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FINITE-TIME BLOW-UP OF CLASSICAL SOLUTION TO THE TWO-FLUID MODEL WITH DENSITY-DEPENDENT VISCOSITY

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ABSTRACT. This paper concerns the initial-boundary value problem for a compressible two-fluid model with density-dependent viscosities (possibly degenerating in vacuum), subject to Dirichlet boundary conditions. We prove that the two-fluid system with non-monotone pressure will blow-up in finite time under the assumption that the initial densities include an isolated mass group.

1. INTRODUCTION

The utilization of the two-phase flow model has been extensively employed in various industries such as the petroleum industry, low temperature industrial biomedical microtechnology and other related fields. For more background, we refer the readers to [1, 10, 11, 28, 33]. In this paper, we study the compressible two-fluid model with the following form:

$$(1.1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t n + \operatorname{div}(n \mathbf{u}) = 0, \quad (x, t) \in \Omega \times \mathbb{R}_+, \\ \partial_t((\rho + n)\mathbf{u}) + \operatorname{div}((\rho + n)\mathbf{u} \otimes \mathbf{u}) + \nabla P = \operatorname{div}[2h(\rho, n)\mathbb{D}(\mathbf{u})] + \nabla(g(\rho, n)\operatorname{div} \mathbf{u}). \end{cases}$$

The densities of two different fluids are represented by $\rho(x, t)$ and $n(x, t)$ respectively, $\mathbf{u}(x, t)$ stands for the mixed velocity. The shear viscosity coefficient $h(\rho, n)$ and the bulk viscosity coefficient $g(\rho, n)$ satisfy the following physical restrictions:

$$(1.2) \quad h(\rho, n) \geq 0, \quad 2h(\rho, n) + dg(\rho, n) \geq 0.$$

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We consider the non-monotone pressure $P(\rho, n) \in C^2([0, \infty)^2)$ satisfying $P(0, 0) = 0$ and with the form

$$(1.3) \quad c_1(\rho^\gamma + n^\alpha - 1) \leq P(\rho, n) \leq c_2(\rho^\gamma + n^\alpha + 1),$$

where the constants $c_1, c_2, \gamma, \alpha, \mu$ satisfy

$$(1.4) \quad c_1 > 0, \ c_2 > 0, \ \gamma \geq 1, \ \alpha \geq 1, \ \max\{\gamma, \alpha\} \geq 2, \ 0 < \mu < 1.$$

The strain tensor is given by $\mathbb{D}(\mathbf{u}) := \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^t}{2}$. We consider the system (1.1) in a bounded smooth domain $\Omega \subset \mathbb{R}^d (d \geq 2)$, and pose the usual Dirichlet boundary condition

$$(1.5) \quad \mathbf{u}|_{\partial\Omega} = 0, \text{ for } t \geq 0.$$

Moreover, we complement (1.1) with the initial conditions

$$(1.6) \quad \rho(x, 0) = \rho_0(x), \ n(x, 0) = n_0(x), \ \mathbf{u}(x, 0) = \mathbf{u}_0(x).$$

It is well known that the compressible Navier-Stokes (CNS) model holds a pivotal role in the field of mechanics. Consequently, we first recall some classical advancements in the CNS model. For monotonic pressure, Kazhikhov and Shelukhin [16] established the existence result for the model in one-dimensional space with initial density bounded away from zero. Then Matsumura and Nishida [23] proved the global existence with small initial data in multidimensional space. For non-monotonic pressure, Feireisl [7] proved the global existence of weak solutions on a compact set. Recently Bresch and Jabin [2] focused on more general pressure laws that are not thermodynamically stable and indicated global weak solutions to the system. Readers can see more results and physical background in [8, 21].

