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MONODROMY THROUGH NARROW BIFURCATION LOCUS OF THE MANDELBROT SET

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Abstract. We study the behavior of the itinerary sequence of each point of the Julia set of $z \mapsto z^2 + c$ when the parameter c in the shift locus is allowed to pass through points in the bifurcation locus, which we call *narrow*. We first show the combinatorial and geometric properties of narrow characteristic arcs. We also show how the itinerary sequence changes in an algorithmic way by using the lamination models proposed by Keller [12]. Finally, we found an equivalence relation on the set of 0-1 sequences such that the changing rule is a shift invariant up to the equivalence relation. This generalizes Atela's works ([2], [1]), which dealt with the special case of the generalized rabbit polynomials.

1. Introduction

The shift locus of complex polynomials of degree $d \geq 2$ is a collection of polynomials where every critical point escapes to infinity under iterations of itself. The reason we call it a shift polynomial is the following theorem.

Theorem 1.1. *Suppose P is a shift polynomial of degree $d \geq 2$ and J_P be a Julia set of P. Then there exists a homeomorphism between J_P and Σ_d, a set of one-sided infinite sequence of d symbols. Furthermore $P|_{J_P} : J_P \to J_P$ is conjugated by this homeomorphism to the one-sided shift map $\sigma : \Sigma_d \to \Sigma_d$.* i.e.,

$$\begin{array}{ccc} J_P & \longrightarrow & \Sigma_d \\ \big\downarrow P & & \big\downarrow \sigma \\ J_P & \longrightarrow & \Sigma_d \end{array}$$

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We define the topology of Σ_d as the product topology of $\{0, 1, \cdots, d-1\}^{\mathbb{N}}$, so it is homeomorphic to a Cantor set.

The most famous and the simplest one is the exterior of the Mandelbrot set, $\mathbb{C} - \mathcal{M}$, which is the shift locus of the quadratic polynomials $\mathcal{S}_2$. Since the Mandelbrot set seems to be fractal, a shift locus of degree d, $\mathcal{S}_d$, also has a fractal structure, and it has been a challenge to understand its topology and geometry. In 1994, Blanchard, Devaney, and Keen [3] showed that there is an action of the fundamental group of the shift locus $\pi_1(\mathcal{S}_d)$ on the set of shift automorphisms of d symbols. Furthermore they proved that this action is surjective. *i.e.,* for every shift automorphism ϕ, there exists an element in $\pi_1(\mathcal{S}_d)$ which acts on the Julia set in exactly the same way as ϕ.

$$\text{BDK} : \pi_1(\mathcal{S}_d) \twoheadrightarrow Aut(\Sigma_d, \sigma)$$

Here a shift automorphism is an automorphism of Σ_d that commutes with the one-sided shift map σ.

Unlike $d \geq 3$, the quadratic case is of no interests in the usual sense, since the shift locus $\mathcal{S}_2$ is conformal to $\mathbb{C} - \mathbb{D}$ as proved by Douady and Hubbard. Hence $\pi_1(\mathcal{S}_2)$ is $\mathbb{Z}$ and it is generated by a loop γ wrapped around $\mathcal{M}$ as in the figure 1. Under the BDK map, this generator acts on Σ_2 as a symbol change of 0 and 1 which is the only nontrivial shift automorphism in Σ_2.

$$\begin{aligned} \text{BDK} : \pi_1(\mathcal{S}_2) \;\cong\; \langle\gamma\rangle &\longrightarrow Aut(\Sigma_2, \sigma) \\ \gamma \qquad &\longmapsto (0,1) \text{ symbol swap} \end{aligned}$$

Let J_α be a Cantor Julia set of $P_\alpha : z \mapsto z^2 + c$, where the parameter angle of c is α. Then as c moves around the loop, J_0 permutes itself and comes back to the position where it started.

Narrow characteristic arcs. Milnor proved in [17] that there is a combinatorial way (called "Orbit portrait") to describe the dynamical properties of parabolic points. Also, Douady, Hubbard and Lavaurs proved in [8] that every parabolic point has exactly two parameter rays that land at the root of a hyperbolic component. These pair of angles are called *companion angles*, denoted as α and $\overline{\alpha}$. The arc whose endpoints form a pair of companion angles is called a *characteristic arc.* The collection of all characteristic arcs becomes a lamination, so called *Quadratic Minor Lamination.* In [18], Schleicher proposed the notion of narrow characteristic arcs. Simply put, a narrow characteristic arc of period n is an arc of width $\frac{1}{2^n-1}$. We give a necessary and sufficient condition for the narrowness for a characteristic arc in terms of its orbit portrait. Using this, we propose a geometric condition for simply renormalizable arcs.

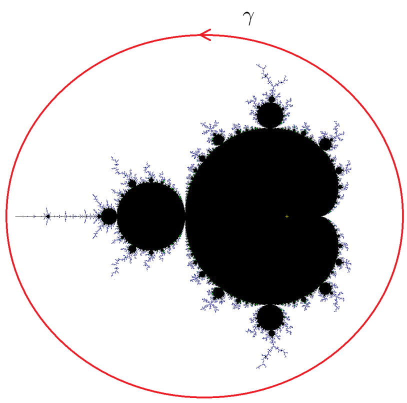


FIGURE 1. γ is a loop which moves around the Mandelbrot set $\mathcal{M}$. γ is a generator of the shift locus $\mathcal{S}_2 \cong \mathbb{C} - \overline{\mathbb{D}}$.

THEOREM 1.2 (Corollary 3.14). *Every narrow arc/component is not simply renormalizable. Every prime period arc/component is not simply renormalizable. Every satellite arc/component is not narrow unless its root meets in the main cardioid M_0.*

We note that Schleicher defines the *internal address* of each hyperbolic component of $\mathcal{M}$ in [18]. Using this, he completely determines when the corresponding polynomial is renormalizable, including both simple and crossed cases.

Kneading sequence changes at the bifurcation locus. The set of external angles for all points in the Julia set J_α is equal to that of $J_{\overline{\alpha}}$. Therefore, in terms of kneading and itinerary sequences, it is natural to ask what happens if we change α to $\overline{\alpha}$. Figure 2 shows some examples of parameter rays that are narrow. We proved in theorem 5.8 for narrow cases there is an algorithm how the itinerary sequences change.

In [2] and [1], Atela obtained such an algorithm for the case of generalized rabbit polynomials (in the papers, Atela called them *main bifurcation points.*) Generalized rabbit polynomials or main bifurcation points correspond to the set of characteristic arcs of parameter angle $\left(\frac{1}{2^n-1}, \frac{2}{2^n-1}\right)$ for $n \geq 2$. Atela constructs the dynamic graphs, and every itinerary sequence is lifted to the infinite path of the graph, and the rule is found by differing the coloring of a pair of vertices. We have generalized this result to any narrow characteristic arc. Our method differs from Atela's in that we use the lamination model proposed by Keller in [12]. This method allows us to more explicitly decribe how itinerary sequences change.

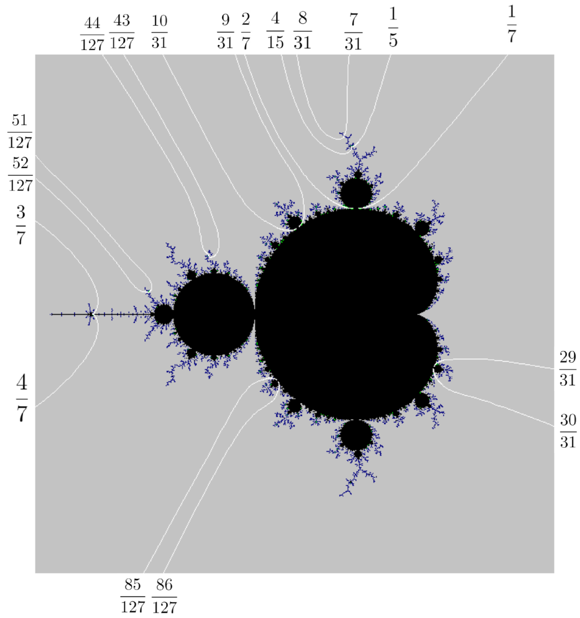


FIGURE 2. Examples of parameter ray pairs landing at the bifurcation locus, all of which narrow. We examine what happens when c moves along a pair of parameter rays.

THEOREM 1.3 (Theorem 5.8). *Let $s = s_1 s_2 s_3 \cdots$ be a* 0-1 *sequence. Also, let $v = v^\alpha$ be the repetition word of α, $e = e^\alpha$ is a characteristic symbol of α and* $\mathrm{PER}(\alpha) = |v| + 1$ *be the period of α. Then we define $\varphi_\alpha(s)$ as below.*

1. $i = 0$.
2. *Set $w = s_i \cdots s_{i+|v|}$.*
   (a) *If $w = 0v$ or $1v$, set $i \leftarrow i + |v| + 1$ and go back to (2).*
   (b) *If not, but if there is a previous $0v$ or $1v$, check $s_i = e$.*
      (i) *If $s_{i+|v|+1} = e$, swap all $0v$ and $1v$.*
      (ii) *If $s_{i+|v|+1} \neq e$, trace back to the previous $0v$ (or $1v$) and go back to (b).*
3. *If w is neither $0v$ nor $1v$, then set $i \leftarrow i + 1$ and go back to (2).*

Suppose $(\alpha, \overline{\alpha})$ is narrow. Choose $p \in J_\alpha$ and let θ be the dynamic angle of p. If θ is not precritical angle, then $\varphi_\alpha(I^\alpha(\theta)) = I^{\overline{\alpha}}(\theta)$.

The power of the theorem is that (1) we do not have to lift the given sequence to an infinite path of the graph as suggested in [2], and (2) we can always get the target itinerary sequence just by looking at the given sequence from the beginning.

As in [2], we also provide the dynamic graph method for understanding φ_α in the appendix A. Such a method is applicable regardless of the narrowness of α, although it is hard to get the target sequence explicitly.

Currently, the algorithm φ_α covers only the points that are not precritical angles. To extend the domain to the whole $J_\alpha \cong \{0,1\}^{\mathbb{N}}$, we give a marking at each precritical angle, which will be introduced in the next subsection.

Equivalence relation on itinerary sequences. The itinerary sequence function $I^\alpha : S^1 \to \Sigma_2$ is not surjective, and moreover is not defined for some angles that are preimages of α under angle doubling map. To try to make any itinerary sequence compatible with the points in Cantor type Julia set, we first extend the angle of S^1 by $\mathbf{E}_\alpha$ (see Definition 6.1).

Furthermore, φ_α is neither injective nor surjective, and sometimes multivalued. We examine when such behaviors occurs and solve the issues by defining the equivalence relation as follows.

Definition 1.4 (Definition 6.5). *Let $s_1, s_2 \in \Sigma_2$ and $x_1, x_2 \in J_\alpha$ be the corresponding point. Suppose θ_1, θ_2 are extended angle of x_1, x_2, respectively. We define the equivalence relation $\sim$ as follows,*

$$s_1 \sim s_2 \quad \Leftrightarrow \quad \theta_1 \approx_\alpha \theta_2$$

Quotienting $\mathbf{E}_\alpha$ by this equivalence relation, φ_α became well-defined and shift-invariant. Here the shift-invariant means that it commutes with one-sided shift of sequences.

Theorem 1.5 (Theorem 6.6). *$\varphi_\alpha : \Sigma_2/\sim \to \Sigma_2/\sim$ is well-defined. Moreover it is order 2 element and shift-invariant.*

Future works. In the upcoming paper, we will construct big mapping classes in $\mathrm{Mod}(\mathbb{C} - \{\text{Cantor set}\})$ or $\mathrm{Mod}(S^2 - \{\text{Cantor set}\})$ associated to each point in the bifurcation locus of degree 2. Such big mapping classes generate a subgroup of $\mathrm{Mod}(\mathbb{C} - \{\text{Cantor set}\})$ or $\mathrm{Mod}(S^2 - \{\text{Cantor set}\})$ which features many interesting dynamical properties.

Outline of the paper. We begin with a brief introduction to basic complex dynamics in section 2. In section 3, narrow arcs and the combinatorics of their orbit portrait are discussed in detail, and we prove that the narrow condition is sufficient not to be simply renormalizable. In section 4 we introduce Keller's work in [12] and define the quadrilateral lamination to describe how the itinerary sequence changes, only for angles that are not pre-critical angles under the angle doubling map. Section 5 takes care of the pre-critical angles to fully define φ_α for critical angle α. We extend angles in S^1 with some markers and in section 6 we give an equivalence relation to make φ_α to be shift invariant. In the appendix A we show that for narrow arcs there is a dynamic graph analogous to the one in [2].

2. Preliminary

In this section we briefly summarize basic theories related to the dynamics of quadratic polynomials. For more details, we recommend [16], [10], [8] and [6]. Some figures are drawn by the computer program *'mandel'*, [11] written by Wolf Jung.

2.1. *Quadratic polynomials.*

Any complex polynomial $a_d z^d + \cdots + a_0$ can be conjugated to a *monic* ($a_d = 1$) and *centered* ($a_{d-1} = 0$) polynomial by a linear map. In particular, any quadratic polynomial $P(z) = \alpha z^2 + \beta z + \gamma$ is conjugated to a polynomial of the form $z \mapsto z^2 + c$ for some $c \in \mathbb{C}$, hence quadratic polynomials are determined up to linear conjugacy by the constant term $c \in \mathbb{C}$. Consequently the moduli space of quadratic polynomials is just a complex plane $\mathbb{C}$. First, we briefly recall the definition of the Fatou set and the Julia set.

