is the fixed point of ϕ . Therefore, G is a pants-like COL_3 group.

Then we assume that in $\mathbb{H}$, G is generated by a hyperbolic element ϕ defined by $z \mapsto e^a z$ for some positive real number a . We set $Q^0 = \{e^{aq} : q \in \mathbb{Q}\} \cup \{-e^{aq} : q \in \mathbb{Q}\}$. Observe that Q^0 is preserved by G and ϕ maps the geodesic connecting -1 and 1 to the geodesic connecting $-e^a$ and e^a . Moreover, $p^{-1}(Q^0)$ is a countable dense subset of $\partial\mathbb{D}$. Lemma 6.6 can apply to two closed interval $p^{-1}([-e^a, -1])$ and $p^{-1}([1, e^a])$ since $\{-e^a, -1, 1, e^a\} \subset Q^0$ and both $[-e^a, -1] \cap Q^0$ and $[1, e^a] \cap Q^0$ are countable dense subsets of $[-e^a, -1]$ and $[1, e^a]$, respectively. Therefore, there is a lamination system $\mathcal{L}_0^0$ such that $E(\mathcal{L}_0^0) = p^{-1}(Q^0 \cap ([-e^a, -1] \cup [1, e^a]))$. Then,

$$\mathcal{L}^0 = \bigcup_{k \in \mathbb{Z}} (p^{-1} \phi^k p)(\mathcal{L}_0^0)$$

is a G -invariant very full lamination system such that $E(\mathcal{L}^0) = p^{-1}(Q^0)$.

Now, we set $Q^{\sqrt{2}} = \{e^{a(\sqrt{2}+q)} : q \in \mathbb{Q}\} \cup \{-e^{a(\sqrt{2}+q)} : q \in \mathbb{Q}\}$, and $Q^{\sqrt{3}} = \{e^{a(\sqrt{3}+q)} : q \in \mathbb{Q}\} \cup \{-e^{a(\sqrt{3}+q)} : q \in \mathbb{Q}\}$. In the similar way, we can find two very full G -invariant lamination systems $\mathcal{L}^{\sqrt{2}}$ and $\mathcal{L}^{\sqrt{3}}$ such that $\mathcal{L}^{\sqrt{2}} = p^{-1}(Q^{\sqrt{2}})$ and $\mathcal{L}^{\sqrt{3}} = p^{-1}(Q^{\sqrt{3}})$. Then, by construction, $\{\mathcal{L}^0, \mathcal{L}^{\sqrt{2}}, \mathcal{L}^{\sqrt{3}}\}$ is a pants-like COL_3 collection. Therefore, G is a pants-like COL_3 group.

Finally, we consider the case where G is an infinite dihedral group generated by two elliptic elements of order two. In $\mathbb{H}$, for each non-negative real number a , there is a unique order two elliptic isometry $\phi_{a/2}$ that fixes $ie^{a/2}$. We may assume that G is generated by ϕ_0 and $\phi_{a/2}$ for some positive real number a . Note that $\phi_{a/2} \circ \phi_0$ is the hyperbolic element defined by $z \mapsto e^a z$.

Then we can see that there are exactly two cone points $c_1 = \pi_G(i)$ and $c_2 = \pi_G(ie^{a/2})$ of order two and that there is the only one ideal boundary B in

$\mathbb{H}/G$ which is homeomorphic to the circle. We can find three countable dense subsets P_1 , P_2 and P_3 of B that are pairwise disjoint. First, we consider P_1 . Choose two distinct points p_1^1 and p_1^2 in P_1 . Then, by Proposition 6.8, there are two disjoint regular simple geodesic rays ℓ_1^1 and ℓ_1^2 such that for each $i \in \{1, 2\}$, ℓ_1^i is a regular simple geodesic ray emanating from c_i with the ideal end point p_1^i . Note that by definition, ℓ_1^1 and ℓ_1^2 are simple geodesics in $\mathbb{H}/G$ since c_1 and c_2 are of order two. Hence, there are two bi-infinite geodesics $\tilde{\ell}_1^1$ and $\tilde{\ell}_1^2$ such that for each $k \in \{1, 2\}$, $\tilde{\ell}_1^k$ is the connected component of $\pi_G^{-1}(\ell_1^k)$ passing through $ie^{(k-1)a/2}$. Also, for each $k \in \{1, 2\}$, there are two points p_-^k and p_+^k in $\mathbb{R} - \{0\}$ such that $p_-^k < 0$, $p_+^k > 0$, and $CH(\{p_-^k, p_+^k\}) = \tilde{\ell}_1^k$.

Now, we write $I_1 = [p_-^1, p_+^1]$ and $J_1 = [p_+^1, p_+^2]$. Observe that $L(G) = \{0, \infty\}$ and that $\pi_G^{-1}(P_1)$ is a countable dense subset of $\mathbb{R} - \{0\}$. Then ∂I_1 and ∂J_1 are contained in $\pi_G^{-1}(P_1)$, $\pi_G^{-1}(P_1) \cap I_1$ is dense in I_1 , and $\pi_G^{-1}(P_1) \cap J_1$ is dense in J_1 . So, we can apply Lemma 6.6 to $CH(I_1 \cup J_1)$. Therefore, there is a geodesic lamination Λ_1^0 on $CH(I_1 \cup J_1)$. Then

$$\Lambda_1 = \bigcup_{g \in G} g(\Lambda_1^0)$$

is also a geodesic lamination preserved under the G -action. Therefore, there is a G -invariant very full lamination system $\mathcal{L}_1$, induced by Λ_1 , such that $p(E(\mathcal{L}_1)) = \pi_G^{-1}(P_1)$. Similarly, there are two G -invariant very full lamination systems $\mathcal{L}_2$ and $\mathcal{L}_3$ such that $p(E(\mathcal{L}_2)) = \pi_G^{-1}(P_2)$ and $p(E(\mathcal{L}_3)) = \pi_G^{-1}(P_3)$. By construction, $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$ is a pants-like COL_3 collection. Thus, G is a pants-like COL_3 group. $\square$

From now on, we consider the case where G is a non-elementary Fuchsian group of the second kind. Let $R(G)$ be the set of all ramification points of G in $\Omega(G)$. Note that $R(G) \cup L(G)$ is a closed subset of the Riemann sphere $\hat{\mathbb{C}}$. Hence, the open subset $\Omega(G) - R(G) = \hat{\mathbb{C}} - (R(G) \cup L(G))$ of $\hat{\mathbb{C}}$ is a Riemann surface. Set $U_G = \Omega(G) - R(G)$. Then $\pi_1(U_G)$ is a free group of infinite rank. Moreover, by the uniformization theorem, the universal cover of U_G is $\mathbb{H}$, and hence U_G carries a complete hyperbolic metric. Observe that there is no isometrically embedded half plane and funnel in U_G . Therefore, $G(U_G)$ is a Fuchsian group of the first kind. Now, we define

$$F_G = \{h \in \text{Aut}(\mathbb{H}) : \pi_{U_G} \circ h = g \circ \pi_{U_G} \text{ for some } g \in G\}.$$

Note that $G(U_G)$ is the deck transformation group of π_{U_G} and that F_G is isomorphic to $\pi_1(U_G/G)$ since $\pi_G^d \circ \pi_{U_G}$ is the universal covering map of U_G/G .

Now, we define a homomorphism $\pi_{U_G}^*$ from F_G to G so that for each $h \in F_G$, $\pi_{U_G} \circ h = \pi_{U_G}^*(h) \circ \pi_{U_G}$. Then we have an exact sequence

$$\{1\} \longrightarrow G(U_G) \xrightarrow{i} F_G \xrightarrow{\pi_{U_G}^*} G \longrightarrow \{1\}$$

where i is the inclusion map. Hence, $F_G/G(U_G)$ is isomorphic to G and since $\hat{\mathbb{R}} = L(G(U_G)) \subset L(F_G)$, F_G is a Fuchsian group of the first kind. Therefore, there is no isometrically embedded half plane and funnel in $U_G/G = \mathbb{H}/F_G$ and there is no cone point in U_G/G by construction. Hence, U_G/G is a complete hyperbolic surface of which the fundamental group is a Fuchsian group of the first kind.

By Lemma 6.3, there are three pairwise transverse geometric pants decompositions Λ_1, Λ_2 , and Λ_3 in U_G/G . Observe that U_G/G is the doubling of $(\mathbb{H}/G)^o$ about the ideal boundaries as the following diagram commutes.

$$\begin{array}{ccccc} \mathbb{H} - R(G) & \hookrightarrow & U_G & \hookrightarrow & \Omega(G) \\ \downarrow \pi_G & & \downarrow \pi_G^d & & \downarrow \pi_G^d \\ (\mathbb{H}/G)^o & \hookrightarrow & U_G/G & \hookrightarrow & \Omega(G)/G \end{array}$$

where the horizontal arrows are the inclusion maps. Note that the inclusion maps are not isometries and that the map J induced from the complex conjugation is an isometry on U_G/G fixing every point of $(\hat{\mathbb{R}} - L(G))/G$. Therefore, each ideal boundary component of $(\mathbb{H}/G)^o$ is realized as a simple geodesic in U_G/G . Hence $B = (\hat{\mathbb{R}} - L(G))/G$ is a geodesic lamination in U_G/G .

Let P be the completion of a connected component of the complement of Λ_k for some $k \in \{1, 2, 3\}$. Since there are no cone points in U_G/G , there are three types of geometric pairs of pants which have only simple closed geodesics and punctures as geometric boundaries. See Figure 12.

Let b be a connected component of B . If b is a simple closed curve which corresponds to the ideal boundary of a funnel in $\mathbb{H}/G$, then for each $i \in \{1, 2, 3\}$, $b \cap \Lambda_i$ is b itself or a finite subset of b . If b is a simple bi-infinite geodesic, corresponding to the ideal boundary of a half-plane, then for each $i \in \{1, 2, 3\}$, $b \cap \Lambda_i$ is a countable discrete closed subset of b since any bi-infinite geodesic $\tilde{b}$ such that $(\pi_G^d \circ \pi_{U_G})(\tilde{b}) = b$ has no cusp point as an end point. In particular, when b intersects with P , each connected component of $b \cap P$ is a properly embedded arc in P or a geodesic boundary of P . On the other hands, if c is a simple closed curve in Λ_i for some $i \in \{1, 2, 3\}$, then $c \cap ((\mathbb{H}/G)^o \cup B)$ is c itself or a disjoint union of properly embedded arcs.