The subject of two-phase flow has become increasingly important in a wide variety of engineering systems and biological systems. From the mathematical point, researchers have noted the similar mathematic structure between the CNS model and the two-phase model. Therefore, in light of the results observed in the CNS model, there has been a significant surge in research focused on investigating the well-posedness of the two-phase model in recent years. For the system (1.1) with standard polytropic pressure $P(\rho, n) = \rho^\gamma + n^\alpha$ and constant viscosity, the existence of global weak solutions in the three dimensional space has been established in [3, 27]. Due to the observed phenomena, the two-fluid model with non-monotonic pressure is more close to the real industry applications, such as, the model of gas-kick flow scenarios in oil wells is related to a compressible gas-liquid model with non-monotonic pressure $P(n, m) = C(\frac{n}{\rho_l - m})^\gamma$ (see [6]). The mathematical analysis of global solutions to the two-phase system with non-monotone pressure has been established in [4, 6, 14, 15, 17, 20, 24, 29, 32]. More precisely, Novotný and Pokorný [24] proved the existence of weak solutions for large initial data on an arbitrarily large time interval where more general pressure functions are considered. In the one-dimensional space, the global existence

and uniqueness of the classical solution for the two-fluid model with density-dependent viscosity was proved by Chen et al. [4]. Li et al. [20] established the global existence of weak solutions to the initial value problem (IVP) for the drift-flux system in the periodic domain $T^d := R^d/Z^d (d \geq 2)$. In addition, there has been important progress made recently about the outflow/inflow problem for viscous multi-phase flow [9, 12, 25].

Currently, extensive research has been dedicated to the blow-up phenomenon of compressible fluid. In the context of the CNS model, Xin [30] established the blow-up phenomenon when the initial densities possess compact support. Then Rozanova [26] generalized [30] to the case with initial data rapidly decays at far fields. Furthermore, Xin and Yan [31] showed that any classical solutions of viscous compressible fluids lacking heat conduction will blow up in a finite time, provided that the initial data exhibits an isolated mass group. Inspired by the work [31], researchers extended the corresponding result to other model, such as [5, 19]. Jiu, Wang and Xin [13] proved the blow-up in the smooth solutions to the Cauchy problem for both the full CNS equations and isentropic CNS equations, considering constant and degenerate viscosities in arbitrary dimensions, subject to certain limitations on the initial data. In the case of the initial density ρ_0 possesses compact support, Li et al. [18] established the non-existence of any non-trivial classical solutions with finite energy to the Cauchy problem of the full CNS and isentropic CNS equations in the standard inhomogeneous Sobolev space for any short time.

Due to the structural similarity between the CNS model and the two-fluid model, the purpose of this article is to investigate the blow-up phenomenon of the classical solution for the compressible two-fluid model with non-monotonic pressure, under the condition that the initial density allows for the presence of an isolated mass group. To obtain our main results, it is essential to introduce various physical quantities, including mass, momentum, momentum of inertia, internal energy, potential energy, total energy. The crucial step in the proof is to provide estimate for physical measurement $G(t) := \frac{1}{2} \int_{U(t)} |x|^2 (\rho(x, t) + n(x, t)) dx$. Thus we need to control the term $|\int_{U(t)} (2h(\rho, n) + dg(\rho, n)) dx|$. However, compared with [5, 19], the energy inequality cannot be employed for a direct estimate of $|\int_{U(t)} (2h(\rho, n) + dg(\rho, n)) dx|$ due to the disparity in the pressure term. In order to overcome the difficulty, we utilize the Helmholtz free energy function introduced by [4].

The paper is organized as follows. Section 2 is devoted to presenting our main results. In Section 3, we will give some basic properties of the physical quantities and then prove the main results.

2. MAIN RESULTS

Before stating our main results, we give the following definitions.

DEFINITION 2.1. For $T > 0$, a pair $(\rho(x, t), n(x, t), \mathbf{u}(x, t))$ is called a classical solution to the Dirichlet problem for System (1.1)-(1.6) on $\partial\Omega$ if the nonnegative function $\rho, n \in C^1(\Omega \times [0, T])$, and the vector field $\mathbf{u} \in C^1([0, T]; C^2(\Omega))$ satisfies the system (1.1) point-wisely on $\Omega \times [0, T]$, take on the initial condition (1.6) continuously, and satisfy the boundary condition (1.5) continuously.