Definition 2.1 (Fatou set, Julia set). *Let $P : S \to S$ be a quadratic polynomial which maps the Riemann sphere S to itself. Fatou set F_P is a collection of points $z \in S$ such that there is a neighborhood U of z satisfying that the iterations of P at U, $P|_U, P^{\circ 2}|_U, \cdots$ became a normal family. By definition, Fatou set is open. Julia set J_P is the complement of F_P.* i.e., $J_P = S - F_P$.

Also, the set of points whose forward orbit $\{P^{\circ n}(z)\}_{n \in \mathbb{Z}}$ is bounded is called the filled Julia set K_P. As the name implies, the boundary of K_P is J_P. The connectedness of the Julia set is determined by the behavior of the critical point $z = 0$.

Theorem 2.2. *Let $P(z) = z^2 + c$, $c \in S$. Then*

- *J_P is connected $\Leftrightarrow$ $\{P^{\circ n}(0)\}_{n=1,2,\cdots}$ is bounded.*
- *J_P is a Cantor set $\Leftrightarrow$ $\{P^{\circ n}(0)\}_{n=1,2,\cdots}$ escapes to infinity.*

Furthermore if 0 escapes to infinity under iteration of P (the latter case), $P|_{J(P)}$ is conjugate to the one-sided shift on 2 symbols.

The last statement is exactly the theorem 1.1 for the case $d = 2$. The shift locus of degree 2, denoted by $\mathcal{S}_2$, is the collection of all quadratic shift polynomials. Recall that the Mandelbrot set $\mathcal{M}$ is the set of $c \in \mathbb{C}$ for which 0 does not escape to infinity, i.e., it is the connectedness locus by the theorem 2.2. Therefore, $\mathcal{S}_2$ is the complement of $\mathcal{M}$.

For a quadratic polynomial P, suppose some z satisfies $P^{\circ k}(z) = z$. We call the smallest such positive integer k is called a period and $\{z, P(z), \cdots, P^{\circ(k-1)}(z)\}$ as a periodic cycle (abusing the notation, it is sometimes just called a fixed point, even though it is not a single point). We define the multiplier $\rho(z) := \left(P^{\circ k}\right)'(z) = P'(z) \times P'(P(z)) \times \cdots P'(P^{\circ(k-1)}(z))$ for each periodic cycle of period k. Note that the multiplier remains constant in the same periodic cycle. Fixed points are classified according to their multipliers: Each fixed point is called superattracting if $|\rho| = 0$, attracting if $0 < |\rho| < 1$, irrationally indifferent if $\rho = e^{2\pi i\theta}$ with irrational θ, parabolic if $\rho = e^{2\pi i\theta}$ with rational θ, and repelling if $|\rho| > 1$.

DEFINITION 2.3 (Hyperbolic component). *Let $P_c := z^2 + c$ for $c \in \mathbb{C}$. Let X_k be the set of pairs $(c, z) \in \mathbb{C}^2$ such that $P_c^{\circ k}(z) = z$. Consider the function $\rho_k : X_k \to \mathbb{C}$ which sends $(c, z) \mapsto (P_c^{\circ k})'(z)$, the multiplier of such a fixed point. Let A_k be the set of pairs $(c, z) \in X_k$ with $|\rho(c, z)| < 1$ and M_k be $\pi(A_k)$, where $\pi : (c, z) \mapsto c$ is the projection onto the first factor (*i.e., *M_k is the collection of c such that P_c has an attracting fixed point of period k). Then each component M of M_k is an open component of the interior $\mathring{\mathcal{M}}$ of $\mathcal{M}$, and is called "*hyperbolic component*".*

It is still an open question whether each component of $Int(\mathcal{M})$ is hyperbolic or not. This is called the *hyperbolic density conjecture.*

The multiplier map ρ gives an analytic isomorphism from each hyperbolic component M to an open unit disk $\mathbb{D}$, and it extends to a homeomorphism from $\overline{M}$ to $\overline{\mathbb{D}}$. Abusing the notation, let ρ be the extended homeomorphism. Then $c_M := \rho^{-1}(0)$ is called the *center* of M and $r_M := \rho^{-1}(1)$ is called the *root* of M.

2.2. *Parameter angle.*

Douady and Hubbard proved in [8] that $\mathcal{M}$ is connected, by giving an analytic map between $\mathbb{C} - \overline{\mathbb{D}}$ to $\mathbb{C} - \mathcal{M}$. It is not yet known whether $\mathcal{M}$ is locally connected and this is called the *MLC conjecture.* The hyperbolic density conjecture we mentioned above is true if the Mandelbrot set is locally connected.

THEOREM 2.4. *There is an analytic isomorphism Ψ between $\mathbb{C} - \mathcal{M}$ and $\mathbb{C} - \overline{\mathbb{D}}$, and thus the Mandelbrot set $\mathcal{M}$ is connected.*

Identifying the exterior of $\mathcal{M}$ with $\mathbb{C} - \overline{\mathbb{D}}$, we can assign an angle to each point in $\mathbb{C} - \mathcal{M}$. More precisely, for each $c \in \mathbb{C} - \mathcal{M}$, the *external angle* of c is $\arg(\Psi^{-1}(c))/2\pi \in [0, 1]$. Here, the angles are normalized to be between 0 and 1 for convenience and we identify 0 and 1 so that the set of external angles forms a circle of unit circumference. We also call the ray $\Psi^{-1}\big(\{re^{i\alpha} \mid r \leq 1\}\big)$ as an *external ray* $\mathcal{R}_\alpha$. The external angles and external rays are also called *parameter angles* and *parameter rays* here, since they are on the parameter plane. We will define external angles and external rays similarly in the complement of Julia sets, and they will be called *dynamic angles* and *dynamic rays* respectively since they live in the dynamic plane. We denote $P_\alpha := z^2 + c$, where the external angle of $c \in \mathcal{S}_2$ is α, and J_α as its Julia set.

Again from Douady and Hubbard, it is well known that for every parabolic point $c \in \mathcal{M}$ there exists a pair of parameter rays $\mathcal{R}_\alpha, \mathcal{R}_{\overline{\alpha}}$ which land at c and vice versa. Furthermore such α and $\overline{\alpha}$ are periodic under angle doubling, so they are a rational with odd denominators. Such angle pairs are called *companion angles.*

2.3. *External angle and Kneading sequences.*

For the polynomial case, there is a nice coordinate change done by Böttcher in [5].

THEOREM 2.5 (Böttcher coordinates). *Let P be a polynomial of degree d. Then there exists a pair of neighborhoods $V \subset U$ of ∞ and an analytic mapping $\phi : V \to \mathbb{C}$ which sends ∞ to 0 and satisfies that $\phi'(\infty) = 1$ and $(\phi(z))^d = \phi(P(z))$.*

The Böttcher coordinates allow us to understand the dynamics around infinity, which is the semi-conjugation of a monomial map $w \mapsto w^d$. Let $G_P := \log |\phi|$, a Green's function.

LEMMA 2.6 (Properties of Green's function).

1. $G_P : \mathbb{C} \to \mathbb{R}_{\geq 0}$.
2. *The level set of G_P (except at 0) forms a horizontal foliation of $\mathbb{C} - K_P$.*
3. $K_P = G_P^{-1}(0)$.
4. *Critical points of G_P consists of critical points of P and their preimages under P.*

With the lemma 2.6, we can identify a small neighborhood of infinity as a disk, and thus draw a ray of fixed angle. Such ray is called *dynamic ray* of *dynamic angle* θ, and it is orthogonal to every level set of G_P. We also call it $\mathcal{R}_\theta$, abusing the notation. In other words, dynamic rays form the vertical foliation orthogonal to the level sets which form the horizontal foliation.

For periodic α under angle doubling, each dynamic ray of $\mathbb{C} - J_\alpha$ will end at some points in J_α or hit (pre-)critical points and bifurcate. Note that 0 is the only critical point for P_α, which is a preimage of the critical value and its dynamic angle is α. Thus, in the Böttcher coordinate sense, two dynamic rays of angles $\frac{\alpha}{2}, \frac{\alpha+1}{2}$ land at 0.

Now, select any point $p \in J_\alpha$ and let θ be its dynamic angle. Then the kneading sequence of p (or equivalently, θ) is defined as follows.

DEFINITION 2.7 (Kneading sequence). *Assume the above. Divide S^1 into 2 pieces, cut at $\frac{\alpha}{2}$ and $\frac{\alpha+1}{2}$. Let A_α be an open arc $\{\theta \mid \frac{\alpha}{2} < \theta < \frac{\alpha+1}{2}\}$ and B_α is the other open arc. Then the kneading sequence $I^\alpha(\theta) := x_0x_1x_2\cdots$ is a 0-1 sequence possibly with some $*$'s whose entries are given by*

$$x_i = \begin{cases} 0 & \text{if } 2^i\theta \in A_\alpha \\ 1 & \text{if } 2^i\theta \in B_\alpha \\ * & \text{if } 2^i\theta = \dfrac{\alpha}{2} \text{ or } \dfrac{\alpha+1}{2} \end{cases}$$

So we construct a map $J_P \to \Sigma_2$, by sending $p \in J_P$ to its itinerary sequence. We end up this section by introducing some notation.

- A 0-1 word w is a finite sequence $w := w_0 w_1 \cdots w_k$, where $w_i \in \{0,1\}$. Similarly, a 0-1 sequence is an infinite sequence whose entries are all 0 or 1.
- $\{0,1\}^{\mathbb{N}}$ is a union of all 0-1 sequences of infinite length.
- A 0-1 word w is in $\{0,1\}^n$ if and only if the length $|w|$ of w is n.
- $\{0,1\}^* := \bigcup_{n\in\mathbb{N}} \{0,1\}^n$ is the collection of all finite 0-1 words w.
- For $w \in \{0,1\}^*$, $\overline{w} \in \{0,1\}^{\mathbb{N}}$ is an infinite sequence of repeating w's.

These notations will be used in the followiung sections discussing lamination models.

2.4. *Renormalization and Tuning.* In [7], Douady proposed that there exists a homeomorphism of the Mandelbrot set $\mathcal{M}$ into itself. More precisely, let M_0 be the main cardioid component of $\mathcal{M}$ and choose any hyperbolic component $\mathcal{P}$. Then there exists a topological embedding $\psi_{\mathcal{P}} : \mathcal{M} \to \mathcal{M}$ which satisfying the following:

1. $\psi_{\mathcal{P}}(0) = c_{\mathcal{P}}$, where $c_{\mathcal{P}}$ is the center of $\mathcal{P}$.
2. $\psi_{\mathcal{P}}(M_0) = \mathcal{P}$. In other words, $\psi_{\mathcal{P}}$ sends the main cardioid to the hyperbolic compontent $\mathcal{P}$ we selected.
3. $\partial\psi_{\mathcal{P}}(\mathcal{M}) \subset \partial\mathcal{M}$.

This is called a *(simple) renormalization* or *tuning*. For any $\in \mathcal{M}$, $\psi_{\mathcal{P}}(c)$ is said to be M *tuned by* c is denoted by $\mathcal{P} \perp c$. Let $\mathcal{Q}$ be another hyperbolic component. We denote $\mathcal{P} * \mathcal{Q}$ as an image of $\mathcal{Q}$ under $\psi_{\mathcal{P}}$. We note that $\mathcal{P} * \mathcal{Q}$ is also a hyperbolic component, which is included in the wake of $\mathcal{P}$.

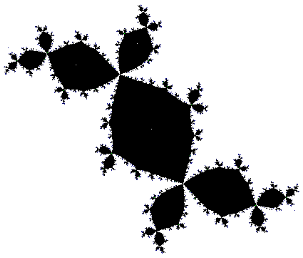
The tuning also affects their Julia sets also. Roughly speaking, the filled Julia set $K_{M\perp c}$ is obtained by first drawing $K_{c_{\mathcal{P}}}$ and replacing each of its bounded Fatou components with a a copy of K_c. See the figures 3a, 3b and 4. All Julia sets in figures correspond to the center of their hyperbolic components.

There are two types of renormalizability, one is simple and the other is crossed. Rather than give the details, we refer to [14] and [15] for further discussions.

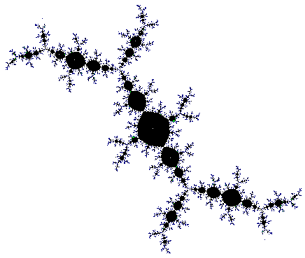
Let $\mathcal{P}$ be any hyperbolic component. Let $\alpha < \overline{\alpha}$ be a pair of parameter angles of the root of $\mathcal{P}$. In the same paper, Douady first proposed the following formula and recently by Blé, Cabrera Epstein and Tiozzo ([4], [9]) independently generalized the formula.

Lemma 2.8 (Douady's angle tuning). *Consider the situation above. Note that α and $\overline{\alpha}$ are both periodic under angle doubling of the same period, and thus in base 2 expansion, they have a repeating word $w_0, w_1 \in \{0,1\}^n$ of the same length, respectively.* i.e.,

$$\alpha = 0.w_0 w_0 \cdots . \qquad \textit{and} \qquad \overline{\alpha} = 0.w_1 w_1 \cdots .$$



(A) Douady's Rabbit. The parameter angles of the hyperbolic component are $\frac{1}{7}$ and $\frac{2}{7}$.



(B) Rabbit tuned by basilica. The parameter angles are $\frac{10}{63}$, $\frac{17}{63}$

FIGURE 3. Figures of rabbit and tuned rabbit. Note that the Fatou component of the rabbit changes to the basilica.