Now, we assume that $P \cap B \neq \emptyset$. Let P' be a connected component of $P \cap ((\mathbb{H}/G)^o \cup B)$. Then, we can see that each manifold boundary of P' is a geodesic boundary of P or consists of an alternative sequence of arcs in B and arcs in ∂P . Hence, each boundary of P' is a piecewise geodesic circle.

From now on, we discuss the shape of P' . First, we consider the case where there is an embedded arc in $P \cap B$ having the end points in one geodesic boundary of P . See Figure 13. In each case, the red line represents the embedded arc and P can have a finite number of parallel arcs in $P \cap B$ to the red line which are also geodesic. These parallel arcs divide P into three types of pieces. See Figure 14. The red lines are arcs in $P \cap B$ and the black lines

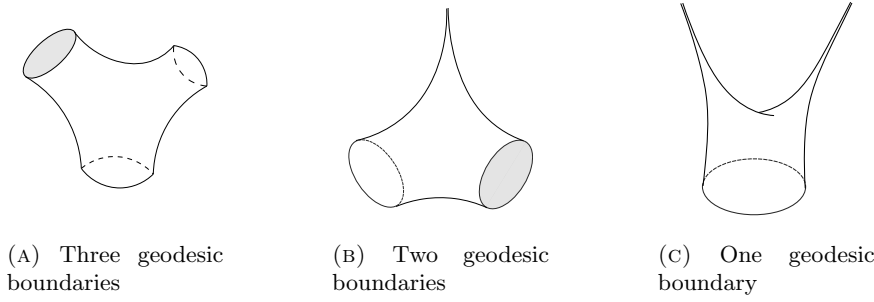


FIGURE 12. In each Λ_i , there are three types of geometric pairs of pants according to the number of closed geodesic boundary components since U_G/G has no cone point.

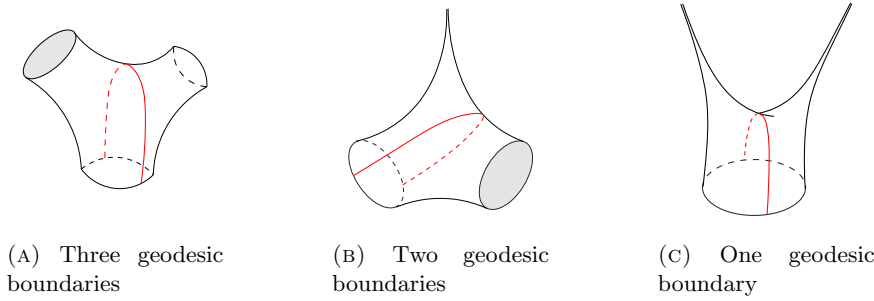


FIGURE 13. The red lines are curves in $P \cap B$.

are arcs in ∂P . The small circle in Figure 14b is a puncture and the round circle in Figure 14c is a geodesic boundary of P . We can see that a piece of the first two types can not intersect with B in the complement of the red lines. But, a piece of the third type has three possibilities. The first is that there is no intersection with B except the red lines. The second case is that the piece has an embedded arc in $P \cap B$ connecting two distinct black lines. See Figure 15a. The additional red line represents the arc. Note that since the arc is not separating, there are even number of parallel arcs. Then, the parallel arcs divide the piece into pieces which are of the type of Figure 14a or of the type of Figure 18a. Finally, in Figure 14c, the geodesic boundary is in B . See Figure 15b. Thus, Figure 14a, Figure 14b, Figure 14c, Figure 15b and Figure 18a are all possible cases of P' when P has an arc in $P \cap B$ having the end points in one geodesic boundary.

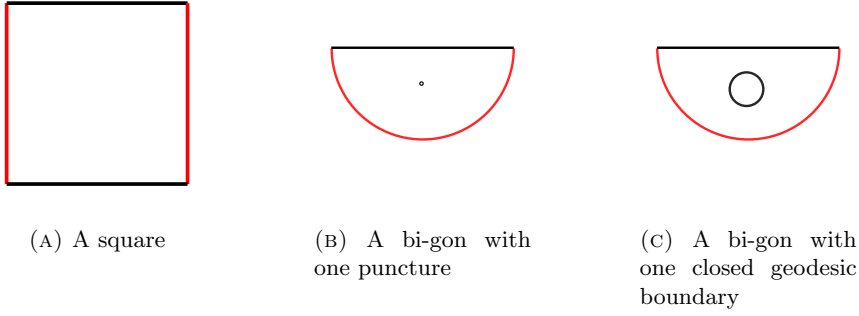


FIGURE 14. There are three types of pieces from Figure 13. The red lines are embedded arcs in $P \cap B$ and the black lines are arcs in ∂P . The small circle in Figure 14b represents a puncture.

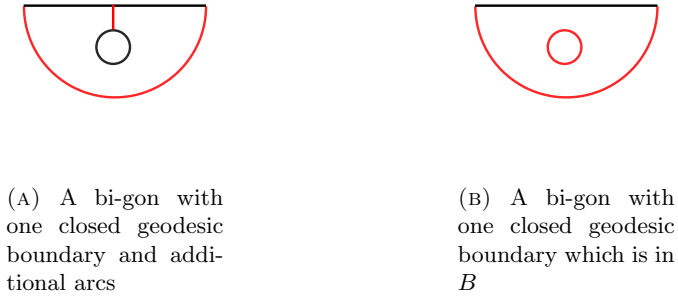
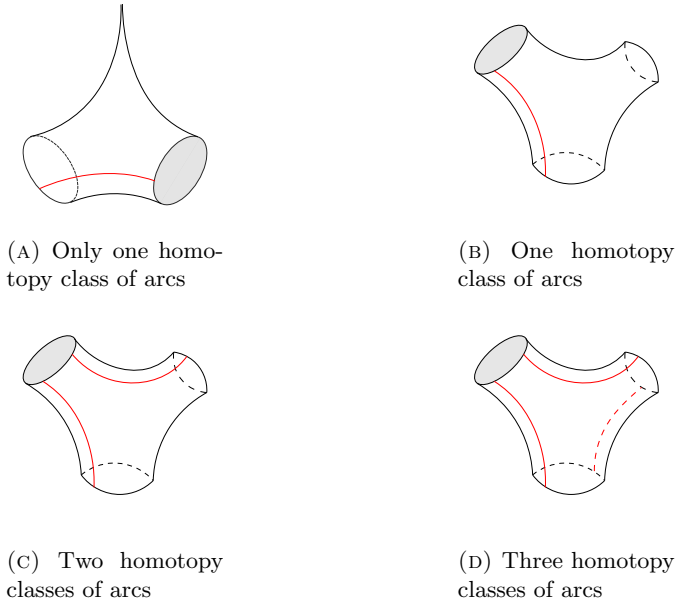


FIGURE 15. The red lines are curves in $P \cap B$.

Now, we assume that there is no properly embedded arc in $P \cap B$ having two end points in one geodesic boundary. Also, suppose that there is a properly embedded arc in $P \cap B$ connecting two distinct geodesic boundaries of P . Then, there are four possible cases according to the number of homotopic types of arcs. See Figure 16. The red lines represent the homotopy classes of the embedded arcs in $P \cap B$.

In Figure 16a, the red line is not separating so there are even number of parallel arcs which are homotopic to the red line. Hence, these parallel arcs divide P into pieces which are of the types of Figure 14a or Figure 17a. Similarly, in Figure 16b, the red line is not separating so there are even number of parallel arcs which are homotopic to the red line. These parallel arcs divide

FIGURE 16. The red lines are curves in $P \cap B$.

P into pieces which are of the types of Figure 14a or Figure 17b. In particular, in a piece of the type of Figure 17b, the geodesic boundary may be in $P \cap B$. Therefore, Figure 17c is also a possible case. Finally, from Figure 16c and Figure 16d, we can get pieces which are of the types of Figure 14a, Figure 18a, or Figure 18b. Therefore, Figure 14a, Figure 18a, Figure 18b, Figure 17a, Figure 17b, and Figure 17c are all possible cases of P' when $P \cap B$ has an arc connecting two distinct boundaries.

Then, we consider the case where $P \cap B$ has no arc. Then, $P \cap B$ consists of closed geodesics. Therefore, we can list up the possible types of P' according to the number of boundaries contained in $P \cap B$. See Figure 19. Thus, we can summarize all possible cases of P' in Figure 20.

Now, for a simple closed geodesic γ in U_G/G , we set the collar width $c(\gamma)$ to be $\ln(\coth(l(\gamma)/4))$ where $l(\gamma)$ is the length of γ with respect to the hyperbolic metric of U_G/G . Then, we write $U(\gamma)$ for the $c(\gamma)$ -neighborhood of γ in U_G/G which is homeomorphic to the annulus. Note that for two disjoint simple closed geodesics α and β in U_G/G , $U(\alpha) \cap U(\beta) = \emptyset$ and if b is a connected component of B and $b \cap \alpha = \emptyset$, then $b \cap U(\alpha) = \emptyset$.

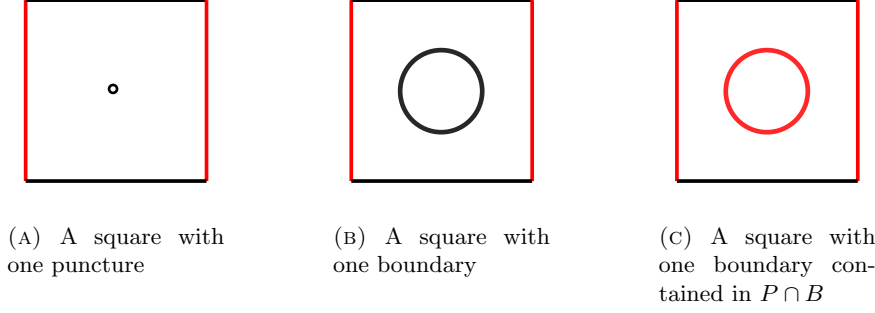


FIGURE 17. The red lines are embedded arcs in $P \cap B$, and the black lines are arcs in ∂P . The small circle represents a puncture.

