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# ON THE NOTION OF APPROXIMATELY LOWER PRE-OSCILLATORY SEQUENCE OF FUNCTIONS

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Abstract. In this paper we introduce the notion of an approximately lower pre-oscillatory (app LPO, for brevity) sequence of functions. When the domain is a compact interval and the sequence consists of absolutely continuous functions, a similar notion is introduced in the paper A. Raguž, Some results in asymptotic analysis of finite-energy sequences of one-dimensional Cahn-Hilliard functional with non-standard two-well potential, Glas. Mat. Ser. III **59(79)** (2024), 125–145. The generalization considered herein is twofold. On the one hand, we consider the case of the domain which is a measurable set of possibly infinite measure, and, on the other, we consider the case of a sequence of measurable functions. We adapt the definition accordingly, and we present some properties of the aforementioned notion of an app LPO sequence of functions. In particular, we study the cases when such an app LPO property is preserved under the outer or the inner composition with a suitable class of functions.

## 1. Introduction

The structure of the paper is as follows. In this introductory section, we outline the structure of the paper. In Section 2, we introduce the notation and terminology, define the notion of an approximately lower pre-oscillatory sequence of functions (referred to as an app LPO sequence of functions), explain the motivation, and describe the context for our considerations. In Section 3 (and Section 4, respectively), we state and prove the main results of the paper. These results pertain to the basic asymptotic properties of an app LPO sequence of functions defined on a domain that is an arbitrary measurable set $\Omega \subseteq \mathbf{R}$ and to the preservation of the app LPO property of

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a sequence of functions under the outer or inner composition of a function, respectively. In Section 5, we discuss the case where the domain $\Omega \subseteq \mathbf{R}$ is a measurable set of finite measure. Appendix A provides an overview of classical results and may be skipped by experienced readers. Appendix B contains results that are most likely known but are not easily found in the literature.

## 2. Notation and Terminology

In this paper we will distinguish between monotonic and strictly monotonic functions. By $I$ ($J$, resp.) we always denote intervals in $\mathbf{R}$ which are, unless otherwise specified, bounded or unbounded, open or closed (or neither open nor closed). If $u : \Omega \longrightarrow \mathbf{R}$ is a given function, where $\Omega \subseteq \mathbf{R}$ is a given non-empty set, we define $\|u\|_{\infty,\Omega} := \sup_{s\in\Omega}|u(s)| \in [0,+\infty]$. By $u \llcorner \omega$ we denote the restriction of $u$ on the set $\omega \subseteq \Omega$, while by $u^{\leftarrow}(\{\xi\})$ we denote the preimage of $\xi$ with respect to $u$, i.e., $u^{\leftarrow}(\{\xi\}) := \{s \in \Omega : u(s) = \xi\}$. We say that $u : \Omega \longrightarrow \mathbf{R}$ is a Lipschitz function on $\Omega$, and we write $u \in \mathrm{Lip}(\Omega)$, if it holds that there exists $M \geq 0$ such that for every $s_1, s_2 \in \Omega$ we have $|u(s_1) - u(s_2)| \leq M|s_1 - s_2|$. By $\mathrm{Lip}(u; \Omega)$ we denote the Lipschitz constant of $u$ on $\Omega$. We say that $u : \Omega \longrightarrow D$ is a bi-Lipschitz function, if the following three conditions are fulfilled: (i) $u : \Omega \longrightarrow D$ is a bijection; (ii) $u : \Omega \longrightarrow D$ is a Lipschitz function on $\Omega$, and (iii) $u^{-1} : D \longrightarrow \Omega$ is a Lipschitz function on $D$. By $\mathrm{card}\, S$ we denote the cardinality of a set $S$. For a given set $A \subseteq \mathbf{R}$, by $\overline{A}$ we denote its closure. If open set $U \subseteq \mathbf{R}$ satisfies $\overline{U} \subseteq \Omega$, we write $U \subseteq\subseteq \Omega$. If $\psi : (a, +\infty) \longrightarrow \mathbf{R}$ is a given function, we recall that the limit inferior of $\psi$ at $+\infty$ is defined by $\liminf_{\xi\longrightarrow+\infty} \psi(\xi) := \inf\{\liminf_{n\longrightarrow+\infty} \psi(\xi_n) : x_n \longrightarrow +\infty \text{ as } n \longrightarrow +\infty\}$.

The following definitions, pertaining to various notions of measurable functions, are stated in accordance with subsection 5.2 in [12] (in particular, cf. Definition 5.38 and subsequent comments therein).

We recall that, if by $(X, \Sigma_X)$ ($(Y, \Sigma_Y)$, resp.) we denote a measurable space, given a non-empty set $X$ ($Y$, resp.), which is equipped with $\sigma$-algebra $\Sigma_X$ ($\Sigma_Y$, resp.), we say that a function $\phi : X \longrightarrow Y$ is $(\Sigma_X, \Sigma_Y)$-measurable if for every $F \in \Sigma_Y$ it holds that $\phi^{\leftarrow}(F) \in \Sigma_X$. If $(X_i, \Sigma_i)$, where $i = 1, 2, 3$, are given measurable spaces, if $\psi : (X_1, \Sigma_1) \longrightarrow (X_2, \Sigma_2)$ is $(\Sigma_1, \Sigma_2)$-measurable, and if $u : (X_2, \Sigma_2) \longrightarrow (X_3, \Sigma_3)$ is $(\Sigma_2, \Sigma_3)$-measurable, then the composition $u \circ \psi : (X_1, \Sigma_1) \longrightarrow (X_3, \Sigma_3)$ is $(\Sigma_1, \Sigma_3)$-measurable. By $\Sigma_{\mathcal{B}}(\mathbf{R})$ we denote the Borel $\sigma$-algebra on $\mathbf{R}$ (i.e, the smallest $\sigma$-algebra which contains all open sets in $\mathbf{R}$), and if $E \in \Sigma_{\mathcal{B}}(\mathbf{R})$, we say that $E$ is a Borel set. By $\lambda : \Sigma_{\mathcal{L}}(\mathbf{R}) \longrightarrow [0, +\infty]$ we denote the Lebesgue measure on $\mathbf{R}$ (where $\Sigma_{\mathcal{L}}(\mathbf{R})$ stands for $\sigma$-algebra of Lebesgue measurable sets on $\mathbf{R}$, i.e., for the completion of $\Sigma_{\mathcal{B}}(\mathbf{R})$ with respect to the Lebesgue measure), while by $\lambda^* : \mathcal{P}(\mathbf{R}) \longrightarrow [0, +\infty]$ (where $\mathcal{P}(\mathbf{R})$ stands for the power set of $\mathbf{R}$) we denote the outer Lebesgue measure on $\mathbf{R}$. A Lebesgue measurable function $u : \mathbf{R} \longrightarrow \mathbf{R}$ is a function which is $(\Sigma_{\mathcal{L}}(\mathbf{R}), \Sigma_{\mathcal{B}}(\mathbf{R}))$-measurable. We point out that we have the strict inclusion $\Sigma_{\mathcal{B}}(\mathbf{R}) \subset \Sigma_{\mathcal{L}}(\mathbf{R})$ (since there are examples of $\Sigma_{\mathcal{L}}(\mathbf{R})$-measurable sets which are not $\Sigma_{\mathcal{B}}(\mathbf{R})$)-measurable (cf. Exercise 5.19 in [12])), and that $(\Sigma_{\mathcal{L}}(\mathbf{R}), \Sigma_{\mathcal{B}}(\mathbf{R}))$-measurability actually differs from $(\Sigma_{\mathcal{L}}(\mathbf{R}), \Sigma_{\mathcal{L}}(\mathbf{R}))$-measurability (since there are examples of $(\Sigma_{\mathcal{L}}(\mathbf{R}), \Sigma_{\mathcal{B}}(\mathbf{R}))$-measurable functions $u$ and $\Sigma_{\mathcal{L}}(\mathbf{R})$-measurable sets $E \subseteq \mathbf{R}$ for which $u^{\leftarrow}(E)$ is not a $\Sigma_{\mathcal{L}}(\mathbf{R})$-measurable set (cf. Exercise 5.41 in [12])). We say that $\Omega \subseteq \mathbf{R}$ is a measurable set if it is Lebesgue measurable (i.e., if $\Omega \in \Sigma_{\mathcal{L}}(\mathbf{R})$). Given a measurable set $\Omega \subseteq \mathbf{R}$, we say that $u : \Omega \longrightarrow \mathbf{R}$ is a measurable function on $\Omega$ if $u$ is measurable in the sense of Lebesgue, meaning that for every open set $U \subseteq \mathbf{R}$ it holds that $u^{\leftarrow}(U)$ is a measurable set (i.e., it holds that $u^{\leftarrow}(U) \in \Omega \cap \Sigma_{\mathcal{L}}(\mathbf{R})$). In particular, $u : \Omega \longrightarrow \mathbf{R}$ is a measurable function on $\Omega$ iff for every $a \in \mathbf{R}$ it follows that $u^{\leftarrow}(a, +\infty)$ ($u^{\leftarrow}[a, +\infty)$, resp.) is a measurable set. We mention here that, in this paper, the definition of measurability of a real function $u$ is always meant irrespective of the image of $u$. For example, if we say that $u : \Omega \longrightarrow D$ is a measurable function, where $\Omega, D \subseteq \mathbf{R}$ are measurable sets, we mean that $u : \Omega \longrightarrow \mathbf{R}$ is a measurable function in the usual sense of Lebesgue, and that $\{u(s) : s \in \Omega\} = D$. By stating that "$u : \Omega \longrightarrow D$ is a bijection such that $u$ is a measurable function", rather than "$u : \Omega \longrightarrow D$ is a measurable bijection", we avoid a possible ambiguity in the statements of our results. As a consequence of the aforementioned definitions, in our terminology, the composition of two measurable functions in general is not a measurable function (a classical counterexample follows from Exercise 3.77 and Exercise 1.33 in [21]). This fact raises the issues which are discussed in Section 4.

If $\Omega \subseteq \mathbf{R}$ is a measurable set, and if $u : \Omega \longrightarrow \mathbf{R}$ is a measurable function, by $\int_\Omega u(\sigma)d\sigma$ ($\int_\Omega u$, resp.) we denote the Lebesgue integral of the function $u$ over the set $\Omega$. In particular, if $a, b \in \mathbf{R}$ and $a \leq b$, by $\int_a^b u(\sigma)d\sigma$ ($\int_a^b u$, resp.) we denote the Lebesgue integral of the function $u$ over the interval $(a, b) := \{\sigma \in \mathbf{R} : a < \sigma < b\}$. A measurable function $u : \Omega \longrightarrow \mathbf{R}$ is said to be Lebesgue integrable on the measurable set $\Omega$ iff $\int_\Omega |u(\sigma)|d\sigma < +\infty$. A measurable function $u : \Omega \longrightarrow \mathbf{R}$ is said to be locally integrable if it is Lebesgue integrable on every compact set $K \subseteq \Omega$. If two measurable functions $u, v : \Omega \longrightarrow \mathbf{R}$ are equal outside of the set $N \subseteq \Omega$ such that $\lambda(N) = 0$, we say that $u$ and $v$ are equal almost everywhere on $\Omega$, and we write $u(s) = v(s)$ (a.e. $s \in \Omega$). Following [6] (cf. Definition 1.14 and the subsequent comments therein), we do not identify two measurable functions $u, v : \Omega \longrightarrow \mathbf{R}$ which are equal almost everywhere on $\Omega$. Instead, if two measurable functions $u, v : \Omega \longrightarrow \mathbf{R}$ are equal almost everywhere on $\Omega$, we say that $v$ is a representative of $u$, and by $[u]$ we denote the set of all representatives of $u$. The set of all Lebesgue integrable functions (locally integrable functions, resp.) on $\Omega$ is denoted by $\mathcal{L}^1(\Omega)$ ($\mathcal{L}^1_{loc}(\Omega)$, resp.). We define $\mathrm{L}^1(\Omega) := \{[u] : u \in \mathcal{L}^1(\Omega)\}$ ($\mathrm{L}^1_{loc}(\Omega) := \{[u] : u \in \mathcal{L}^1_{loc}(\Omega)\}$, resp.), and $\mathrm{L}^\infty(\Omega) := \{[u] : \|u\|_{\infty,\Omega} < +\infty\}$ ($\mathrm{L}^\infty_{loc}(\Omega) := \{[u] : \|u\|_{\infty,K} < +\infty$ for every compact $K \subseteq \Omega\}$, resp.), which is the usual notation in the literature. In particular, if $K \subseteq \mathbf{R}$ is a compact set, then we have $\mathcal{L}^1_{loc}(K) = \mathcal{L}^1(K)$ ($\mathrm{L}^1_{loc}(K) = \mathrm{L}^1(K)$, resp.). By $\mathrm{C}(\Omega)$ ($C^\infty(\Omega)$, resp.) we denote the set of all continuous functions on the open set $\Omega$ (all functions $u : \Omega \longrightarrow \mathbf{R}$ which are differentiable infinitely many times, resp.). $C_c(\Omega)$ ($C_c^\infty(\Omega)$, resp.) denotes the space of all functions in $\mathrm{C}(\Omega)$ (all functions in $C^\infty(\Omega)$, resp.) whose support is compact, while $C_0(\Omega)$ denotes the closure of $C_c(\Omega)$ with respect to the norm $\|\cdot\|_{\infty,\Omega}$. The set of all finite signed Radon measures on $\Omega$, denoted by $\mathcal{M}_b(\Omega)$ (cf. Definition B.107 and Definition B.110 in [21]), is identified with the dual of $\mathrm{C}_0(\Omega)$ (cf. Theorem B.111 in [21]), while by $|\mu|$ we denote the total variation measure of $\mu \in \mathcal{M}_b(\Omega)$ (cf. Definition B.69 and Proposition B.72 in [21]). A more thorough review of these two notions is included in Appendix A. We write "WLG" as an abbreviation instead of the expression "without loss of generality". Throughout the paper, given an open set $\Omega \subseteq \mathbf{R}$, by $\mathrm{BV}(\Omega)$ we denote the set of all real functions of bounded variation on $\Omega$ (cf. [6], p. 166, or [21], Definition 14.1, p. 459), which is defined as follows:

DEFINITION 2.1. *Let $\Omega \subseteq \mathbf{R}$ be an open set. We define the space of functions of bounded variation on $\Omega$, denoted by $\mathrm{BV}(\Omega)$, as the space of all functions $u \in \mathcal{L}^1(\Omega)$ whose distributional first-order derivative is a finite signed Radon measure; that is, there exists a finite signed measure $\mu : \Sigma_{\mathcal{B}}(\Omega) \longrightarrow \mathbf{R}$ such that $\int_\Omega u\varphi' = -\int_\Omega \varphi d\mu$ for all $\varphi \in \mathrm{C}_c^\infty(\Omega)$, where $\int_\Omega \varphi d\mu$ stands for the integral of $\varphi$ over $\Omega$ with respect to the measure $\mu$. The measure $\mu$ is called the weak, or distributional, derivative of $u$ and it is denoted by $Du$. Here $\Sigma_{\mathcal{B}}(\Omega)$ is the Borel $\sigma$-algebra on $\Omega$. By $\mathrm{BV}_{loc}(\Omega)$ we denote the set of all $u \in \mathcal{L}^1_{loc}(\Omega)$ which satisfy $u \in \mathrm{BV}(U)$ for all open sets $U \subseteq\subseteq \Omega$.*

REMARK 2.2. To distinguish between the classical derivative and the distributional derivative, by $u'$ ($Du$, resp.) we denote the classical derivative (the distributional derivative, resp.) of the function $u$. We recall that a distributional derivative is a generalized concept of differentiation that extends the classical derivative to functions that may not be differentiable in the usual sense. If $u \in \mathcal{L}^1_{loc}(\Omega)$, the distributional derivative $Du$ is defined via its action on the so-called test functions $\varphi \in \mathrm{C}_c^\infty(\Omega)$ as follows: $\langle Du, \varphi\rangle := -\langle u, \varphi'\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between a distribution $Du$ (an element of the dual of $\mathrm{C}_c^\infty(\Omega)$) and a test function $\varphi$. Rigorous definition of the notion of a distribution can be found in chapter 10 in [21], or in [3]. Roughly speaking, this definition uses integration by parts and avoids requiring $u$ to be differentiable. Instead, it relies on how $u$ interacts with smooth test functions. If $u$ is differentiable in the classical sense, its distributional derivative coincides with the classical one.

On the other hand, if $I \subseteq \mathbf{R}$ is an interval, by $\mathrm{BPV}(I)$ we denote the set of all real functions of bounded pointwise variation on $I$ (cf. [21], Definition 2.1, p. 29), which is defined as follows:

DEFINITION 2.3. *Given an interval $I \subseteq \mathbf{R}$, a partition of $I$ is a finite set $P := \{s_0, \dots, s_n\} \subset I$, where $s_0 < \cdots < s_n$. The pointwise variation of a function $u : I \longrightarrow \mathbf{R}$ on the interval $I$ is defined by* $\mathrm{Var}(u; I) := \sup \sum_{i=1}^{n} |u(s_i) - u(s_{i-1})|$, *where the supremum is taken over all partitions $P := \{s_0, ..., s_n\}$ of $I$, where $n \in \mathbf{N}$. A function $u : I \longrightarrow \mathbf{R}$ has finite or bounded pointwise variation on $I$ if it holds that* $\mathrm{Var}(u; I) < +\infty$. *When no confusion is possible, we write* $\mathrm{Var}(u)$ *instead of* $\mathrm{Var}(u; I)$. *The set of all functions $u : I \longrightarrow \mathbf{R}$ of bounded pointwise variation is denoted by* $\mathrm{BPV}(I)$. *A function $u : I \longrightarrow \mathbf{R}$ has locally finite or locally bounded pointwise variation if it holds that* $\mathrm{Var}(u; [a, b]) < +\infty$ *for all compact intervals $[a, b] \subseteq I$. The set of all functions $u : I \longrightarrow \mathbf{R}$ of locally bounded pointwise variation is denoted by* $\mathrm{BPV}_{loc}(I)$.