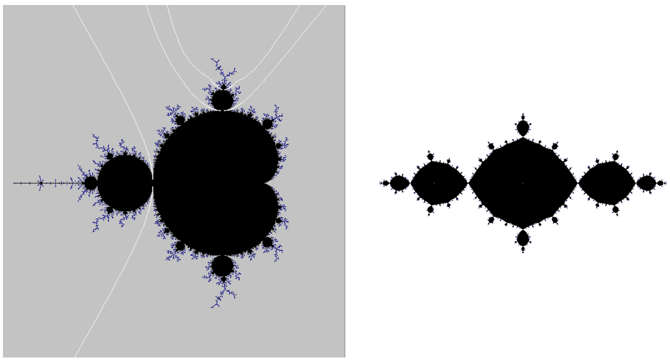


FIGURE 4. The white lines are the parameter rays, landing on hyperbolic components corresponding to basilica, rabbit, and rabbit tuned by basilica. The right figure is a basilica.

Let $x \in \partial\mathcal{M}$ and there is a parameter ray of angle θ that lands at x. Suppose the binary expansion of θ is

$$\theta = 0.t_1t_2t_3\cdots.$$

Then there is a parameter ray of angle $\mathcal{P} \perp \theta$ which lands at $\mathcal{P} \perp x$ and,

$$\mathcal{P} \perp \theta = 0.w_{t_1}w_{t_2}w_{t_3}\cdots.$$

Note that the formula preserves the order. In other words, $a < b \Rightarrow (\mathcal{P} \perp a) < (\mathcal{P} \perp b)$.

There are a few notes about the formula. First of all, sometimes the binary expansion is not 1-1. Especially in S^1, we identify $0 = 1$ and so 0

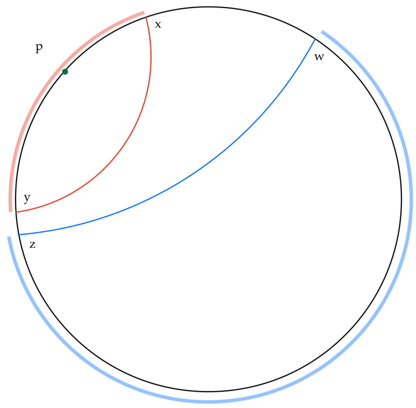


FIGURE 5. Here xy and zw are chords. The point p lies behind xy and the chord xy is behind zw. Note that two arcs (x, y) and (z, w) are disjoint.

gets two different binary expansions $0.000\cdots$. and $0.111\cdots$. According to Douady's formula, these two expansions correspond to the pair of parameter angles of the root of M.

3. CHARACTERISTIC ARCS

In this section, we identify S^1 as the set of real numbers quotient by $\mathbb{Z}$ and choose a real number in $[0, 1)$ as a representative for each point in S^1. There is a canonical cyclic order on S^1, which is inherited from the natural total order on $\mathbb{R}$, under the map $x \mapsto e^{2\pi i x}$.

Let's begin with some notations. For $x, y \in S^1$, we denote (x, y) as an *arc* of S^1, which is a set of z satisfying that $[x, z, y]$ is positively ordered (with respect to the canonical cyclic order).

We denote xy as a *chord* of S^1 whose endpoints are $x, y \in S^1$. Unless specified, we do not care about the order of two points. Usually we will draw it like a curve inside a circle, just like a hyperbolic geodesic in the Poincaré disk. When $x = y$ the chord xy is called *degenerated*.

The length of arc $l(x, y)$ is measured by the usual metric in S^1.

DEFINITION 3.1. *We call a point* p lying behind *the chord* xy *if* p *belongs to the interior of one of* $(x, y), (y, x)$ *with length less than* $1/2$. *(The case where* xy *is a diameter is excluded here, and this is not necessary in the rest of the paper). A chord* xy *is* nested *by* zw *(denoted as* $xy \subset zw$*) if every point* p *lie behind* xy *is also behind* zw.

See also the figure 5.

3.1. *Orbit portrait.* For periodic nonzero $p \in S^1$, in [17] Milnor introduced a combinatorial model for understanding the periodic points in the dynamic plane, called *Orbit portrait*.

Definition 3.2 (Orbit portrait). *Let O_j be the set of angles. Then the collection $\mathcal{O} := \{O_1, \cdots, O_p\}$ is called* orbit portrait *if*

1. *Every O_j is finite and $|O_i| = |O_j|$ for all $i \neq j$.*
2. *For any j, the angle doubling map $\theta \mapsto 2\theta \mod \mathbb{Z}$ is a bijective map between O_j and O_{j+1}, which preserving the cyclic order of $\mathbb{R}/\mathbb{Z}$.*
3. *For any $a \in \cup_{j=1}^{p} O_j$, a is periodic under angle doubling of the same period rp, a multiple of p.*
4. *O_j's are pairwise unlinked. More precisely, for each $i \neq j$, the line/polygon $\mathbf{P}_i$ inscribed in the unit circle S_1 whose vertices are points in O_i has no intersections with $\mathbf{P}_j$.*

p is called the *orbit period*, and rp is called the *ray period* or *angle period*. There are examples of orbit portraits in figure 6.

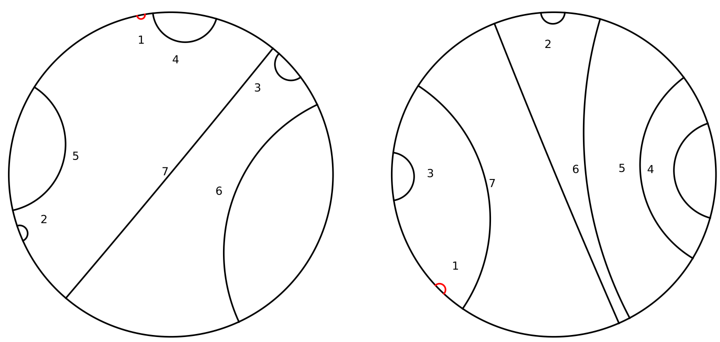


Figure 6. Examples of characteristic arcs. Characteristic arcs for left and right orbit portraits are $\left(\frac{35}{127}, \frac{36}{127}\right)$, $\left(\frac{158}{255}, \frac{161}{255}\right)$, respectively. We denote by the number for each ith image of A_1. Note that the left one is narrow and the right one is not narrow. So the characteristic arc on the right is nested by the 7th orbit.

The last statement implies that the orbit portrait yields a lamination if we draw a line/polygon of the endpoints of each O_j. An orbit portrait captures the dynamics of any repelling periodic point of a given Julia set. In particular, we have the following.

Theorem 3.3 ([10], Theorem 10.5.13). *Suppose $I = (\alpha, \overline{\alpha})$ is a characteristic arc of period n and M is a corresponding hyperbolic component. Let c_M be the center of M so that $p(z) = z^2 + c_M$ is postcritically finite. Then* *the dynamic rays of angle α and $\overline{\alpha}$ land at the root of the Fatou component containing c_M.*

In other words, if there is a polygon gap in the orbit portrait, then some Fatou components containing critical orbit meets at a point.

Complementary arcs for each $A_j \in \mathcal{O}$ are subintervals obtained by cutting at each point in A_j. By the lemma 3.2, the angle doubling map h still be a bijection between the set of complementary arcs of A_j, A_{j+1}, except one. In particular, 2 complementary arcs of length $> 1/2$ cannot exist in one set. Suppose L_j is the complementary arc of length $> 1/2$ if it exists. Then except for L_j, h sends the others diffeomorphically, and L_j covers the whole circle of length 1. The overlapping region should be one of the complementary arcs of A_{j+1}, since the sum of the lengths should be 2 and there are only 1 piece left after taking h. The lemma below elaborates on the above.

LEMMA 3.4 ([17], Lemma 2.5). *The set of complementary arcs of A_j is diffeomorphic to that of A_{j+1} (modulo its period) except for one. For each j there exists L_j whose image under h covers the whole circle, and there exists a unique complementary arc for A_{j+1} which is covered only once.*

A unique minimum length is attained among all complementary arcs, and any complementary arc that realizes the minimum is called *characteristic arc* of $\mathcal{O}$.

Abusing the above notation, we enumerate the orbit under angle doubling of the characteristic arc I as follows. Set $I := A_1$. If $l(A_i) < 1/2$, then A_{i+1} is an image of A_i under angle doubling. Otherwise, we set A_{i+1} as the unique complementary arc obtained from the lemma 3.4.

There is a 1-1 correspondence between the set of characteristic arcs and a bifurcation locus. Each point in the bifurcation locus has a unique hyperbolic component which admits the point as its root. Thus we can identify the characteristic arc with its hyperbolic component. More precisely, if c is a point in the bifurcation locus, then $z^2 + c$ has a parabolic cycle of points. Draw all the rays that land on points in the parabolic cycle, and you get an orbit portrait. Milnor proved that every orbit portrait can be realized as a parabolic cycle of certain quadratic polynomials.

Moreover, if a characteristic arc I is nested by the other characteristic arc J, then these orbit portraits are unlinked to each other. In a dynamical sense, if c belongs to a hyperbolic component corresponding to I (in the sense of Lemma 3.3), then the Julia set of z^2+c has a repelling periodic point whose landing rays form an orbit portrait of J. This is called *orbit forcing.*

One can collect all the characteristic arcs and draw them in the unit disk. These arcs form a lamination which is called *quadratic minor lamination (or simply QML)*, as first proposed by Thurston. We give the picture of QML in the figure 7. Characteristic arcs have properties that allow us to obtain all arcs in an algorithmic way.

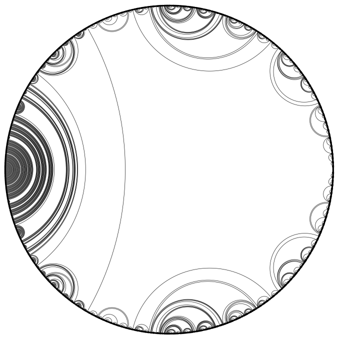


FIGURE 7. QML, in this figure we only draw arcs up to period 10.

LEMMA 3.5. *Every leaf in QML satisfies the following.*

1. *Every periodic angle is an endpoint of a non-degenerate leaf of QML, whose the other endpoint has the same period.*
2. *Every leaf of QML is the limit of leaves with periodic endpoints.*
3. *Suppose two leaves in QML have the same period* p *and one is nested by the other. Then there exists a leaf* L *such that it has period* $< p$ *and it separates two leaves.*

The first 2 property implies that it suffices to find all periodic characteristic arcs to generate QML. Lavaurs proposed in [13] an algorithm to get all characteristic arcs.

LEMMA 3.6 (Lavaurs algorithm, [13]). *Start with a leaf* $(1/3, 2/3)$. *Suppose* C_n *is a collection of all characteristic arcs whose period less than* n. *For* $p = n + 1$, *consider all points of* $k/(2^{n+1} - 1), 1 \leq k \leq 2^{n+1} - 2$. *Connect them in pairs, starting with* $1/(2^{n+1} - 1)$, *using the following rule*

1. *Select the point in the order of increasing angle.*
2. *The other endpoint is the smallest angle among all angles that are not linked with any other leaf in the collection* C_n.

When every point of period $(n + 1)$ *finds its pair, increase* p *by* 1. *Iteratively, the algorithm collects all characteristic arcs.*

Suppose z_1 is a parabolic fixed point of P and $\{z_1, \cdots, z_p\}$ be its periodic orbit under P. There are two types of parabolic points, one is *primitive* and the other one is *satellite*.

Definition 3.7. *Suppose $\alpha \in S^1$ be nonzero periodic and $\overline{\alpha}$ is its companion angle. α is called* satellite *if the orbit of α coincides with that of $\overline{\alpha}$. Otherwise, α is called* primitive.

We briefly remark some properties of primitive/satellite points.

Remark (See Lemma 2.7 in [17]).

1. If α is primitive, there are no polygon gaps in the orbit portrait. With respect to the dynamical plane and the rays, there are at most 2 rays landing on each point in the periodic orbit.
2. If α is satellite, then there is a r-gon gap in the orbit portrait. *i.e.,* exactly r rays land at each point of the periodic orbit.

3.2. *Narrow arcs.* Another criterion to classify a characteristic arc is by its length.

Definition 3.8 (Narrow arc, narrow component, [18]). *Suppose I is a characteristic arc of angular period n. Then I is called* narrow arc *if $l(I) = \frac{1}{2^n-1}$. The hyperbolic component corresponding to I is called* narrow component.

By definition and the Lemma 3.5, if a characteristic arc I is not narrow, then there is a shorter characteristic arc lie behind I whose period is strictly less than n.

Corollary 3.9. *Suppose I is a characteristic arc of period n. If I is not narrow and I satisfies $\frac{1}{2^k-1} < l(I) < \frac{1}{2^{k-1}-1}$, then there exists a narrow arc J nested by I, whose period is k. Furthermore, k is strictly less than n.*

Proof. Since $\frac{1}{2^k-1} < l(I) < \frac{1}{2^{k-1}-1}$, there are 2 or 3 points of period k lie behind I. By lemma 3.5 there must be 2 points and these must be endpoints of some characteristic arc J, because of the unlinked condition of QML. Then $l(J) = \frac{1}{2^k-1}$. So J is narrow of period k. If $k = n$, $l(I) < \frac{1}{2^{n-1}-1}$ implies $l(I) = \frac{1}{2^n-1}$. Hence a contradiction. ∎

Note that such an arc J is the longest narrow arc among all the narrow arcs which lie behind I. Using the Corollary 3.9 we prove the equivalence definition of the narrow condition of I.