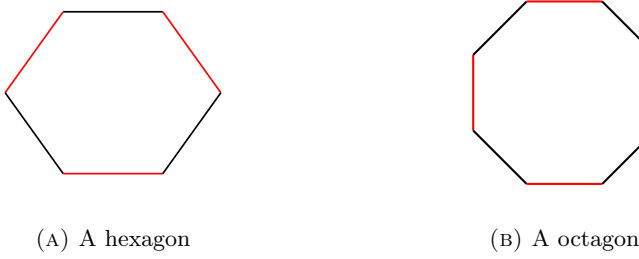


FIGURE 18. The red lines are embedded arcs in $P \cap B$ and the black lines are arcs in ∂P .

Then, we consider the restrictions $\mathcal{C}_i = \Lambda_i \cap ((\mathbb{H}/G)^o \cup B)$ of the geodesic laminations. $\mathcal{C}_i$ are a disjoint union of simple closed curves and properly embedded arcs in $(\mathbb{H}/G)^o \cup B$. First, we remove closed curves in $\mathcal{C}_i \cap B$ from $\mathcal{C}_i$. Then, choose a connected component α of $\mathcal{C}_1$. We define a regular neighborhood $\bar{V}(\alpha)$ in $(\mathbb{H}/G)^o \cup B$ as follows. If α is a simple closed curve, then $\bar{V}(\alpha) = U(\alpha)$ and if α is a properly embedded arc, then there is the simple closed curve γ in Λ_1 containing α and $\bar{V}(\alpha)$ is the connected component of $U(\gamma) \cap ((\mathbb{H}/G)^o \cup B)$ containing α . Hence, each pair of two distinct connected components of $\mathcal{C}_1$ have disjoint regular neighborhoods. Similarly, we can define regular neighborhoods for connected components of $\mathcal{C}_2$ and $\mathcal{C}_3$.

Choose three pairwise disjoint countable dense subsets Q_1, Q_2 , and Q_3 of B . We will replace the arcs in each $\mathcal{C}_i$ properly so that every arc in $\mathcal{C}_i$ has endpoints in Q_i . We first consider $\mathcal{C}_1$. Let $\gamma : [0, 1] \rightarrow \mathcal{C}_1$ be a proper

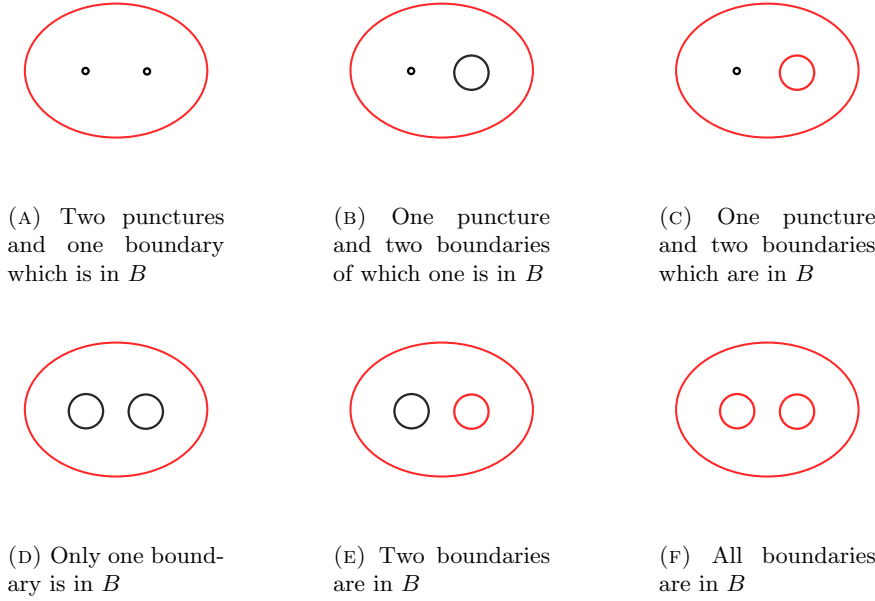


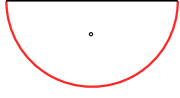
FIGURE 19. The red circles are closed geodesics in $P \cap B$ and the black circles are closed geodesic in $\partial P - B$. The small circles represent punctures.

embedding, parametrizing an arc of $\mathcal{C}_1$. Then there is an isotopy H through proper embeddings in $(\mathbb{H}/G)^\circ \cup B$ satisfying the following:

1. $H(s, 0) = \gamma(s)$,
2. $H([0, 1] \times [0, 1]) \subset \overline{V}(\gamma([0, 1]))$,
3. for all $t \in [0, 1]$, $\{H(0, t), H(1, t)\} \subset B$, and
4. $\{H(0, 1), H(1, 1)\} \subset Q_1$.

Set $\gamma'(s) = H(s, 1)$. Then $\gamma'([0, 1])$ is a properly embedded arc in $(\mathbb{H}/G)^\circ \cup B$ whose endpoints lie in Q_1 , and it is contained in $\overline{V}(\gamma([0, 1]))$. We replace $\gamma([0, 1])$ in $\mathcal{C}_1$ by $\gamma'([0, 1])$. Since the regular neighborhoods are disjoint, this replacement can be performed independently for all arcs in $\mathcal{C}_1$. Thus, after properly replacing arcs, we may assume that every arc in $\mathcal{C}_1$ has endpoints in Q_1 . Applying the same argument to $\mathcal{C}_2$ and $\mathcal{C}_3$, we may assume that for each $i \in \{1, 2, 3\}$, every arc in $\mathcal{C}_i$ has endpoints in Q_i .

For each arc α in $\mathcal{C}_i$, there is a unique simple bi-infinite geodesic ℓ_α in $\mathbb{H}/G$ which has the same homotopy type as α and connects the two endpoints of α . To see this, we first observe that, since α is a simple arc containing no



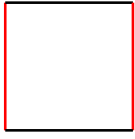
(A) A bi-gon with one puncture



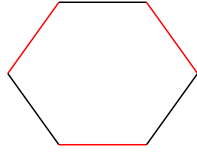
(B) A bi-gon with one closed geodesic boundary



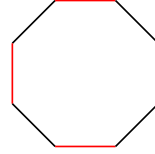
(C) A bi-gon with one closed geodesic boundary which is in B



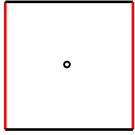
(D) A square



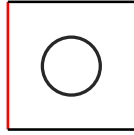
(E) A hexagon



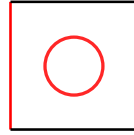
(F) An octagon



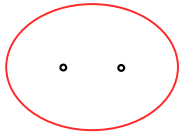
(G) A square with one puncture



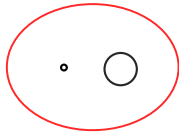
(H) A square with one boundary



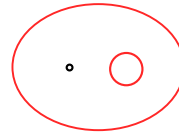
(I) A square with one boundary which is in $P \cap B$



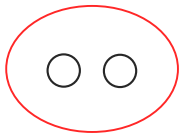
(J) Two punctures and one boundary which is in B



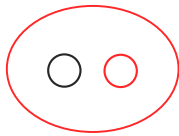
(K) One puncture and two boundaries of which one is in B



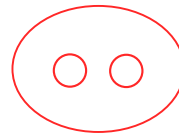
(L) One puncture and two boundaries which are in B



(M) Only one boundary is in B



(N) Two boundaries are in B



(O) All boundaries are in B

FIGURE 20. The red circles are closed geodesics in $P \cap B$ and the black lines are in ∂P . The small circles represent punctures.

cone points in $(\overline{\mathbb{H}} \setminus L(G))/G$, $\pi_G^{-1}(\alpha)$ is a disjoint G -invariant collection of lifts of α , each of which is a proper arc in $\overline{\mathbb{H}} \setminus L(G)$. Choose a lift $\tilde{\alpha}$ of α joining two distinct points e_1, e_2 in $\partial\mathbb{H} \setminus L(G)$. Then there is a unique geodesic $\tilde{\ell}_\alpha$ in $\mathbb{H}$ joining e_1 and e_2 .

To see that $\pi_G(\tilde{\ell}_\alpha)$ is the desired ℓ_α , we first show that $\tilde{\ell}_\alpha$ does not contain a ramification point. Assume, for contradiction, that x is a ramification point of index m . Let g be the elliptic element of G rotating about x through angle $2\pi/m$. If $m > 2$, then $g(\tilde{\ell}_\alpha)$ intersects $\tilde{\ell}_\alpha$ transversely at x , and so $\{e_1, e_2\}$ and $g(\{e_1, e_2\})$ are linked on $\partial\mathbb{H}$. It follows that $\tilde{\alpha}$ intersects its nontrivial translate $g(\tilde{\alpha})$, contradicting the fact that α is simple. If $m = 2$, then $g(e_1) = e_2$, which implies that α is a loop based at $\pi_G(e_1)$, contradicting the fact that α is an arc. Hence $\tilde{\ell}_\alpha$ does not contain a ramification point.

Therefore, $\pi_G(\tilde{\ell}_\alpha)$ is a bi-infinite geodesic in the geometric interior of $\mathbb{H}/G$, joining the two endpoints of α . Set $\ell_\alpha = \pi_G(\tilde{\ell}_\alpha)$. Then, by the same strategy as in the proof of Proposition 6.8, we can construct the desired homotopy from α to ℓ_α , avoiding cone points.

We say that ℓ_α is the **geodesic realization** of α . We define Λ_i^0 to be the disjoint union of the geodesic realizations of the components of $\mathcal{C}_i$.

Fix $i \in \{1, 2, 3\}$. Let P' be the Cauchy completion of a connected component of $\mathbb{H}/G - \Lambda_i^0$. By construction, we can see that P' is a geometric pair of pants or each possible shape of P' corresponds to one of Figure 20. Now, the red arcs are a part of ideal boundaries of $\mathbb{H}/G$, the black arcs are simple bi-infinite geodesics in $\mathbb{H}/G$ connecting ideal points in Q_i and the small circles are punctures or cone points. But, the black circles have two possibilities. One is that the geodesic is a regular simple closed curve in $\mathbb{H}/G$. The other is that the geodesic comes from a geodesic segment connecting two cone points of order two.