In light of [31], we introduce the following definition of the isolated mass group.

DEFINITION 2.2. Let V, U be two bounded open subsets of Ω and $V \subset U$. The pair (U, V) is called an isolated mass group of initial density $\rho_0(x), n_0(x)$ if it holds that

$$\begin{cases} V \subset \bar{V} \subset U \subset \bar{U} \subset \Omega, U \text{ is connected,} \\ \rho_0(x) = 0, n_0(x) = 0, x \in U \setminus V, \\ \int_V \rho_0(x) dx > 0, \int_V n_0(x) dx > 0. \end{cases}$$

Let (U, V) be an isolated mass group of $\rho_0(x), n_0(x)$ in Ω . and $H(\rho_0, n_0)$ be the Helmholtz free energy function. Then we set

Mass

$$M_0 := \int_V \rho_0(x) dx + \int_V n_0(x) dx := M_{\rho_0} + M_{n_0},$$

Momentum

$$\mathbb{P}_0 := \int_V \rho_0(x) \mathbf{u}_0(x) dx + \int_V n_0(x) \mathbf{u}_0(x) dx := \mathbb{P}_{\rho_0} + \mathbb{P}_{n_0},$$

Momentum weight

$$F_0 := \int_V \rho_0(x) \mathbf{u}_0(x) \cdot x dx + \int_V n_0(x) \mathbf{u}_0(x) \cdot x dx := F_{\rho_0} + F_{n_0},$$

Momentum of inertia

$$G_0 := \frac{1}{2} \int_V \rho_0(x) |x|^2 dx + \frac{1}{2} \int_V n_0(x) |x|^2 dx := G_{\rho_0}(t) + G_{n_0}(t),$$

Total energy

$$E_0 := \int_V \frac{1}{2} (\rho_0(x) + n_0(x)) |\mathbf{u}_0(x)|^2 + H(\rho_0(x), n_0(x)) dx.$$

Moreover, we assume that

$$(2.1) \quad \frac{\mathbb{P}_0^2}{M_0} + dc_1(|B_R(0)|^{1-\gamma} M_{\rho_0}^\gamma + |B_R(0)|^{1-\alpha} M_{n_0}^\alpha - |B_R(0)|) > 0.$$

It is well known that the Helmholtz free energy function $H(\rho, n)$ corresponding to pressure P is a solution of the following partial differential equation of the first order in $(0, \infty)^2$:

$$(2.2) \quad P(\rho, n) = \rho \frac{\partial H(\rho, n)}{\partial \rho} + n \frac{\partial H(\rho, n)}{\partial n} - H(\rho, n).$$

To derive the pressure estimate, we adopt the approach presented in reference [4], employing the Helmholtz free energy function $H(\rho, n)$ that corresponds to $P(\rho, n)$:

$$(2.3) \quad H(\rho, n) := (\rho + n) \int_1^{\rho+n} \frac{P(s \frac{\rho}{\rho+n}, s \frac{n}{\rho+n})}{s^2} ds, \text{ if } \rho + n > 0, \quad H(0, 0) = 0.$$

We assume that the viscosity coefficients $h(\rho, n)$ and $g(\rho, n)$ satisfy the following form

$$(2.4) \quad h(\rho, n) = A\rho^\lambda + Bn^\beta, \quad g(\rho, n) = C\rho^\tau + Dn^q,$$

where $A > 0$, $B > 0$, $2A + dC \geq 0$, $2B + dD \geq 0$, and one of the following conditions:

(i) $\lambda = \tau$, $\beta = q$, $2A + dC = 0$, $2B + dD = 0$;

(ii) λ , $\tau \in (0, \gamma]$, β , $q \in (0, \alpha]$.

Our main results are stated as follows.