REMARK 2.4. If $I$ is a singleton, then it admits no partitions. In such a case, we set $\mathrm{Var}(u; I) := 0$. If $\inf I \in I$ and/or $\sup I \in I$, then, in the definition of $\mathrm{Var}(u; I)$, it suffices to consider partitions $P := \{s_0, \dots, s_n\}$ such that $s_0 = \inf I$ and/or $s_n = \sup I$ (cf. Remark 2.3 in [21]). If $I$ is a compact interval, then it holds that $\mathrm{BPV}_{loc}(I) = \mathrm{BPV}(I)$.

Connection between Definition 2.1 and Definition 2.3 is explained in Appendix A (cf. Theorem 6.20). Given an interval $I \subseteq \mathbf{R}$, by $\mathrm{AC}(I)$ we denote the set of all absolutely continuous functions on $u : I \longrightarrow \mathbf{R}$ (cf. Definition 6.12). We adopt the standard notation for the Sobolev space $\mathrm{H}^1(0, 1)$. We define

$$\mathrm{H}^1(0, 1) := \{\theta \in \mathrm{L}^2(0, 1) : D\theta \in \mathrm{L}^2(0, 1)\} ,$$

where $\mathrm{L}^2(0, 1) := \{[u] : u \in \mathcal{L}^2(0, 1)\}$, and where $\mathcal{L}^2(0, 1)$ denotes the set of all measurable functions $u : (0, 1) \longrightarrow \mathbf{R}$ such that $\int_0^1 |u|^2 < +\infty$. We recall that every $\theta \in \mathrm{H}^1(0, 1)$ admits an absolutely continuous representative (cf. Theorem in 7.16 in [21]). The reader is cautioned that, without further mention, we consistently regard each element of $\mathrm{H}^1(0, 1)$ as its absolutely continuous representative.

The presentation is quite technical, and we aimed to keep technical details to a minimum. The reader is encouraged to read Section 2 first, followed by the appendices, before diving into technicalities of the main part of the paper. Most of the technical work references the results of [6] and [21], but we have attempted to provide a self-sufficient presentation of the topic as far as possible. In this paper, we use notation compatible with that in [6] and [21]. We mention that, in the recent literature, there is occasionally a discrepancy in the notation used (for example, in [2], $\theta \in \mathrm{L}^1([a, b])$ is said to belong to BPV($[a,b]$) if it allows a representative which is a BPV($[a,b]$)-function (cf. p. 165 therein)). The statement of the Banach-Zaretsky theorem (cf. Theorem 6.16) already illustrates the fact that the choice of the suitable representative affects validity of the results considered, whereby the set BPV($[a,b]$) in general can not be replaced by BV($[a,b]$). The importance of difference between sets BV($I$) and BPV($I$) can also be inferred, for example, from the statements of Corollary 5.3 and Corollary 5.6 (whose proof uses the set BPV($I$)) on the one hand, and on the other, from the statements of Corollary 4.5 and Corollary 5.10 (which pertain to the set BV($I$)).

Our analysis in the forthcoming sections is somewhat related to the classical counterexamples in analysis and measure theory. Cantor's function $c$ (cf. Example 1.31 in [21]) is a well-known example of a function with the following properties: $c \in \mathrm{C}([0,1])$, $c$ is increasing on $[0,1]$, $c \in \mathrm{BPV}([0,1])$, $c'(s) = 0$ (a.e. $s \in [0,1]$) (therefore $c'$ is Lebesgue integrable on $[0,1]$), but the fundamental theorem of calculus (cf. Theorem 6.14) does not hold for $c$ (cf. [5] for a detailed presentation of properties of Cantor's function). As a consequence of the Banach-Zaretsky theorem, Cantor's function $c$ does not satisfy the Luzin (N) property (cf. Definition 3.34 and Example 3.35 in [21]), that is, $c$ fails to map sets of measure zero onto sets of measure zero. Weil's function is even more pathological, and, in our context, it can be viewed as a refinement of Cantor's function (cf. Remark 6.15). A more comprehensive list of similar counterexamples in analysis and measure theory can be found in [11] and [16].

While the importance of the Luzin (N) property (cf. Definition 6.1) has been well-established (cf. Theorem 6.16), the Banach (S) property (cf. Definition 7.1) is less explored in the literature, yet it plays an important role in the considerations in this paper. Herein we also introduce an intermediate property, dubbed (F) property (cf. Definition 7.5), which, in a manner of speaking, lies between the Luzin (N) property and the Banach (S) property (cf. Proposition 7.6).

To simplify the statements of our main results, we introduce the following terminology (cf. [26]).

Definition 2.5. *Consider a non-empty measurable set $\Omega \subseteq \mathbf{R}$. We say that $(K_{\eta_j})$ increases to $\Omega$ with respect to the measure $\lambda$ as $j \longrightarrow +\infty$, and we write $K_{\eta_j} \nearrow \Omega$ as $j \longrightarrow +\infty$, if there exists a decreasing sequence of positive real numbers $(\eta_j)$ such that $\eta_j \longrightarrow 0$ as $j \longrightarrow +\infty$, and a sequence of measurable sets $(K_{\eta_j})$ such that $K_{\eta_j} \subseteq \Omega$, and such that there exists $j_0 \in \mathbf{N}$ such that for every $j \geq j_0$ we have $\lambda(\Omega \backslash K_{\eta_j}) \leq \eta_j$.*

Remark 2.6. We note that in the definition above we do not require that $(\eta_j)$ is strictly decreasing, nor do we require that $K_{\eta_j} \subseteq K_{\eta_{j+1}}$. For simplicity, in the rest of the paper we omit explicitly stating that the aforementioned property of sets $(K_{\eta_j})$ is meant with respect to the measure $\lambda$. When no confusion is possible, we write $K_\eta \nearrow \Omega$ as $\eta \searrow 0$ instead of $K_{\eta_j} \nearrow \Omega$ as $\eta_j \searrow 0$.

At this point we recall the definition of an LPO sequence, which was introduced in [26] (cf. Definition 2.2, (i) therein).

DEFINITION 2.7. *Consider a measurable set* $\Omega \subseteq \mathbf{R}$. *We say that a sequence of functions* $u_n : \Omega \longrightarrow \mathbf{R}$ *is a lower pre-oscillatory sequence (an LPO sequence, for short) on* $\Omega$ *if* $(u_n)$ *is a sequence of measurable functions, and if there exists a sequence of measurable sets* $(K_\eta)$ *such that* $K_\eta \nearrow \Omega$ *as* $\eta \searrow 0$, *and such that there exists* $0 < \overline{\eta} < \lambda(\Omega)$ *with the following property: for every* $0 < \eta \leq \overline{\eta}$ *we have* $\liminf_{\xi \longrightarrow +\infty} \liminf_{n \longrightarrow +\infty} \operatorname{card} |u_n \llcorner K_\eta|^{\leftarrow}(\{\xi\}) = 0$.

The motivation for the introduction of the notion of such an LPO property comes from the asymptotic analysis of certain classes of finite-energy sequences of one-dimensional Cahn-Hilliard functional (cf. [1], [20], [25], [26]). We recall that Cahn-Hilliard functional in its simplest form is defined by

$$J_0^\varepsilon(u) := \int_0^1 \Big(\varepsilon u'^2(s) + \varepsilon^{-1} W(u(s))\Big) ds \,, \tag{2.1}$$

where a small parameter $\varepsilon$ tends to zero (meaning that small parameter $\varepsilon$ is defined only for countably many values $\varepsilon = \varepsilon_n$ for a sequence $(\varepsilon_n)$ such that $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow +\infty$), $u \in \mathrm{H}^1(0,1)$, $W$ is a non-negative continuous function with the suitable behavior at infinity such that $W(\zeta) = 0$ if and only if $\zeta \in \{-1, 1\}$ holds true (in short, $W$ is the two-well potential with symmetrically placed wells). We say that a sequence $(u_\varepsilon)$ in $\mathrm{H}^1(0,1)$ is a finite-energy sequence (or an FE sequence) for $(J_0^\varepsilon)$ if it holds that $\limsup_{\varepsilon \longrightarrow 0} J_0^\varepsilon(z_\varepsilon) < +\infty$. The aforementioned Cahn-Hilliard functionals are primary examples of integral functionals with singularly perturbed non-convex integrands, whose minimizers exhibit rather intrinsic behavior, involving oscillation and/or concentration effects, as small parameter $\varepsilon$ tends to zero. Further examples of similar integral functionals are studied in [24] and [28].

We also recall that we say that an FE sequence $(u_\varepsilon)$ for (2.1) is an uniformly normal FE sequence (or an UN FE sequence) for (2.1) on $(0,1)$ if there exists $\varepsilon_0 > 0$ and a measurable set $G \subseteq (0,1)$ such that $\lambda(G) > 0$ with the property: for every measurable set $A \subseteq G$ such that $\lambda(A) > 0$ it holds that $\sup_{0<\varepsilon\leq\varepsilon_0} \inf_A |u_\varepsilon| < +\infty$ (cf. Definition 2.1 in [26]). Thus, the uniform normality is a kind of generalized boundary condition imposed on $(u_\varepsilon)$. In Proposition 4.2 in [26] it was shown that every UN FE sequence for (2.1) is an LPO sequence on $(0,1)$ as $\varepsilon \longrightarrow 0$. While the conclusion of Proposition 4.2 in [26] is correct, it is not of interest because of the following observations.

PROPOSITION 2.8. *Consider a family* $\mathcal{F} := \{A_\alpha : \alpha \in I\}$ *of disjoint measurable sets in* $\mathbf{R}$, *where* $I$ *is a set of indices. If for every* $\alpha \in I$ *we have* $\lambda(A_\alpha) > 0$, *then* $\mathcal{F}$ *is at most countable.*

PROOF. Since the Lebesgue measure is $\sigma$-finite, there exists a sequence of measurable sets $(X_n)$ in $\mathbf{R}$ such that $\mathbf{R} = \cup_{n=1}^{+\infty} X_n$, where for every $n \in \mathbf{N}$ it holds that $\lambda(X_n) < +\infty$. We set $\mathcal{F}_{n,k} := \{A \in \mathcal{F} : \lambda(A \cap X_n) \geq \frac{1}{k}\}$, where $k \in \mathbf{N}$. Then for every $A \in \mathcal{F}$ there exist $k \in \mathbf{N}$ and $n \in \mathbf{N}$ such that $A \in \mathcal{F}_{n,k}$, which gives $\cup_{n=1}^{+\infty} \cup_{k=1}^{+\infty} \mathcal{F}_{n,k} = \mathcal{F}$. To prove that $\mathcal{F}$ is a countable family, it suffices to show that $\mathcal{F}_{n,k}$ is a finite family for every $k \in \mathbf{N}$ and every $n \in \mathbf{N}$. To this end, we argue as follows. If finitely many sets $A_{\alpha_1}, \ldots, A_{\alpha_m}$ belong to $\mathcal{F}_{n,k}$, we get $\frac{m}{k} \leq \sum_{i=1}^{m} \lambda(A_{\alpha_i} \cap X_n) = \lambda((\cup_{i=1}^{m} A_{\alpha_i}) \cap X_n) \leq \lambda(X_n)$, getting $m \leq k\lambda(X_n)$. Thus the cardinality of $\mathcal{F}_{n,k}$ does not exceed $k\lambda(X_n)$, which completes the proof. □

REMARK 2.9. If an uncountable family $\mathcal{C}$ of measurable subsets of $\mathbf{R}$ with strictly positive Lebesgue measure is given, then there exists an uncountable sub-collection $\mathcal{D}$ of $\mathcal{C}$ such that each pair of sets belonging to $\mathcal{D}$ has an intersection with strictly positive Legesgue measure. This observation follows from results proved in [13]. On the other hand, such a conclusion is not true in the case of non-measurable sets. More precisely, there exists a continuum of pairwise disjoint subsets of the interval $[0,1]$ such that each of these subsets has the outer Lebesgue measure equal to 1 (cf. [22]).

THEOREM 2.10. *Consider a measurable set $\Omega \subseteq \mathbf{R}$ such that $\lambda(\Omega) > 0$, and an arbitrary sequence of measurable functions $u_n : \Omega \longrightarrow \mathbf{R}$. Then $(u_n)$ is an LPO sequence on $\Omega$.*

PROOF. By Proposition 2.8 for every fixed $n \in \mathbf{N}$ the set of values $y \in \mathbf{R}$ such that $|u_n|^{\leftarrow}(\{y\})$ has positive measure is at most countable. Hence for all $k \in \mathbf{N}$ there exists $y_{k,n} \in [k, k+1)$ such that $\lambda(|u_n|^{\leftarrow}(\{y_{k,n}\})) = 0$. We define $\Omega_0 := \Omega \setminus \bigcup_{n \in \mathbf{N}} \bigcup_{k \in \mathbf{N}} |u_n|^{\leftarrow}(\{y_{k,n}\})$. Then it holds that $\lambda(\Omega \setminus \Omega_0) = 0$ and the sets $|u_n|(\Omega_0)$ and $\{y_{k,n}, k \in \mathbf{N}, n \in \mathbf{N}\}$ are mutually disjoint. We observe that there exists a subsequence $y_{k_m,m}$ such that $\lim_{m \longrightarrow +\infty} y_{k_m,m} = +\infty$ and such that $\lim_{m \longrightarrow +\infty} k_m = +\infty$, and such that sets $\cup_{n=1}^{+\infty} |u_n(\Omega_0)|$ and $\{\xi_m : m \in \mathbf{N}\}$, where $\xi_m := y_{k_m,m}$, are mutually disjoint. Thus, for every $m \in \mathbf{N}$ it holds that $\xi_m \notin \cup_{n=1}^{+\infty} |u_n(\Omega_0)|$, and we get $\liminf_{n \longrightarrow +\infty} \operatorname{card} |u_n \llcorner \Omega_0|^{\leftarrow}(\{\xi_m\}) = 0$, which completes the proof. □

Therefore, it is necessary to replace the notion of an LPO sequence of functions by a more suitable notion, which can capture the underlying asymptotic behavior of a given sequence of measurable functions $(u_n)$ at infinity. In this paper we introduce the notion of an approximately LPO sequence (an app LPO sequence, for short) of functions (cf. Definition 2.16), which is a modification of the LPO property, and we propose it as a tool for studying the aforementioned asymptotic behavior at infinity. To begin with, we recall the notions of one-sided right approximate limits of a given function (which in general can be non-measurable) $g : (a, +\infty) \longrightarrow \mathbf{R}$ at point $s_0 \in (a, +\infty)$ (cf. Section 2.9.12 in [8]).

Definition 2.11. *Let* $g : (a, +\infty) \longrightarrow \mathbf{R}$ *be a given function, and let* $s_0 \in (a, +\infty)$ *be a given point.*

(i) *We write* $L_0 = \text{app liminf}_{s \longrightarrow s_0+} g(s)$ *if it holds that*

$$L_0 := \sup\{L \in \{-\infty\} \cup \mathbf{R} : \lim_{r \longrightarrow 0+} \frac{\lambda^*(\{g \geq L\} \cap (s_0, s_0 + r))}{r} = 1\} . \tag{2.2}$$

*where* $\lambda^*$ *is the outer Lebesgue measure on* $\mathbf{R}$.

(ii) *We write* $\overline{L}_0 = \text{app limsup}_{s \longrightarrow s_0+} g(s)$ *if it holds that*

$$\overline{L}_0 := \inf\{L \in \mathbf{R} \cup \{+\infty\} : \lim_{r \longrightarrow 0+} \frac{\lambda^*(\{g \leq L\} \cap (s_0, s_0 + r))}{r} = 1\} . \tag{2.3}$$

(iii) *If* $L_0 = \overline{L}_0$*, we write* $L_0 = \text{app lim}_{s \longrightarrow s_0+} g(s)$ *and we say that* $L_0$ *is one-sided right approximate limit of* $g$ *at* $s_0$.

Definition 2.12. *Let* $g : (a, +\infty) \longrightarrow \mathbf{R}$ *be a given function, and let* $s_0 \in (a, +\infty)$ *be a given point.*

(i) *One-sided left approximate limit inferior (limit superior, resp) of* $g$ *at* $s_0$ *is defined by replacing* $(s_0, s_0 + r)$ $(s_0+$*, resp.) by* $(s_0 - r, s_0)(s_0-$*, resp.) in (2.2) ((2.3), resp.).*

(ii) *A proper (or two-sided) approximate limit inferior (limit superior, resp.) of* $g$ *at* $s_0$ *exists if both corresponding one-sided approximate limit inferiors (limit superiors, resp.) exist and have the same value, which is denoted by* $\text{app liminf}_{s \longrightarrow s_0} g(s)$ $(\text{app limsup}_{s \longrightarrow s_0} g(s)$*, resp.).*

(iii) *Finally, if two-sided approximate limit inferior and two-sided approximate limit superior of* $g$ *at* $s_0$ *are equal, the joint value is denoted by* $\text{app lim}_{s \longrightarrow s_0} g(s)$ *and it is said to be the approximate limit of* $g$ *at* $s_0$.

Remark 2.13. In the case when $g$ is a measurable function, in the definitions above the outer Lebesgue measure $\lambda^*$ is replaced by the Lebesgue measure $\lambda$ (cf. subsection 1.7.2 in [5]). However, since this paper considers approximate limits of functions that are not necessarily measurable, we adopt the definitions involving the outer Lebesgue measure $\lambda^*$.

The notion of an approximate limit and its variants are used to define the approximately continuity and the approximate differentiability, and were first utilized by A. Denjoy and A. Ya. Khinchin to study of the properties of the Lebesgue integral and the Denjoy-Khinchin integral (cf. Theorem 14.11 in [14]). If the pointwise limit of $g$ exists at $s_0$, it coincides with the corresponding two-sided approximate limit of $g$ at $s_0$, but the converse is not true (cf. Remark 2.15). If $\Omega \subseteq \mathbf{R}$ is a measurable set and if $g : \Omega \longrightarrow \mathbf{R}$ is a measurable function, then it holds that $\text{app lim}_{z \longrightarrow s} g(z) = g(s)$ (a.e. $s \in \Omega$), in which case we say that $g$ is an approximately continuous function at almost every point of $\Omega$. The converse is also true: the almost everywhere approximate continuity of a function is in fact a characterization of measurability (this result is known as the Stepanov-Denjoy theorem, cf. Theorem 2.9.13 in [8]).