Proposition 3.10. *Suppose $I = A_1$ is a characteristic arc of period n. Then I is narrow if and only if I is not nested by A_i for all $2 \leq i < n$.*

Proof. We denote I as A_1. Suppose to the contrary that A_1 is narrow and it is nested by A_k for some $2 \leq k \leq n-1$. $l(A_1) = \frac{1}{2^n-1}$ implies $l(A_i) = \frac{2^{i-1}}{2^n-1}$. Thus A_n is the only arc of length $> 1/2$ and $A_i \to A_{i+1}$ is just an angle doubling for all $i \leq n-1$. Let $x < y$ be the endpoint of A_1. The preimage of x and y under angle doubling are $\frac{x}{2}, \frac{x+1}{2}$ and $\frac{y}{2}, \frac{y+1}{2}$, respectively. Since $A_{k-1} \to A_k$ is an angle doubling, the endpoints of A_{k-1} must contain

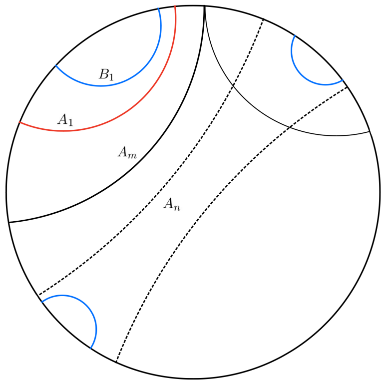


FIGURE 8. In the middle there are two candidate chords of A_n. Then A_m cannot nest the preimage of A_1 without intersecting A_n.

either $\frac{x}{2}, \frac{y}{2}$ or $\frac{x+1}{2}, \frac{y+1}{2}$. But A_n is one of the long sides of a quadrilateral whose vertices are preimages, A_n and A_{k-1} intersect. Therefore we conclude that if A_1 is narrow, then it is never nested or only nested by A_n.

Now, suppose that I is not narrow and satisfies $\frac{1}{2^m-1} < l(A_1) < \frac{1}{2^{m-1}-1}$. By the Corollary 3.9, there exists a unique longest narrow arc B_1 of period $m < n$. Denote B_j be its jth forward orbit. We split it into 2 cases. See also the figure 8.

- $l(A_1) > \frac{1}{2^{m-1}}$.

  Then $\frac{1}{2^{m-1}} < l(A_1) < \frac{1}{2^{m-1}-1}$ gives $\frac{2^{i-1}}{2^{m-1}} < l(A_i) < \frac{2^{i-1}}{2^{m-1}-1}$ for $1 \le i \le m$ and thus A_{m-1} is the first A_i whose length is greater than $1/2$. Endpoints of A_{m-1} cannot be in the preimage of A_1 unless one of the complementary arcs of A_m is shorter than A_1, which is impossible. Since A_1 nests B_1, the preimage of B_1 is a (proper) subset of A_1. This implies that A_m nests B_1. Since A_1 is the shortest among all other A_j, A_m cannot be located between B_1 and A_1. Therefore A_m nests A_1 and m is strictly less than n, so we are done.
- $l(A_1) < \frac{1}{2^{m-1}}$.

  Same strategy as above, we get A_{m+1} nesting A_1. So we are done if $m+1 < n$. Suppose $m+1 = n$. Then again A_m is the first A_i whose length is greater than $1/2$. Let S_m be $S^1 - A_m$. If A_m contains B_1, then A_{m-1} must contain one of the preimages of B_1 since $A_{m-1} \to A_m$ is just an angle doubling map. A_m is strictly longer than $l(A_1)/2$ and hence contain the preimage of A_1 under angle doubling (not in a forward orbit), which induces an intersection between A_n. This implies that S_m contains B_1 and therefore A_m must nest A_1.

Same analogy as in the latter case, if we replace A_m with A_n, we get A_n must nest A_1. ∎

The proposition implies that the characteristic arc I is always nested by some A_m with m less than or equal to the period of I, and $m = n$ if and only if I is narrow.

LEMMA 3.11. *Suppose* $I = (\frac{a}{2^n-1}, \frac{a+1}{2^n-1})$. *Then a must be an odd number.*

PROOF. By assumption, $I = A_1$ is narrow. So A_n nests A_1. Consider the preimage of A_1 under angle doubling map. It consists of 2 short arcs, and if you draw the polygon whose vertices are the endpoints of such arcs, A_n must be the longer side that nests A_1. This implies that the endpoints of A_n are $\frac{a+1}{2(2^n-1)}$ and $\frac{a}{2(2^n-1)} + \frac{1}{2}$. Suppose a is even. Then the denominator of both endpoints of A_n are even. But this is impossible, since A_n should be periodic under angle doubling, which forces that every endpoint of A_i's has to have an odd denominator. Therefore, a must be odd unless $\frac{a+1}{2(2^n-1)}$ became strictly preperiodic. ∎

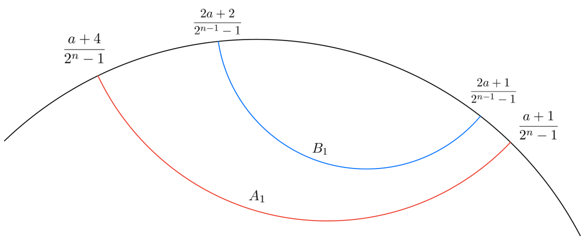


FIGURE 9. Examples in the remark.

Remark. 1. The lemma 3.10 says that A_m is the first one that nests A_1.

2. In the latter case, only $l(A_1)$ which satisfies the condition $\frac{1}{2^{n-1}-1} < l(A_1) < \frac{1}{2^{n-2}-1}$ is $l(A_1) = \frac{3}{2^n-1}$. There are only 1 case for A_1, B_1 as below, because of the lemma 3.11.

$$\frac{a+1}{2^n-1} < \frac{2a+1}{2^{n-1}-1} < \frac{2a+2}{2^{n-1}-1} < \frac{a+4}{2^n-1}$$

These are shown in the figure 9.

3.3. *Simply renormalizable arcs.* Suppose $\mathcal{P}$ and $\mathcal{Q}$ are hyperbolic components, and I is a characteristic arc corresponding to the hyperbolic component $\mathcal{P} * \mathcal{Q}$. (Recall the subsection 2.4)

DEFINITION 3.12. *A hyperbolic component $\mathcal{P}$ is* simply renormalizable *if the corresponding hyperbolic component is simply renormalizable.* i.e., *there exists two hyperbolic component $\mathcal{Q}, \mathcal{R}$ (not necessarily distinct) that $\mathcal{P} = \mathcal{Q} * \mathcal{R}$. We also call a characteristic arc I is* simply renormalizable *if the corresponding hyperbolic component is simply renormalizable.*

We first note that $\mathcal{Q} \mapsto \mathcal{P} * \mathcal{Q}$ gives a tuning from any point $c \in \mathcal{Q}$, and hence $p(z) = z^2 + c$ with $c \in \mathcal{P} * \mathcal{Q}$ is renormalizable. Choose c as the center of $\mathcal{P} * \mathcal{Q}$ and look at the orbit portrait of I. Since $\mathcal{P} * \mathcal{Q}$ is contained in the wake of $\mathcal{P}$, the orbit portrait $\mathcal{O}_{\mathcal{P}}$ corresponds to $\mathcal{P}$, and is unlinked with the orbit portrait of I. Furthermore, the orbit portrait $\mathcal{O}_{\mathcal{P}}$ partition the arcs $I = A_1, A_2, \cdots, A_n$ into the same number of m subsets $\{A_1, A_{m+1}, \cdots A_{(k-1)m+1}\}$, $\{A_2, A_{m+2}, \cdots, A_{(k-1)m+2}\}$, $\cdots \{A_m, A_{2m}, \cdots, A_{km}(= A_n)\}$, where m, k, and $n = km$ are the periods of $\mathcal{Q}$, $\mathcal{P}$, and $\mathcal{P} * \mathcal{Q}$, respectively.

The small Julia set $K(1)$ corresponds to c_M. The local map $p^{\circ m}$ is hybrid equivalent to some polynomial, whose constant term belongs to $\mathcal{Q}$. So locally there is an orbit portrait corresponding to $\mathcal{Q}$ and also for the other $K(i)$'s.

By using Douady's angle tuning formula 2.8, we can recover each orbit portrait into the whole orbit portrait of $\mathcal{I}$. Also, each small orbit portrait is separated by chords in $\mathcal{O}_{\mathcal{P}}$.

i.e., If given characteristic arc I is simply renormalizable, then there exists a characteristic arc J which satisfies the following,

- I is nested by J.
- The period of J is a integer multiple of period of I. The orbit portrait of J divides the orbits in I into subsets of equal cardinality.

This condition gives a sufficient condition for renormalizability.

LEMMA 3.13. *Suppose I is a renormalizable arc of period n. Then $I := A_1$ is nested by A_i for some i strictly less than n. Therefore, A_1 cannot be narrow.*

PROOF. With the above discussions, the characteristic arc I is in the small partition and forms an orbit portrait. Let this orbit portrait be $\mathcal{O}_{inn}$, and denote each arc by $I = B_1, \cdots, B_k$. By the proposition 3.10, I is nested by B_l for some $l \leq k$. Using Douady's angle tuning formula 2.8, A_1 is nested by $A_{(l-1)m+1}$, which is obviously different from A_n. Here $km = n$. Therefore A_1 cannot be a narrow arc by the proposition 3.10. ∎

COROLLARY 3.14. *Every narrow arc/component is not simply renormalizable. Every prime period arc/component is not simply renormalizable. Every satellite arc/component is not narrow unless its root meets in the main cardioid M_0.*

However the reverse is not true. For example, there are non-narrow characteristic arcs of prime period. It is easy to verify that to be renormalized, the period must have a divisor other than 1 and itself, and thus it is not simply renormalized.

Unfortunately, there are many examples which is non-prime, non-narrow and not simply renormalizable. We found such examples by using the internal address of hyperbolic components, proposed by Schleicher in [18]. In fact, the internal address gives a complete classification of renormalizability, including cross-renormalizability. We end the section up with a question.

QUESTION 1. *What other geometric conditions should be required to be non-renormalizable?*

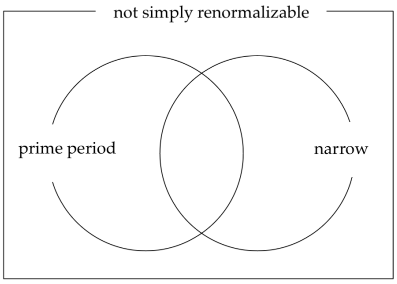


FIGURE 10. Inclusion relations between narrow, prime period and not simply renormalizable conditions. Note that there are plenty of not simply renormalizable arcs which are non-narrow, non-prime period.

4. Invariant Laminations

Most of the content in this section is done by Keller, we refer the reader to [12] for further details. Here we provide his notations and theorems.

4.1. *Lamination model generated by a long chord.* Let h be an angle doubling map in S^1. *i.e.,* $h(x) = 2x \mod 1$. Fix $\alpha \in S^1$ and assume that $\alpha \neq 0$ is periodic under h. In other words, there exists the smallest $n \in \mathbb{N}$, a period of α, such that $h^n(\alpha) = \alpha$. Let $\mathrm{PER}(\alpha)$ be a period of α and v^α be its repeating word in the kneading sequence $\hat{\alpha} := I^\alpha(\alpha)$. Note that v^α always starts with 0, since α is on $A_\alpha := \frac{\alpha}{2}\frac{\alpha+1}{2}$. Since α is periodic, one of $\frac{\alpha}{2}, \frac{\alpha+1}{2}$ should be equal to $h^{\mathrm{PER}(\alpha)-1}(\alpha)$. Such a point is denoted by $\dot{\alpha}$, and the other one is denoted by $\ddot{\alpha}$, which is preperiodic.

Consider the branches of the inverse h^{-1} whose domain is $S^1 - \{\alpha\}$. Denote one branch as l_0^α if the image is $(\frac{\alpha}{2}, \frac{\alpha+1}{2})$, and the other as l_1^α if the

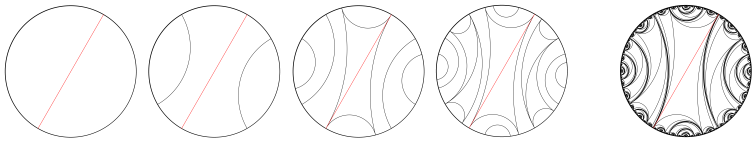


FIGURE 11. $\alpha = \frac{1}{3}$. Starting with the red line $\frac{1}{6}\frac{2}{3}$ preimages will be added. Figures show the step 1 to step 4, and the rightmost one shows $\mathcal{L}_\alpha$.

image is $(\frac{\alpha+1}{2}, \frac{\alpha}{2})$. To extend the domain to the whole S^1, define $l_s^{\alpha,t}(\alpha) = \dot{\alpha}$ if $t = s$, otherwise $l_s^{\alpha,t}(\alpha) = \ddot{\alpha}$ for $t, s \in \{0, 1\}$.

Graphically, the map $l_s^{\alpha,t}$ determines which endpoint of the arc $(\frac{\alpha}{2}, \frac{\alpha+1}{2})$ or $(\frac{\alpha}{2}, \frac{\alpha+1}{2})$ is the preimage of α, depending on the parity of s and t.

Let $w = w_1w_2\cdots w_n$ be a finite length word with $0, 1$ symbols. Iteratively, we can define $l_w^{\alpha,t} := l_{w_1}^{\alpha,t} \circ \cdots l_{w_n}^{\alpha,t}$. If the input is in $S^1 - \{\alpha, h(\alpha), \cdots, h^{n-1}(\alpha)\}$, then the superscript t is irrelevant.

First, note that the branched inverse is considered as a prefix attaching map on the $\{0, 1\}^{\mathbb{N}}$.