THEOREM 2.3. *Let $(\rho(x, t), n(x, t), \mathbf{u}(x, t))$ be a classical solution to the Dirichlet problem of the compressible two-phase model (1.1) with initial data (1.6). Suppose that the initial density $\rho_0(x)$, $n_0(x)$ admit an isolated mass group (U, V) and $h(\rho, n)$, $g(\rho, n)$ satisfy (1.2) if one of the following conditions holds:*

(1) *Condition (i) holds, and M_0 is finite;*

(2) *Condition (ii) holds, and M_0 , E_0 are finite.*

Then the classical solution $(\rho(x, t), n(x, t), \mathbf{u}(x, t))$ will blow up in finite time.

REMARK 2.4. If M_0 , E_0 are finite and $h(\rho, n)$ and $g(\rho, n)$ satisfy: $\lambda = \tau$, $2A + dC = 0$, β , $q \in (0, \alpha]$ or λ , $\tau \in (0, \gamma]$, $\beta = q$, $2B + dD = 0$, the Theorem 2.3 still holds.

3. PROOF OF THEOREM 2.3

Let $(\rho(x, t), n(x, t), \mathbf{u}(x, t))$ be a classical solution to the compressible two-fluid model (1.1) on $\Omega \times (0, T)$, where T is the maximal time of existence. We denote by $X(a, t)$ the particle path starting from a when $t = 0$, i.e.

$$(3.1) \quad \begin{cases} \frac{d}{dt} X(a, t) &= \mathbf{u}(X(a, t), t), \\ X(a, 0) &= a, \end{cases}$$

and set

$$(3.2) \quad U(t) = \{X(a, t) | a \in U\}, \quad V(t) = \{X(a, t) | a \in V\}.$$

The pair $(U(t), V(t))$ is an isolated mass group of the density $\rho(x, t)$, $n(x, t)$ in Ω at time t , and it will not disappear for any $t < T$. In fact, $\mathbf{u} \in$

$C^1([0, T]; C^2(\Omega))$ ensures that $X(a, t)$ is well-defined (existence and uniqueness) by the classical theory of ordinary differential system. Furthermore, $V \subset \bar{V} \subset U \subset \bar{U} \subset \Omega$ implies that

$$V(t) \subset \overline{V(t)} \subset U(t) \subset \overline{U(t)} \subset \Omega.$$

Since $\rho_0(x) = 0$, $n_0(x) = 0$ in $U \setminus V$, we can immediately obtain the following lemma from mass equation (1.1)₁ and (1.1)₂.

LEMMA 3.1. *Suppose $(\rho(x, t), n(x, t), \mathbf{u}(x, t))$ is a classical solution to system (1.1)-(1.6) with a suitable boundary condition on $\partial\Omega$, it holds that*

$$(3.3) \quad \rho(x, t) = 0, \quad n(x, t) = 0 \text{ in } U(t) \setminus V(t).$$

Similar to [19], we assume that the density ρ, n equal to 0 on $\partial U(t)$, otherwise we could choose $R(t)$ instead of $U(t)$ satisfying $V(t) \subset \bar{V(t)} \subset R(t) \subset \bar{R(t)} \subseteq U(t)$, then ρ, n equal to 0 on $\partial R(t)$.

The following lemma is the famous transport formula. It can be found in [22]:

LEMMA 3.2. *Let $U(t)$ be defined as (3.2), for any $f(x, t) \in C^1(\mathbb{R}^d \times \mathbb{R}^+)$, we have*

$$(3.4) \quad \frac{d}{dt} \int_{U(t)} f(x, t) dx = \int_{U(t)} f_t(x, t) dx + \int_{\partial U(t)} f(x, t) (\mathbf{u} \cdot \nu) dx,$$

where ν is the unit out normal to $\partial U(t)$.

Before showing Lemma 3.3, we set $G(t) := \frac{1}{2} \int_{U(t)} (\rho + n) |x|^2 dx$, $F(t) := \int_{U(t)} (\rho + n) \mathbf{u} \cdot x dx$, $E(t) := \int_{U(t)} [\frac{1}{2}(\rho + n) |\mathbf{u}|^2 + H(\rho, n)] dx := E_k(t) + I(t)$.