While the notion of approximate limit is a classical topic in textbooks in measure theory (cf. [4], [6], [8], [23], [27] [29]), it is not particularly often used within the context of the study of the behavior of a function (or a sequence of functions) at infinity. We were not able to find such examples in the available literature. In the next step, we define the notions of an approximate limit inferior at $+\infty$, an approximate limit superior at $+\infty$ and an approximate limit at $+\infty$ of a given function (which is not necessarily measurable).

Definition 2.14. *If a function $g : (a, +\infty) \longrightarrow \mathbf{R}$ is given, where $a > 0$, we define*

$$\text{app limsup}_{s\longrightarrow+\infty} g(s) := \text{app limsup}_{\sigma\longrightarrow 0+} \tilde{g}(\sigma) \ , \tag{2.4}$$

$$\text{app liminf}_{s\longrightarrow+\infty} g(s) := \text{app liminf}_{\sigma\longrightarrow 0+} \tilde{g}(\sigma) \ , \tag{2.5}$$

*where $\tilde{g}(\sigma) := g(\frac{1}{\sigma})$, $\sigma \in (0, \frac{1}{a})$. If (2.4) and (2.4) have the joint value $L_0$, we write $L_0 = \text{app lim}_{s\longrightarrow+\infty} g(s)$, and we say that $g$ has the approximate limit at $+\infty$ which is equal to $L_0$.*

Remark 2.15. If $g$ and $h$ are two functions defined on $(a, +\infty)$ such that $g = h$ almost everywhere, we immediately observe their respective approximate limits at $+\infty$ defined by (2.4) and (2.5) coincide. If the pointwise limit of $g$ at $+\infty$ exists, it coincides with the corresponding approximate limit of $g$ at $+\infty$. The converse is not true. For example, if we set $g(s) = \chi_{\mathbf{R}\setminus\mathbf{Q}}(s)$, where $s \in \mathbf{R}$, then for every $s_0 \in \mathbf{R}$ we have $\text{app lim}_{s\longrightarrow s_0} g(s) = 1$, and $\text{app lim}_{s\longrightarrow+\infty} g(s) = 1$, while $g$ has no pointwise limit at any point $s_0 \in \mathbf{R}$, nor does $g$ have pointwise limit at infinity. We conclude that the approximate limit at $+\infty$, as defined in Definition 2.14, is a proper notion for studying the asymptotic behavior of a given function $g : (a, +\infty) \longrightarrow \mathbf{R}$ at $+\infty$. In quite the same way we can define the corresponding approximate limits of a function $g : (-\infty, -a) \longrightarrow \mathbf{R}$ at $-\infty$, where $a > 0$.

Definition 2.16. *Consider a measurable set $\Omega \subseteq \mathbf{R}$. We say that a sequence of functions $u_n : \Omega \longrightarrow \mathbf{R}$ is an approximately lower pre-oscillatory sequence (an app LPO sequence, for short) on $\Omega$ if $(u_n)$ is a sequence of measurable functions, and if there exists a sequence of measurable sets $(K_\eta)$ such that $K_\eta \nearrow \Omega$ as $\eta \searrow 0$, and such that there exists $0 < \overline{\eta} < \lambda(\Omega)$ with the following property: for every $0 < \eta \leq \overline{\eta}$ we have*

$$\text{app liminf}_{\xi\longrightarrow+\infty} \liminf_{n\longrightarrow+\infty} \text{card}\, |u_n \llcorner K_\eta|^{\leftarrow}(\{\xi\}) = 0 \ . \tag{2.6}$$

*If we want to specify the choice of the sequence $(K_\eta)$, we say that $(u_n)$ is an app LPO sequence via the choice of the sequence of subsets $(K_\eta)$.*

In accordance with the latter definition, we introduce the following class of functions:

DEFINITION 2.17. *Consider a measurable set $\Omega \subseteq \mathbf{R}$. We say that a function $u : \Omega \longrightarrow \mathbf{R}$ is an approximately lower pre-oscillatory function (an app LPO function, for short) on $\Omega$ if $u$ is a measurable function, and if there exists a sequence of measurable sets $(K_\eta)$ such that $K_\eta \nearrow \Omega$ as $\eta \searrow 0$, and such that there exists $0 < \overline{\eta} < \lambda(\Omega)$ with the following property: for every $0 < \eta \leq \overline{\eta}$ we have*

$$\text{app}\ \liminf_{\xi \longrightarrow +\infty} \text{card}\, |u \llcorner K_\eta|^{\leftarrow}(\{\xi\}) = 0\ . \tag{2.7}$$

*If we want to specify the choice of the sequence $(K_\eta)$, we say that $u$ is an app LPO function via the choice of the sequence of subsets $(K_\eta)$.*

We observe that, if the sequence $(u_n)$ is an app LPO sequence on $\Omega$, and if there exists a measurable function $u : \Omega \longrightarrow \mathbf{R}$ such that for every $n \in \mathbf{N}$ we have $u_n = u$, it follows that $u$ is an app LPO function on $\Omega$.

REMARK 2.18. We make the following initial remarks.

(i) By the aforementioned definition of an app LPO sequence of functions $(u_n)$, it is tacitly understood that $(u_n)$ is a sequence of measurable functions. This fact is extensively used in the statements of our results in Section 4.

(ii) In the statements of our results in Section 3 and Section 4, we always assume that $\lambda(\Omega) > 0$. This is because, according to Definition 2.7, if a measurable set $\Omega \subseteq \mathbf{R}$ satisfies $\lambda(\Omega) = 0$, and if $u_n : \Omega \longrightarrow \mathbf{R}$ is an arbitrary sequence of measurable functions, then $(u_n)$ is an app LPO sequence on $\Omega$.

(iii) If $\omega \subseteq \Omega$ is an arbitrary measurable set such that $\lambda(\omega) > 0$, and if $(u_n)$ is an app LPO sequence on $\Omega$, then it holds that $(u_n \llcorner \omega)$ ($(u_n \chi_\omega)$, resp.) is an app LPO sequence on $\omega$ (on $\Omega$, resp.). On the other hand, if $u_n : \omega \longrightarrow \mathbf{R}$ is an app LPO sequence on $\omega$, then the extension $\overline{u}_n : \Omega \longrightarrow \mathbf{R}$ defined by $\overline{u}_n(s) := u_n(s)$, if $s \in \omega$, and $\overline{u}_n(s) := 0$, if $s \in \Omega \backslash \omega$, is an app LPO sequence on $\Omega$.

(iv) Consider a non-empty measurable set $\Omega \subseteq \mathbf{R}$ such that $\lambda(\Omega) > 0$ and two measurable functions $u : \Omega \longrightarrow \mathbf{R}$ and $v : \Omega \longrightarrow \mathbf{R}$ such that $u(s) = v(s)$ (a.e. $s \in \Omega$). Then there exists a measurable set $E \subseteq \Omega$ such that $\lambda(E) = 0$, and such that for every $\xi \in \mathbf{R}$ we have $\text{card}\, |u \llcorner \Omega \backslash E|^{\leftarrow}(\{\xi\}) = \text{card}\, |v \llcorner \Omega \backslash E|^{\leftarrow}(\{\xi\})$. As a consequence, $u$ is an app LPO function on $\Omega$ iff $v$ is an app LPO function on $\Omega$.

If $u : \Omega \longrightarrow \mathbf{R}$ is a given function, the mapping $\xi \mapsto \text{card}\, u^{\leftarrow}(\{\xi\})$ is known as the Banach indicatrix of $u$ (also known as the multiplicity function of $u$) on $\Omega$ (cf. [21], subchapter 2.7). For the research involving various properties of the Banach indicatrix function of classes of measurable functions and related questions, see, for example, [19] and references therein. In this paper we do not analyze measurability of the Banach indicatrix. This issue is addressed in Theorem 2.60 in [21]. From the geometric viewpoint, condition (2.6) is a kind of asymptotic flatness condition at infinity imposed on the sequence of functions $(u_n)$. In Section 2 in [25] a few similar properties based on the Banach indicatrix are introduced and analyzed (cf. Definition 2 therein), and functional-analytic aspects are discussed. In short, we can view app LPO sequences $(u_n)$ as a class of sequences which allow some kind of leakage to infinity of as $n \longrightarrow +\infty$, resulting possibly in concentration at infinity, or in rapid oscillations, but which do not totally charge infinity. An elementary example of an app LPO sequence of functions $(u_n)$ is a sequence of functions which satisfies $\limsup_{n\longrightarrow+\infty} \|u_n\|_{\mathrm{L}^\infty(\Omega)} < +\infty$ (cf. Lemma 2.19). Therefore, the app LPO property (2.6) can be understood as an extension of the notion of boundedness of a sequence in $\mathrm{L}^\infty$-norm (cf. Remark 2.21).

Lemma 2.19. *Consider a non-empty measurable set $\Omega \subseteq \mathbf{R}$ such that $\lambda(\Omega) > 0$ and a sequence of measurable functions $u_n : \Omega \longrightarrow \mathbf{R}$. If $(u_n)$ satisfies $\limsup_{n\longrightarrow+\infty} \|u_n\|_{\mathrm{L}^\infty(\Omega)} < +\infty$, then $(u_n)$ is an app LPO sequence on $\Omega$.*

Proof. In the first step we show that, if $u \in \mathrm{L}^\infty(\Omega)$, then there exists a measurable set $\Omega_0 \subseteq \Omega$ such that $\lambda(\Omega\backslash\Omega_0) = 0$ and such that

$$\text{app liminf}_{\xi\longrightarrow+\infty}\text{card}\,|u\llcorner\Omega_0|^{\leftarrow}(\{\xi\}) = 0\,. \tag{2.8}$$

We note that there exists a measurable set $E \subseteq \Omega$ such that $\lambda(E) = 0$ and such that $\|u\|_{\mathrm{L}^\infty(\Omega)} = \|u\|_{\infty,\Omega\backslash E}$. We set $\Omega_0 := \Omega\backslash E$. Then $\Omega_0 \subseteq \Omega$ is a measurable set which satisfies $\lambda(\Omega\backslash\Omega_0) = 0$. Moreover, if we set $u_0 := u\llcorner\Omega_0$ and $L_0 := \text{app liminf}_{\xi\longrightarrow+\infty}\text{card}\,|u_0|^{\leftarrow}(\{\xi\})$, it holds that

$$L_0 = \sup\{L \in \{-\infty\}\cup\mathbf{R} : \lim_{r\longrightarrow0+} r^{-1}\lambda^*(\{\tilde{C}_0 \geq L\}\cap(0,r)) = 1\}\,,$$

where $\tilde{C}_0(z) := C_0(\frac{1}{z})$ and $C_0(\xi) := \text{card}\,|u_0|^{\leftarrow}(\{\xi\})$. We immediately observe that we have $L_0 \geq 0$. On the other hand, if we choose $L > 0$, for every $R > \|u\|_{\mathrm{L}^\infty(\Omega)}$ it results that $\lambda^*\{\xi > R : C_0(\xi) \geq L\} = 0$. In turn, we get $\lim_{R\longrightarrow+\infty} R\lambda^*\{\xi > R : C_0(\xi) \geq L\} = 0$, which gives

$$\lim_{r\longrightarrow0+} r^{-1}\lambda^*\{z \in (0,r) : \tilde{C}_0(z) \geq L\} = 0\,. \tag{2.9}$$

Hence, by (2.9) we conclude that $L_0 = 0$, getting (2.8). In the second step, we consider a sequence of measurable functions $u_n : \Omega \longrightarrow \mathbf{R}$ such that there exist $n_0 \in \mathbf{N}$ and $C \geq 0$ such that $\sup_{n\geq n_0}\|u_n\|_{\mathrm{L}^\infty(\Omega)} \leq C$. By the first step, for every $n \geq n_0$ there exists a measurable set $E_n \subseteq \Omega$ such that $\lambda(E_n) = 0$ and such that for every $L > 0$ we have

$$\lim_{r\longrightarrow0+} r^{-1}\lambda^*\{z \in (0,r) : \tilde{C}_n(z) \geq L\} = 0\,,$$

where $\tilde{C}_n(z) := C_n(\frac{1}{z})$ and $C_n(\xi) := \text{card}\,|u_n\llcorner\Omega\backslash E_n|^{\leftarrow}(\{\xi\})$. We set $E_0 := \cup_{n=n_0}^{+\infty}$ and $\Omega_0 := \Omega\backslash E_0$, whereby we obtain $\lambda(\Omega\backslash\Omega_0) = 0$ and

$$\lim_{r\longrightarrow0+} r^{-1}\lambda^*\{z \in (0,r) : \tilde{C}_{n,0}(z) \geq L\} = 0\,, \tag{2.10}$$

where $n \geq n_0$, $\tilde{C}_{n,0}(z) := C_{n,0}(\frac{1}{z})$ and $C_{n,0}(\xi) := \operatorname{card} |u_n \llcorner \Omega_0|^{\leftarrow}(\{\xi\})$. We set $A_{L,n} := \{z \in (0,r) : \tilde{C}_{n,0}(z) \geq L\}$, where $L > 0$ and $n \geq n_0$. Since there exists $n_1 \geq n_0$ such that for every $n \geq n_1$ it holds that $\liminf_{n \longrightarrow +\infty} A_{L,n} \subseteq A_{L,n}$, we get $\lambda^*(\liminf_{n \longrightarrow +\infty} A_{L,n}) \leq \lambda^*(A_{L,n})$. Finally, by (2.10), the inclusion $\{z \in (0,r) : \liminf_{n \longrightarrow +\infty} \tilde{C}_{n,0}(z) \geq L\} \subseteq \liminf_{n \longrightarrow +\infty} A_{L,n}$ yields the claim. □

Corollary 2.20. *Consider a non-empty measurable set $\Omega \subseteq \mathbf{R}$ such that $\lambda(\Omega) > 0$ and a sequence of measurable functions $u_n : \Omega \longrightarrow \mathbf{R}$ with the following property: there exists $\eta_0 > 0$ such that for every $0 < \eta \leq \eta_0$ we have $\limsup_{n \longrightarrow +\infty} \|u_n\|_{\mathrm{L}^\infty(K_\eta)} < +\infty$, where a sequence of measurable sets $(K_\eta)$ satisfies $K_\eta \nearrow \Omega$ as $\eta \searrow 0$. Then $(u_n)$ is an app LPO sequence on $\Omega$.*

Proof. The assertion follows immediately from the definition of an app LPO sequence and from Lemma 2.19. □

Remark 2.21. We present several examples to illustrate key concepts. In particular, we demonstrate that the set of app LPO sequences strictly contains the set of sequences of functions bounded in $\mathrm{L}^\infty$. Additionally, we show that there exist smooth functions that are not app LPO functions. The details of the calculations in the examples below are left to the interested reader.

(i) We recall that a sequence $g_n : \mathbf{R} \longrightarrow \mathbf{R}$ defined by $g_n(s) := n\chi_{(0,\frac{1}{n})}(s)$ is an example of a sequence of functions which, due to the concentration at the point $s = 0$, converges to zero (a.e. $s \in \mathbf{R}$) as $n \longrightarrow +\infty$, but neither strong nor weak convergence in $\mathrm{L}^1(\mathbf{R})$ occurs as $n \longrightarrow +\infty$ (cf. [9], Chapter 6, Exercise 6.9). The sequence satisfies $\|g_n\|_{\mathrm{L}^\infty(\mathbf{R})} \geq n$. If we set $K_\eta := (-\infty, 0) \cup (\eta, +\infty)$, it results that $K_\eta \nearrow \mathbf{R}$ as $\eta \searrow 0$ and $\|g_n\|_{\mathrm{L}^\infty(K_\eta)} = 0$, where $0 < \eta < 1$ and $n \geq \frac{1}{\eta}$. Therefore, by Corollary 2.20, $(g_n)$ is an obvious example of a sequence which is not bounded in $\mathrm{L}^\infty(\mathbf{R})$ and which is an app LPO sequence on $\mathbf{R}$.

(ii) An elementary example of a sequence of measurable functions $u_n : \Omega \longrightarrow \mathbf{R}$ which is an app LPO sequence on $\Omega$, but which is not bounded in $\mathrm{L}^\infty(K_\eta)$ (where $(K_\eta)$ is an arbitrary choice of a sequence of measurable sets such that $K_\eta \nearrow \Omega$ as $\eta \searrow 0$) is as follows. We set $\Omega := [0, +\infty)$, and, for $n \in \mathbf{N}$, we define $u_n(s) := \max\{s, n\}$, where $s \in \Omega$. Then for every $n \in \mathbf{N}$ it results $\|u_n\|_{\mathrm{L}^\infty(K_\eta)} \geq n$, which proves that a sequence $(u_n)$ is not bounded in $\mathrm{L}^\infty(K_\eta)$. Furthermore, for every $\xi \geq 1$ we have $\operatorname{card} |u_n|^{\leftarrow}(\{\xi\}) = (+\infty)\chi_{\{n\}}(\xi) + \chi_{(n,+\infty)}(\xi)$, which shows that a sequence $(u_n)$ is an app LPO sequence on $\Omega$.