Finally, we construct a geodesic lamination in each case of P' so that for any complement A of the lamination in P' , the closure of each connected component of $\pi_G^{-1}(A)$ is an ideal polygon. The case where P' is a geometric pair of pants is done by Lemma 5.8 so we consider the cases in Figure 20. The case of Figure 20d is already done by Lemma 6.6. Here, the countable dense subset Q in Lemma 6.6 comes from Q_i . Now, we consider the cases of Figure 20e and Figure 20f. See Figure 21. By adding the blue bi-infinite geodesics, the blue geodesics bound half-planes and the blue and black geodesics bound an ideal hexagon and an ideal octagon in each cases. In each half-plane, we can construct a geodesic lamination by Lemma 6.5. The countable dense subset Q in Lemma 6.5 also comes from Q_i . Then we are done.

Then, we consider the case where P' is of the type of Figure 20a and the small circle corresponds to a cone point of order two. Choose a point p of Q_i in the interior of the red line. Then there is a unique regular simple geodesic ray emanating from the cone point with the ideal end point p . See

Figure 22a. The blue line is the geodesic ray. Then, since the completion of the complement of the geodesic ray is of the type of 20d, we can similarly construct a geodesic lamination.

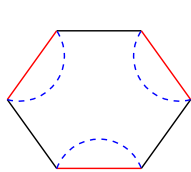
Now, we consider the case where P' is of the type of Figure 20a and the small circle is a cone point of which the order is greater than 2 or a puncture. Choose a point p of Q_i in the interior of the red line. Contrary to the case of order two, there is a bi-infinite geodesic of which the end points are p . See Figure 22. The blue line is the geodesic. This blue line divides the bi-gon into a piece of the type of Figure 20d and a piece which is an ideal monogon or a monogon with one puncture. Then, in the piece of the type of Figure 20d, we can construct a geodesic lamination as the previous cases. If the other piece is an ideal monogon, we are done. If the piece is a monogon with one puncture, we need to add one more bi-infinite geodesic emanating to the puncture as in Figure 4b.

Then, we consider the case of Figure 20b. Choose a point p of Q_i in the interior of the red line. Then, there is a blue bi-infinite geodesic of which the end points are p . See Figure 22c. The blue line is the geodesic. This blue line divides the bi-gon into a piece of the type of Figure 20d and a piece which is a monogon with one hole. In the piece of the type of Figure 20d, we can construct a geodesic lamination as the previous cases. In the monogon with one hole, we need to add one more geodesic as in Figure 4c. Then we are done.

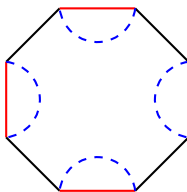
In the case of Figure 20c, there is a unique simple closed geodesic that bounds the funnel. See Figure 23a. The blue circle is the geodesic. This line divides the bi-gon into a piece of the type of Figure 20b and the funnel. In the piece of the type of Figure 20b, we can construct a geodesic lamination by the previous case. We only need to consider the funnel. Choose a point p of Q_i in the closed ideal boundary. Then there is a unique bi-infinite geodesic of which end points are p . See Figure 23b. The orange line is the bi-infinite geodesic. Then, the orange line divides the funnel into a monogon with one hole and a half-plane. Therefore, we can construct a geodesic lamination as previous cases.

For remaining cases, observe that by adding proper bi-infinite geodesic connecting two ideal points in Q_i and dividing P' along the geodesic, we can reduce the construction problem into the previous cases. See Figure 24. The blue line is the bi-infinite geodesic which we want. Observe that each geodesic divides P' into pieces in which we can construct geodesic laminations. Therefore, in each case, we can construct a geodesic lamination in P' .

Now, by adding a geodesic lamination in each complementary region of Λ_i^0 , we obtain a geodesic lamination in $\mathbb{H}/G$ whose corresponding lamination system $\mathcal{L}_i$ is very full. Observe that, by construction, for each pair of distinct indices $i, j \in \{1, 2, 3\}$, the set $E(\mathcal{L}_i) \cap E(\mathcal{L}_j)$ consists of cusp points of

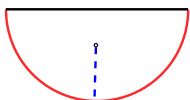


(A) A hexagon

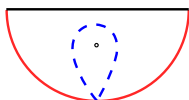


(B) An octagon

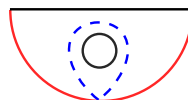
FIGURE 21



(A) A bigon with a cone point of order two

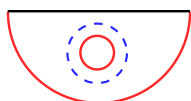


(B) A bigon with a puncture or a cone point not of order two

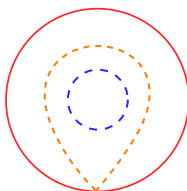


(C) A bigon with one geodesic boundary

FIGURE 22



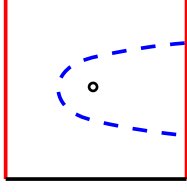
(A) A bigon with one closed ideal boundary



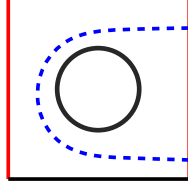
(B) A funnel

FIGURE 23

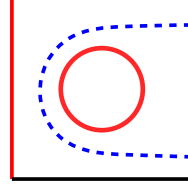
G . Therefore, the collection $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$ of G -invariant very full lamination systems is a pants-like COL_3 collection. Thus, G is a pants-like COL_3 group.



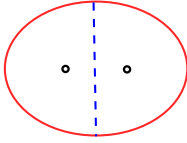
(A) A square with one marked point



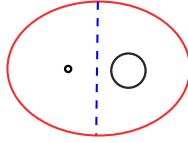
(B) A square with one geodesic boundary



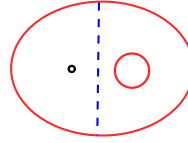
(C) A square with one closed ideal boundary



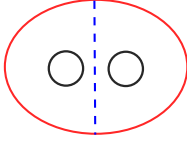
(D) Two marked points and one closed ideal boundary



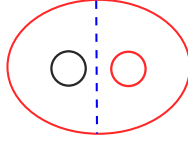
(E) One marked point and one closed ideal boundary



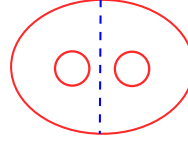
(F) One marked points and two closed ideal boundaries



(G) Only one ideal boundary



(H) Two ideal boundaries



(I) All boundaries are ideal boundaries

FIGURE 24. The blue lines are bi-infinite geodesics connecting two points in Q_i . The marked points which are drawn as the small circles are puncture or cone points.

THEOREM 6.10. *Let G be a non-elementary Fuchsian group of the second kind. Then G is a pants-like COL_3 group.*

Theorem B Let G be a Fuchsian group. Suppose that $\mathbb{H}/G$ is not a geometric pair of pants. Then G is a pants-like COL_3 group.

PROOF. By Theorem 6.4, Theorem 6.9 and Theorem 6.10, we are done.

□

We conclude this section by explaining how the proof of Theorem B can be modified to obtain quite full laminations. See [4] for the definition of quite fullness and related terminology. The only additional issue arises from geometric pairs of pants containing punctures: in such a pair of pants, leaves from distinct laminations may exit through the same puncture, and this has to be taken into account when modifying the laminations constructed in Lemma 5.8.

For the cases shown in Figure 7c, where the pair of pants has either two punctures or one puncture and one cone point, the modification is simple. It is enough to remove the gray geodesics from the pieces of the type shown in Figure 4b; then the complementary region lifts to a crown in the universal cover whose pivot is a cusp point corresponding to the puncture.

It remains to consider the punctured case shown in Figure 7b. In this case, to construct the desired geometric pair of pants together with the required lamination, we use a surface D bounded by two bi-infinite geodesics and containing one puncture; topologically, D is a once-punctured closed disk with two points removed from its boundary. Such a surface can be obtained by gluing two adjacent boundary geodesics of an ideal quadrilateral. We then glue the two boundary geodesics of D . As in Section 5.2.3, choose shearing points on these two boundary geodesics. Depending on the relative positions of the shearing points, the completion P of the resulting surface is either a thrice-punctured sphere or a punctured pair of pants of the type shown in Figure 7b. In the latter case, the lengths of the boundary geodesics are determined by the relative positions of the shearing points, together with the distances between the sides of the initial ideal quadrilateral. Thus, the bi-infinite geodesic arising as the gluing locus (which is the image of the boundary of D) provides the desired replacement lamination in P , whose complementary regions in the geometric interior of P lift to crowns in the universal cover.

Applying this replacement to each such pair of pants, and, in the case shown in Figure 22b, further removing the gray geodesics from the pieces of the type shown in Figure 4b, we obtain quite full laminations. As a consequence, Theorem B can be strengthened to yield the second statement of Corollary 1.3 as follows:

COROLLARY 6.11. *Given a Fuchsian group G , we can construct three quite full circle laminations satisfying the disjoint endpoint condition and preserved by G , provided that $\mathbb{H}^2/G$ is not a geometric pair of pants.*

7. APPENDIX

In this appendix, we prove Lemma 7.3. This lemma refines [3, Lemma 7.7], which treats only the case where G is discrete and torsion-free. Before proving the lemma, we need the following lemma and definition.

LEMMA 7.1. *Let G be a laminar group and $\mathcal{L}$ a G -invariant very full lamination system. Assume that a point p in S^1 is a cusp point of G . Then there is a sequence $\{U_n\}_{n=1}^\infty$ of nondegenerate open intervals such that for each $n \in \mathbb{N}$, $\overline{U_{n+1}} \subset U_n$ and $U_n = I_n \cup \{p\} \cup J_n$ for some two disjoint elements I_n and J_n of $\mathcal{L}$, and $\bigcap_{n=1}^\infty U_n = \{p\}$. In particular, $\{I_n\}_{n=1}^\infty$ and $\{J_n\}_{n=1}^\infty$ are quasi-rainbows at p .*

PROOF. We can take a parabolic element g of G fixing p as p is a cusp point of G . Also there is a leaf ℓ of $\mathcal{L}$ having p as an end point and we denote the other end point by q . Then since g is a parabolic element, $g(q) \neq q$ and $g(q) \neq p$ so there is an element I in ℓ containing the point $g(q)$. Because g is orientation preserving, $g(I) \subsetneq I$ and since g is parabolic, the sequence $\{g^n(I)\}_{n=1}^\infty$ is a quasi-rainbow at p . Likewise $g^{-1}(q) \in I^*$ as $q \in g(I)^*$. Since g^{-1} is orientation preserving, $g^{-1}(I^*) \subsetneq I^*$ so since g is parabolic, the sequence $\{g^{-n}(I^*)\}_{n=1}^\infty$ is a quasi-rainbow at p . Now for each $n \in \mathbb{N}$, we define U_n as the union of the sets $g^n(I)$, $\{p\}$ and $g^{-n}(I^*)$. Since I and I^* are disjoint and $g^n(I) \subsetneq I$ and $g^{-n}(I^*) \subsetneq I^*$ for all $n \in \mathbb{N}$, then $\{g^n(I), g^{-n}(I^*)\}$ is a distinct pair for all $n \in \mathbb{N}$. Hence for each $n \in \mathbb{N}$, U_n is a nondegenerate open interval because $\partial g^n(I) \cap \partial g^{-n}(I^*) = \{p\}$, and $\overline{U_{n+1}} \subset U_n$ as $g^{n+1}(q) \in g^n(I)$ and $g^{-n-1}(q) \in g^{-n}(I^*)$. Finally $\bigcap_{n=1}^\infty U_n = \{p\}$ since the sequences $\{g^n(I)\}_{n=1}^\infty$ and $\{g^{-n}(I^*)\}_{n=1}^\infty$ are quasi-rainbows at p . Thus we are done. $\square$

For brevity we make a definition for the sequence $\{U_n\}_{n=1}^\infty$ of the lemma.