LEMMA 3.3. *Suppose $(\rho(x, t), n(x, t), \mathbf{u}(x, t))$ is a classical solution to system (1.1)-(1.6) in $\Omega \times (0, T)$ with a suitable boundary condition on $\partial\Omega$, then for each $0 < t < T$, we have*

$$(3.5) \quad \int_{U(t)} \rho dx = M_{\rho_0}, \quad \int_{U(t)} n dx = M_{n_0},$$

$$(3.6) \quad \int_{U(t)} (\rho + n) \mathbf{u} dx = \mathbb{P}_0,$$

$$(3.7) \quad \frac{d}{dt} G(t) = F(t),$$

$$(3.8) \quad \frac{d}{dt} F(t) = 2E_k(t) + d \int_{U(t)} P dx - \int_{U(t)} [(2h(\rho, n) + dg(\rho, n)) \operatorname{div} \mathbf{u}] dx,$$

$$(3.9) \quad \int_{U(t)} \left[\frac{1}{2} (\rho + n) |\mathbf{u}|^2 + H(\rho, n) \right] dx + \int_0^t \int_{U(s)} \left[h(\rho, n) \left(\frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^t}{2} \right) \cdot \nabla \mathbf{u} + g(\rho, n) |\operatorname{div} \mathbf{u}|^2 \right] dx ds = E_0.$$

PROOF. According to (1.1), Lemma 3.2 and the assumption: $\rho, n|_{\partial U(t)} = 0, P(0, 0) = 0$, we obtain the conservation of mass and momentum :

$$\begin{aligned}\frac{d}{dt} \int_{U(t)} \rho(x, t) dx &= \int_{U(t)} (\rho_t + \operatorname{div}(\rho \mathbf{u})) dx = 0, \\ \frac{d}{dt} \int_{U(t)} n(x, t) dx &= \int_{U(t)} (n_t + \operatorname{div}(n \mathbf{u})) dx = 0, \\ \frac{d}{dt} \int_{U(t)} (\rho + n) \mathbf{u} dx &= \int_{U(t)} \left((\rho + n)_t + \operatorname{div}((\rho + n) \mathbf{u} \otimes \mathbf{u}) \right) dx \\ &= \int_{U(t)} \left(\operatorname{div}[2h(\rho, n)\mathbb{D}(\mathbf{u})] + \nabla(g(\rho, n)\operatorname{div} \mathbf{u}) - \nabla P \right) dx = 0,\end{aligned}$$

which imply (3.5) and (3.6). Meanwhile, we can obtain (3.7) and (3.8) by using integration by parts as:

$$\begin{aligned}\frac{d}{dt} G(t) &= \frac{d}{dt} \int_{U(t)} \frac{1}{2} (\rho + n) |x|^2 dx = \frac{1}{2} \int_{U(t)} ((\rho + n) |x|^2)_t + \operatorname{div}((\rho + n) |x|^2 \mathbf{u}) dx \\ &= \frac{1}{2} \int_{U(t)} \left((\rho + n)_t + \operatorname{div}((\rho + n) \mathbf{u}) \right) |x|^2 + 2(\rho + n) \mathbf{u} \cdot x dx \\ &= \int_{U(t)} (\rho + n) \mathbf{u} \cdot x dx = F(t),\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt} F(t) &= \frac{d}{dt} \int_{U(t)} (\rho + n) \mathbf{u} \cdot x dx = \int_{U(t)} \left(((\rho + n) \mathbf{u})_t + \operatorname{div}((\rho + n) \mathbf{u} \otimes \mathbf{u}) \right) \cdot x \\ &\quad + (\rho + n) |\mathbf{u}|^2 dx \\ &= \int_{U(t)} (\rho + n) |\mathbf{u}|^2 + \left(\operatorname{div}[2h(\rho, n)\mathbb{D}(\mathbf{u})] + \nabla(g(\rho, n)\operatorname{div} \mathbf{u}) - \nabla P \right) \cdot x dx \\ &= 2E_k(t) + d \int_{U(t)} P dx - \int_{U(t)} \left((2h(\rho, n) + dg(\rho, n)) \operatorname{div} \mathbf{u} \right) dx.\end{aligned}$$