(iii) On the other hand, the function $u : \mathbf{R} \longrightarrow \mathbf{R}$ defined by $u(s) := s(1 + \sin(s^3))$ is an example of a smooth function such that for every $\xi \in \mathbf{R}$ we have $\operatorname{card} u^{\leftarrow}(\{\xi\}) = +\infty$. Moreover, if we consider an arbitrary sequence of measurable sets $(K_\eta)$ such that $K_\eta \nearrow \mathbf{R}$ as $\eta \searrow 0$, and arbitrary real numbers $L_0 > 0$ and $R_0 > 0$, then for every $L \geq L_0$ and every $R \geq R_0$ we get $\lambda^*\{\xi > R : \text{card}\,|u\llcorner K_\eta|^{\leftarrow}(\{\xi\}) \geq L\} = +\infty$. This observation, quite in a similar way as in the proof of Lemma 2.19, eventually gives app $\liminf_{\xi \longrightarrow +\infty} \text{card}\,|u\llcorner K_\eta|^{\leftarrow}(\{\xi\}) = +\infty$. Thus, $u$ is not an app LPO function on $\mathbf{R}$. In particular, it follows that the sequence $v_n : \mathbf{R} \longrightarrow \mathbf{R}$ defined by $v_n(s) := \frac{s}{n}(1+\sin(s^3))$ is an example of a sequence of functions which is not an app LPO sequence on $\mathbf{R}$, yet its pointwise limit on $\mathbf{R}$ as $n \longrightarrow +\infty$ is an app LPO function on $\mathbf{R}$.

(iv) Conversely, if we consider a sequence of measurable functions $w_n : \mathbf{R} \longrightarrow \mathbf{R}$ such that $w_n(s) \longrightarrow u(s)$ (a.e. $s \in \mathbf{R}$) as $n \longrightarrow +\infty$ and such that $w_n(\mathbf{R})$ is a countable set for every $n \in \mathbf{N}$, it follows that $(w_n)$ is an example of an app LPO sequence on $\mathbf{R}$ such that its limit is not an app LPO function on $\mathbf{R}$.

(v) Finally, we note that function $z_n : \mathbf{R} \longrightarrow \mathbf{R}$ defined by $z_n(s) := \max\{\min\{u(s), n\}, -n\}$, where $n \in \mathbf{N}$, is an app LPO function on $\mathbf{R}$ for every $n \in \mathbf{N}$, the sequence $(z_n)$ is not an app LPO sequence on $\mathbf{R}$ as $n \longrightarrow +\infty$, and the limit of $(z_n)$ as $n \longrightarrow +\infty$ is not an app LPO function on $\mathbf{R}$.

Next, we introduce an elementary sufficient condition for a measurable function $C$ to satisfy $L_0 = 0$ in (2.4) (in which case we say that $C$ decays to zero at $+\infty$ from below in a measure-theoretical sense). The following result can also be viewed as a measure-theoretic analogue of the so-called Barbălat's Lemma (cf. [7]).

PROPOSITION 2.22. *Consider strictly positive measurable functions* $C_0 : [\rho_0, +\infty) \longrightarrow \mathbf{R}$ *and* $Z : [\rho_0, +\infty) \longrightarrow \mathbf{R}$ *which satisfy* $\int_{\rho_0}^{+\infty} Z(\xi)C_0(\xi)d\xi \leq M_0 < +\infty$ *and* $\int_{\rho_0}^{+\infty} Z^\theta(\xi)d\xi = +\infty$, *where* $\rho_0 > 0$ *and* $0 < \theta < 1$. *Then it holds that* app $\liminf_{\xi \longrightarrow +\infty} C_0(\xi) = 0$.

PROOF. We set $L_0 :=$ app $\liminf_{\xi \longrightarrow +\infty} C_0(\xi)$, and we assume that $L_0 > 0$. Then there exists $t_0 > 0$ such that for every $t \in [t_0, L_0)$ it holds that $\lim_{r \longrightarrow 0+} r^{-1}\lambda\{z \in (0, r) : \tilde{C}_0(z) \geq t\} = 1$, where $\tilde{C}_0(z) := C_0(\frac{1}{z})$, and we get

$$\lim_{R \longrightarrow +\infty} R\lambda(A_{t,R}) = 1 \,, \tag{2.11}$$

where $A_{t,R} := \{\zeta > R : C_0(\zeta) \geq t\}$. From integrability of $Z \cdot C_0$, we deduce that for every $t > 0$ we have $\lim_{R \longrightarrow +\infty} \int_{A_{t,R}} Z(\xi)C_0(\xi)d\xi = 0$. At this point we consider a strictly decreasing sequence $(t_k)$ of strictly positive real numbers such that $t_k \searrow 0$ as $k \longrightarrow +\infty$. We observe that we have $(R, +\infty) = \cup_{k=1}^{+\infty} A_{t_k,R}$. Since for every $R > 0$ we have $\int_R^{+\infty} Z^\theta(\xi)d\xi = +\infty$, we infer that for every $M > 0$ there exists $R_0 = R_0(M) > 0$ such that for every $R \geq R_0$ we have $\int_R^{+\infty} Z^\theta(\xi)d\xi \geq M$. On the other hand, by the monotone convergence theorem we get $\int_{A_{t_k,R}} Z^\theta(\xi)d\xi \nearrow \int_{\cup_{k=1}^{+\infty} A_{t_k,R}} Z^\theta(\xi)d\xi$ as $k \longrightarrow +\infty$, whereby it results $\lim_{k\longrightarrow+\infty} \int_{A_{t_k,R}} Z^\theta(\xi)d\xi \geq M$. Hence, for every $\Delta > 0$ there exists $k_0 = k_0(\Delta) \in \mathbf{N}$ such that for every $k \geq k_0$ it holds that $\int_{A_{t_k,R}} Z^\theta(\xi)d\xi \geq M + \Delta$. Next, we consider $\theta' < 0$ such that $\frac{1}{\theta} + \frac{1}{\theta'} = 1$, and we apply the reverse Hölder inequality, getting

$$\begin{aligned}\int_{A_{t_k,R}} Z(\xi)C_0(\xi)d\xi &\geq \Big(\int_{A_{t_k,R}} Z^\theta(\xi)d\xi\Big)^{\frac{1}{\theta}} \Big(\int_{A_{t_k,R}} C_0^{\theta'}(\xi)d\xi\Big)^{\frac{1}{\theta'}} \\ &\geq (M+\Delta)^{\frac{1}{\theta}} \Big(\int_{A_{t_k,R}} C_0^{\theta'}(\xi)d\xi\Big)^{\frac{1}{\theta'}},\end{aligned}$$

where $R \geq R_0$ and $k \geq k_0$. Since $M > 0$ was arbitrary, as we pass to the limit as $R \longrightarrow +\infty$, we derive $\lim_{R\longrightarrow+\infty} \int_{A_{t_k,R}} C_0^{\theta'}(\xi)d\xi = +\infty$. In effect, from $\theta' < 0$ and $t_k > 0$ it results that $\lim_{R\longrightarrow+\infty} t_k^{\theta'} \int_{A_{t_k,R}} d\xi = +\infty$, and $\lim_{R\longrightarrow+\infty} \lambda(A_{t_k,R}) = +\infty$, which contradicts (2.11). ∎

In the final result of this section, we illustrate an application of the notion of an app LPO sequence. Specifically, the following theorem represents an improved version of of Proposition 4.2 in [26]. In view of Lemma 2.19, it can also be interpreted as an extension of Theorem 1.3 in [20], generalizing the case $q = \frac{1}{2}$ to the case $q > \frac{1}{2}$ (see below). In particular, it provides a sufficient condition for the existence of app LPO FE sequences. By Lemma 3.2, choosing absolutely continuous representatives in $\mathrm{H}^1(0,1)$ entails no loss of generality in the statement of Theorem 2.23.

Theorem 2.23. *Consider a two-well potential $W$ such that for every $0 < r \leq \frac{1}{2}$ we have $\int_0^{+\infty} V^r(\xi)d\xi < +\infty$, and such that there exists $\frac{1}{2} < q^- < +\infty$ such that for every $q \in (q^-, +\infty)$ we have $\int_0^{+\infty} V^q(\xi)d\xi = +\infty$, where $V : [0,+\infty) \longrightarrow [0,+\infty)$ is defined by $V(\xi) := \min\{W(\zeta) : |\zeta| = \xi\}$. Then every UN FE sequence $(u_\varepsilon)$ for $(J_0^\varepsilon)$ is an app LPO sequence on $(0,1)$ as $\varepsilon \longrightarrow 0$.*

Proof. By Corollary 2.20 WLG we can assume that for every sequence of measurable sets such that $K_\eta \nearrow [0,1]$ as $\eta \searrow 0$ there exists $\eta_0 > 0$ such that for every $0 < \eta \leq \eta_0$ we have $\lim_{\varepsilon\longrightarrow0} \|u_\varepsilon\|_{\mathrm{L}^\infty(K_\eta)} = +\infty$. We choose a sequence $\overline{u}_\varepsilon \in \mathrm{C}^1([0,1])$ as in Proposition 5.2 in [25]. Hence, $\xi \mapsto \operatorname{card} |\overline{u}_\varepsilon \llcorner A|^{\leftarrow}(\{\xi\})$ is a measurable function for an arbitrary measurable set $A \subseteq [0,1]$. (cf. [12], Theorem 6.81, p. 385 , or [6], Theorem 3.9, p. 122). We observe that it holds that $(u_\varepsilon)$ is an UN FE sequence iff $(\overline{u}_\varepsilon)$ is an UN FE sequence. Also, for every sequence of measurable sets such that $K_\eta \nearrow [0,1]$ as $\eta \searrow 0$ there exists $\eta_0 > 0$ such that for every $0 < \eta \leq \eta_0$ we have $\lim_{\varepsilon\longrightarrow0} \|\overline{u}_\varepsilon\|_{\mathrm{L}^\infty(K_\eta)} = +\infty$. As in the proof of Proposition 4.2 in [26], for an arbitrary $1 < p < +\infty$ and for every $0 < \eta \leq \eta_0 < 1$ there exists

$\rho_0(\eta) > 0$ such that

$$+\infty > M_1 \geq \int_{\rho_0(\eta)}^{+\infty} \sqrt{V^p(\xi)} \cdot \overline{C}_\eta(\xi) d\xi \ , \tag{2.12}$$

where $\overline{C}_\eta(\xi) := \liminf_{\varepsilon \longrightarrow 0} \operatorname{card} |\overline{u}_\varepsilon \llcorner \tilde{K}_\eta|^{\leftarrow}(\{\xi\})$, and where a sequence of suitably chosen measurable sets $(\tilde{K}_\eta)$ satisfies $\tilde{K}_\eta \nearrow [0,1]$ as $\eta \searrow 0$. We set $p^- := 2q^-$, we choose $p_0 > p^-$ and we define $\overline{C}_{0,\eta}(\xi) := \overline{C}_\eta(\xi)$, if $\overline{C}_\eta(\xi) > 0$ ($\overline{C}_{0,\eta}(\xi) := (\sqrt{V(\xi)})^{1-p_0}$, if $\overline{C}_\eta(\xi) = 0$, resp.), where $\xi > \rho_0(\eta)$. Since it holds that $\int_{\rho_0(\eta)}^{+\infty} \sqrt{V(\xi)} d\xi < +\infty$, we obtain

$$+\infty > M_0 \geq \int_{\rho_0(\eta)}^{+\infty} \sqrt{V^{p_0}(\xi)} \cdot \overline{C}_{0,\eta}(\xi) d\xi \ . \tag{2.13}$$

We set $Z(\xi) := \sqrt{V^{p_0}(\xi)}$, getting $\int_{\rho_0(\eta)}^{+\infty} Z^\theta(\xi) d\xi = +\infty$, where $0 < \frac{p^-}{p_0} < \theta < 1$. By Proposition 2.22 it follows that app $\operatorname{liminf}_{\xi \longrightarrow +\infty} \overline{C}_{0,\eta}(\xi) = 0$, and from $\overline{C}_{0,\eta}(\xi) \geq \overline{C}_\eta(\xi)$ we get app $\operatorname{liminf}_{\xi \longrightarrow +\infty} \overline{C}_\eta(\xi) = 0$. Hence, it follows that $(\overline{u}_\varepsilon)$ is an app LPO sequence on $(0,1)$ as $\varepsilon \longrightarrow 0$. To proceed, we choose an arbitrary subsequence $(u_{\varepsilon_n})$ of the sequence $(u_\varepsilon)$ such that $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow +\infty$, and, as we apply Proposition 3.5 to the sequences $(u_{\varepsilon_n})$ and $(\overline{u}_{\varepsilon_n})$, we conclude that $(u_{\varepsilon_n})$ is an app LPO sequence on $(0,1)$. Finally, the assertion follows by an application of Lemma 3.3, (i). □

## 3. Basic asymptotic properties of app LPO sequences: the case of the general domain $\Omega$

In this section we present some basic properties of app LPO sequences, proofs of which do not go beyond elementary set-theoretic arguments. In the following, we consider a general non-empty measurable set $\Omega \subseteq \mathbf{R}$ such that $\lambda(\Omega) > 0$, including the case $\lambda(\Omega) = +\infty$.

Lemma 3.1. *Consider a non-empty measurable set $\Omega \subseteq \mathbf{R}$ such that $\lambda(\Omega) > 0$ and a sequence of measurable functions $u_n : \Omega \longrightarrow \mathbf{R}$. Then the following conclusions hold.*

(i) *$(u_n)$ is an app LPO sequence on $\Omega$ iff $(u_n \llcorner \Omega \backslash E)$ is an app LPO sequence on $\Omega \backslash E$ for every measurable set $E \subset \Omega$ such that $\lambda(E) = 0$.*

(ii) *If $(u_n)$ is an app LPO sequence on $\Omega$ via the choice of the sequence of subsets $(K_{\eta_l})$ such that $K_{\eta_l} \nearrow \Omega$ as $l \longrightarrow +\infty$, then $(u_n)$ is an app LPO sequence via the choice of an arbitrary subsequence of subsets $(K_{\eta_l})$.*

(iii) *If $(u_n)$ is an app LPO sequence on $\Omega$ via the choice of the sequence of subsets $(K_\eta^{(1)})$ and via the choice of the sequence of subsets $(K_\eta^{(2)})$, then $(u_n)$ is an app LPO sequence via the choice of the sequence of subsets $(K_\eta)$, where $K_\eta := K_\eta^{(1)} \cap K_\eta^{(2)}$.*

(iv) *If there exists a sequence of subsets* $(K_{\eta_l})$ *such that* $K_{\eta_l} \nearrow \Omega$ *as* $l \longrightarrow +\infty$ *with the property that for every subsequence* $(K_{\eta_{l_m}})$ *it holds that* $(u_n)$ *is an app LPO sequence on* $\Omega$ *via the choice of the sequence of subsets* $(K_{\eta_{l_m}})$*, then it follows that* $(u_n)$ *is an app LPO sequence on* $\Omega$ *via the choice of the sequence of subsets* $(K_{\eta_l})$*.*

(v) *If* $(u_n)$ *is an app LPO sequence via the choice of the sequence of subsets* $(K_\eta)$*, then there exists a subsequence* $(K_{\eta_m})$ *and an increasing sequence of sets* $\tilde{K}_{\eta_m}$ *such that* $\tilde{K}_{\eta_m} \subseteq K_{\eta_m}$ *and such that* $(u_n)$ *is an app LPO sequence via the choice of the sequence of subsets* $(\tilde{K}_{\eta_m})$*.*

Proof. Assertions (i) and (ii) follow immediately from the definition of app LPO sequence. Assertion (iii) follows since we have $\lambda(\Omega \backslash K_\eta) \leq \lambda(\Omega \backslash K_\eta^{(1)}) + \lambda(\Omega \backslash K_\eta^{(2)})$, while assertion (iv) can be easily deduced by assuming the opposite. On the other hand, by the assertion (ii), we can choose a subsequence $(K_{\eta_m})$ of the sequence $(K_\eta)$ such that $\sum_{m=1}^{+\infty} \lambda(\Omega \backslash K_{\eta_m}) < +\infty$ and such that there exists $m_0 \in \mathbf{N}$ such that for every $m \geq m_0$ we have

$$\text{app liminf}_{\xi \longrightarrow +\infty} \liminf_{n \longrightarrow +\infty} \text{card}\, |u_n \llcorner K_{\eta_m}|^{\leftarrow}(\{\xi\}) = 0 \; .$$

Furthermore, by iteration of (iii), we infer that an increasing sequence of measurable sets $\tilde{K}_{\eta_m} := \cap_{i=m}^{\infty} K_{\eta_i}$ has the following properties: $\tilde{K}_{\eta_m} \subseteq K_{\eta_m}$, $\tilde{K}_{\eta_m} \nearrow \Omega$ as $m \longrightarrow +\infty$ and

$$\text{app liminf}_{\xi \longrightarrow +\infty} \liminf_{n \longrightarrow +\infty} \text{card}\, |u_n \llcorner \tilde{K}_{\eta_m}|^{\leftarrow}(\{\xi\}) = 0 \; ,$$

which completes the proof of (v). □

Lemma 3.2. *Consider a non-empty measurable set* $\Omega \subseteq \mathbf{R}$ *such that* $\lambda(\Omega) > 0$*, and two sequences of measurable functions* $u_n : \Omega \longrightarrow \mathbf{R}$ *and* $v_n : \Omega \longrightarrow \mathbf{R}$*. If there exists* $n_0 \in \mathbf{N}$ *such that for every* $n \geq n_0$ *it holds that* $u_n(s) = v_n(s)$ *(a.e.* $s \in \Omega$*), then* $(u_n)$ *is an app LPO sequence on* $\Omega$ *iff* $(v_n)$ *is an app LPO sequence on* $\Omega$*. In particular, if there exists a measurable function* $u : \Omega \longrightarrow \mathbf{R}$ *such that for every* $n \geq n_0$ *it holds that* $u_n(s) = u(s)$ *(a.e.* $s \in \Omega$*), then* $(u_n)$ *is an app LPO sequence on* $\Omega$ *iff* $u$ *is an app LPO function on* $\Omega$*.*

Proof. By assumption, it follows that for every $n \geq n_0$ there exists a measurable set $E_n$ such that $\lambda(E_n) = 0$ and such that $u_n \llcorner \Omega \backslash E_n = v_n \llcorner \Omega \backslash E_n$. If we set $E := \cup_{n=n_0}^{+\infty} E_n$, we get $\lambda(E) = 0$ and $u_n \llcorner \Omega \backslash E = v_n \llcorner \Omega \backslash E$, where $n \geq n_0$ is arbitrary. Hence, we recover the assertion by Lemma 3.1, (i). □

To proceed, we note that, if $(u_n)$ is an app LPO sequence, in general it does not follow that every subsequence of $(u_n)$ is also an app LPO sequence. This is a consequence of the fact that lim inf as $n \longrightarrow +\infty$ appears in the definition of the app LPO property (2.6). The simple counterexample is as follows. We define $u_{2k-1} := v_k$ ($u_{2k} := z_k$, resp.), where $k \in \mathbf{N}$, and where

$(v_k)$ is an app LPO sequence of functions, while $(z_k)$ is a sequence of functions which is not an app LPO sequence. Then $(u_n)$ is an app LPO sequence, but its subsequence $(u_{n_k})$, where $n_k := 2k$, is not an app LPO sequence.