DEFINITION 7.2. *Let $\mathcal{L}$ be a lamination system. Let p be a point in S^1 and $\{U_n\}_{n=1}^\infty$ be a sequence of nondegenerate open intervals containing p . We call the sequence $\{U_n\}_{n=1}^\infty$ angel wings at p if for each $n \in \mathbb{N}$, $\overline{U_{n+1}} \subset U_n$ and $U_n = I \cup \{p\} \cup J$ for some two disjoint elements I and J of $\mathcal{L}$, and $\bigcap_{n=1}^\infty U_n = \{p\}$.*

So Lemma 7.1 says that every cusp point of a group has angel wings if the group has a very full invariant lamination system. Now we prove Lemma 7.3.

LEMMA 7.3. *Let G be a pants-like COL_3 group and $\mathcal{C} = \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$ be a pants-like collection for G . Suppose that there are two sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in S^1 such that $\{x_n\}_{n=1}^\infty$ converges to a point x in S^1 , $\{y_n\}_{n=1}^\infty$ converges to a point y in S^1 and $x \neq y$, and that $\{g_n\}_{n=1}^\infty$ is an approximation sequence of G such that $\{g_n(x_n)\}_{n=1}^\infty$ converges to a point x' in S^1 , $\{g_n(y_n)\}_{n=1}^\infty$ converges to a point y' in S^1 and $x' \neq y'$. Further assume that we can take a sequence $\{a_n\}_{n=1}^\infty$ in S^1 so that for each $n \in \mathbb{N}$, $a_n \in \text{Fix}_{h_n}$ for*

some hyperbolic element h_n of G , $\{a_n\}_{n=1}^\infty$ converges to x and $\{g_n(a_n)\}_{n=1}^\infty$ converges to y' . Then the sequence $\{g_n\}_{n=1}^\infty$ has the convergence property.

PROOF. We will show that we can take a subsequence satisfying the condition of the convergence property with respect to x and y' . First we choose a number ϵ in $\mathbb{R}$ so that $0 < \epsilon < \frac{1}{3} \min\{d_{\mathbb{C}}(x, y), d_{\mathbb{C}}(x', y')\}$. Then for each $p \in \{x, y, x', y'\}$, we denote the ϵ -neighborhood of p by $N_\epsilon(p) = \{z \in S^1 : d_{\mathbb{C}}(p, z) < \epsilon\}$. Note that $N_\epsilon(x) \cap N_\epsilon(y) = \emptyset$ and $N_\epsilon(x') \cap N_\epsilon(y') = \emptyset$. We can take a subindex α of $\text{id}_{\mathbb{N}}$ so that for each $n \in \mathbb{N}$, $\{x_{\alpha(n)}, a_{\alpha(n)}\} \subset N_\epsilon(x)$, $y_{\alpha(n)} \in N_\epsilon(y)$, $g_{\alpha(n)}(x_{\alpha(n)}) \in N_\epsilon(x')$, and $\{g_{\alpha(n)}(y_{\alpha(n)}), g_{\alpha(n)}(a_{\alpha(n)})\} \subset N_\epsilon(y')$. Now we divide into four cases depending on whether x and y' are cusp points.

First we consider the case where both x and y' are not cusp points. Then at least two of $\mathcal{C}$ have a rainbow at x and also at least two of $\mathcal{C}$ have a rainbow at y' . So at least one of $\mathcal{C}$ has rainbows at x and y' simultaneously. Hence we can take two rainbows $\{I_n\}_{n=1}^\infty$ at x and $\{J_n\}_{n=1}^\infty$ at y' in some $\mathcal{L}$ of $\mathcal{C}$ so that for each $\overline{I_{n+1}} \subset I_n$ and $\overline{J_{n+1}} \subset J_n$ for all $n \in \mathbb{N}$, and $\overline{I_n} \subset N_\epsilon(x)$ and $\overline{J_n} \subset N_\epsilon(y')$ for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, there is a number $m(n)$ in $\mathbb{N}$ such that for all $m \in \mathbb{N}$ with $m(n) \leq m$, $\{x_{\alpha(m)}, a_{\alpha(m)}\} \subset I_n$ and $\{g_{\alpha(m)}(y_{\alpha(m)}), g_{\alpha(m)}(a_{\alpha(m)})\} \subset J_n$. A subindex β of α can be given by $\beta(n) = \alpha(\sum_{k=1}^n m(k))$ so that for each $n \in \mathbb{N}$, $\{x_{\beta(n)}, a_{\beta(n)}\} \subset I_n$ and $\{g_{\beta(n)}(y_{\beta(n)}), g_{\beta(n)}(a_{\beta(n)})\} \subset J_n$.

Now we show that the sequence $\{g_{\beta(n)}\}_{n=1}^\infty$ has the convergence property with respect to x and y' . Let K be a compact subset of $S^1 - \{x\}$ and U be an open neighborhood of y' on S^1 . Then since $\{I_n\}_{n=1}^\infty$ and $\{J_n\}_{n=1}^\infty$ are rainbows, there is a number N in $\mathbb{N}$ such that $K \subset I_N^*$ and $J_N \subseteq U$. Note that for all $n \in \mathbb{N}$ with $N \leq n$, $\{x_{\beta(n)}, a_{\beta(n)}\} \subset I_N$ and $\{g_{\beta(n)}(y_{\beta(n)}), g_{\beta(n)}(a_{\beta(n)})\} \subset J_N$ as $I_n \subseteq I_N$ and $J_n \subseteq J_N$. Choose a number $n_0 \in \mathbb{N}$ with $N \leq n_0$. Because $N_\epsilon(x)$ does not intersect with $N_\epsilon(y)$ and $y_{\beta(n_0)} \in I_N^*$, $g_{\beta(n_0)}(y_{\beta(n_0)}) \in g_{\beta(n_0)}(I_N)^* \cap J_N$ so by the unlinkedness of $g_{\beta(n_0)}(I_N)^*$ and J_N , there are three possible cases: $g_{\beta(n_0)}(I_N)^* \subset J_N$; $g_{\beta(n_0)}(I_N) \subset J_N$; $g_{\beta(n_0)}(I_N) \subset J_N^*$. If $g_{\beta(n_0)}(I_N) \subset J_N$, then $g_{\beta(n_0)}(x_{\beta(n_0)}) \in J_N$ and so $g_{\beta(n_0)}(x_{\beta(n_0)}) \in N_\epsilon(y')$. However this is a contradiction since $N_\epsilon(x')$ and $N_\epsilon(y')$ are disjoint. If $g_{\beta(n_0)}(I_N) \subset J_N^*$, then $g_{\beta(n_0)}(I_N) \cap J_N = \emptyset$ but $g_{\beta(n_0)}(a_{\beta(n_0)}) \in g_{\beta(n_0)}(I_N) \cap J_N$. So it is a contradiction. Therefore $g_{\beta(n_0)}(I_N)^* \subset J_N$ and this implies that $g_{\beta(n_0)}(K) \subset g_{\beta(n_0)}(I_N)^* \subset J_N \subset U$. Thus we can conclude that $\{g_{\beta(n)}\}_{n=1}^\infty$ has the convergence property and so $\{g_{\alpha(n)}\}_{n=1}^\infty$ has the convergence property.

Next we consider the case where x is not a cusp point and y' is a cusp point. Since x is not a cusp point, at most one of $\mathcal{C}$ has x as an end point so there are at least two of $\mathcal{C}$ have rainbows at x . On the other hand, y' has angel wings in each of $\mathcal{C}$. So there is a lamination system $\mathcal{L}$ in $\mathcal{C}$ which has a rainbow at x and angel wings at y' . Hence, in some $\mathcal{L}$ in $\mathcal{C}$, we can take a rainbow

$\{I_n\}_{n=1}^\infty$ at x and angel wings $\{U_n\}_{n=1}^\infty$ at y' so that for each $n \in \mathbb{N}$, $\overline{I_{n+1}} \subset I_n$, $\overline{I_n} \subset N_\epsilon(x)$ and $\overline{U_n} \subset N_\epsilon(y')$. Then for each $n \in \mathbb{N}$, there is a number $m(n)$ in $\mathbb{N}$ such that for all $m \in \mathbb{N}$ with $m(n) \leq m$, $\{x_{\alpha(m)}, a_{\alpha(m)}\} \subset I_n$ and $\{g_{\alpha(m)}(y_{\alpha(m)}), g_{\alpha(m)}(a_{\alpha(m)})\} \subset U_n$. A subindex β of α can be given by $\beta(n) = \alpha(\sum_{k=1}^n m(k))$ so that for each $n \in \mathbb{N}$, $\{x_{\beta(n)}, a_{\beta(n)}\} \subset I_n$ and $\{g_{\beta(n)}(y_{\beta(n)}), g_{\beta(n)}(a_{\beta(n)})\} \subset U_n$.