Taking the time derivative on $E(t)$, and combining with (2.2), we get

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{d}{dt} \int_{U(t)} \left[\frac{1}{2}(\rho+n)|\mathbf{u}|^2 + H(\rho, n) \right] dx \\
&= \int_{U(t)} \left[\left(\frac{1}{2}(\rho+n)|\mathbf{u}|^2 + H(\rho, n) \right)_t + \operatorname{div} \left(\left(\frac{1}{2}(\rho+n)|\mathbf{u}|^2 + H(\rho, n) \right) \mathbf{u} \right) \right] dx \\
&= \int_{U(t)} \left(((\rho+n)\mathbf{u})_t + \operatorname{div}((\rho+n)\mathbf{u} \otimes \mathbf{u}) \right) \cdot \mathbf{u} - \frac{1}{2} \left((\rho+n)_t + \operatorname{div}((\rho+n)\mathbf{u}) \right) |\mathbf{u}|^2 \\
&\quad + H_\rho \rho_t + H_n n_t + (H_\rho \nabla \rho + H_n \nabla n) \cdot \mathbf{u} + H \operatorname{div} \mathbf{u} dx \\
&= \int_{U(t)} \left[(\operatorname{div}(2h\mathbb{D}(\mathbf{u})) + \nabla(g \operatorname{div} \mathbf{u}) - \nabla P) \cdot \mathbf{u} - P \operatorname{div} \mathbf{u} \right] dx \\
&= - \int_{U(t)} \left[2h(\rho, n) \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{u}) + g(\rho, n) |\operatorname{div} \mathbf{u}|^2 \right] dx,
\end{aligned}$$

integrating the above equation and we obtain that

$$\int_{U(t)} \left[\frac{1}{2}(\rho+n)|\mathbf{u}|^2 + H(\rho, n) \right] dx - E(0) = - \int_0^t \int_{U(s)} \left[h(\rho, n) \left(\frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^t}{2} \right) \cdot \nabla \mathbf{u} + g(\rho, n) |\operatorname{div} \mathbf{u}|^2 \right] dx ds,$$

which implies (3.9). $\square$

Inspired by [5], we will estimate $|\int_0^t \int_{U(s)} (2h(\rho, n) + dg(\rho, n)) \operatorname{div} \mathbf{u} \, dx ds|$.

LEMMA 3.4. *Suppose $(\rho(x, t), n(x, t), \mathbf{u}(x, t))$ is a classical solution to system (1.1)-(1.6) in $\Omega \times (0, T)$ with a suitable boundary condition on $\partial\Omega$. If $h(\rho, n)$ and $g(\rho, n)$ satisfy the conditions in Theorem 2.1, then for each $0 < t < T$, it holds that*

$$(3.10) \quad \left| \int_0^t \int_{U(s)} (2h(\rho, n) + dg(\rho, n)) \operatorname{div} \mathbf{u} \, dx ds \right| \leq C_3 t^{\frac{1}{2}},$$

where C_3 is a positive constant independent of t .

PROOF. By virtue of Hölder inequality, we get

$$\begin{aligned}
(3.11) \quad & \left| \int_0^t \int_{U(s)} (2h(\rho, n) + dg(\rho, n)) \operatorname{div} \mathbf{u} \, dx ds \right| \\
& \leq \int_0^t \int_{U(s)} (2h(\rho, n) + dg(\rho, n)) |\operatorname{div} \mathbf{u}| \, dx ds \\
& \leq \left(\int_0^t \int_{U(s)} (2h(\rho, n) + dg(\rho, n)) |\operatorname{div} \mathbf{u}|^2 \, dx ds \right)^{\frac{1}{2}} \left(\int_0^t \int_{U(s)} [2h(\rho, n) + dg(\rho, n)] \, dx ds \right)^{\frac{1}{2}} \\
& \leq E_0^{\frac{1}{2}} \left(\int_0^t \int_{U(s)} [2h(\rho, n) + dg(\rho, n)] \, dx ds \right)^{\frac{1}{2}}.
\end{aligned}$$

When $h(\rho, n)$, $g(\rho, n)$ satisfy Condition (i), the result is obviously true.