LEMMA 3.3. *Consider a non-empty measurable set* $\Omega \subseteq \mathbf{R}$ *such that* $\lambda(\Omega) > 0$ *and a sequence of measurable functions* $u_n : \Omega \longrightarrow \mathbf{R}$. *Then the following conclusions hold.*

(i) *If there exists a subsequence* $(u_{n_m})$ *of the sequence* $(u_n)$ *such that* $(u_{n_m})$ *is an app LPO sequence on* $\Omega$*, then* $(u_n)$ *is an app LPO sequence on* $\Omega$.

(ii) $(u_n)$ *is an app LPO sequence on* $\Omega$ *iff there exists a sequence of measurable sets* $(K_\eta)$ *such that* $K_\eta \nearrow \Omega$ *as* $\eta \searrow 0$ *and* $\eta_0 > 0$ *such that for every* $0 < \eta \leq \eta_0$ *it holds that* $(u_n)$ *is an app LPO sequence on* $K_\eta$.

PROOF. The assertion (i) follows from the inequality

$$\liminf_{n\longrightarrow+\infty} \operatorname{card} |u_n \llcorner K_\eta|^{\leftarrow}(\{\xi\}) \leq \liminf_{m\longrightarrow+\infty} \operatorname{card} |u_{n_m} \llcorner K_\eta|^{\leftarrow}(\{\xi\}) .$$

Regarding the assertion (ii), it is clear that, if $(u_n)$ is an app LPO sequence on $\Omega$, then $(u_n)$ is an app LPO sequence on $K_\eta$. To prove the converse, we consider a sequence $(\eta_0^{(k)})$ such that $\eta_0 > \eta_0^{(k)} > 0$ and such that for every $0 < \eta < \eta_0^{(k)}$ there exists a measurable set $(K_\eta^{(k)})$ such that $K_\eta^{(k)} \subseteq K_\eta$, $\lambda(K_\eta \backslash K_\eta^{(k)}) \leq \eta_0^{(k)}$ and

$$\text{app liminf}_{\xi\longrightarrow+\infty} \liminf_{n\longrightarrow+\infty} \operatorname{card} |u_n \llcorner K_\eta^{(k)}|^{\leftarrow}(\{\xi\}) = 0 .$$

In effect, there exists a sequence $(\eta_k)$ of strictly positive real numbers such that $\eta_k \searrow 0$ as $k \longrightarrow +\infty$, and a sequence of measurable sets $(K_{\eta_k}^{(k)})$ in $\Omega$ such that for every $k \in \mathbf{N}$ we have that $\lambda(\Omega \backslash K_{\eta_k}^{(k)}) \leq \lambda(\Omega \backslash K_{\eta_k}) + \lambda(K_{\eta_k} \backslash K_{\eta_k}^{(k)}) \leq 2\eta_k$, whereby it holds that

$$\text{app liminf}_{\xi\longrightarrow+\infty} \liminf_{n\longrightarrow+\infty} \operatorname{card} |u_n \llcorner K_{\eta_k}^{(k)}|^{\leftarrow}(\{\xi\}) = 0 .$$

Finally, we observe that, by construction, we have $K_{\eta_k}^{(k)} \nearrow \Omega$ as $k \longrightarrow +\infty$, which shows that $(u_n)$ is an app LPO sequence on $\Omega$. ∎

REMARK 3.4. If we do not assume measurability of $(u_n)$ in the statement of Lemma 3.3, (ii), then we can only conclude that there exists a measurable set $\Omega_0 \subseteq \Omega$ such that $\lambda(\Omega \backslash \Omega_0)$ and such that $(u_n \llcorner \Omega_0)$ is a sequence of measurable functions, where $\Omega_0 := \cup_{j=1}^{+\infty} K_{\eta_j}$ and $K_{\eta_j} \nearrow \Omega$ as $j \longrightarrow +\infty$, and so it follows that $(u_n)$ is an app LPO sequence on $\Omega_0$, but not necessarily on $\Omega$.

Next, we sharpen Lemma 3.2.

PROPOSITION 3.5. *Consider a non-empty measurable set* $\Omega \subseteq \mathbf{R}$ *such that* $\lambda(\Omega) > 0$*, and two sequences of measurable functions* $u_n : \Omega \longrightarrow \mathbf{R}$ *and*

$v_n : \Omega \longrightarrow \mathbf{R}$ *such that* $\lim_{n\longrightarrow+\infty} \lambda\{s \in \Omega : u_n(s) \neq v_n(s)\} = 0$. *Then* $(u_n)$ *is an app LPO sequence on* $\Omega$ *iff* $(v_n)$ *is an app LPO sequence on* $\Omega$.

Proof. We set $\Omega_n := \{s \in \Omega : u_n(s) = v_n(s)\}$. Then $\varepsilon_n := \lambda(\Omega\backslash\Omega_n)$ satisfies $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow +\infty$, and there exists a subsequence $(\varepsilon_{n_k})$ such that $\sum_{k=1}^{+\infty} \varepsilon_{n_k} < +\infty$. Thus, for every $0 < \delta < 1$ there exists $k_0 = k_0(\delta) \in \mathbf{N}$ such that $\sum_{k=k_0}^{+\infty} \lambda(\Omega\backslash\Omega_{n_k}) \leq \delta$. Next, we set $\omega_\delta := \cap_{k=k_0}^{+\infty} \Omega_{n_k}$, getting $u_{n_k} \llcorner \omega_\delta = v_{n_k} \llcorner \omega_\delta$ for every $k \geq k_0(\delta)$ and $\lambda(\Omega\backslash\omega_\delta) \leq \delta$. Finally, we apply Lemma 3.3, (ii). □

Remark 3.6. We note that, if there exists a subsequence $(u_{n_m})$ ($(v_{n_m})$, resp.) of the sequence $(u_n)$ ($(v_n)$, resp.) such that $\lim_{m\longrightarrow+\infty} \lambda\{s \in \Omega : u_{n_m}(s) \neq v_{n_m}(s)\} = 0$, then it is not true that $(u_n)$ is an app LPO sequence on $\Omega$ iff $(v_n)$ is an app LPO sequence on $\Omega$. In particular, the condition $\liminf_{n\longrightarrow+\infty} \lambda\{s \in \Omega : u_n(s) \neq v_n(s)\} = 0$ does not imply the that the conclusion of Proposition 3.5 is true.

Lemma 3.7. *Consider a non-empty measurable set* $\Omega \subseteq \mathbf{R}$ *such that* $\lambda(\Omega) > 0$ *and a sequence of measurable functions* $u_n : \Omega \longrightarrow \mathbf{R}$. *Suppose that every subsequence of* $(u_n)$ *admits a further subsequence which is an app LPO sequence on* $\Omega$. *Then we have the following:*

(i) $(u_n)$ *is an app LPO sequence on* $\Omega$,
(ii) *every subsequence of* $(u_n)$ *is an app LPO sequence on* $\Omega$.

Proof. By Lemma 3.3, (i), it remains to show the assertion (ii). To this end, we consider an arbitrary subsequence $(u_{n_k})$ of the sequence $(u_n)$, and we extract its subsequence $(u_{n_{k_j}})$ which is an app LPO sequence on $\Omega$. But then Lemma 3.3, (i), once more implies that $(u_{n_k})$ is an app LPO sequence on $\Omega$, which proves the assertion (ii). □

The last theorem of this section refines Lemma 3.7.

Theorem 3.8. *Consider a non-empty measurable set* $\Omega \subseteq \mathbf{R}$ *such that* $\lambda(\Omega) > 0$ *and a sequence of measurable functions* $u_n : \Omega \longrightarrow \mathbf{R}$. *If there exists a sequence of measurable sets* $(K_\eta)$ *such that* $K_\eta \nearrow \Omega$ *as* $\eta \searrow 0$ *and* $\eta_0 > 0$ *such that for every* $0 < \eta \leq \eta_0$ *there exists a subsequence* $(u_{n_m})$ *(which possibly depends on* $\eta$*) such that* $(u_{n_m})$ *is an app LPO sequence on* $K_\eta$, *then it follows that* $(u_n)$ *is an app LPO sequence on* $\Omega$. *If every subsequence of the sequence* $(u_n)$ *satisfies the aforementioned property, then every subsequence of* $(u_n)$ *is an app LPO sequence on* $\Omega$.

Proof. The assertion follows immediately from Lemma 3.3. □

## 4. Composition of functions and app LPO property: the case of the general domain $\Omega$

In this section we deal with the problem of preservation of the app LPO property of a sequence of functions under the outer and the inner composition with a given function. The proofs of results in this section require using the intermediate level well-known results from measure theory (cf. Appendix A and Appendix B).

Proposition 4.1. *Consider a non-empty measurable set $\Omega \subseteq \mathbf{R}$ such that $\lambda(\Omega) > 0$ and a sequence of measurable functions $u_n : \Omega \longrightarrow \mathbf{R}$. Then we have the following:*

(i) *if $(u_n)$ is an app LPO sequence on $\Omega$, then for every strictly increasing bijection $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ it holds that $\varphi \circ |u_n|$ is an app LPO sequence on $\Omega$,*

(ii) *if there exists a strictly increasing bijection $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ such that $(\varphi \circ |u_n|)$ is an app LPO sequence on $\Omega$, then $(u_n)$ is an app LPO sequence on $\Omega$.*

Proof. We note that $\varphi \circ |u_n|$ is a measurable function on $\Omega$, since, by monotonicity of $\varphi$, for every $\alpha \geq 0$ we have

$$\{s \in \Omega : \varphi(|u_n(s)|) \geq \alpha\} = \{s \in \Omega : |u_n(s)| \geq \varphi^{-1}(\alpha)\} \tag{4.14}$$

(cf. Remark B.22 in [21]). By Definition 2.16, there exists a sequence of measurable sets $(K_\eta)$ such that $K_\eta \nearrow \Omega$ as $\eta \searrow 0$ and $\eta_0 > 0$ such that for every $0 < \eta \leq \eta_0$ we have app $\text{liminf}_{\xi \longrightarrow +\infty} C_\eta(\xi) = 0$, where we set $C_\eta(\xi) := \liminf_{n \longrightarrow +\infty} \operatorname{card} |u_n \llcorner K_\eta|^{\leftarrow}(\{\xi\})$, and where $\xi \in [0, +\infty)$. We note that, by the assumption, we have $\varphi(0) = 0$, and $\varphi$ can be extended to a strictly increasing bijection from $\mathbf{R}$ to $\mathbf{R}$ (which is not relabeled) such that $\lim_{x \longrightarrow -\infty} \varphi(x) = -\infty$, whereby we set $\varphi(-\infty) := -\infty$. By Definition 2.14 it holds that $\tilde{C}_\eta(z) := C_\eta(\frac{1}{z})$ satisfies app $\text{liminf}_{z \longrightarrow 0+} \tilde{C}_\eta(z) = 0$, i.e.,

$$\sup\{L \in \{-\infty\} \cup \mathbf{R} : \lim_{r \longrightarrow 0+} \frac{\lambda^*(\{\tilde{C}_\eta \geq L\} \cap (0, r))}{r} = 1\} = 0 \ .$$

Since $\varphi$ is a strictly increasing bijection, we get $\{\tilde{C}_\eta \geq L\} = \{\varphi \circ \tilde{C}_\eta \geq \varphi(L)\}$, $\{L : L \in \{-\infty\} \cup \mathbf{R}\} = \{\varphi(L) : L \in \{-\infty\} \cup \mathbf{R}\}$, and

$$\sup\{\varphi(L) \in \{-\infty\} \cup \mathbf{R} : \lim_{r \longrightarrow 0+} \frac{\lambda^*(\{\varphi \circ \tilde{C}_\eta \geq \varphi(L)\} \cap (0, r))}{r} = 1\} = \varphi(0) \ ,$$

which amounts to app $\text{liminf}_{z \longrightarrow 0+} \varphi(\tilde{C}_\eta(z)) = 0$, getting the assertion (i). The assertion (ii) follows in quite the same way. ∎

Remark 4.2. Regarding the statement (ii) of Proposition 4.1, we note that measurability of $|u_n|$ can be obtained from the identity (4.14), where $\alpha \geq 0$, and from the fact that $\varphi$ is a bijection from $[0, +\infty)$ to $[0, +\infty)$.

In particular, if $(u_n)$ is a sequence of non-negative functions, assumption of measurability of $(u_n)$ in the statement (ii) can be avoided.

While the statement and proof concerning the outer composition of the approximate LPO sequence are quite straightforward, the corresponding result for the inner composition is more involved. We emphasize that the results presented in the remainder of this section, as well as in the next, are derived under rather restrictive assumptions. Moreover, in statement (i) of Proposition 4.3, we do not assume that $\psi$ is a measurable function, for simplicity. This is because the measurability of $\psi$ follows from the assumptions imposed on $\psi^{-1}$. Similar observations extend to the other results pertaining to the inner composition and the app LPO property, namely Proposition 4.4, Corollary 4.5, Corollary 5.3, Corollary 5.7, Corollary 5.8, Corollary 5.9 and Corollary 5.10. We point out that, if $I, J \subseteq \mathbf{R}$ are intervals, then there exist a continuous increasing function $\psi : I \longrightarrow J$ and a measurable function $g : J \longrightarrow \mathbf{R}$ such that $g \circ \psi : I \longrightarrow \mathbf{R}$ is not measurable (compare Exercise 3.77 and Exercise 1.33 in [21]). In the results below, we focus on the analysis of the case where the function $\psi$, which appears as the inner function in the composition, is an injection. In particular, the formulation of Proposition 4.3 (Proposition 4.4, resp.) requires the notion of the Banach (S) property (the Luzin (N) property, resp.) which is introduced in Appendix B (cf. Definition 7.1) (in Appendix A (cf. Definition 6.1), resp.).

PROPOSITION 4.3. *Consider two non-empty measurable sets $\Omega \subseteq \mathbf{R}$, $D \subseteq \mathbf{R}$ such that $\lambda(\Omega) > 0$, $\lambda(D) > 0$, and a sequence of functions $u_n : \Omega \longrightarrow \mathbf{R}$. Then we have the following:*

(i) *if $(u_n)$ is an app LPO sequence on $\Omega$, then for every bijection $\psi : D \longrightarrow \Omega$ such that $\psi^{-1}$ is a measurable function which satisfies the Banach (S) property on $\Omega$ it holds that $(u_n \circ \psi)$ is an app LPO sequence on $D$,*
(ii) *if there exists a bijection $\psi : D \longrightarrow \Omega$ such that $\psi$ is a measurable function which satisfies the Banach (S) property on $D$, and such that $(u_n \circ \psi)$ is an app LPO sequence on $D$, then $(u_n)$ is an app LPO sequence on $\Omega$.*

PROOF. In order to prove (i) ((ii), resp.), in the first step we address measurability of $u_n \circ \psi$ ($(u_n)$, resp.). For every open set $U \subseteq \mathbf{R}$, we have $(u_n \circ \psi)^{\leftarrow}(U) = \psi^{-1}(u_n^{\leftarrow}(U))$ ($u_n^{\leftarrow}(U) = \psi((u_n \circ \psi)^{\leftarrow}(U))$, resp.), where, since $u_n$ ($u_n \circ \psi$, resp.) is a measurable function, it follows that $u_n^{\leftarrow}(U)$ ($(u_n \circ \psi)^{\leftarrow}(U)$, resp.) is a measurable set. Next, by Lemma 7.2 and Proposition 6.4, (ii), we infer that for a measurable set $A := u_n^{\leftarrow}(U)$ ($B := (u_n \circ \psi)^{\leftarrow}(U)$, resp.), it follows that $\psi^{-1}(A)$ ($\psi(B)$, resp.) is also a measurable set. Hence, $u_n \circ \psi$ ($u_n$, resp.) is a measurable function.

In the second step, we argue as follows. If $(u_n)$ is an app LPO sequence on $\Omega$ (if $(u_n \circ \psi)$ is an app LPO sequence on $D$, resp.), there exists $\eta_0 > 0$ such that for every $0 < \eta \leq \eta_0$ there exists a measurable set $K_\eta \subseteq \Omega$ ($Q_\eta \subseteq D$, resp.) such that $\lambda(\Omega \backslash K_\eta) \leq \eta$ ($\lambda(D \backslash Q_\eta) \leq \eta$, resp.), and such that we have

$$\text{app liminf}_{\xi \longrightarrow +\infty} \liminf_{n \longrightarrow +\infty} \text{card} \, |u_n \llcorner K_\eta|^{\leftarrow}(\{\xi\}) = 0$$

$$\Big(\text{app liminf}_{\xi \longrightarrow +\infty} \liminf_{n \longrightarrow +\infty} \text{card} \, |(u_n \circ \psi) \llcorner Q_\eta|^{\leftarrow}(\{\xi\}) = 0 \,, \text{ resp.}\Big) \,.$$

We define $Q_\eta := \psi^{-1}(K_\eta)$ ($K_\eta := \psi(Q_\eta)$, resp.). From $\lim_{\eta \longrightarrow 0} \lambda(\Omega \backslash K_\eta) = 0$ ($\lim_{\eta \longrightarrow 0} \lambda(D \backslash Q_\eta)$, resp.) we get

$$\lim_{\eta \longrightarrow 0} \lambda(\psi^{-1}(\Omega \backslash K_\eta)) = 0 \qquad \Big(\lim_{\eta \longrightarrow 0} \lambda(\psi(D \backslash Q_\eta)) = 0 \,, \text{ resp.}\Big).$$

Since it holds that $\psi^{-1}(\Omega \backslash K_\eta) = D \backslash Q_\eta$ ($\psi(D \backslash Q_\eta) = \Omega \backslash K_\eta$, resp.), this means that $K_\eta \nearrow \Omega$ ($Q_\eta \nearrow D$, resp.) as $\eta \searrow 0$ implies $Q_\eta \nearrow D$ ($K_\eta \nearrow \Omega$, resp.) as $\eta \searrow 0$. On the other hand, from definition of $K_\eta$ and $Q_\eta$, for every $\xi \in [0, +\infty)$ we deduce $|u_n \llcorner K_\eta|^{\leftarrow}(\{\xi\}) = |(u_n \circ \psi) \llcorner Q_\eta|^{\leftarrow}(\{\xi\})$, getting the assertion (i) ((ii), resp.). ∎

To state further results, we introduce the following abbreviations for certain properties of a bijection $\psi : D \longrightarrow \Omega$:

(N.1) $\psi$ satisfies the Luzin (N) property on $D$,
(N.2) $\psi^{-1}$ satisfies the Luzin (N) property on $\Omega$,
(B.1) $\psi \in \text{BV}(D)$,
(B.2) $\psi^{-1} \in \text{BV}(\Omega)$,
(S.1) $\psi$ satisfies the Banach (S) property on $D$,
(S.2) $\psi^{-1}$ satisfies the Banach (S) property on $\Omega$.