LEMMA 4.1. *Let $p(z) = z^2 + c$ so that the external angle of c is α. Suppose the itinerary sequence of $x \in J_p$ is $x_1x_2\cdots$ and let θ be its external angle. For any $0, 1$ finite length word $w = w_1w_2\cdots w_n$, $l_w^{\alpha,t}(\theta) = w_1 \cdots w_n x_1 x_2 \cdots$.*

PROOF. This is straightforward from the definition of l. Recall that we attach a superscript 0 or 1 if the branch is defined on A_α or B_α (defined in 2.7). ☐

Now construct the lamination model for the long chord $S_\lambda^\alpha := \frac{\alpha}{2}\frac{\alpha+1}{2} = \dot{\alpha}\ddot{\alpha}$. Here λ is the empty word. Let $S_w^{\alpha,t} := l_w^{\alpha,t}(S_\lambda^\alpha)$ for finite $0, 1$ word w. As α is periodic, $l_{tv^\alpha}^{\alpha,t}(\dot{\alpha}) = \dot{\alpha}$ and $l_{(1-t)v^\alpha}^{\alpha,t}(\ddot{\alpha}) = \ddot{\alpha}$.

To be precise, we define the α regular word $w \in \{0, 1\}^*$.

DEFINITION 4.2 (α-regular word). *Fix a periodic $\alpha \in S^1$. $w \in \{0, 1\}^*$ is called α-regular if w does not end with $0v^\alpha$ nor $1v^\alpha$.*

We call *step k* lamination generated by α if we collect all the jth preimages of the long chord S_λ^α for all $j < k$. We also denote $\mathcal{L}_\alpha$ as a collection of all chords $S_w^{\alpha,t}$, $w \in \{0, 1\}^*$ and $t \in \{0, 1\}$. By construction, it became an invariant under h, both forward and backward. See the figure 11.

Note that $S_{0v^\alpha}^{\alpha,0}, S_{0v^\alpha}^{\alpha,1}, S_\lambda^\alpha$ form a triangle and similarly, $S_{1v^\alpha}^{\alpha,0}, S_{1v^\alpha}^{\alpha,1}, S_\lambda^\alpha$ also form a triangle. We call them Δ_{0v^α} and Δ_{1v^α}, respectively. Under the inverse map of h, for every finite word $u \in \{0, 1\}^*$, there are infinitely many triangles with mutually disjoint interiors and possibly shared vertices/edges. We call

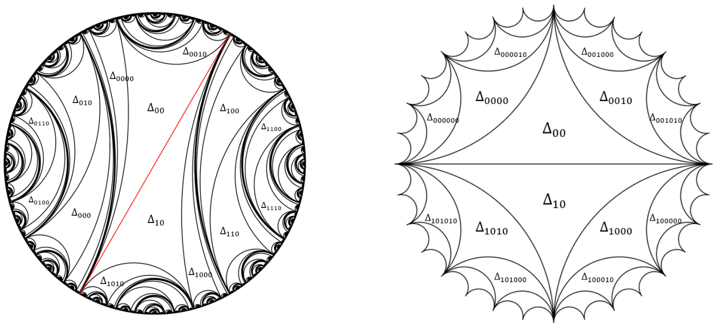


FIGURE 12. Here we have $\alpha = \frac{1}{3}$ and some indexes of triangle gaps. If we focus on the triangle gaps starting from the long chord(red line in the left figure), then it forms an infinite binary tree.

these triangle gaps $\Delta_{utv^\alpha}, t \in \{0,1\}$, named after the label of the chord. More specifically, three sides of Δ_{utv^α} are $S^{\alpha,0}_{utv^\alpha}, S^{\alpha,1}_{utv^\alpha}, S^{\alpha,t}_{u}$.

Remark. Let $u \in \{0,1\}^*$. Consider two triangle gaps Δ_{utv^α} and $\Delta_{utv^\alpha 0 v^\alpha}$. They share one side $S^{\alpha,0}_{utv^\alpha}$. Similarly, Δ_{utv^α} and $\Delta_{utv^\alpha 1 v^\alpha}$ share a side $S^{\alpha,1}_{utv^\alpha}$. So, for each α-regular word w, infinitely many triangles are attached from the Δ_{wtv^α}. Iteratively, it will form an infinite binary tree of triangles. *i.e.,* if you have a $0,1$ one-sided sequence $x_1 x_2 \cdots$, then there is an associated path of triangle gaps $\Delta_{wx_1v^\alpha}, \Delta_{wx_1v^\alpha x_2 v^\alpha}, \cdots$ for each α regular word w. See the figure 12.

To distinguish sides of given triangle gap, we adopt the following notion.

DEFINITION 4.3 (Characteristic symbol). *Let α be a nonzero periodic in S^1. The symbol t with $d(S^{\alpha,t}_{0v^\alpha}) < 1/4$ is called the characteristic symbol of α, denoted as e^α.*

Note that $S^{\alpha,t}_{0v^\alpha}$ is one side of the triangle gap Δ_{0v^α}. As the definition says, the characteristic symbol indicates which side of the triangle (except the long chord) is shorter than the other. More generally, suppose we have an α-regular word w. Then from S^α_w there is a triangle gap Δ_{w0v^α}, whose sides are $S^{\alpha,0}_{w0v^\alpha}, S^{\alpha,1}_{w0v^\alpha}$, and S^α_w. The characteristic symbol e says $d(S^{\alpha,e}_{w0v^\alpha}) < d(S^{\alpha,1-e}_{w0v^\alpha})$.

But for the other triangle gap Δ_{w1v^α} by symmetry $d(S^{\alpha,1-e}_{w1v^\alpha}) < d(S^{\alpha,e}_{w1v^\alpha})$. So the characteristic symbol is a tool to determine which one is shorter and it works like as a parity between super/subscripts.

There are many ways to find which $t \in \{0,1\}$ is a characteristic symbol of α, we refer one way as below.

Lemma 4.4 (Proposition 2.41 in [12]). *Let α be a non-zero periodic in S^1. $e^\alpha = 0$ if $\dot{\alpha}\dot{\overline{\alpha}}$ separates $\ddot{\alpha}\ddot{\overline{\alpha}}$ and $0 \in S^1$. Symmetrically, $e^\alpha = 1$ if $\ddot{\alpha}\ddot{\overline{\alpha}}$ separates $\dot{\alpha}\dot{\overline{\alpha}}$ and $0 \in S^1$.*

$S^{\alpha,e^\alpha}_{u0v^\alpha}$ for some $u \in \{0,1\}^*$ became smaller and degenerated eventually as $|u|$ grows, since the subscript e^α always chooses the shorter side.

4.2. *Limit of laminations.*

Definition 4.5 (Accumulation of chords). *Suppose x_iy_i is a sequence of chords with $x_i \to x$ and $y_i \to y$ in a usual topology of S^1. Then we say that x_iy_i accumulates to xy.*

Now we can consider the limit lamination of the given lamination $\mathcal{L}_\alpha$. *i.e.,* the set of all accumulation chords of $\mathcal{L}_\alpha$. We call it $\partial\mathcal{L}_\alpha$. It is easy to check that $\partial\mathcal{L}_\alpha$ is again an invariant lamination under h.

Since $d(S^{\alpha,e^\alpha}_{u0v^\alpha})$ goes to 0, taking the limit of $S^{\alpha,e}_w$ it eventually degenerates. So we have the following corollary.

Corollary 4.6. *$\partial\mathcal{L}_\alpha$ for non-zero periodic $\alpha \in S^1$ is a limit of $S^{\alpha,1-e}_w$ for all $w \in \{0,1\}$.*

With the previous remark, in the collection of $S^{\alpha,1-e}_w$ every chord whose one endpoint is α has a form of $S^{\alpha,1-e}_{(v^\alpha(1-e))^n v^\alpha}$, and they are associated by one binary path $(1-e)(1-e)\cdots$. So the other endpoint is $l^\alpha_{(v^\alpha(1-e))^n v^\alpha}(\ddot{\alpha})$. Furthermore since it follows only the $(1-e)$ marker, the triangle path became thinner and thus $\alpha, l^\alpha_{v^\alpha}(\ddot{\alpha}), \cdots, l^\alpha_{(v^\alpha(1-e))^n v^\alpha}(\ddot{\alpha}), \cdots$ are positively ordered. By the corollary 4.6, it accumulates and converges. We take the following definition.

Definition 4.7 (Associated periodic point). *Let α be a nonzero periodic in S^1. The point $\overline{\alpha} := \lim_{n\to\infty} l^\alpha_{(v^\alpha(1-e))^n v^\alpha}(\ddot{\alpha})$ is well defined and is called* associated periodic point with α.

In other words, $\overline{\alpha}$ is one endpoint of the accumulation chord whose the other endpoint is α. The associated periodic point $\overline{\alpha}$ shares the properties of α as follows.

Proposition 4.8. *$PER(\alpha) = PER(\overline{\alpha})$ and $v^\alpha = v^{\overline{\alpha}}$.*

We will not prove the proposition. Instead, we will note some properties of the associated periodic points.

Remark.

1. For a non-zero periodic α in S^1, the associated periodic point $\overline{\alpha}$ of α and its companion angle coincide. Hence, the proposition 4.8 holds, and furthermore $\overline{\overline{\alpha}} = \alpha$. (See the corollary 4.13.)

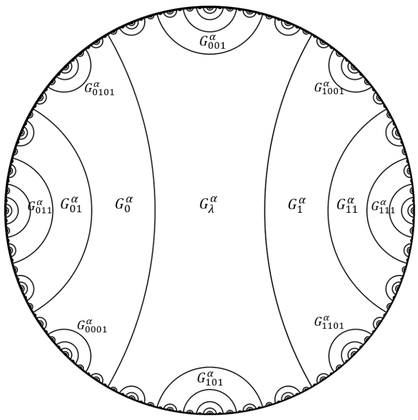


FIGURE 13. $\partial\mathcal{L}_\alpha$ for $\alpha = \frac{1}{3}$. Note that all the gaps are infinite gaps, whose intersection with S^1 is a Cantor set.

2. Since the angle doubling map h preserves the cyclic order, we conclude that for $w \in \{0,1\}^*$,
$$\lim_{n\to\infty} l^{\alpha}_{w(v^\alpha(1-e))^n v^\alpha}(\ddot{\alpha}) = l^{\alpha,1-e}_{w}(\overline{\alpha})$$
and hence
$$\lim_{n\to\infty} S^{\alpha,1-e}_{w(v^\alpha(1-e))^n v^\alpha} = l^{\alpha,1-e}_{w}(\alpha\overline{\alpha})$$
This is just a preimage argument for the accumulation chord $\alpha\overline{\alpha} \in \partial\mathcal{L}_\alpha$.
3. Under this observation we give a nice formula for $\overline{\alpha}$. Let $m = \mathrm{PER}(\alpha)$.
$$\overline{\alpha} = \alpha + \frac{2^m}{2^m-1}\left(l^{\alpha}_{v^\alpha}(\ddot{\alpha}) - \alpha\right)$$

$\dot{\alpha}\dot{\overline{\alpha}}$ and $\ddot{\alpha}\ddot{\overline{\alpha}}$ are the preimage of $\alpha\overline{\alpha}$, and one of them is the chord whose endpoints coincide with the arc A_n in the orbit portrait, where $n = \mathrm{PER}(\alpha)$. So we have the following.

COROLLARY 4.9. *Let $(\alpha, \overline{\alpha})$ is a characteristic arc of period n. Then* $e^\alpha = \alpha \times (2^n - 1) \mod 2$. *Therefore, every narrow characteristic arc has its characteristic symbol $e^\alpha = 1$.*

PROOF. The proof is exactly the same with Lemma 3.11. Suppose $\alpha = \frac{a}{2^n-1}$. If a is even, then $\ddot{\overline{\alpha}} = \overline{\alpha}/2 > \alpha/2 = \dot{\alpha}$. Thus $\ddot{\alpha}\ddot{\overline{\alpha}}$ and $0 \in S^1$ and $e^\alpha = 0$. Symmetric proof for odd a. ∎

From the chord $\alpha\overline{\alpha}$ in the limit lamination $\partial\mathcal{L}_\alpha$, we can generate whole $\partial\mathcal{L}_\alpha$.

THEOREM 4.10 (Proposition 2.42 in [12].). *Let α be a non-zero periodic in S^1 and $e = e^\alpha$ be the characteristic symbol of α. Then*

$$\partial\mathcal{L}_\alpha = \text{ Closure of } \left\{ l_w^{\alpha,1-e}(\alpha\overline{\alpha}) \mid w \in \{0,1\}^* \right\}$$

One can easily guess that all gaps in $\partial\mathcal{L}_\alpha$ are made up of collection of triangles starting at a chord in $\mathcal{L}_\alpha$. As in the previous remark, triangles form an infinite binary tree. Together with the observation of the correspondence between the triangle path and the binary sequence, we deduce the following.

PROPOSITION 4.11. *Suppose α and e are as above and $m = PER(\alpha)$. Then for every α-regular word $w \in \{0,1\}^*$ there is a corresponding* infinite *gap G_w^α in $\partial\mathcal{L}_\alpha$, whose boundaries are $l^{\alpha,1-e}_{ws_1v^\alpha s_2v^\alpha\cdots s_n}$ for $s_i \in \{0,1\}$.*

We call $G^\alpha_{v^\alpha}$ as a *critical value gap* because it contains α as its boundary gap, which is an external angle of critical value. Also, G^α_λ, an infinite gap corresponding to the empty word λ is called a *critical point gap.*

PROOF. Fix a α-regular word w. Then the gap G_w^α is associated to the infinite binary tree, which is a collection of triangle gaps starting from S_w^α. *i.e.,* G_w^α can be viewed as a union of $\Delta^\alpha_{ws_1v^\alpha s_2v^\alpha\cdots s_nv^\alpha}$, with $s_1\cdots s_n \in \{0,1\}^*$. Furthermore, each chord accumulated by the long sides of such triangles became a boundary of the given gap G_w^α. ∎

Choose an infinity gap G_w^α of the α-regular word w. By the proposition 4.11, this gap is bounded by infinitely many chords $l^{\alpha,1-e}_{ws_1v^\alpha s_2v^\alpha\cdots s_n}$. So the infinity gap can be rewritten as a Cantor set, starting from the circle and deleting all the points out which lie behind the chords $l^{\alpha,1-e}_{ws_1v^\alpha s_2v^\alpha\cdots s_n}$. Thus, every surviving point $p \in G_w^\alpha$ has an itinerary sequence of form $ws_1v^\alpha s_2v^\alpha\cdots$, where $s_1s_2\cdots$ is a 0-1 sequence. Note that if given itinerary sequence ends with $\overline{0v^\alpha}$ or $\overline{1v^\alpha}$, then it converges to an endpoint of some boundary chord. According to the above observation we get the following.