When $h(\rho, n)$, $g(\rho, n)$ satisfy Condition (ii), according to (1.3) and (2.3), we have

$$(3.12) \quad c_1(\rho+n) \int_1^{\rho+n} \frac{\left(\frac{s\rho}{\rho+n}\right)^\gamma + \left(\frac{sn}{\rho+n}\right)^\alpha - 1}{s^2} ds \leq H(\rho, n) \leq c_2(\rho+n) \int_1^{\rho+n} \frac{\left(\frac{s\rho}{\rho+n}\right)^\gamma + \left(\frac{sn}{\rho+n}\right)^\alpha + 1}{s^2} ds.$$

By virtue of the fact that $\left(\frac{\rho}{\rho+n}\right)^{\gamma-1} < 1$, $\left(\frac{n}{\rho+n}\right)^{\alpha-1} < 1$, we have

$$\rho + n + \frac{\rho^\gamma}{(\rho+n)^{\gamma-1}(\gamma-1)} + \frac{n^\alpha}{(\rho+n)^{\alpha-1}(\alpha-1)} \leq \frac{\gamma\rho}{\gamma-1} + \frac{\alpha n}{\alpha-1}.$$

Combine with (3.12), we get the upper and lower bounds of the Helmholtz function $H(\rho, n)$

$$c_1\left(\frac{\rho^\gamma}{\gamma-1} + \frac{n^\alpha}{\alpha-1} + 1 - \frac{\gamma\rho}{\gamma-1} - \frac{\alpha n}{\alpha-1}\right) \leq H(\rho, n) \leq c_2\left(\frac{\rho^\gamma}{\gamma-1} + \frac{n^\alpha}{\alpha-1} - 1 + \rho + n\right).$$

Integrating on domain, we have

$$c_1 \int_{U(t)} \left(\frac{\rho^\gamma}{\gamma-1} + \frac{n^\alpha}{\alpha-1} + 1 - \frac{\gamma\rho}{\gamma-1} - \frac{\alpha n}{\alpha-1}\right) dx \leq \int_{U(t)} H(\rho, n) dx \leq c_2 \int_{U(t)} \left(\frac{\rho^\gamma}{\gamma-1} + \frac{n^\alpha}{\alpha-1} - 1 + \rho + n\right) dx.$$

Then according to (1.4), there exist two positive numbers C_1 and C_2 such that

$$(3.13) \quad \begin{aligned} & \int_{U(t)} [2h(\rho, n) + dg(\rho, n)] dx \\ & \leq C_1 \int_{U(t)} (\rho^\gamma + n^\alpha + 1) dx \\ & \leq C_1 \max\{\gamma-1, \alpha-1\} \int_{U(t)} \left(\frac{\rho^\gamma}{\gamma-1} + \frac{n^\alpha}{\alpha-1} + 1 - \frac{\gamma\rho}{\gamma-1} - \frac{\alpha n}{\alpha-1}\right) dx \\ & \quad + C_1 \max\{\gamma-1, \alpha-1\} \int_{U(t)} \left(\frac{\gamma\rho}{\gamma-1} + \frac{\alpha n}{\alpha-1}\right) dx \\ & \leq C_2(E_0 + 1). \end{aligned}$$

Integrating (3.13) on time, we get

$$\int_0^t \int_{U(s)} [2h(\rho, n) + dg(\rho, n)] dx ds \leq C_2(E_0 + 1)t.$$

Which implies

$$(3.14) \quad \left| \int_0^t \int_{U(s)} (2h(\rho, n) + dg(\rho, n)) \operatorname{div} u \, dx ds \right| \leq C_3 t^{\frac{1}{2}},$$

$$C_3 := (C_2(E_0 + 1)E_0)^{\frac{1}{2}}.$$

□

In the light of [13], we derive the estimate of $G(t)$.