Proposition 4.4. *Consider two non-empty measurable sets $\Omega \subseteq \mathbf{R}$, $D \subseteq \mathbf{R}$ such that $\lambda(\Omega) > 0$, $\lambda(D) > 0$, and a sequence of functions $u_n : \Omega \longrightarrow \mathbf{R}$.*

(i) *If $(u_n)$ is an app LPO sequence on $\Omega$, then for every bijection $\psi : D \longrightarrow \Omega$ such that $\psi^{-1}$ is a Lipschitz function on $\Omega$ it holds that $(u_n \circ \psi)$ is an app LPO sequence on $D$.*
(ii) *If there exists a bijection $\psi : D \longrightarrow \Omega$ such that $\psi$ is a Lipschitz function on $D$, and such that $(u_n \circ \psi)$ is an app LPO sequence on $D$, then $(u_n)$ is an app LPO sequence on $\Omega$.*
(iii) *If $\Omega$ is an open set, if $(u_n)$ on $\Omega$ is an app LPO sequence, and if $\psi : D \longrightarrow \Omega$ is an arbitrary bijection which satisfies (N.1), (N.2) and (B.2), then the following assertion holds:*
(A) *there exists a sequence of measurable sets $(D_k)$ such that for every $k \in \mathbf{N}$ we have the following properties: $D_k \subseteq D_{k+1} \subseteq D$, $\lambda(D_{k+1} \backslash D_k) > 0$, $(u_n \circ \psi)$ is an app LPO sequence on $D_k$.*
(iv) *If $D$ is an open set, if there exists a bijection $\psi : D \longrightarrow \Omega$ which satisfies (N.1), (N.2) and (B.1), and such that $(u_n \circ \psi)$ is an app LPO sequence on $D$, then the following assertion holds:*

(B) *there exists a sequence of measurable sets* $(\Omega_k)$ *such that for every* $k \in \mathbf{N}$ *we have the following properties:* $\Omega_k \subseteq \Omega_{k+1} \subseteq \Omega$, $\lambda(\Omega_{k+1} \backslash \Omega_k) > 0$, $(u_n)$ *is an app LPO sequence on* $\Omega_k$.

PROOF. To begin with, we note that, by Corollary 6.9, (i), from assumptions (i) ((iii), resp.), it follows that $u_n \circ \psi$ is a measurable function on $D$ ($D_k$, resp.). On the other hand, by Corollary 6.9, (ii), from assumptions (ii) ((iv), resp.), it follows that $u_n$ is a measurable function on $\Omega$ ($\Omega_k := \psi(D_k)$, resp.). Regarding the proof of (i) ((ii), resp.), we observe that it holds that $\lambda(\psi^{-1}(A)) \leq \mathrm{Lip}(\psi^{-1}; \Omega)\lambda(A)$ ($\lambda(\psi(B)) \leq \mathrm{Lip}(\psi; D)\lambda(B)$, resp.), where $A \subseteq \Omega$ ($B \subseteq D$, resp.) is an arbitrary measurable set and where $\mathrm{Lip}(\psi^{-1}; \Omega)$ ($\mathrm{Lip}(\psi; D)$, resp.) is the Lipschitz constant of $\psi^{-1}$ on $\Omega$ ($\psi$ on $D$, resp.). Hence, the assertion (i) ((ii), resp.) follows as in the second step of the proof of Proposition 4.3.

Regarding (iii), by Theorem 6.12 in [6], it follows that for every $\varepsilon > 0$ there exists a Lipschitz function $\Phi_\varepsilon : \Omega \longrightarrow \mathbf{R}$ such that $\lambda\{s \in \Omega : \psi^{-1}(s) \neq \Phi_\varepsilon(s)\} \leq \varepsilon$. We choose a strictly decreasing sequence $(\varepsilon_j)$ such that $\varepsilon_j \searrow 0$ as $j \longrightarrow +\infty$. We set $\Omega_{\varepsilon_j} := \{s \in \Omega : \psi^{-1}(s) \neq \Phi_{\varepsilon_j}(s)\}$ and $\Omega_j := \cup_{i=1}^{j} \Omega_{\varepsilon_i}$. Then for every $j \in \mathbf{N}$ it follows that $\Omega_j \subseteq \Omega_{j+1}$ and $\lambda(\Omega_{j+1} \backslash \Omega_j) = \lambda(\Omega_{\varepsilon_{j+1}}) > 0$. Furthermore, we estimate $\mathrm{Lip}(\psi^{-1}; \Omega_j) \leq \sum_{i=1}^{j} \mathrm{Lip}(\psi^{-1}; \Omega_{\varepsilon_i})$, which shows that $\psi^{-1}$ is Lipschitz function on $\Omega_j$, where $j \in \mathbf{N}$ is arbitrary. Since $\psi^{-1}$ satisfies the Luzin (N) property on $\Omega$, it follows that $D_j := \psi^{-1}(\Omega_j)$ is a measurable set for every $j \in \mathbf{N}$, whereby for every $j \in \mathbf{N}$ we have $D_j \subseteq D_{j+1}$. On the other hand, since $\psi$ has the Luzin (N) property on $D$, it follows that $\lambda(D_j) > 0$ for every $j \in \mathbf{N}$. We note that for every $j \in \mathbf{N}$ and every $k \in \mathbf{N}$ we have $|u_n \llcorner (K_\eta \cap \Omega_j)|^{\leftarrow}(\{\xi\}) = |(u_n \circ \psi) \llcorner (Q_\eta \cap D_j)|^{\leftarrow}(\{\xi\})$, where, for every $j \in \mathbf{N}$ we have $K_\eta \cap \Omega_{\varepsilon_j} \nearrow \Omega_{\varepsilon_j}$ and $Q_\eta \cap D_j \nearrow D_j$ as $\eta \searrow 0$. Since $\psi^{-1} \in \mathrm{Lip}(\Omega_j)$, by (i), it follows that $(u_n \circ \psi)$ is an app LPO sequence on $D_j$ for every $j \in \mathbf{N}$. Finally, we observe that $\psi(D_{j+1} \backslash D_j) = \Omega_{j+1} \backslash \Omega_j$, and, since $\psi$ satisfies the Luzin (N) property on $D$, $\lambda(\Omega_{j+1} \backslash \Omega_j) > 0$ implies $\lambda(D_{j+1} \backslash D_j) > 0$. Hence, we get the assertion (iii). If we define $D_j := \cup_{i=1}^{j} D_{\varepsilon_i}$, $\Omega_j := \psi(D_j)$, where $D_{\varepsilon_j} := \{s \in D : \psi(s) = \Psi_{\varepsilon_j}(s)\}$, and where $\Psi_{\varepsilon_j} : D \longrightarrow \mathbf{R}$ is a suitably chosen Lipschitz continuous function, the assertion (iv) is proved similarly as the assertion (iii). ◻

COROLLARY 4.5. *Consider two non-empty measurable sets* $\Omega \subseteq \mathbf{R}$, $D \subseteq \mathbf{R}$ *such that* $\lambda(\Omega) > 0$, $\lambda(D) > 0$, *and a sequence of functions* $u_n : \Omega \longrightarrow \mathbf{R}$.

(i) *If* $\psi : D \longrightarrow \Omega$ *is a bi-Lipschitz bijection, then we have the following equivalence:* $(u_n)$ *is an app LPO sequence on* $\Omega$ *iff* $(u_n \circ \psi)$ *is an app LPO sequence on* $D$.

(ii) *Suppose that* $\Omega$ *and* $D$ *are open sets, and that* $\psi : D \longrightarrow \Omega$ *is a bijection which satisfies (N.1), (N.2), (B.1) and (B.2). Then the assertions (A) and (B) in Proposition 4.4 are equivalent.*

Proof. The claim (i) ((ii), resp.) follows immediately from claims (i) and (ii) ((iii) and (iv), resp.) of Proposition 4.4. □

## 5. The case of the domain $\Omega$ of finite Lebesgue measure

In this section we address the case when $\lambda(\Omega) < +\infty$, which provides a very different, and much simpler, setting for studying the app LPO property (2.6).

Proposition 5.1. *If $\Omega \subseteq \mathbf{R}$ is a measurable set such that $0 < \lambda(\Omega) < +\infty$, it follows that every measurable function $u : \Omega \longrightarrow \mathbf{R}$ is an app LPO function on $\Omega$.*

Proof. Indeed, by Luzin's Theorem, for every $0 < \eta < \lambda(\Omega)$ there exists a compact set $K_\eta \subseteq \Omega$ such that $u \in \mathrm{C}(K_\eta)$ and $\lambda(\Omega \backslash K_\eta) \leq \eta$. Since any continuous function attains its extremal values on any non-empty compact set, we get $\sup_{s \in K_\eta} |u(s)| = \max_{s \in K_\eta} |u(s)| \leq C_\eta < +\infty$. By Lemma 2.19 it results that $u$ is an app LPO function on $K_\eta$, which, by Lemma 3.3, (ii), shows that $u$ is an app LPO function on $\Omega$. □

Proposition 5.2. *Consider a measurable set $\Omega \subseteq \mathbf{R}$ such that $0 < \lambda(\Omega) < +\infty$. If for given a sequence of measurable functions $u_n : \Omega \longrightarrow \mathbf{R}$ there exists $n_0 \in \mathbf{N}$ such that $\sup_{n \geq n_0} |u_n(s)| < +\infty$ (a.e. $s \in \Omega$), then it follows that $(u_n)$ is an app LPO sequence on $\Omega$. In particular, we have the following:*

(i) *$\sup_{n \geq n_0} |u_n|$ is an app LPO function on $\Omega$,*
(ii) *for every $n_1 \in \mathbf{N}$ $\inf_{n \geq n_1} |u_n|$ is an app LPO function on $\Omega$,*
(iii) *$\limsup_{n \longrightarrow +\infty} |u_n|$ is an app LPO function on $\Omega$,*
(iv) *$\liminf_{n \longrightarrow +\infty} |u_n|$ is an app LPO function on $\Omega$.*

Proof. We note that $\sup_{n \geq n_0} |u_n|$ is a measurable function on $\Omega$, and we apply Proposition 5.1, getting (i). In particular, from the proof of Proposition 5.1 it follows that $(u_n)$ is an app LPO sequence on $\Omega$. Assertions (ii), (iii) and (iv) follow by the same argument. □

Corollary 5.3. *Consider two compact intervals $I, J \subseteq \mathbf{R}$, $I = [a, b]$, $J = [c, d]$, such that $\lambda(I) > 0$, $\lambda(J) > 0$, and a sequence of measurable functions $u_n : I \longrightarrow \mathbf{R}$. Then we have the following:*

(i) *if $(u_n)$ is an app LPO sequence on $I$, then for every a strictly monotonic bijection $\psi : J \longrightarrow I$ such that $\psi^{-1}$ satisfies the Luzin (N) property on $I$ it holds that $(u_n \circ \psi)$ is an app LPO sequence on $J$,*
(ii) *if there exists a strictly monotonic bijection $\psi : J \longrightarrow I$ such that $\psi$ satisfies the Luzin (N) property on $J$, and such that $(u_n \circ \psi)$ is an app LPO sequence on $J$, then $(u_n)$ is an app LPO sequence on $I$.*

Proof. It is enough to show that, under the assumption in (i) ((ii), resp.), it follows that $\psi^{-1}$ ($\psi$, resp.) satisfies the Banach (S) property on $I$ ($J$, resp.). Indeed, we observe that, by Proposition 6.4.5 in [10], $\psi$ ($\psi^{-1}$, resp.) is continuous on $J$ ($I$, resp.). Hence, $\psi^{-1}$ ($\psi$, resp.) is also continuous, strictly monotonic and bounded on $I$ ($J$, resp.). By Proposition 2.11 in [21], it follows that $\psi^{-1} \in \mathrm{BPV}([a,b])$ ($\psi \in \mathrm{BPV}([c,d])$, resp.). Since by the assumption in (i) ((ii), resp.), $\psi^{-1}$ ($\psi$, resp.) satisfies the Luzin (N) property, Banach-Zaretsky Theorem (cf. Theorem 6.16) implies that $\psi^{-1}$ ($\psi$, resp.) is absolutely continuous on $[a,b]$ (on $[c,d]$, resp.). By Theorem 7.13 it results that $\psi^{-1}$ ($\psi$, resp.) satisfies the Banach (S) property on $[a,b]$ (on $[c,d]$, resp.), and Proposition 4.3 applies. □

Proposition 5.4. *Consider a measurable set* $\Omega \subseteq \mathbf{R}$ *such that* $0 < \lambda(\Omega) < +\infty$*, and a sequence of measurable functions* $u_n : \Omega \longrightarrow \mathbf{R}$. *Suppose that the following holds:*

(i) *there exists a sequence of measurable sets* $(\Omega_k)$ *such that* $\Omega_k \subseteq \Omega_{k+1} \subseteq \Omega$ *and such that* $\Omega = \cup_{k=1}^{\infty} \Omega_k \cup N$*, where* $\lambda(N) = 0$*,*

(ii) *for every* $k \in \mathbf{N}$ *it holds that* $(u_n)$ *is an app LPO sequence on* $\Omega_k$.

*Then it follows that* $(u_n)$ *is an app LPO sequence on* $\Omega$.

Proof. From the monotone convergence theorem it follows that $\lambda(\Omega) = \lim_{k\longrightarrow+\infty} \lambda(\Omega_k)$. Then, since we have $\lambda(\Omega\backslash\Omega_k) = \lambda(\Omega) - \lambda(\Omega_k)$, from $\lambda(\Omega) < +\infty$ it results that $\lim_{k\longrightarrow+\infty} \lambda(\Omega\backslash\Omega_k) = 0$, which shows that $\Omega_k \nearrow \Omega$ as $k \longrightarrow +\infty$. Now the assertion follows from Lemma 3.3, (ii). □

Remark 5.5. If we do not assume measurability of $(u_n)$ in the statement of Proposition 5.4, then we can only conclude that there exists a measurable set $\Omega_0 \subseteq \Omega$ such that $\lambda(\Omega\backslash\Omega_0)$ and such that $(u_n \llcorner \Omega_0)$ is a sequence of measurable functions, where $\Omega_0 := \cup_{k=1}^{+\infty} \Omega_k$, and so it follows that $(u_n)$ is an app LPO sequence on $\Omega_0$, but not necessarily on $\Omega$.

Corollary 5.6. *Conclusion (i) ((ii), resp.) of Corollary 5.3 holds in the case of non-empty open intervals* $I = (a,b)$ *and* $J = (c,d)$ *such that* $0 < \lambda(I) < +\infty$ *and* $0 < \lambda(J) < +\infty$.