PROPOSITION 4.12. *$p \in G_w^\alpha$ if and only if $I^\alpha(p)$ ends with $0v^\alpha$-$1v^\alpha$ sequences. Furthermore, if $I^\alpha(p)$ ends with $\overline{0v^\alpha}$ or $\overline{1v^\alpha}$, then p is an endpoint of the boundary chord.*

The converse of the latter statement does not hold at all because the endpoint of the boundary chord could be the preimage of α, whose itinerary sequence eventually contains $*$.

4.3. *Translation into quadratic polynomials.* In this section $\alpha \in S^1$ is nonzero periodic under h. For such α there is an associated angle $\overline{\alpha}$. As we mentioned in the preliminary section, such a point is called a *companion angle.* Thus two external rays of angle $\alpha, \overline{\alpha}$ in the parameter space end up at the same point r_{M_α}, which is a root of a hyperbolic component M_α. From now on, we drop M and denote the center and root as c_α and r_α, respectively, if α is well understood. *i.e.,* $z \mapsto z^2 + r_\alpha$ has a parabolic fixed point for any periodic α.

Corollary 4.13 (Corollary 4.15 in [12]). *For periodic $\alpha \in S^1$ the parameter rays of the angles $\alpha, \overline{\alpha}$ land at the same root point r_M for some M. Therefore, the associated angle $\overline{\alpha}$ of the nonzero periodic α is identical to the companion angle.*

At r_M, the Julia set of $z^2 + r_M$ is connected. Let $\alpha, \overline{\alpha}$ be a pair corresponding to M. Let $\approx_\alpha$ be an equivalence relation on S^1 which satisfies

$$\theta_1 \approx_\alpha \theta_2 \quad \Leftrightarrow \quad \text{Dynamic rays of angle } \theta_1, \theta_2 \text{ lands at the same point.}$$

If we draw a line/polygon whose endpoints/vertices are in the same class of $\approx_\alpha$, then it becomes a lamination. We call this a *landing pattern* of α, or the *pinched disk model of K_α ($= K_{\overline{\alpha}}$).* We remark the following theorem.

Theorem 4.14 (Corollary 4.2 in [12]). *Suppose α is nonzero periodic in S^1. Then the landing pattern of α is equal to $\partial\mathcal{L}_\alpha$.*

Suppose a parameter c follows along the parameter ray $\mathcal{R}_\alpha$ and finally lands on the root r_M. At each moment c is on the $\mathcal{R}_\alpha$, the lamination model of J_c is the one generated by the long chord $\frac{\alpha}{2}\frac{\alpha+1}{2} = \dot{\alpha}\ddot{\alpha}$. If we pinch it from the long chord and its 1st preimages and 2nd and so on, then the limiting process would be a Julia set of $z \mapsto z^2 + c_\alpha$, where c_α is a parameter outside of the Mandelbrot set whose parameter angle is α.

If c eventually lands at r_M, we choose a lamination model of J_{r_M} as its landing pattern. Then such a choice of limit lamination allows us to consider the landing of the parameter as in the lamination model as below. *i.e.,*

$$\lim_{c \in \mathcal{R}_\alpha,\ c \to r_M} \mathcal{L}_\alpha = \partial\mathcal{L}_\alpha$$

The limit on the left is given by the topology derived from the accumulation of chords. It gives a way to understand the boundary of the shift locus of degree 2 in terms of laminations.

5. Action on itinerary sequences

We now introduce the quadrilateral lamination $\mathcal{Q}$.

Definition 5.1 (quadrilateral lamination). *Suppose $\overline{\alpha}$ is an associated angle of α. The central gap associated to $(\alpha, \overline{\alpha})$, denoted as $Q_\lambda^{(\alpha,\overline{\alpha})}$, is a gap whose endpoint is $\alpha/2, (\alpha+1)/2, \overline{\alpha}/2, (\overline{\alpha}+1)/2$.* i.e., *long chords $S_\lambda^\alpha, S_\lambda^{\overline{\alpha}}$ of $\mathcal{L}_\alpha$ and $\mathcal{L}_{\overline{\alpha}}$ are diagonals of the central gap $Q_\lambda^{(\alpha,\overline{\alpha})}$.*

Quadrilateral lamination *associated with $(\alpha, \overline{\alpha})$, denoted as $\mathcal{Q}^\alpha$, is the collection of preimages of $Q = Q_\lambda^{(\alpha,\overline{\alpha})}$ under h.*

The central gap is actually a rectangle, since its diagonals are diameters. So the there are two types of sides, one is longer than the other. We provide a figure 14 of quadraliteral lamination $\mathcal{Q}^\alpha$ for $\alpha = 1/3$.

Remark that α and its associated angle $\overline{\alpha}$ form a characteristic arc. Also, $h(Q)$ is equal to the characteristic arc $\alpha\overline{\alpha}$ and the shorter sides are the preimage l^{α,e^α} and the other is the preimage of $l^{\alpha,1-e^\alpha}$ because of their length. So we have the following.

PROPOSITION 5.2. *All chords in $\partial\mathcal{L}_\alpha$ are in the $\mathcal{Q}^\alpha$. The others have a form of $l_w^{\alpha,e}(\alpha\overline{\alpha})$.* i.e., *every chord in $\mathcal{Q}^\alpha$ has a form of $l_w^{\alpha,t}(\alpha\overline{\alpha})$, $t \in \{0,1\}$ and $w \in \{0,1\}^*$.*

PROOF. Note that the quadrilateral gap Q^α is surrounded by 4 different chords, one pair of long sides and the other pair of short sides. Since $\alpha/2, (\alpha+1)/2$ are antipodal to each other, the long side must be some form of $\frac{\alpha}{2}\frac{\overline{\alpha}}{2}$ or $\frac{\alpha}{2}\frac{\overline{\alpha}+1}{2}$ and the other one must be the short side. Without loss of generality, suppose $\frac{\alpha}{2}\frac{\overline{\alpha}}{2}$ is a long side. Since it is a preimage of $\alpha\overline{\alpha}$ and it is longer than the other preimage $\frac{\alpha}{2}\frac{\overline{\alpha}+1}{2}$, it coincides with $l_s^{\alpha,1-e}(\alpha\overline{\alpha})$. The exact same argument gives $l_s^{\alpha,e}(\alpha\overline{\alpha}) = \frac{\alpha}{2}\frac{\overline{\alpha}+1}{2}$. Here s is just a constant, used only to specify where the chord lies beyond S^α or not. Hence all chords are the preimages of $l_w^{\alpha,t}(\alpha\overline{\alpha})$ with $t \in \{0,1\}$ and $w \in \{0,1\}^*$. □

We call every chord in $l_w^{\alpha,e^\alpha}(\alpha\overline{\alpha})$ as *short chord*, and we denote every chord $l_w^{\alpha,1-e^\alpha}(\alpha\overline{\alpha})$ as *long chord*. Therefore, $\mathcal{Q}^\alpha$ is a lamination generated by $\alpha\overline{\alpha}$.

COROLLARY 5.3. *$\mathcal{Q}^\alpha$ is a lamination.*

Adding short chords gives another description of infinite gaps in $\partial\mathcal{L}_\alpha$. Choose the infinite gap G_λ^α at the center. It is bounded by the long chords. Adding the pair of short chords in the center gap divides the infinite gap G_λ^α into 3 pieces, center gap and 2 more pieces. If we add more short chords, 2 more quadrilateral gaps are attached on both short chords of Q^α. Iteratively, G_λ^α consists of infinitely many quadrilateral gaps, forming an infinite binary tree. See also the figure 14.

We associate each quadrilateral gap Q_w with a word $w \in \{0,1\}^*$. Such a word is a prefix, *i.e.,* $l_w^\alpha(Q) = Q_w$. By the observation, if $w = w'0v^\alpha$ or $w'1v^\alpha$, then the quadrilateral gap Q_w is located right next to the $Q_{w'}$, sharing one of the short chords of $Q_{w'}$.

As in the definition, all quadrilateral gaps in $\mathcal{Q}$ have diagonals which are the preimages of S_λ^α and $S_\lambda^{\overline{\alpha}}$. So when α changes to $\overline{\alpha}$, only the diagonal changes. This implies that if a point p lies behind the short chord of Q^α, the first symbol x of $I^\alpha(p)$ changes to $1-x$ when it comes to $I^{\overline{\alpha}}(p)$. Hence we get the following,

LEMMA 5.4. *Suppose p is not an endpoint of $\partial\mathcal{L}_\alpha$. Then the nth symbol s of $I^\alpha(p)$ changes to $(1-s)$ if and only if p lies behind the $(n-1)$th preimage of the short chord.*

PROOF. After taking $h^{(n-1)}$, the nth symbol of $I^\alpha(h^{(n-1)}(p))$ became the first one, since h acts on the itinerary sequence as a one-sided shift. As

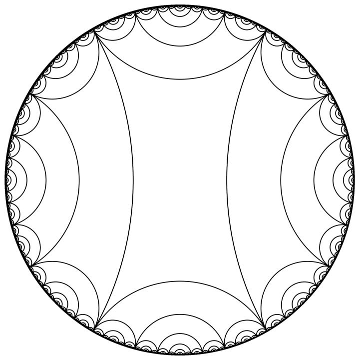


FIGURE 14. $\mathcal{Q}^\alpha$ with $\alpha = \frac{1}{3}$. Compare with the figure 13 also.

diagonals of Q are the long chords, the point lies on the short side if and only if the first symbol changes. ∎

By the proposition 5.2, every infinity gap in $\partial\mathcal{L}_\alpha$ can be decomposed into quadrilateral gap Q^α and its preimages, by $l_w^{\alpha,e}(\alpha\overline{\alpha})$. We call every chord of form $l_w^{\alpha,1-e}(\alpha\overline{\alpha})$ as a *long chord* and $l_w^{\alpha,e}(\alpha\overline{\alpha})$ as a *short chord*.

Consider the chords of $\mathcal{L}_\alpha$. Those are the diagonals of (pre-)quadrilateral gaps. When α changes to $\overline{\alpha}$, then the diagonal of each (pre-)quadrilateral gap changes to the other diagonal, which is the chord of $\mathcal{L}_{\overline{\alpha}}$. Translating into the itinerary sequence, it implies that the symbol $0, 1$ followed by v^α swaps each other.

THEOREM 5.5. *Suppose p is a point in G_w^α for some α-regular word w and $I^\alpha(p) = ws_1v^\alpha s_2v^\alpha\cdots$ for some $s_i \in \{0,1\}$. Then $I^{\overline{\alpha}}(p) = w'(1-s_1)v^\alpha(1-s_2)v^\alpha\cdots$.* i.e., *$0v^\alpha, 1v^\alpha$ are swapped each other in the $0v^\alpha$-$1v^\alpha$ sequence part. Here w' is another α-regular word.*

PROOF. The first statement is obvious by the lemma 5.4. Also the actual position of p does not move, it still lies on the same infinity gap, but different encoding with respect to $\partial\mathcal{L}_{\overline{\alpha}}$. ∎

Although the encoding of the infinity gap differs when α changes to $\overline{\alpha}$, its depth does not change. Therefore $|w| = |w'|$. *i.e.,* φ_α is a level preserving map.

So the only thing we left is how the encoding of the chosen infinite gap changes whenever α alters to $\overline{\alpha}$. In other words, we need a rule for α-regular words. Recall that the only situation where the symbol changes is when the point p or the gap G lies on some preimage of the short side of Q^α. Therefore, whenever the infinite gap in $\partial\mathcal{L}_\alpha$ lies behind the short side, some symbols in the encoded α-regular word w change.

LEMMA 5.6. *Suppose α is narrow. Let p lie behind one short side of the center gap of α. Then $I^\alpha(p)$ starts with $0v^\alpha$ or $1v^\alpha$ and the first symbol depends on which short side p lies behind.*

PROOF. Since α is narrow, every point lies behind the chord $\alpha\overline{\alpha}$ follows v^α. This is because under angle doubling such points always lie behind $A_1, \cdots, A_{n-1}$. Two short sides of the center gap are the preimage of the characteristic arc. Thus the first symbol of itinerary sequence is 0 or 1 followed by v^α, and the first symbol depends only on which preimage p belongs to. □

PROPOSITION 5.7. *Suppose $I = \alpha\overline{\alpha}$ is narrow. Then the infinite gap of the shortest prefix among all gaps which lie behind the short edge $l^{\alpha,e^\alpha}_{tv^\alpha}$ is $G^\alpha_{tv^\alpha e^\alpha}$, $t \in \{0,1\}$.*

PROOF. We only prove for the case $t = 0$. The other case holds by symmetry. Let Q be the center gap. By the observation, the quadrilateral gap Q_{0v^α} with the prefix $0v^\alpha$ shares the short chord of the center gap. Let l_1 be a chord of Q_{0v^α} opposite to the short chord, and choose any quadrilateral gap Q which lies behind l_1. Then since it lies behind the short side of the center gap, by lemma 5.6 the prefix of Q starts with $0v^\alpha$. Take h^n to be Q, then the endpoints of $h^n(Q)$ are contained in the arc A_n. Moreover, since $h^n(l_1) = \ddot{\alpha}\ddot{\overline{\alpha}}$, all endpoints lie behind the chord $\ddot{\alpha}\ddot{\overline{\alpha}}$. Thus any quadrilateral gap lie behind the short chord starts with $tv^\alpha e^\alpha$. □

Note that the infinite gap $G^\alpha_{tv^\alpha(1-e^\alpha)}$, $t \in \{0,1\}$ do not lie behind the short chord of the center gap. See the figure 13.