LEMMA 3.5. *Suppose $(\rho(x, t), n(x, t), \mathbf{u}(x, t))$ is a classical solution to system (1.1)-(1.6) in $\Omega \times (0, T)$. If $h(\rho, n)$ and $g(\rho, n)$ satisfy the conditions in Theorem 2.1, then for each $0 < t < T$, it holds that*

$$(3.15) \quad G(t) \geq G_0 + F_0 t + \frac{1}{2}(C_4 + dC_5)t^2 - \frac{2}{3}C_3 t^{\frac{3}{2}}.$$

PROOF. Owing to (3.7) and (3.8), we take time derivative get

$$\frac{d^2}{dt^2}G(t) = \frac{d}{dt}F(t) = \int_{U(t)} [(\rho + n)|\mathbf{u}|^2 + dP - (2h(\rho, n) + dg(\rho, n))\operatorname{div} \mathbf{u}] dx.$$

Given the Hölder inequality, we can get

$$(3.16) \quad \int_{U(t)} (\rho + n)|\mathbf{u}|^2 dx \geq \frac{\mathbb{P}_0^2}{M_0} := C_4.$$

By virtue of the mass conservation equation and the fact that $U(t)$ is always in a bounded domain $\Omega \subset B_R(0)$ for some $R > 0$, we obtain that

$$(3.17) \quad \begin{aligned} \int_{U(t)} P dx &\geq c_1 \int_{U(t)} (\rho^\gamma + n^\alpha - 1) dx \\ &\geq c_1 (|U(t)|^{1-\gamma} M_{\rho_0}^\gamma + |U(t)|^{1-\alpha} M_{n_0}^\alpha - |U(t)|) \\ &\geq c_1 (|B_R(0)|^{1-\gamma} M_{\rho_0}^\gamma + |B_R(0)|^{1-\alpha} M_{n_0}^\alpha - |B_R(0)|) := C_5. \end{aligned}$$

Combining (3.16), (3.17) and Lemma 3.4, we have

$$\frac{d}{dt}G(t) \geq F_0 + (C_4 + dC_5)t - C_3 t^{\frac{1}{2}},$$

which infers that

$$(3.18) \quad G(t) \geq G_0 + F_0 t + \frac{1}{2}(C_4 + dC_5)t^2 - \frac{2}{3}C_3 t^{\frac{3}{2}}.$$

□

Now we are ready to prove the Theorem 2.3.

Proof of Theorem 2.3. Combining the previous assumption (2.1), we deduce the following inequality from Lemma 3.5

$$G(t) \geq G_0 + F_0 t + \frac{1}{2}(C_4 + dC_5)t^2 - \frac{2}{3}C_3 t^{\frac{3}{2}} \rightarrow +\infty \text{ as } t \rightarrow +\infty.$$

While on the other side, we have

$$(3.19) \quad G(t) = \frac{1}{2} \int_{U(t)} |x|^2 (\rho(x, t) + n(x, t)) dx \leq \frac{1}{2} R^2 (M_{\rho_0} + M_{n_0}),$$

Combining (3.18) with (3.19) yields that

$$(3.20) \quad G_0 + F_0 t + \frac{1}{2}(C_4 + dC_5)t^2 - \frac{2}{3}C_3 t^{\frac{3}{2}} \leq \frac{1}{2} R^2 (M_{\rho_0} + M_{n_0}),$$

Let $t \rightarrow +\infty$, we find the left side of (3.20) goes to infinity, while the right side of (3.20) is bounded. Which means that the time span of the classical solutions $(\rho(x, t), n(x, t), \mathbf{u}(x, t))$ is finite. Therefore, the proof of Theorem 2.1 is completed.

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