Proof. We consider a representation of $J$ ($I$, resp.) by a sequence of compact intervals $(J_m)$ ($(I_m)$, resp.) such that for every $m \in \mathbf{N}$ we have $J_m \subseteq J_{m+1} \subseteq J$ ($I_m \subseteq I_{m+1} \subseteq I$, resp.), and such that $J = \cup_{m=1}^{+\infty} J_m$ ($I = \cup_{m=1}^{+\infty} I_m$, resp.). By Proposition 6.4.5 in [10], we deduce that $\psi^{-1}$ ($\psi$, resp.) is continuous, strictly monotonic and bounded on the compact interval $\psi(J_m)$ ($\psi^{-1}(I_m)$, resp.), where $m \in \mathbf{N}$ is arbitrary. Hence, by Corollary 5.3, (i) ((ii), resp.), it follows that $(u_n \circ \psi)$ ($(u_n)$, resp.) is an app LPO sequence on $J_m$ ($I_m$, resp.), while, by Proposition 5.4, we conclude that $(u_n \circ \psi)$ ($(u_n)$, resp.) is an app LPO sequence on $J$ ($I$, resp.). □

Corollary 5.7. *Consider two non-empty measurable sets $\Omega \subseteq \mathbf{R}$, $D \subseteq \mathbf{R}$ such that $< \lambda(\Omega) < +\infty$, $0 < \lambda(D) < +\infty$, and a sequence of functions $u_n : \Omega \longrightarrow \mathbf{R}$.*

(i) *If $\Omega$ is an open set, and if $(u_n)$ is an app LPO sequence on $\Omega$, then for every bijection $\psi : D \longrightarrow \Omega$ such that $\psi$ which satisfies (N.1), (N.2) and (B.2), it follows that $(u_n \circ \psi)$ is an app LPO sequence on $D$.*
(ii) *If $D$ is an open set, and if there exists a bijection $\psi : D \longrightarrow \Omega$ which satisfies (N.1), (N.2) and (B.1), and such that $(u_n \circ \psi)$ is an app LPO sequence on $D$, then it follows that $(u_n)$ is an app LPO sequence on $\Omega$.*

Proof. The claim (i) ((ii), resp.) follows by Proposition 5.4 and by Proposition 4.4 (iii), ((iv), resp.)), since the measurability of $u_n \circ \psi$ ($u_n$, resp.)) can be obtained by Corollary 6.9 as in the proof of Proposition 4.4. Indeed, by using the same notation as in the proof of Proposition 4.4 (iii), ((iv), resp.)), we observe that $\Omega_0 := \cup_{j=1}^{+\infty}\Omega_j$ ($D_0 := \cup_{j=1}^{+\infty}D_j$, resp.) satisfies $\lambda(\Omega\backslash\Omega_0) = 0$ ($\lambda(D\backslash D_0) = 0$, resp.). We set $D_0 := \cup_{j=1}^{+\infty}D_j$ ($\Omega_0 := \cup_{j=1}^{+\infty}\Omega_j$, resp.), whereby $D\backslash D_0 = \psi^{-1}(\Omega\backslash\Omega_0)$ ($\Omega\backslash\Omega_0 = \psi(D\backslash D_0)$, resp.). Since $\psi^{-1} : \Omega \longrightarrow D$ ($\psi : D \longrightarrow \Omega$, resp.) satisfies the Luzin (N) property on $\Omega$ (on $D$, resp.), it results $\lambda(D\backslash D_0) = 0$ ($\lambda(\Omega\backslash\Omega_0) = 0$, resp.). Hence, while by Proposition 4.4 (iii), ((iv), resp.)) it follows that for every $j \in \mathbf{N}$ we have that $(u_n \circ \psi)$ is an app LPO sequence on $D_j$ ($(u_n)$ is an app LPO sequence on $\Omega_j$, resp.), by Proposition 5.4 we conclude that $(u_n \circ \psi)$ is an app LPO sequence on $D_0$ ($(u_n)$ is an app LPO sequence on $\Omega_0$, resp.). In effect, we proved the assertion (i) ((ii), resp.). ◻

Corollary 5.8. *Consider two non-empty open sets $\Omega \subseteq \mathbf{R}$, $D \subseteq \mathbf{R}$ such that $< \lambda(\Omega) < +\infty$, $0 < \lambda(D) < +\infty$, and a sequence of functions $u_n : \Omega \longrightarrow \mathbf{R}$. If $\psi : D \longrightarrow \Omega$ is a bijection which satisfies (N.1), (N.2), (B.1) and (B.2), then we have the following equivalence: $(u_n)$ is an app LPO sequence on $\Omega$ iff $(u_n \circ \psi)$ is an app LPO sequence on $D$.*

Proof. The assertion follows directly from Corollary 6.9, (iii), and from Corollary 5.7. ◻

Corollary 5.9. *Consider two non-empty measurable sets $\Omega \subseteq \mathbf{R}$, $D \subseteq \mathbf{R}$ such that $\lambda(\Omega) > 0$, $\lambda(D) > 0$ and a sequence of functions $u_n : \Omega \longrightarrow \mathbf{R}$.*

(i) *If $\Omega$ is an open set such that $\lambda(\Omega) < +\infty$, and if $\psi : D \longrightarrow \Omega$ is a bijection which satisfies (N.1), (S.2), and (B.2), then we have the following implication: if $(u_n)$ is an app LPO sequence on $\Omega$, then $(u_n \circ \psi)$ is an app LPO sequence on $D$.*
(ii) *If $D$ is an open set such that $\lambda(D) < +\infty$, and if $\psi : D \longrightarrow \Omega$ is a bijection which satisfies (S.1), (N.2), and (B.1), then we have the*

*following implication: if $(u_n \circ \psi)$ is an app LPO sequence on $D$, then $(u_n)$ is an app LPO sequence on $\Omega$.*

PROOF. The assertion (i) ((ii), resp.) follows directly from Proposition 7.6, from Lemma 7.2, and from Corollary 5.7, (i) ((ii), resp.). □

COROLLARY 5.10. *Consider two non-empty open sets $\Omega \subseteq \mathbf{R}$, $D \subseteq \mathbf{R}$ such that $\lambda(\Omega) > 0$, $\lambda(D) > 0$ and a sequence of functions $u_n : \Omega \longrightarrow \mathbf{R}$.*

(i) *If $\lambda(\Omega) < +\infty$, if $\psi : D \longrightarrow \Omega$ is a bijection which satisfies (N.1), (S.2), (B.1) and (B.2), then we have the following equivalence: $(u_n)$ is an app LPO sequence on $\Omega$ iff $(u_n \circ \psi)$ is an app LPO sequence on $D$.*

(ii) *If $\lambda(D) < +\infty$, if $\psi : D \longrightarrow \Omega$ is a bijection which satisfies (S.1), (N.2), (B.1) and (B.2), then we have the following equivalence: $(u_n)$ is an app LPO sequence on $\Omega$ iff $(u_n \circ \psi)$ is an app LPO sequence on $D$.*

PROOF. The assertion (i) ((ii), resp.) follows directly from Proposition 7.6, from Lemma 7.2, and from Corollary 5.8, (i) ((ii), resp.). □

## 6. APPENDIX A

For convenience of the less experienced reader, in Appendix A we collected a number of the classical results in real analysis of functions of one variable and measure theory, which are relevant to understanding the choice of our definition of an app LPO sequence of functions. We refer the reader to the textbook [30] for further analysis of measurable sets, properties of measurable functions, and the related examples and counterexamples involving Cantor-type functions.

DEFINITION 6.1. *Consider a measurable set $\Omega \subseteq \mathbf{R}$. We say that a function $g : \Omega \longrightarrow \mathbf{R}$ satisfies the Luzin (N) property on $\Omega$ if the following holds: for every measurable set $E \subseteq \Omega$ it holds that $\lambda(E) = 0$ implies $\lambda(g(E)) = 0$.*

REMARK 6.2. Regarding Definition 6.1, we make the following remarks.

(i) We note that in Definition 6.1 we do not assume that $g$ is a measurable function,

(ii) By the completeness of the Lebesgue measure $\lambda$, in Definition 6.1, the condition "$\lambda(g(E)) = 0$" can be replaced by the condition "$\lambda^*(g(E)) = 0$" without changing the notion considered, where $\lambda^*$ is the outer Lebesgue measure on $\mathbf{R}$.

DEFINITION 6.3. *Consider a measurable set $\Omega \subseteq \mathbf{R}$. We say that a function $g : \Omega \longrightarrow \mathbf{R}$ preserves measurability of sets in $\Omega$ if for every measurable set $E \subseteq \Omega$ it holds that $g(E)$ is a measurable set.*

Proposition 6.4. *Consider a measurable set* $\Omega \subseteq \mathbf{R}$ *such that* $\lambda(\Omega) > 0$ *and a function* $g : \Omega \longrightarrow \mathbf{R}$*. Then we have the following:*

(i) *if g preserves measurability of sets in* $\Omega$*, then g satisfies the Luzin (N) property on* $\Omega$*,*
(ii) *if g is a measurable function and if g satisfies the Luzin (N) property on* $\Omega$*, then g preserves measurability of sets in* $\Omega$*,*
(iii) *if g is a measurable function, then g preserves measurability of sets in* $\Omega$ *iff g has the Luzin (N) property on* $\Omega$*.*

Proof. If $g$ preserves measurability of sets on $\Omega$, we consider a measurable set $A \subseteq \Omega$ such that $\lambda(A) = 0$, and we show that $\lambda(g(A)) = 0$. To this end, we will show that every subset $D \subseteq g(A)$ is measurable, and the claim follows (here we used well-known fact that every set of positive measure contains a non measurable subset, cf. Corollary 5.22 in [12] or Corollary 3.39 [30]). Since $D \subseteq g(A)$, we get $g^{\leftarrow}(D) \subseteq A$. Hence, by the completeness of $\lambda$, we have $\lambda(g^{\leftarrow}(D)) = 0$. In particular, $g^{\leftarrow}(D)$ is a measurable set. On the other hand, from $D = g(g^{\leftarrow}(D))$ we conclude that $D$ is also a measurable set, which furnishes the proof of (i).

If $g$ is a measurable function which satisfies the Luzin (N) property on $\Omega$, we consider an arbitrary measurable subset $A \subseteq \Omega$, and we show that $g(A)$ is also a measurable set. We observe that we have $A = \cup_{n=1}^{+\infty} A_n$, where $A_n := A \cap (-n, n)$, and where $n \in \mathbf{N}$. Since it holds that $g(A) = \cup_{n=1}^{+\infty} g(A_n)$, it suffices to show that for every $n \in \mathbf{N}$ it holds that $g(A_n)$ is a measurable set. By the Luzin theorem, for every $\varepsilon > 0$ there exists a compact set $F_\varepsilon^n \subseteq A_n$ such that $\lambda(A_n \backslash F_\varepsilon^n) \leq \varepsilon$ and such that $g \in \mathrm{C}(F_\varepsilon^n)$. Then we have $A_n = \cup_{k=1}^{+\infty} F_{\frac{1}{k}}^n \cup M_n$, where $\lambda(M_n) = 0$. By continuity of $g$ on $F_{\frac{1}{k}}^n$ it follows that $g(F_{\frac{1}{k}}^n)$ is a compact set for every $k \in \mathbf{N}$. On the other hand, by the Luzin (N) property of $g$, it follows that $\lambda(g(M_n)) = 0$. Hence, $g(A_n) = \cup_{k=1}^{+\infty} g(F_{\frac{1}{k}}^n) \cup g(M_n)$ is a measurable set, whereby the proof of (ii) is completed. □

Remark 6.5. We note that the proof of the assertion (i) in Proposition 6.4 does not require $g$ to be measurable, while the proof of the assertion (ii) does require measurability of $g$ (since we are using the Luzin theorem). When function $g$ is a bijection, the setting is simpler, as discussed in Proposition 6.7.

Lemma 6.6. *Consider a measurable set* $\Omega \subseteq \mathbf{R}$*, and two measurable functions* $g_1, g_2 : \Omega \longrightarrow \mathbf{R}$*. Assume that the following assumptions are fulfilled:*

(i) $g_1(s) = g_2(s)$ *(a.e.* $s \in \Omega$*),*
(ii) $g_1$ *satisfies the Luzin (N) property on* $\Omega$*,*
(iii) $g_2$ *satisfies the Luzin (N) property on* $\Omega$*.*

*Then for every measurable set* $A \subseteq \Omega$ *we have* $\lambda(g_1(A)) = \lambda(g_2(A))$*.*

Proof. By (i) there exists a measurable set $N \subseteq \Omega$ such that where $\lambda(N)$ and such that for every $s \in \Omega \backslash N$ we have $g_1(s) = g_2(s)$, getting $g_1(A \backslash N) =$

$g_2(A \backslash N)$. On the other hand, by (ii) ((iii), resp.), we get $\lambda(g_1(A \cap N)) = 0$ $(\lambda(g_2(A \cap N)) = 0$, resp.), which proves the assertion. □

To proceed, we address the problem of the measurability of the inverse of a given bijection. We point out that, since the definition of measurability requires the preimage (but not necessarily the image) of a measurable set to be measurable, the aforementioned problem is nontrivial. Here we focus on bijections $\psi$ which are measurable in the sense of Lebesgue. In this setting it is not difficult to formulate rather strong assumptions on $\psi$ that guarantee the measurability of its inverse. For example, if $\psi : D \longrightarrow \Omega$ is a bi-Lipschitz function, where $D \subseteq \mathbf{R}$ and $\Omega \subseteq \mathbf{R}$ are measurable sets, then $\psi^{-1} : \Omega \longrightarrow D$ is obviously a measurable function. We note, however, that even smooth bijections from $\mathbf{R}$ to $\mathbf{R}$ need not have a locally Lipschitz inverse. An elementary counterexample is $\psi(s) := s^3$. We observe that, if $\psi : \mathbf{R} \longrightarrow \mathbf{R}$ is a continuous bijection, then (by an application of the Intermediate Value Theorem) it results that $\psi$ is a strictly monotonic function on $\mathbf{R}$. Hence, $\psi$ is a strictly monotonic continuous function. By Proposition 6.4.5 in [10], it follows that $\psi^{-1} : \mathbf{R} \longrightarrow \mathbf{R}$ is continuous and therefore measurable. In the next proposition we obtain a more general sufficient condition for measurability of the inverse of an injection defined on a general domain in $\mathbf{R}$ (compare also Corollary 7.14).

PROPOSITION 6.7. *Consider a measurable set $D \subseteq \mathbf{R}$ and an injection $\psi : D \longrightarrow \mathbf{R}$. Then we have the following:*

(i) *if $\psi$ preserves measurability of sets in $D$, then $\psi^{-1}$ is a measurable function,*
(ii) *if $\Omega := \psi(D)$ is a measurable set, and if $\psi^{-1} : \Omega \longrightarrow D$ is a measurable function which satisfies the Luzin (N) property on $\Omega$, then $\psi$ is a measurable function.*

PROOF. To prove (i) ((ii), resp.), we consider an open set $U \subseteq \mathbf{R}$ and we note that we have $(\psi^{-1})^{\leftarrow}(U) = \psi(U \cap D)$ $(\psi^{\leftarrow}(U) = \psi^{-1}(U \cap \Omega)$, resp.). Since for every open set $U \subseteq \mathbf{R}$ we have that $U \cap D$ $(U \cap \Omega \subseteq \mathbf{R}$, resp.) is a measurable set, by Proposition 6.4, (i) ((ii), resp.) it follows that $(\psi^{-1})^{\leftarrow}(U)$ $(\psi^{\leftarrow}(U)$, resp.) is a measurable set. □

REMARK 6.8. We point out that, by the Luzin-Suslin theorem (cf. Theorem 15.2 in [17]), if $D \subseteq \mathbf{R}$ is a Borel set, and if $\psi : D \longrightarrow \mathbf{R}$ is an injection, then $\Omega := \psi(D)$ is a Borel set. Such a conclusion is not true for measurable sets $D$ and $\Omega$, and, in the statement (ii) of Proposition 6.7, it is necessary to assume that $\Omega := \psi(D)$ is a measurable set.

COROLLARY 6.9. *Consider two non-empty measurable sets $\Omega \subseteq \mathbf{R}$ and $D \subseteq \mathbf{R}$, a function $g : \Omega \longrightarrow \mathbf{R}$, and a bijection $\psi : D \longrightarrow \Omega$. Then the following conclusions hold.*

(i) *If $g$ is a measurable function on $\Omega$, and if $\psi^{-1}$ is a measurable function which satisfies the Luzin (N) property on $\Omega$, then $g \circ \psi$ is a measurable function on $D$.*
(ii) *If $g \circ \psi$ is a measurable function on $\Omega$, if $\psi$ is a measurable function which satisfies the Luzin (N) property on $D$, then $g$ is a measurable function on $\Omega$.*
(iii) *In particular, if $\psi$ satisfies the Luzin (N) property on $D$, and if $\psi^{-1}$ satisfies the Luzin (N) property on $\Omega$, then we have the following equivalence: $g$ is a measurable function on $\Omega$ iff $g \circ \psi$ is a measurable function on $D$.*

Proof. Regarding (i) ((ii), resp.), we consider an arbitrary open set $U \subseteq \mathbf{R}$, and we observe that we have $(g\circ\psi)^{\leftarrow}(U) = \psi^{-1}(g^{\leftarrow}(U))$ ($g = (g\circ\psi)\circ\psi^{-1}$ and $g^{\leftarrow}(U) = \psi((g \circ \psi)^{\leftarrow}(U))$, resp.). By measurability of $g$ ($g \circ \psi$, resp.), it follows that $g^{\leftarrow}(U)$ ($(g \circ \psi)^{\leftarrow}(U)$, resp.) is a measurable set, while by measurability $\psi^{-1}$ ($\psi$, resp.), by the Luzin (N) property of $\psi^{-1}$ ($\psi$, resp.) and by Proposition 6.4, (ii), it follows that $\psi^{-1}(g^{\leftarrow}(U))$ ($\psi((g \circ \psi)^{\leftarrow}(U))$, resp.) is also a measurable set. ∎

Remark 6.10. It is not necessary to assume in (i) that $\psi$ is a measurable function, since, by Proposition 6.7, this follows from the assumption that $\psi^{-1}$ is a measurable function which satisfies the Luzin (N) property on $\Omega$.

Corollary 6.11. *(Preservation of Luzin (N) property under composition). Consider two non-empty measurable sets $\Omega \subseteq \mathbf{R}$ and $D \subseteq \mathbf{R}$, a measurable function $g : \Omega \longrightarrow \mathbf{R}$, and a bijection $\psi : D \longrightarrow \Omega$. Then we have the following conclusion: If $\psi$ satisfies the Luzin (N) property on $D$, if $\psi^{-1}$ is a measurable function which satisfies the Luzin (N) property on $\Omega$, and if $g$ satisfies the Luzin (N) property on $\Omega$, then $g \circ \psi$ is a measurable function which satisfies the Luzin (N) property on $D$.*

Proof. By Corollary 6.9, (i), it follows that $g \circ \psi$ is a measurable function on $D$. To show that $g \circ \psi$ satisfies the Luzin (N) property on $D$, we consider a measurable set $M \subseteq D$ such that $\lambda(M) = 0$. Then we have $(g \circ \psi)(M) = g(\psi(M))$, and the claim follows since $\psi$ ($g$, resp.) satisfy the Luzin (N) property on $D$ ($\Omega$, resp.). ∎

In the next definition, we recall the notion of an absolutely continuous function (cf. [21], chapter 3).

Definition 6.12. *Let $I \subseteq \mathbf{R}$ be an interval. A function $g : I \longrightarrow \mathbf{R}$ is said to be an absolutely continuous function on $I$, and we write $g \in \mathrm{AC}(I)$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite number of disjoint intervals $(a_j, b_j)$, $j = 1, ..., N$, with $[a_j, b_j] \subseteq I$, it holds that $\sum_{j=1}^{N} |b_j - a_j| \leq \delta$ implies $\sum_{j=1}^{N} |g(b_j) - g(a_j)| \leq \varepsilon$. By $\mathrm{AC}_{loc}(I)$ we denote the set of all functions $g : I \longrightarrow \mathbf{R}$ which satisfy $g \in \mathrm{AC}([a, b])$ for every interval $[a, b] \subseteq I$.*

In order to state and apply the classical characterization of $AC$-functions, we recall the notion of an equi-integrable measurable function.