Similarly we show that the sequence $\{g_{\beta(n)}\}_{n=1}^\infty$ has the convergence property with respect to x and y' . Let K be a compact subset of $S^1 - \{x\}$ and U be an open neighborhood of y' on S^1 . Then since $\{I_n\}_{n=1}^\infty$ is a rainbow and $\{U_n\}_{n=1}^\infty$ is angel wings, there is a number N in $\mathbb{N}$ such that $K \subset I_N^*$ and $U_N \subseteq U$. Note that for all $n \in \mathbb{N}$ with $N \leq n$, $\{x_{\beta(n)}, a_{\beta(n)}\} \subset I_N$ and $\{g_{\beta(n)}(y_{\beta(n)}), g_{\beta(n)}(a_{\beta(n)})\} \subset U_N$ as $I_n \subseteq I_N$ and $U_n \subseteq U_N$. Now we write $I_n = (r_n, s_n)_{S^1}$ and $U_n = (u_n, v_n)_{S^1}$ for all $n \in \mathbb{N}$. Note that $(u_n, y')_{S^1}$ and $(y', v_n)_{S^1}$ are in $\mathcal{L}$ for all $n \in \mathbb{N}$. Now, we choose a number $n_0 \in \mathbb{N}$ with $N < n_0$. Then $r_{n_0} \in (y_{\beta(n_0)}, a_{\beta(n_0)})_{S^1}$ and $s_{n_0} \in (a_{\beta(n_0)}, y_{\beta(n_0)})_{S^1}$ as $a_{\beta(n_0)} \in (r_{n_0}, s_{n_0})_{S^1}$ and $y_{\beta(n_0)} \in (s_{n_0}, r_{n_0})_{S^1}$ by the choice of ϵ . So one of $g_{\beta(n_0)}([y_{\beta(n_0)}, a_{\beta(n_0)}]_{S^1})$ and $g_{\beta(n_0)}([a_{\beta(n_0)}, y_{\beta(n_0)}]_{S^1})$ must be contained in U_{n_0} because $\{g_{\beta(n_0)}(y_{\beta(n_0)}), g_{\beta(n_0)}(a_{\beta(n_0)})\} \subset U_{n_0}$.

First we consider the case where

$$[g_{\beta(n_0)}(y_{\beta(n_0)}), g_{\beta(n_0)}(a_{\beta(n_0)})]_{S^1} = g_{\beta(n_0)}([y_{\beta(n_0)}, a_{\beta(n_0)}]_{S^1}) \subset U_{n_0}.$$

If $g_{\beta(n_0)}(r_{n_0}) \in (u_{n_0}, y')_{S^1}$, then by the unlinkedness of $g_{\beta(n_0)}(I_{n_0})$ and $(u_{n_0}, y')_{S^1}$, there are two possible cases: $g_{\beta(n_0)}(I_{n_0}) \subset (u_{n_0}, y')_{S^1}$; $g_{\beta(n_0)}(I_{n_0})^* \subset (u_{n_0}, y')_{S^1}$. However if $g_{\beta(n_0)}(I_{n_0}) \subset (u_{n_0}, y')_{S^1}$, then $g_{\beta(n_0)}(x_{\beta(n_0)}) \in (u_{n_0}, y')_{S^1} \subset U_{n_0} \subset N_\epsilon(y')$ so this is a contradiction. Therefore $g_{\beta(n_0)}(I_{n_0})^* \subset (u_{n_0}, y')_{S^1}$ and so

$$g_{\beta(n_0)}(K) \subset g_{\beta(n_0)}(I_N)^* \subset g_{\beta(n_0)}(I_{n_0})^* \subset (u_{n_0}, y')_{S^1} \subset U_{n_0} \subset U_N \subset U.$$

Then if $g_{\beta(n_0)}(r_{n_0}) \in (y', v_{n_0})_{S^1}$, then by the unlinkedness of $g_{\beta(n_0)}(I_{n_0})$ and $(y', v_{n_0})_{S^1}$, there are two possible cases: $g_{\beta(n_0)}(I_{n_0}) \subset (y', v_{n_0})_{S^1}$; $g_{\beta(n_0)}(I_{n_0})^* \subset (y', v_{n_0})_{S^1}$. However if $g_{\beta(n_0)}(I_{n_0}) \subset (y', v_{n_0})_{S^1}$, then $g_{\beta(n_0)}(x_{\beta(n_0)}) \in (y', v_{n_0})_{S^1} \subset U_{n_0} \subset N_\epsilon(y')$ so this is a contradiction. Therefore $g_{\beta(n_0)}(I_{n_0})^* \subset (y', v_{n_0})_{S^1}$ and so

$$g_{\beta(n_0)}(K) \subset g_{\beta(n_0)}(I_N)^* \subset g_{\beta(n_0)}(I_{n_0})^* \subset (y', v_{n_0})_{S^1} \subset U_{n_0} \subset U_N \subset U.$$

Finally we assume that $g_{\beta(n_0)}(r_{n_0}) = y'$. Note that

$$g_{\beta(n_0)}(r_N) \in (g_{\beta(n_0)}(y_{\beta(n_0)}), g_{\beta(n_0)}(a_{\beta(n_0)}))_{S^1}$$

since $a_{\beta(n_0)} \in I_N$ and $y_{\beta(n_0)} \in I_N^*$ by the choice of ϵ , and so $g_{\beta(n_0)}(r_N) \in (g_{\beta(n_0)}(y_{\beta(n_0)}), y')_{S^1} \subset (u_{n_0}, y')_{S^1}$ since

$$[y', g_{\beta(n_0)}(a_{\beta(n_0)})]_{S^1} = g_{\beta(n_0)}([r_{n_0}, a_{\beta(n_0)}])_{S^1} \subset g_{\beta(n_0)}(\overline{I_{n_0}}) \subset g_{\beta(n_0)}(I_N).$$

Hence by the unlinkedness of $g_{\beta(n_0)}(I_N)$ and $(u_{n_0}, y')_{S^1}$, there are two possible cases: $g_{\beta(n_0)}(I_N) \subset (u_{n_0}, y')_{S^1}$; $g_{\beta(n_0)}(I_N)^* \subset (u_{n_0}, y')_{S^1}$. However if $g_{\beta(n_0)}(I_N) \subset (u_{n_0}, y')_{S^1}$, then $g_{\beta(n_0)}(x_{\beta(n_0)}) \in (u_{n_0}, y')_{S^1} \subset U_{n_0} \subset N_\epsilon(y')$ so this is a contradiction. Therefore $g_{\beta(n_0)}(I_N)^* \subset (u_{n_0}, y')_{S^1}$ and so

$$g_{\beta(n_0)}(K) \subset g_{\beta(n_0)}(I_N)^* \subset (u_{n_0}, y')_{S^1} \subset U_{n_0} \subset U_N \subset U.$$

Thus these implies that $\{g_{\beta(n)}\}_{n=1}^\infty$ has the convergence property and so $\{g_{\alpha(n)}\}_{n=1}^\infty$ has the convergence property. In the similar way we can prove that $\{g_{\beta(n)}\}_{n=1}^\infty$ has the convergence property in the case where $g_{\beta(n_0)}([a_{\beta(n_0)}, y_{\beta(n_0)}]_{S^1}) \subset U_{n_0}$. Hence $\{g_{\alpha(n)}\}_{n=1}^\infty$ also has the convergence property.

Now we consider the case where x is a cusp point and y' is not a cusp point. It follows at once from the previous case that $\{g_{\alpha(n)}^{-1}\}_{n=1}^\infty$ has the convergence property. Indeed it implies that $\{g_{\alpha(n)}\}_{n=1}^\infty$ has the convergence property by Remark 4.1. Thus this case is also proved.

Finally we consider the most complicated case when both x and y' are cusp points. Note that by Lemma 3.9, $a_n \neq x$ and $g_n(a_n) \neq y'$ for all $n \in \mathbb{N}$. Also we claim that there is at most one number n in $\mathbb{N}$ such that $g_n(x) = y'$. If there are two numbers n and m in $\mathbb{N}$ with $n \neq m$ such that $g_n(x) = y'$ and $g_m(x) = y'$, then $g_m \circ g_n^{-1}(y') = g_m(x) = y'$ and $y' \in \text{Fix}_{g_m \circ g_n^{-1}}$. However $g_m \circ g_n^{-1}$ is hyperbolic as $\{g_n\}_{n=1}^\infty$ is an approximation sequence and so it is a contradiction by Lemma 3.9. Therefore there is a subindex β of α such that $g_{\beta(n)}(x) \neq y'$ for all $n \in \mathbb{N}$.

Choose a lamination system $\mathcal{L}$ in $\mathcal{C}$. Then we can take two angel wings $\{I_n\}_{n=1}^\infty$ at x and $\{J_n\}_{n=1}^\infty$ at y' in $\mathcal{L}$ so that for each $n \in \mathbb{N}$, $\overline{I_n} \subset N_\epsilon(x)$ and $\overline{J_n} \subset N_\epsilon(y')$. Then for each $n \in \mathbb{N}$, there is a number $m(n)$ in $\mathbb{N}$ such that for all $m \in \mathbb{N}$ with $m(n) \leq m$, $\{x_{\beta(m)}, a_{\beta(m)}\} \subset I_n$ and $\{g_{\beta(m)}(y_{\beta(m)}), g_{\beta(m)}(a_{\beta(m)})\} \subset J_n$. A subindex γ of β can be given by $\gamma(n) = \beta(\sum_{k=1}^n m(k))$ so that for each

$n \in \mathbb{N}$, $\{x_{\gamma(n)}, a_{\gamma(n)}\} \subset I_n$ and $\{g_{\gamma(n)}(y_{\gamma(n)}), g_{\gamma(n)}(a_{\gamma(n)})\} \subset J_n$.

Now we write $I_n = (r_n, s_n)_{S^1}$ and $J_n = (u_n, v_n)_{S^1}$ for all $n \in \mathbb{N}$. Note that for each $n \in \mathbb{N}$, the nondegenerate open intervals $(r_n, x)_{S^1}$, $(x, s_n)_{S^1}$, $(u_n, y')_{S^1}$, and $(y', v_n)_{S^1}$ are in $\mathcal{L}$. Then we show that the sequence $\{g_{\gamma(n)}\}_{n=1}^\infty$ has the convergence property with respect to x and y' . Let K be a compact subset of $S^1 - \{x\}$ and U be an open neighborhood of y' on S^1 . Then since $\{I_n\}_{n=1}^\infty$ and $\{J_n\}_{n=1}^\infty$ are two angel wings, there is a number N in $\mathbb{N}$ such that $K \subset I_N^*$ and $J_N \subset U$. Note that for all $n \in \mathbb{N}$ with $N \leq n$, $\{x_{\gamma(n)}, a_{\gamma(n)}\} \subset I_N$ and $\{g_{\gamma(n)}(y_{\gamma(n)}), g_{\gamma(n)}(a_{\gamma(n)})\} \subset J_N$ as $I_n \subset I_N$ and $J_n \subset J_N$.