Using this, we can now detect how the prefix of the infinite gap changes. Let w be a α-regular word and $n = \mathrm{PER}(\alpha)$. We start by checking whether the first n symbols form a t_1v^α. If not, we just slide one symbol over. If they do, then we check the next n symbols to see if they form t_2v^α or not. We do this until they do not form tv^α. Suppose it ends at t_kv^α. The next step is to check the next following symbol. If the symbol is e^α, then all such t_iv^α change to $(1-t_i)v^\alpha$. This is because it indicates that such a gap lies behind the boundary chord $l^{\alpha,1-e^\alpha}_{t_1v^\alpha\cdots t_kv^\alpha}(\alpha\overline{\alpha})$ of the center gap.

What if the symbol is not e^α? Then we go back to the previous v^α, and check which symbol follows after it. In this case we have the sequence $\cdots t_{k-1}v^\alpha t_kv^\alpha(1-e^\alpha)\cdots$, so we check whether t_k is e^α or not. If not, then we go back and forth.

For example, let $\alpha = \frac{1}{7}$. Then $v^\alpha = 00$ and $e^\alpha = 1$. If we have a α-regular prefix 01000000001010010, it will change to a different prefix as below.

$$0|1|0|0|000\ 0001|0|1001|0 \Rightarrow 0|1|0|0|100\ 1001|0|0001|0$$

Note that the prefix starts with 100 from the second symbol, but 0 follows, which is $1 - e^\alpha$. So the second symbol does not change, and we slide one symbol over.

In summary, we have the following algorithm which depicts how itinerary sequence changes when the parameter passes through a narrow bifurcation locus.

THEOREM 5.8. *Let $s = s_1s_2s_3\cdots$ be a 0-1 sequence and α be narrow. Also, let $v = v^\alpha$ be the repetition word of α, $e = e^\alpha$ be a characteristic symbol of α, and $\mathrm{PER}(\alpha) = |v| + 1$ be the period of α. Then we define $\varphi_\alpha(s)$ as follows.*

1. *$i = 0$.*
2. *Set $w = s_i \cdots s_{i+|v|}$.*
   (a) *If $w = 0v$ or $1v$, set $i \leftarrow i + |v| + 1$ and go back to (2).*
   (b) *If not, but if there is a previous $0v$ or $1v$, check $s_i = e$.*
      (i) *If $s_{i+|v|+1} = e$, swap all $0v$ and $1v$.*
      (ii) *If $s_{i+|v|+1} \neq e$, trace back to the previous $0v$ (or $1v$) and return to (b).*
3. *If w is neither $0v$ nor $1v$, then set $i \leftarrow i + 1$ and go back to (2).*

If s is $I^\alpha(\theta)$, then $\varphi_\alpha(s) = I^{\overline{\alpha}}(\theta)$.

PROOF. We have seen that for any point $p \in G^\alpha_w$ for any α-regular word w. The remaining part is the case where p is the limit of infinite gaps. Then I^α does not end with $0v^\alpha$-$1v^\alpha$ sequences. Therefore, by truncating from the beginning of $I^\alpha(p)$, it is approximated by the prefixes that nests p. Therefore, $\varphi_\alpha(I^\alpha(p)) = I^{\overline{\alpha}}(p)$. □

Notice that if α is not narrow, then the algorithm fails. Because it is not narrow, there exists $m < PER(\alpha)$ where A_m nests A_1. This implies that there is an infinite gap G^α_w with shorter length $|w| < |tv^\alpha e^\alpha|$, and such a gap might change its symbols. Also, if the point lie behind the nonnarrow arc, then the itinerary sequence does not have to begin with v^α, but only with the first $(m - 1)$ symbols of v^α.

Another note is that v^α has an ambiguity for some sequences, especially the itinerary sequence ending with $\overline{0v^\alpha}$ or $\overline{1v^\alpha}$. For example, let $\alpha = \frac{1}{7}$. Then $v^\alpha = 00$ and $e^\alpha = 1$. Now consider the sequence $s = \overline{0} = 0000\cdots$. According to the rule, step (2) does not end and we need to distinguish at which $0v$-$1v$ sequence starts. But the problem is that we cannot distinguish what is the prefix of this sequence because all 3 cases (the empty word, 0, 00) could be its prefix. Each case is obtained by approximating the prefixes. For the case of the empty word prefix, it is approximated by 0, 0001, 000 0001, 000 000 0001, so that after taking φ_α it gives $\overline{100}$.

In the next section we will resolve this ambiguity by constructing an equivalence relation.

6. Equivalence relations on Σ_2

Definition of itinerary sequence avoids that some sequence is not realized in Σ_2. For example, the $\overline{0} = 000\cdots$ cannot be realized by $I^\alpha(\theta)$, $\theta \in S^1$. This is because if we keep taking the inverse image so that each image contains an angle whose itinerary sequence starts with $0\cdots 0$, it converges to the set $\{1/3, 2/3\}$ and both angles contain $*$ in their sequence.

Therefore, the algorithm we propose is not defined on the whole Σ_2 and by the ambiguity φ_α is a multi-valued function. Also, because of the detection of a piece of word and swap, it is far from being shift invariant.

Therefore, our goals in this section are the following.

1. Extend the angles so that I^α maps surjectively on Σ_2.
2. Extend the domain of φ_α to Σ_2.
3. Construct the equivalence relation on Σ_2 to make our map φ_α became a shift map.

6.1. *Extended angles.* Consider $z^2 + 2$. The parameter angle of $c = 2$ is 0, and since it is in $\mathcal{S}_2$, the Julia set is a Cantor set. Let p_1, p_2 be points in the Julia set corresponding to the sequences $\overline{0} = 000\cdots$, $\overline{1} = 111\cdots$, respectively. What are their external angles? Since those two points are fixed point under shift, the angle must be the fixed point of angle doubling map. Therefore, both admit 0 as their external angles.

The main reason is that if we extend Böttcher coordinate to the whole $\mathbb{C} - J_P$ for the shift polynomial P, we keep solving the equation $z \mapsto \sqrt{\phi(P^{\circ 2}(z))}$. Thus, the branched region becomes smaller and smaller and may converge to some points whose angles are precritical. As in the example above, the region where the first symbol of itinerary is equal to 0 is an open interval $(0, 1/2)$, and the region that both first and second symbols are 0 is an open interval $(0, 1/4)$. Keep doinig this, we deduce that the region whose itinerary sequence starts with $\underbrace{0\cdots 0}_{n}$ is $(0, 1/2^n)$. *i.e.,* The point p has a dynamic angle of 0, and such a sequence of intervals converges to 0. But we can do the exact same thing for 1, by taking $(1/2, 1), (3/4, 1), (7/8, 1), \cdots$ and it converges to $1 = 0$.

To distinguish these two points, we define the extended angle.

Definition 6.1. *Let $P := P_\alpha(z) = z^2 + c$ be a shift polynomial satisfying that the parameter angle of c is α, which is periodic. Suppose $x \in J_P$ corresponds to $s = s_1 s_2 \cdots$ and let θ be the external angle of x. Then we put a marker $+$ or $-$ on θ if it satisfies the following.*

1. θ^+ *if* $\lim_{\theta' \to \theta, \theta' > \theta} I^\alpha(\theta') = s$,
2. θ^- *if* $\lim_{\theta' \to \theta, \theta' < \theta} I^\alpha(\theta') = s$,
3. θ *if both limits (from the left and right) coincide.*

We call such an angle $\theta^{\pm}$ *as an* extended angle. *Note that if* θ *is not a preperiodic angle of* α*, then the external angle has no marker.*

For example, if $\alpha = 0$ as in the above, 0^+ is an external angle of sequence $\overline{0}$, whereas 0^- is for $\overline{1}$.

Extended angles have an order that is inherited from the usual order in S^1, by considering $\theta^- < \theta < \theta^+$. With this order, $I^\alpha(\theta^\pm)$ is well defined and compatible with the definition. So, we extend the domain to $\mathbf{E}_\alpha$ as follow.

$\{\theta$ is not a preimage of $\alpha\} \cup \{\theta^\pm$ is a preimage of α, with a corresponding marker$\}$.

PROPOSITION 6.2. *Let* α *be nonzero periodic in* S^1*. Then* $I^\alpha : \mathbf{E}_\alpha \to \Sigma_2$ *is surjective.*

PROOF. This is straightforward from the definition. Pick any sequence in Σ_2. Since we start with the Julia set J_α of angle α, there is a point $x \in J_\alpha$ corresponding to α. Then its extended angle is in $\mathbf{E}_\alpha$. ∎

By definition, φ_α is a map that commutes the diagram.

$$\begin{array}{ccc} \mathbf{E}_\alpha & \xrightarrow{Id} & \mathbf{E}_\alpha \\ \downarrow{\scriptstyle I^\alpha} & & \downarrow{\scriptstyle I^{\overline{\alpha}}} \\ \Sigma_2 & \xrightarrow{\varphi_\alpha} & \Sigma_2 \end{array}$$

Still φ_α is far from injectivity. For example, consider $\alpha = 2/7$ and let $\theta_1 = \frac{1}{7}^+, \theta_2 = \frac{2}{7}^+, \theta_3 = \frac{4}{7}^+$. Then all three $I^\alpha(\theta_i) = \overline{0}$. However it is possible to distinguish the prefix if we know not only the sequence but also the extended angle to which infinite gap it belongs to. In the example, $\theta_1 \in G^\alpha_\lambda, \theta_2 \in G^\alpha_0$, and $\theta_3 \in G^\alpha_{00}$. Their prefixes appear when each marker changes, such as $I^{\frac{2}{7}}\left(\frac{4}{7}^-\right) = \overline{001} = 00\ \overline{100}$. This is described in the figure 15.

6.2. *Parabolic implosion.* Yet φ_α is a multi-valued in the sense of mapping between sequences. In this section we will solve this problem.

We first define a Julia equivalence on $\mathbf{E}_\alpha$. This is again introduced by Keller in [12].

DEFINITION 6.3 (Julia equivalence). *For periodic* $\alpha \in S^1$ *and* $\theta, \theta' \in S^1$*, we denote* $\theta \approx^\alpha \theta'$ *if there exists a finite sequence of points* $\theta = z_1, z_2 \cdots, z_{n-1}, z_n = \theta'$ *such that* $z_i z_{i+1}$ *are chords of* $\partial\mathcal{L}^\alpha$*.*

It is easy to verify that it is indeed an equivalence relation. It can be translated into the language of the itinerary sequences.

PROPOSITION 6.4. $x \approx^\alpha y \iff I^\alpha(x) = I^\alpha(y)$.

The equivalence is extended to the $\mathbf{E}_\alpha$ from S^1, by dropping the markings. By definition, $\theta_1 \approx^\alpha \theta_2$ if two external(dynamics) rays $\mathcal{R}_\theta, \mathcal{R}_{\theta'}$ land on the same point of the Julia set of $z^2 + r_\mathcal{P}$, where $r_\mathcal{P}$ is the root of the hyperbolic

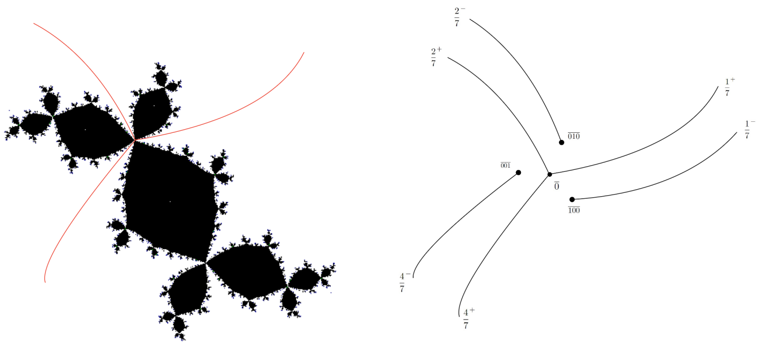


FIGURE 15. Figure on the left illustrates the dynamical plane and actual rays landing at the fixed point of $z^2 + c$, where $c \approx -0.1226 + 0.7449i$, a rabbit. Figure on the right is a schematic version of the left one, before c lands at the root. Black dots are points in the Julia set. Suppose $\alpha = \frac{2}{7}$. As c moves along $\mathcal{R}^\alpha$ and eventually lands at the root of the hyperbolic component, these 4 points collapse into a 1 point. Here, the dynamic rays should in fact bifurcate, but we draw them separately to make them easier to distinguish. Such bifurcation rays were also discussed in [1].

component $\mathcal{P}$ corresponding to α and its companion angle. Therefore, the extension is exactly an equivalence relation of Σ_2.