We fix a number $n_0 \in \mathbb{N}$ with $N < n_0$. We claim that $g_{\gamma(n_0)}(K) \subset U$. Note that one of $g_{\gamma(n_0)}([a_{\gamma(n_0)}, y_{\gamma(n_0)}]_{S^1})$ and $g_{\gamma(n_0)}([y_{\gamma(n_0)}, a_{\gamma(n_0)}]_{S^1})$ must be contained in J_{n_0} as $\{g_{\gamma(n_0)}(a_{\gamma(n_0)}), g_{\gamma(n_0)}(y_{\gamma(n_0)})\} \subset J_{n_0}$. There are two cases depending on the location of $a_{\gamma(n_0)} : a_{\gamma(n_0)} \in (r_{n_0}, x)_{S^1}$; $a_{\gamma(n_0)} \in (x, s_{n_0})_{S^1}$.

First we consider the case where $a_{\gamma(n_0)} \in (r_{n_0}, x)_{S^1}$. Then $y_{\gamma(n_0)} \in (x, r_{n_0})_{S^1}$ by the choice of ϵ . Hence one of $g_{\gamma(n_0)}(r_{n_0})$ and $g_{\gamma(n_0)}(x)$ belongs to J_{n_0} since $x \in (a_{\gamma(n_0)}, y_{\gamma(n_0)})_{S^1}$ and $r_{n_0} \in (y_{\gamma(n_0)}, a_{\gamma(n_0)})_{S^1}$.

In the case where $g_{\gamma(n_0)}(r_{n_0}) \in J_{n_0}$, there are three possible cases:

$$g_{\gamma(n_0)}(r_{n_0}) \in (u_{n_0}, y')_{S^1}, g_{\gamma(n_0)}(r_{n_0}) = y' \text{ or } g_{\gamma(n_0)}(r_{n_0}) \in (y', v_{n_0})_{S^1}.$$

If $g_{\gamma(n_0)}(r_{n_0}) \in (u_{n_0}, y')_{S^1}$, then

$$g_{\gamma(n_0)}((r_{n_0}, x)_{S^1}) \subset (u_{n_0}, y')_{S^1} \text{ or } g_{\gamma(n_0)}((r_{n_0}, x)_{S^1})^* \subset (u_{n_0}, y')_{S^1}$$

by the unlinkedness of $g_{\gamma(n_0)}((r_{n_0}, x)_{S^1})$ and $(u_{n_0}, y')_{S^1}$. When $g_{\gamma(n_0)}((r_{n_0}, x)_{S^1}) \subset (u_{n_0}, y')_{S^1}$, we get that $g_{\gamma(n_0)}(x) \in (u_{n_0}, y')_{S^1}$ since $g_{\gamma(n_0)}(x) \neq y'$, and that $g_{\gamma(n_0)}(\ell((x, s_{n_0})_{S^1}))$ lies on $(u_{n_0}, y')_{S^1}$. However if $g_{\gamma(n_0)}((x, s_{n_0})_{S^1}) \subset (u_{n_0}, y')_{S^1}$, then $g_{\gamma(n_0)}(I_{n_0}) = g_{\gamma(n_0)}((r_{n_0}, s_{n_0})_{S^1}) \subset (u_{n_0}, y')_{S^1} \subset J_{n_0}$ so this is a contradiction since $g_{\gamma(n_0)}(x_{\gamma(n_0)}) \notin J_{n_0}$. Therefore $g_{\gamma(n_0)}((s_{n_0}, x)_{S^1}) \subset (u_{n_0}, y')_{S^1}$. Then

$$\begin{aligned} g_{\gamma(n_0)}(K) &\subset g_{\gamma(n_0)}(I_N^*) \subset g_{\gamma(n_0)}(I_{n_0}^*) \\ &\subset g_{\gamma(n_0)}((s_{n_0}, x)_{S^1}) \subset (u_{n_0}, y')_{S^1} \subset J_{n_0} \subset J_N \subset U. \end{aligned}$$

So this case is proved. Then when $g_{\gamma(n_0)}((r_{n_0}, x)_{S^1})^* \subset (u_{n_0}, y')_{S^1}$, we can see that

$$\begin{aligned} g_{\gamma(n_0)}(K) &\subset g_{\gamma(n_0)}(I_N^*) \subset g_{\gamma(n_0)}(I_{n_0}^*) \\ &\subset g_{\gamma(n_0)}((r_{n_0}, x)_{S^1}^*) \subset (u_{n_0}, y')_{S^1} \subset J_{n_0} \subset J_N \subset U. \end{aligned}$$

Therefore the case where $g_{\gamma(n_0)}(r_{n_0}) \in (u_{n_0}, y')_{S^1}$ is done.

Now if $g_{\gamma(n_0)}(r_{n_0}) = y'$, then there are three cases by the unlinkedness of $g_{\gamma(n_0)}((r_{n_0}, x)_{S^1})$, $(u_{n_0}, y')_{S^1}$ and $(y', v_{n_0})_{S^1}$: $g_{\gamma(n_0)}((r_{n_0}, x)_{S^1}) \subsetneq (y', v_{n_0})_{S^1}$; $(y', v_{n_0})_{S^1} \subset g_{\gamma(n_0)}((r_{n_0}, x)_{S^1}) \subsetneq (u_{n_0}, y')_{S^1}$; $(u_{n_0}, y')_{S^1} \subset g_{\gamma(n_0)}((r_{n_0}, x)_{S^1})$.

We claim that the first case is not possible. If $g_{\gamma(n_0)}((r_{n_0}, x)_{S^1}) \subsetneq (y', v_{n_0})_{S^1}$, then $x \in (y', v_{n_0})_{S^1}$ since $g_{\gamma(n_0)}(r_{n_0}) = y'$. Then $g_{\gamma(n_0)}(\ell((x, s_{n_0})_{S^1}))$ must lie on $(y', v_{n_0})_{S^1}$. However if $g_{\gamma(n_0)}((x, s_{n_0})_{S^1}) \subset (y', v_{n_0})_{S^1}$, then

$$\begin{aligned} g_{\gamma(n_0)}(x_{\gamma(n_0)}) &\in g_{\gamma(n_0)}(I_{n_0}) \\ &= g_{\gamma(n_0)}((r_{n_0}, s_{n_0})_{S^1}) \\ &\subset (y', v_{n_0})_{S^1} \subset J_{n_0} \subset N_\epsilon(y'). \end{aligned}$$

This is a contradiction by the choice of ϵ . Also if $g_{\gamma(n_0)}((x, s_{n_0})_{S^1})^* \subset (y', v_{n_0})_{S^1}$, then

$$\begin{aligned} (y', g_{\gamma(n_0)}(x))_{S^1} &= g_{\gamma(n_0)}((r_{n_0}, x)_{S^1}) \\ &\subset g_{\gamma(n_0)}((x, s_{n_0})_{S^1}^*) = (g_{\gamma(n_0)}(s_{n_0}), g_{\gamma(n_0)}(x))_{S^1} \subset (y', v_{n_0})_{S^1}, \end{aligned}$$

and this implies that $g_{\gamma(n_0)}(s_{n_0}) = y'$. However $g_{\gamma(n_0)}(s_{n_0}) = y' = g_{\gamma(n_0)}(r_{n_0})$ so this is a contradiction. Therefore the claim is proved.

Now we consider the case where

$$(y', v_{n_0})_{S^1} \subset g_{\gamma(n_0)}((r_{n_0}, x)_{S^1}) \subsetneq (u_{n_0}, y')_{S^1}^* = (y', u_{n_0})_{S^1}.$$

This case is also impossible. Observe that $g_{\gamma(n_0)}(x) \in [v_{n_0}, u_{n_0}]_{S^1}$ and $g_{\gamma(n_0)}(y_{\gamma(n_0)})$ must belong to $(u_{n_0}, y')_{S^1}$ in this case. Since $y' = g_{\gamma(n_0)}(r_{n_0}) \in g_{\gamma(n_0)}((r_N, x)_{S^1})$, the leaf $\ell((u_{n_0}, y')_{S^1})$ lies on $g_{\gamma(n_0)}((r_N, x)_{S^1})$. If $(u_{n_0}, y')_{S^1}^* \subset g_{\gamma(n_0)}((r_N, x)_{S^1})$, then $(g_{\gamma(n_0)}(x), g_{\gamma(n_0)}(r_N))_{S^1} = g_{\gamma(n_0)}((r_N, x)_{S^1})^* \subset (u_{n_0}, y')_{S^1}$ and $g_{\gamma(n_0)}(x) \in [u_{n_0}, y']_{S^1}$. This is a contradiction since $[v_{n_0}, u_{n_0}]_{S^1}$ and $[u_{n_0}, y']_{S^1}$ are disjoint. If $(u_{n_0}, y')_{S^1} \subset g_{\gamma(n_0)}((r_N, x)_{S^1})$, then $g_{\gamma(n_0)}(y_{\gamma(n_0)}) \in (u_{n_0}, y')_{S^1} \subset g_{\gamma(n_0)}((r_N, x)_{S^1})$ and this is also a contradiction since $y_{\gamma(n_0)} \notin (r_N, x)_{S^1}$.

Therefore $(u_{n_0}, y')_{S^1}^* \subset g_{\gamma(n_0)}((r_{n_0}, x)_{S^1})$. Then

$$g_{\gamma(n_0)}(K) \subset g_{\gamma(n_0)}(I_{n_0}^*) \subset g_{\gamma(n_0)}((r_{n_0}, x)_{S^1}^*) \subset (u_{n_0}, y')_{S^1} \subset J_{n_0} \subset J_N \subset U.$$

Thus the case where $g_{\gamma(n_0)}(r_{n_0}) = y'$ is also proved.

Then we consider the case where $g_{\gamma(n_0)}(r_{n_0}) \in (y', v_{n_0})_{S^1}$. Then there are two cases by the unlinkedness of $g_{\gamma(n_0)}((r_{n_0}, x)_{S^1})$ and $(y', v_{n_0})_{S^1}$:

- $g_{\gamma(n_0)}((r_{n_0}, x)_{S^1}) \subset (y', v_{n_0})_{S^1}$;
- $g_{\gamma(n_0)}((r_{n_0}, x)_{S^1})^* \subset (y', v_{n_0})_{S^1}$.

If $g_{\gamma(n_0)}((r_{n_0}, x)_{S^1}) \subset (y', v_{n_0})_{S^1}$, then $g_{\gamma(n_0)}(x) \in (y', v_{n_0}]_{S^1} \subset (y', v_N)_{S^1}$ and so the leaf $g_{\gamma(n_0)}(\ell((x, s_{n_0})_{S^1}))$ must lie on $(y', v_N)_{S^1}$. However if $g_{\gamma(n_0)}((x, s_{n_0})_{S^1}) \subset (y', v_N)_{S^1}$, then $g_{\gamma(n_0)}((r_{n_0}, s_{n_0})_{S^1}) \subset (y', v_N)_{S^1}$ and this is a contradiction since $g_{\gamma(n_0)}(x_{\gamma(n_0)}) \notin J_N$. Therefore $g_{\gamma(n_0)}((x, s_{n_0})_{S^1})^* \subset (y', v_N)_{S^1}$ and so

$$g_{\gamma(n_0)}(K) \subset g_{\gamma(n_0)}(I_N^*) \subseteq g_{\gamma(n_0)}((x, s_{n_0})_{S^1}^*) \subset (y', v_N)_{S^1} \subset J_N \subset U.$$

Hence the case where $g_{\gamma(n_0)}(r_{n_0}) \in (y', v_{n_0})_{S^1}$ is also proved. Thus the case where $g_{\gamma(n_0)}(r_{n_0}) \in J_{n_0}$ is proved.

Now we consider the case where $g_{\gamma(n_0)}(x) \in J_{n_0}$. Then $g_{\gamma(n_0)}(x) \in (u_{n_0}, y')_{S^1}$ or $g_{\gamma(n_0)}(x) \in (y', v_{n_0})_{S^1}$ as $g_{\gamma(n_0)}(x) \neq y'$. For brevity, we write

$$L = g_{\gamma(n_0)}((r_{n_0}, x)_{S^1}) \text{ and } R = g_{\gamma(n_0)}((x, s_{n_0})_{S^1}).$$

If $g_{\gamma(n_0)}(x) \in (u_{n_0}, y')_{S^1}$, then the leaves $\ell(L)$ and $\ell(R)$ must lie on $(u_{n_0}, y')_{S^1}$. However if $L \subset (u_{n_0}, y')_{S^1}$ and $R \subset (u_{n_0}, y')_{S^1}$, then $g_{\gamma(n_0)}(I_{n_0}) \subset (u_{n_0}, y')_{S^1}$ and this is a contradiction since $g_{\gamma(n_0)}(x_{\gamma(n_0)}) \notin J_{n_0}$. Hence $L^* \subset (u_{n_0}, y')_{S^1}$ or $R^* \subset (u_{n_0}, y')_{S^1}$. In both cases, $g_{\gamma(n_0)}(I_{n_0}^*) \subset (u_{n_0}, y')_{S^1}$. Therefore

$$g_{\gamma(n_0)}(K) \subset g_{\gamma(n_0)}(I_N^*) \subset g_{\gamma(n_0)}(I_{n_0}^*) \subset (u_{n_0}, y')_{S^1} \subset J_{n_0} \subset J_N \subset U.$$

Similarly if $g_{\gamma(n_0)}(x) \in (y', v_{n_0})_{S^1}$, then the leaves $\ell(L)$ and $\ell(R)$ must lie on $(y', v_{n_0})_{S^1}$. However if $L \subset (y', v_{n_0})_{S^1}$ and $R \subset (y', v_{n_0})_{S^1}$, then $g_{\gamma(n_0)}(I_{n_0}) \subset (y', v_{n_0})_{S^1}$ and this is a contradiction since $g_{\gamma(n_0)}(x_{\gamma(n_0)}) \notin J_{n_0}$. Hence $L^* \subset (y', v_{n_0})_{S^1}$ or $R^* \subset (y', v_{n_0})_{S^1}$. In both cases $g_{\gamma(n_0)}(I_{n_0}^*) \subset (y', v_{n_0})_{S^1}$. Therefore

$$g_{\gamma(n_0)}(K) \subset g_{\gamma(n_0)}(I_N^*) \subset g_{\gamma(n_0)}(I_{n_0}^*) \subset (y', v_{n_0})_{S^1} \subset J_{n_0} \subset J_N \subset U.$$

Thus the case where $g_{\gamma(n_0)}(x) \in J_{n_0}$ also follows and so the case where $a_{n_0} \in (r_{n_0}, x)_{S^1}$ is done. Similarly $g_{\gamma(n_0)}(K) \subset U$ when $a_{n_0} \in (x, s_{n_0})_{S^1}$. This implies that for any $n \in \mathbb{N}$ with $N < n$, $g_{\gamma(n)}(K) \subset U$. Therefore the sequence $\{g_{\gamma(n)}\}_{n=1}^\infty$ has the convergence property and so $\{g_{\alpha(n)}\}_{n=1}^\infty$ has the convergence property. Thus the original sequence $\{g_n\}_{n=1}^\infty$ has the convergence property. $\square$

REFERENCES

- [1] Juan Alonso, Hyungryul Baik, and Eric Samperton. On laminar groups, Tits alternatives and convergence group actions on S^2 . *J. Group Theory*, 22(3):359–381, 2019.
- [2] Venancio Álvarez and José M. Rodríguez. Structure theorems for Riemann and topological surfaces. *J. London Math. Soc. (2)*, 69(1):153–168, 2004.
- [3] Hyungryul Baik. Fuchsian groups, circularly ordered groups and dense invariant laminations on the circle. *Geom. Topol.*, 19(4):2081–2115, 2015.
- [4] Hyungryul Baik, Hongtaek Jung, and KyeongRo Kim. Groups acting on veering pairs and kleinian groups. *Journal of the London Mathematical Society*, 111(1):e70052, 2025.
- [5] Hyungryul Baik and KyeongRo Kim. Laminar groups and 3-manifolds. In Ken’ichi Ohshika and Athanase Papadopoulos, editors, *In the Tradition of Thurston: Geometry and Topology*, pages 365–421. Springer International Publishing, Cham, 2020.
- [6] Ara Basmajian and Dragomir Šarić. Geodesically complete hyperbolic structures. *Math. Proc. Cambridge Philos. Soc.*, 166(2):219–242, 2019.
- [7] Alan F. Beardon. *The geometry of discrete groups*, volume 91 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Corrected reprint of the 1983 original.
- [8] Ian Biringer, Nir Lazarovich, and Arielle Leitner. On the Chabauty space of $PSL_2\mathbb{R}$, I: lattices and grafting. *arXiv preprint arXiv:2110.14401*, 2021.
- [9] B. H. Bowditch. Convergence groups and configuration spaces. In *Geometric group theory down under (Canberra, 1996)*, pages 23–54. de Gruyter, Berlin, 1999.
- [10] Peter Buser. *Geometry and spectra of compact Riemann surfaces*, volume 106 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1992.
- [11] D Calegari. Foliations and the geometrization of 3-manifolds. *Lecture note for the course ‘Foliations and 3-Manifolds’*, 2001.
- [12] Danny Calegari. The geometry of $\mathbf{R}$ -covered foliations. *Geom. Topol.*, 4:457–515, 2000.
- [13] Danny Calegari. Universal circles for quasigeodesic flows. *Geom. Topol.*, 10:2271–2298, 2006.
- [14] Danny Calegari. *Foliations and the geometry of 3-manifolds*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2007.
- [15] Danny Calegari and Nathan M. Dunfield. Laminations and groups of homeomorphisms of the circle. *Invent. Math.*, 152(1):149–204, 2003.
- [16] Andrew Casson and Douglas Jungreis. Convergence groups and Seifert fibered 3-manifolds. *Invent. Math.*, 118(3):441–456, 1994.
- [17] Sérgio Fenley. Ideal boundaries of pseudo-Anosov flows and uniform convergence groups with connections and applications to large scale geometry. *Geom. Topol.*, 16(1):1–110, 2012.
- [18] David Fisher. Groups acting on manifolds: around the Zimmer program. In *Geometry, rigidity, and group actions*, Chicago Lectures in Math., pages 72–157. Univ. Chicago Press, Chicago, IL, 2011.
- [19] Steven Frankel, Saul Schleimer, and Henry Segerman. From veering triangulations to link spaces and back again, 2019.

- [20] David Gabai. Convergence groups are Fuchsian groups. *Ann. of Math. (2)*, 136(3):447–510, 1992.
- [21] Étienne Ghys. Actions de réseaux sur le cercle. *Invent. Math.*, 137(1):199–231, 1999.
- [22] Joel Hass and Peter Scott. Shortening curves on surfaces. *Topology*, 33(1):25–43, 1994.
- [23] A. Hinkkanen. Abelian and nondiscrete convergence groups on the circle. *Trans. Amer. Math. Soc.*, 318(1):87–121, 1990.
- [24] Irwin Kra. *Automorphic forms and Kleinian groups*. W. A. Benjamin, Inc., Reading, Mass., 1972. Mathematics Lecture Note Series.
- [25] Kathryn Mann. Spaces of surface group representations. *Invent. Math.*, 201(2):669–710, 2015.
- [26] Kathryn Mann and Maxime Wolff. A characterization of Fuchsian actions by topological rigidity. *Pacific J. Math.*, 302(1):181–200, 2019.
- [27] John G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.
- [28] Ian Richards. On the classification of noncompact surfaces. *Trans. Amer. Math. Soc.*, 106:259–269, 1963.
- [29] Robert Suzzi Valli. Figure eight geodesics on 2-orbifolds. *Comput. Methods Funct. Theory*, 16(1):105–125, 2016.
- [30] Ser Peow Tan, Yan Loi Wong, and Ying Zhang. Generalizations of McShane’s identity to hyperbolic cone-surfaces. *J. Differential Geom.*, 72(1):73–112, 2006.
- [31] William P. Thurston. *Geometry and topology of three-manifolds*, 1980.
- [32] William P. Thurston. Three-manifolds, Foliations and Circles, I. *arXiv:math/9712268v1 [math.GT]*, 1997.
- [33] Pekka Tukia. Homeomorphic conjugates of Fuchsian groups. *J. Reine Angew. Math.*, 391:1–54, 1988.

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