DEFINITION 6.5. *Let $s_1, s_2 \in \Sigma_2$ and $x_1, x_2 \in J_\alpha$ be the corresponding point. Suppose that θ_1, θ_2 are extended angles of x_1, x_2, respectively. We define the equivalence relation $\sim$ as follows,*

$$s_1 \sim s_2 \quad \Leftrightarrow \quad \theta_1 \approx^\alpha \theta_2$$

As c follows the parameter ray $\mathcal{R}_\alpha$ and eventually lands at the root of the hyperbolic components, some points in the Julia set J_α gathers into 1 point, which corresponds to the equivalence class. It can be considered as a parabolic implosion, because as the parameter c perturbs from the root r_M in the direction of the angle α, some periodic cycles pop out of the point.

THEOREM 6.6. *$\varphi_\alpha : \Sigma_2/\sim \,\to \Sigma_2/\sim$ is well-defined. Furthermore, it is an order 2 element and shift-invariant.*

PROOF. It has order 2 by the definition. Although $\varphi_\alpha : \Sigma_2 \to \Sigma_2$ is multi-valued, but by the construction above the equivalence gathers all the points causing the problem into one class. We check them one by one.

1. Lift each sequence into a subset of the extended angle $\mathbf{E}_\alpha$. Then φ_α is $I^{\overline{\alpha}} \circ (I^\alpha)^{-1}$. By the proposition 6.4, φ_α is well defined on $\Sigma_2/\sim$.
2. By construction of φ_α, it is an element of order 2. In fact, $\varphi_\alpha^{-1} = \varphi_{\overline{\alpha}}$.

3. Shift invariance is straightforward due to the fact that $\partial\mathcal{L}_\alpha$ is invariant under angle doubling. Since the shift map on Σ_2 corresponds to the angle doubling of the angle of p, φ_α commutes with one-sided shift.

∎

We end the section with the remark. Blanchard, Devaney, and Keen proved the following theorem in [3],

THEOREM 6.7 (BDK map). *There is a surjective map from $\pi_1(\mathcal{S}_d)$ to $Aut(\Sigma_d, \sigma)$, a shift automorphism of d symbols.*

The only nontrivial shift automorphism of 2 symbols is a 0-1 swap, mentioned earlier in the introduction. So every φ_α is not a shift automorphism of Σ_2. With the equivalence in the definition 6.5, we now have a shift invariance.

However, there is another shift invariance on the subset of Σ_2. Suppose we ignore the prefix part. Then it becomes a $0v^\alpha$-$1v^\alpha$ sequence, and thus commute with $\sigma^{\circ\,\mathrm{PER}(\alpha)}$, a $\mathrm{PER}(\alpha)$ times shift. *i.e.,* if 0 sends to $0v^\alpha$ and 1 sends to $1v^\alpha$ then it behaves like a shift map and the action of φ_α became similar to the nontrivial shift automorphism of 2 symbols.

With the Douady's angle formula 2.8, if we restrict our interest to the Fatou component containing 0, the 0-1 symbol swap is translated into $0v^\alpha$-$1v^\alpha$ swap after the tuning. We will focus on this phenomenon and construct a big mapping class of $\mathrm{Mod}(\mathbb{C} - \{\text{Cantor set}\})$ or $\mathrm{Mod}(S^2 - \{\text{Cantor set}\})$ in the upcoming paper.

APPENDIX A. SYMBOLIC DYNAMICS AND PUZZLE PIECES

In this section we only care about the narrow components unless otherwise specified. The main purpose of the appendix is to construct the dynamical graph, which is analogous to that of Atela's paper [2]. Such dynamical graph is built up by puzzle pieces of the unit disk, divided by chords of the orbit portrait of α, with some additional chords. Speaking in advance, such a construction is also valid for non-narrow cases also, except the canonical indexing of the puzzle.

A.1. *Markov puzzle pieces.* In section 3, we prove the necessary and sufficient condition between narrowness and no nesting of $I = A_1$. This fact allows us to number the puzzle piece in a canonical way.

Suppose $I = (\alpha, \overline{\alpha}), \alpha < \overline{\alpha}$ is a narrow arc or period n and $I = A_1, A_2, \cdots, A_n$ as defined above. Let a_i as a chord whose endpoints agree with those of A_i's. We add a_0 and S, which is an antipodal chord of a_n and a diameter $\frac{\alpha}{2}\frac{\alpha+1}{2}$. They separate the unit disk into $(n+2)$ or $(n+3)$ pieces if I is satellite or primitive, respectively. (Recall that if I is narrow, I is satellite if and only if I is attached to the main cardioid M_0. In this case we get $n+2$ puzzle pieces.)

We number each piece as follows. We also give an example in the figure 16.

1. There are two puzzle pieces, one enclosed by a_n, S and the other enclosed by a_0, S. We denote them by Π_0^0, Π_0^1.
2. Suppose a_i is the innermost arc. We number the puzzle piece whose boundary is a union of a_i and the arc A_i with Π_i.
3. Suppose $A_{i_1} \subset A_{i_2} \subset \cdots \subset A_{i_k}$ with $i_k \neq n$. We denote the puzzle piece between A_{i_j} and $A_{i_{j+1}}$ as $\Pi_{i_{j+1}}$.
4. Finally there are 2 pieces left, one next to a_n and the other next to a_0. We call them as Π_U and Π_D, respectively. Note that if I is a satellite, then Π_U is a polygon gap, and the number of sides is equal to the period of I.

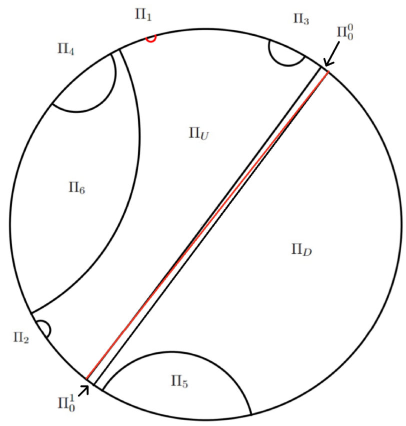


FIGURE 16. Puzzle pieces of $I = (\frac{37}{127}, \frac{38}{127})$.

The angle doubling map $h : S^1 \to S^1$ maps each puzzle piece to a set of puzzle pieces. We draw a directed graph as follows.

- Vertices/states : Each puzzle piece.
- Edges : a directed edge from Π to Π' if $\Pi' \subset h(\Pi)$.

Note that Π_0^0 and Π_0^1 both sends to Π_1. We call this as a transition graph $\mathcal{G}_\alpha$ associated with the angle α. A path in $\mathcal{G}_\alpha$ is a one-sided sequence of states where each consecutive 2 states are connected by an edge compatible with its direction.

LEMMA A.1. *$\mathcal{G}_\alpha$ and $\mathcal{G}_{\overline{\alpha}}$ are isomorphic.*

PROOF. Since α and $\overline{\alpha}$ are the endpoints of I, every puzzle is identical except Π_0^0 and Π_0^1. But these are the preimages of Π_1, so the outgoing edge of

Π_0^* are same, both go to Π_1. The incoming edges are also identical. Suppose they are not. Then there is a puzzle piece Π whose image contains Π_0^0 but not Π_0^1. This implies that the boundary of $h(\Pi)$ contains S. Since the boundary of any piece can admit A_i's or a_i's, and h sends boundary to boundary, S cannot be the boundary. Thus, every incoming and outgoing edge of both Π_0^0 and Π_0^1 is identical, and thus we can consider it as a unified one state Π_0. Define a transition graph $\mathcal{G}'$ of puzzle pieces with Π_0 instead of Π_0^0, Π_0^1. Obviously the same $\mathcal{G}'$ is obtained from α or $\overline{\alpha}$, we get the same transition graph. □

The difference between these two transition graphs appears when we assign a symbol $\in \{0, 1\}$ to each puzzle piece. We assign a symbol to each puzzle piece when we create an itinerary sequence. *i.e.,* a puzzle piece Π is assigned by 0 if Π is contained in A_α, 1 if it is contained in B_α. Note that Π_0^0, Π_1 and Π_U are always marked by 0, Π_0^1, Π_D is 1 on the other side. If we focus only on the arc boundaries of each puzzle piece, then the arc boundary of $\Pi_0^{0,\alpha}$ is the same as that of $\Pi_0^{1,\overline{\alpha}}$.

The symbol assigning map {puzzle pieces} $\mapsto \{0, 1\}$ induces a map {Paths in $\mathcal{G}_\alpha\} \to \Sigma_2$. *i.e.,* Each path gives a description of itinerary sequences. In fact, each sequence in Σ_2 is realized.

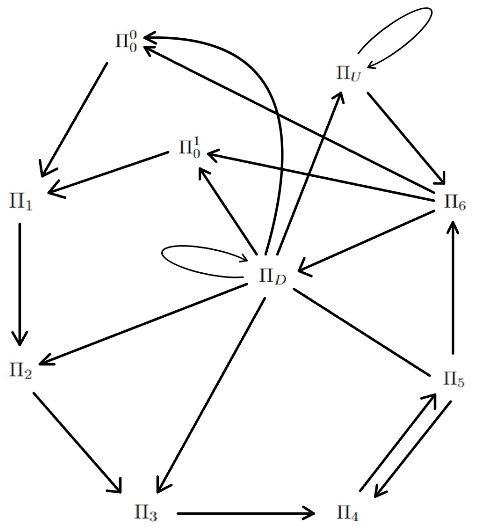


FIGURE 17. The transition graph $\mathcal{G}_\alpha$ for $\alpha = \frac{37}{127}$

THEOREM A.2. *Let $\mathbb{P}$ be the collection of all paths of the transition graph $\mathcal{G}_\alpha$. Then the symbol assigning map is a surjective map $\mathbb{P} \twoheadrightarrow \Sigma_2$.*

We will prove this by showing each sequence is a $I^\alpha(\theta)$ for some $\theta \in \mathbf{E}_\alpha$.

DEFINITION A.3. *Let $x \in \{0,1\}^*$ be a* 0-1 *word and $\alpha \in S^1$ be nonzero periodic. D^α_x is defined as a subset of $\mathbf{E}_\alpha$ that $\{\theta \in \mathbf{E}_\alpha \mid I^\alpha(\theta)$ starts with $x\}$. Also one can define for any sequence $\mathbf{x} = \{x_1, x_2, \cdots\} \in \{0,1\}^{\mathbb{N}}$ as $D_{\mathbf{x}} := \bigcap_{n=1}^{\infty} D_{x_1 \cdots x_n}$.*

LEMMA A.4 (Theorem 2.15 in [12].).
1. *If x is α-regular, then $D^\alpha_x = D^\alpha_{x0} \cup D^\alpha_{x1}$ and $D^\alpha_{x0} \cap D^\alpha_{x1} = S^\alpha_x$.*
2. *If x is not α-regular, then $D^\alpha_x = D^\alpha_{x0} \cup D^\alpha_{x1} \cup \Delta^\alpha_x$. In this case, $D^\alpha_{x0} \cap \Delta^\alpha_x = S^{\alpha,0}_x$ and $D^\alpha_{x1} \cap \Delta^\alpha_x = S^{\alpha,1}_x$.*

The above lemma implies that such D_x has a boundary of chords, and as $|x| \to \infty$ it splits into 2 pieces by another chord generated by the long chord $\frac{\alpha}{2}\frac{(\alpha+1)}{2}$. Thus we get the following corollary.

COROLLARY A.5. *Choose any $\mathbf{x} \in \{0,1\}^{\mathbb{N}}$. For any $\theta_1, \theta_2 \in D_{\mathbf{x}}$, $\theta_1\theta_2$ is unlinked with the orbit portrait of α.*

PROOF. $\partial\mathcal{L}_\alpha$ is generated by $\alpha\overline{\alpha}$, hence it contains every chord in the orbit portrait of α. By the lemma A.4, the boundary of each $D_{\mathbf{x}}$ consists of chords of $\mathcal{L}_\alpha$ for each word $x \in \{0,1\}^*$. Taking the limit, $D_{\mathbf{x}}$ is bounded by chords in $\partial\mathcal{L}_\alpha$, therefore the convex hull of them is unlinked with the orbit portrait. ∎

PROOF. (Proof of theorem A.2) Choose any sequence $x \in \Sigma_2$. Then by the corollary A.5, it is either properly contained in a puzzle or ∂D_x contains boundaries of some puzzle pieces. Choose any angle θ in D_x. Note that θ is an extended angle. The path on the transition graph can be obtained by keeping track of which puzzle piece the angle $h^{\circ m}(\theta)$ belongs to. ∎

The puzzle piece Π_0 admits a preimage of A_1 as its boundary. Note that the itinerary sequence symbol changes only at this region. In other words, we have the following rule for how the itinerary sequence changes when α varies to $\overline{\alpha}$.

THEOREM A.6 (Itinerary sequence rule). *For $\theta \in \mathbf{E}_\alpha$, we can construct $\varphi_\alpha : I^\alpha(\theta) \mapsto I^{\overline{\alpha}}(\theta)$ by the following rule.*

First lift $I^\alpha(\theta)$ to a path $\gamma \in \mathbb{P}$ in $\mathcal{G}_\alpha$. By starting from θ and its forward orbit, there is a piece corresponding to each orbit. This gives a path in $\mathcal{G}_\alpha$. Whenever γ passes either Π^0_0 or Π^1_0, swap each other. Then this path is a lift of $I^\alpha(\theta)$ in $\mathcal{G}_{\overline{\alpha}}$.

By the construction, φ_α is order 2 *element.*

A final notice is that the construction we have introduced here is not limited in the narrow case as we mentioned at the beginning. The only limitation is that we could not find any canonical ways to number/index each state/puzzle piece for non-narrow cases. However, every theorem in the appendix A still holds for the non-narrow cases, analogous to the Atela's work in [2]. Nevertheless, it still exists that the ambiguity of φ_α and we cannot explicitly find how the sequence varies when we have inputs only.

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