Definition 6.13. *Given a measurable function $\varphi : E \longrightarrow \mathbf{R}$, where $E \subseteq \mathbf{R}$ is a measurable set, we say that $\varphi$ is equi-integrable on $E$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable set $F \subseteq E$ it holds that $\lambda(F) \leq \delta$ implies $\int_F |\varphi| \leq \varepsilon$.*

In the next two theorems, we provide a brief review of basic properties of absolutely continuous functions. The first theorem is a compact statement of both Theorem 3.29 and Corollary 3.43 in [21], and it is a version of the fundamental theorem of calculus. For a more detailed exposition on this topic, we refer the reader to chapter 3 of [21] (in particular, to Theorem 3.20, Theorem 3.25, Corollary 3.33 and Theorem 3.41), and to Theorem 3.35 in [9].

Theorem 6.14. *Let $I \subseteq \mathbf{R}$ be an interval and let $g : I \longrightarrow \mathbf{R}$ be a given function. Then the following equivalence holds: $g \in \mathrm{AC}(I)$ if and only if*

(i) *$g$ is continuous on $I$,*
(ii) *$g$ is differentiable almost everywhere in $I$, $g' \in \mathcal{L}^1_{loc}(I)$, and $g'$ is equi-integrable on $I$,*
(iii) *for all $s, s_0 \in I$ such that $s_0 \leq s$ we have*

$$g(s) = g(s_0) + \int_{s_0}^{s} g'(\sigma) d\sigma \ .$$

*Moreover, the equivalence holds if (iii) is replaced by*

(iii') *$g$ satisfies the Luzin (N) property on $I$.*

Remark 6.15. By Weil's Theorem (cf. Theorem 2.40 in [21]), there exists a Lipschitz continuous function $u : [0, 1] \longrightarrow \mathbf{R}$, which is everywhere differentiable, monotonic on no interval in $[0, 1]$, and such that $u'$ is bounded on $[0, 1]$, but $u'$ is not Riemann integrable over any interval in $[0, 1]$. Consequently, the statement (iii) of Theorem 6.14 is necessarily meant with respect to the Lebesgue integral over $[s_0, s]$, and not with respect to the Riemann integral (cf. [30], subsection 5.4, for a detailed analysis of the connection between the Lebesgue integral and the Riemann integral). This fact raises additional technicalities in the proof of Proposition 7.12. A thorough comparison between different types of integrals is given in [18].

The second theorem is a re-statement of Theorem 6.3.1 in [15] in terms of our notation (compare also Corollary 3.49 in [21], which is a local version of the theorem below).

Theorem 6.16. *(Banach-Zaretsky) Let $a, b \in \mathbf{R}$ and $a < b$. A function $g : [a, b] \longrightarrow \mathbf{R}$ belongs to $\mathrm{AC}([a, b])$ if and only if the following holds:*

(i) *$g$ is continuous on $[a, b]$,*
(ii) *$g \in \mathrm{BPV}([a, b])$,*

(iii) *g satisfies the Luzin (N) property on* $[a, b]$.

In the sequel, we define the notions necessary for the introduction of functions of bounded variation in some detail (cf. Definition 2.1). Further related terminology and results can be found in subchapters B.6 and B.9 in [21]. Let $X$ be a non-empty set and $\Sigma$ a $\sigma$-algebra of subsets of $X$. We say that a function $\mu : \Sigma \longrightarrow [-\infty, +\infty]$ is a signed measure on $\Sigma$ if it has the following properties: (i) $\mu(\emptyset) = 0$; (ii) $\mu$ takes at most one of the two values $-\infty$ and $+\infty$, that is, either $\mu : \Sigma \longrightarrow [-\infty, +\infty)$ or $\mu : \Sigma \longrightarrow (-\infty, +\infty]$; (iii) for any countable collection $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\Sigma$, we have $\mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$, where the series on the right converges absolutely or conditionally in the extended real line. By the Jordan decomposition theorem (cf. Theorem B.71 in [21]), any signed measure $\mu$ can be uniquely written as the difference of two non-negative measures $\mu^+$ and $\mu^-$, i.e., $\mu = \mu^+ - \mu^-$. The measure $\mu^+$ ($\mu^-$, resp.) is called the upper variation (lower variation, resp.) of $\mu$, while the measure $|\mu| := \mu^+ + \mu^-$ is called the total variation measure of $\mu$. If $|\mu|$ is a finite measure, we say that the signed measure $\mu$ is finite.

Definition 6.17. *Consider a topological space $X$ endowed with the Borel $\sigma$-algebra $\Sigma_{\mathcal{B}}(X)$. A measure $\nu : \Sigma_{\mathcal{B}}(X) \longrightarrow [0, +\infty]$ is a Radon measure if the following conditions are fulfilled:*

(i) *for every compact set $K \subseteq X$ we have $\nu(K) < +\infty$,*

(ii) *for every open set $A \subseteq X$ it holds that*

$$\nu(A) = \sup\{\nu(K) : K \subseteq A,\ K \text{ compact}\},$$

(iii) *for every $A \in \Sigma_{\mathcal{B}}(X)$ it holds that*

$$\nu(A) = \inf\{\nu(U) : A \subseteq U,\ U \text{ open}\}.$$

The latter definition simplifies if $X$ is locally compact Hausdorff space (cf. Proposition B.108 in [21]).

Definition 6.18. *We say that $\mu : \Sigma_{\mathcal{B}}(X) \longrightarrow [-\infty, +\infty]$ is a signed Radon measure if $\mu$ is a signed measure on $\Sigma_{\mathcal{B}}(X)$, and if its total variation measure is a Radon measure.*

By $\mathcal{M}_b(X)$ we denote the space of all finite signed Radon measures $\mu : \Sigma_{\mathcal{B}}(X) \longrightarrow \mathbf{R}$. It can be verified that $\mathcal{M}_b(X)$ is a Banach space with the norm $\|\mu\|_{\mathcal{M}_b(X)} := |\mu|(X)$. Given an open set $\Omega \subseteq \mathbf{R}$, we are able to define the set $\mathrm{BV}(\Omega)$ as the set of all integrable functions $u$ on $\Omega$ such that its distributional derivative $Du$ is a finite signed Radon measure (compare Definition 2.1). We note that $\mathrm{BPV}(\Omega)$ and $\mathrm{BV}(\Omega)$ are quite similar classes of functions, but nevertheless different. If $\Omega \subseteq \mathbf{R}$ is an open set, by Definition 5.1 (Definition 5.2, resp.) in [6], if $f \in \mathrm{BV}(\Omega)$ ($f \in \mathrm{BV}_{loc}(\Omega)$, resp.), then $f$ is measurable, since by definition of $BV(\Omega)$ ($\mathrm{BV}_{loc}(\Omega)$, resp.)

we have $\mathrm{BV}(\Omega) \subseteq \mathcal{L}^1(\Omega)$ ($\mathrm{BV}_{loc}(\Omega) \subseteq \mathcal{L}^1_{loc}(\Omega)$, resp.) (cf. Definition 14.1 in [21]). On the other hand, if $I \subseteq \mathbf{R}$ is an interval, by Definition 2.1 (Remark 2.3, resp.) in [21], the space of functions $\mathrm{BPV}(I)$ ($\mathrm{BPV}_{loc}(I)$, resp.) is defined as the set of all functions $f : I \longrightarrow \mathbf{R}$ which satisfy $\mathrm{Var}(f; I) < +\infty$ ($f \in \mathrm{BPV}([a, b])$ for every compact interval $[a, b] \subseteq I$, resp.). Furthermore, if $\Omega \subseteq \mathbf{R}$ an arbitrary open set, the space of functions $\mathrm{BPV}(\Omega)$ is in [21] defined as follows. If $\mathrm{Var}(f; \Omega) := \sum_{n=1}^{+\infty} \mathrm{Var}(f; I_n) < +\infty$, we write $f \in \mathrm{BPV}(\Omega)$, where $\Omega = \cup_{n=1}^{+\infty} I_n$ is re-written as a countable union of pairwise disjoint open intervals $(I_n)$. We mention that, if $a, b \in \mathbf{R}$ and $a < b$, then $\mathrm{BPV}_{loc}([a, b]) = \mathrm{BPV}([a, b])$. In effect, we have the following:

LEMMA 6.19. *Let $I \subseteq \mathbf{R}$ be an interval. If $f \in \mathrm{BPV}(I)$, then*

(i) *$f$ is a measurable function,*
(ii) *$f$ is bounded, i.e., $\|f\|_{\infty,I} < +\infty$.*

PROOF. If $f \in \mathrm{BPV}(I)$, by the Jordan decomposition theorem (cf. Theorem 5.2.15 in [15]), we have $f(s) = g(s) - h(s)$, where $s \in I$, and where $g : I \longrightarrow \mathbf{R}$ and $h : I \longrightarrow \mathbf{R}$ are increasing functions on $I$. Since $g$ ($h$, resp.) is increasing, we conclude that for every $\alpha \geq 0$ the set $g^{\leftarrow}(\alpha, +\infty)$ ($h^{\leftarrow}(a, +\infty)$, resp.) is either an interval or empty set, and therefore is measurable. Hence, $g$ and $h$ are measurable, and so is $f$. By Proposition 2.12 in [21], we get $\|f\|_{\infty,I} < +\infty$. □

We note that, by Lemma 6.19, if $I \subseteq \mathbf{R}$ is an interval, then we have the embeddings $\mathrm{BPV}(I) \hookrightarrow \mathrm{L}^\infty(I)$ and $\mathrm{BPV}_{loc}(I) \hookrightarrow \mathrm{L}^\infty_{loc}(I)$. If $\Omega \subseteq \mathbf{R}$ is an open set, the difference between the spaces $\mathrm{BPV}(\Omega) \hookrightarrow \mathrm{L}^\infty_{loc}(\Omega)$ and $\mathrm{BV}(\Omega) \hookrightarrow \mathrm{L}^1(\Omega)$ is specifically addressed in Theorem 7.2. in [21] (for consideration of the case $\Omega = \mathbf{R}$ compare also Theorem 3.27, (b), in [9]). Herein we quote such a result.

THEOREM 6.20. *Let $\Omega \subseteq \mathbf{R}$ be an open set. If $u \in \mathcal{L}^1(\Omega)$, and if $u \in \mathrm{BPV}(\Omega)$, then $u \in \mathrm{BV}(\Omega)$ and*

$$|Du|(\Omega) \leq \mathrm{Var}(u; \Omega) .$$

*Conversely, if $u \in \mathrm{BV}(\Omega)$, then $u$ admits a right continuous representative $\overline{u}$ such that $\overline{u} \in \mathrm{BPV}(\Omega)$ and such that*

$$\mathrm{Var}(\overline{u}; \Omega) = |Du|(\Omega) .$$

REMARK 6.21. If $\Omega \subseteq \mathbf{R}$ is an open set, and if $u \in \mathcal{L}^1_{loc}(\Omega)$, we set

$$V(u; \Omega) := \sup\{\int_\Omega u\phi' : \phi \in \mathrm{C}^\infty_c(\Omega),\ \|\phi\|_{\infty,\Omega} \leq 1\} .$$

By the Riesz representation theorem in $\mathrm{C}_0(\Omega)$ (cf. Theorem B.111 in [21]), we have the following (cf. Exercise 14.3. in [21]):

(i) if $Du \in \mathcal{M}_b(\Omega)$, then $|Du|(\Omega) = V(u; \Omega)$,

(ii) if $V(u;\Omega) < +\infty$, then $Du \in \mathcal{M}_b(\Omega)$.

In particular, if $u \in \mathcal{L}^1(\Omega)$, then $u \in \mathrm{BV}(\Omega)$ iff $V(u;\Omega) < +\infty$.

## 7. Appendix B

For easier reading, we assembled in Appendix B a number of results related to the Banach (S) property and absolute continuity. We keep the arguments in the proofs as simple as possible. Textbook [21] is a recommended source for further information about absolutely continuous functions.

We believe that most of the results in Appendix B are probably known, but not easily found in the literature.

Definition 7.1. *Consider a measurable set $\Omega \subseteq \mathbf{R}$. We say that a function $g : \Omega \longrightarrow \mathbf{R}$ satisfies the Banach (S) property on $\Omega$ if the following holds: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable set $E \subseteq \Omega$ it holds that $\lambda(E) \leq \delta$ implies $\lambda^*(g(E)) \leq \varepsilon$.*

Lemma 7.2. *Consider a measurable set $\Omega \subseteq \mathbf{R}$ and a function $g : \Omega \longrightarrow \mathbf{R}$. Then we have the following:*

(i) *if $g$ satisfies the Banach (S) property on $\Omega$, then it also satisfies the Luzin (N) property on $\Omega$,*

(ii) *if $g$ is a measurable function which satisfies the Banach (S) property on $\Omega$, then $g$ preserves measurability of sets in $\Omega$. Therefore, in the case of measurable function $g$, in Definition 7.1, the condition "$\lambda^*(g(E)) \leq \varepsilon$" can be replaced WLG by the condition "$\lambda(g(E)) \leq \varepsilon$".*

Proof. To prove (i), we assume the opposite. Then there exists a measurable set $E_0 \subseteq [a,b]$ such that $\lambda(E_0) = 0$ and such that $\lambda^*(g(E_0)) > \varepsilon_0 > 0$. By the Banach (S) property it follows that for $\varepsilon_0 > 0$ there exists $\delta_0 = \delta_0(\varepsilon_0) > 0$ such that for every $0 < \delta \leq \delta_0$ and for every measurable set $E \subseteq [a,b]$ we have that $\lambda(E) \leq \delta$ implies $\lambda^\star(g(E)) \leq \varepsilon_0$. If we choose $E := E_0$, then for an arbitrary $\delta > 0$ we have $\lambda(E_0) \leq \delta$, which, by the Banach (S) property gives $\lambda^*(g(E_0)) \leq \varepsilon_0$, which is a contradiction. Now (ii) follows from (i) and from Proposition 6.4, (ii). ∎

Remark 7.3. We note that in Lemma 7.2, (i), $g$ need not be a measurable function, while in Lemma 7.2, (ii), it is necessary to assume that $g$ is a measurable function.

In the following proposition, we provide a characterization of the Banach (S) Property.

Proposition 7.4. *Consider a measurable set $\Omega \subseteq \mathbf{R}$ such that $\lambda(\Omega) > 0$ and a measurable function $g : \Omega \longrightarrow \mathbf{R}$. Then the following equivalence holds: $g$ satisfies the Banach (S) property on $\Omega$ iff for every sequence of measurable sets $(E_m)$ in $\Omega$ such that $\lim_{m\longrightarrow+\infty} \lambda(E_m) = 0$, it follows that $\lim_{m\longrightarrow+\infty} \lambda(g(E_m)) = 0$.*

Proof. To prove necessity, let $\varepsilon > 0$ be given. Then there exists $\delta = \delta_\varepsilon > 0$ such that for every measurable set $E \subseteq \Omega$ such that $\lambda(E) \leq \delta$ it follows that $\lambda(g(E)) \leq \varepsilon$. On the other hand, since $\lim_{m\longrightarrow+\infty} \lambda(E_m) = 0$, for the aforementioned $\delta_\varepsilon > 0$ there exists $m_0 = m_0(\varepsilon) \in \mathbf{N}$ such that for every $m \geq m_0$ we have $\lambda(E_m) \leq \delta_\varepsilon$. Hence, $\lambda(g(E_m)) \leq \varepsilon$. To prove sufficiency, we assume the opposite, i.e., that $g$ does not satisfy the Banach (S) property. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ there exists a measurable set $E^0_\delta \subseteq \Omega$ such that $\lambda(E^0_\delta) \leq \delta$ and $\lambda(g(E^0_\delta)) \geq \varepsilon_0 > 0$. We choose the sequence $\delta_m := \frac{1}{m}$ and we set $E_m := E^0_{\frac{1}{m}}$, where $m \in \mathbf{N}$. It results that $\lim_{m\longrightarrow+\infty} \lambda(E_m) = 0$ and $\liminf_{m\longrightarrow+\infty} \lambda(g(E_m)) \geq \varepsilon_0 > 0$, which is a contradiction. □

In the next definition, we introduce a new property of functions, which we were not able to find in the literature.

Definition 7.5. *Consider an arbitrary measurable set* $\Omega \subseteq \mathbf{R}$ *such that* $\lambda(\Omega) > 0$, *and a function* $g : \Omega \longrightarrow \mathbf{R}$. *We say that* $g$ *satisfies (F) property on* $\Omega$, *if for every measurable set* $A \subseteq \Omega$ *we have that* $\lambda(A) < +\infty$ *implies* $\lambda(g(A)) < +\infty$.

Proposition 7.6. *Consider an arbitrary measurable set* $\Omega \subseteq \mathbf{R}$ *such that* $\lambda(\Omega) > 0$ *and a function* $g : \Omega \longrightarrow \mathbf{R}$. *Then we have the following:*

(i) *if* $g$ *is a measurable function which satisfies (F) property on* $\Omega$, *then* $g$ *satisfies the Luzin (N) property on* $\Omega$,

(ii) *if* $g$ *is a measurable function which satisfies the Banach (S) property on* $\Omega$, *then* $g$ *satisfies (F) property on* $\Omega$.

Proof. To show (i), we note that, if $g$ satisfies (F) property on $\Omega$, then for every measurable set $A \subseteq \Omega$ it follows that $g(A)$ is measurable. If $A \subseteq \Omega$ is an arbitrary measurable set, we define $A_n := A \cap (-n, n)$, where $n \in \mathbf{N}$, getting $A = \cup_{n=1}^{+\infty} A_n$ and $g(A) = \cup_{n=1}^{+\infty} g(A_n)$. Hence, it results that $g(A)$ is a measurable set. Now the assertion (i) follows from Proposition 6.4, (i). Next, we address the proof of (ii). Since $g$ satisfies the Banach (S) property on $\Omega$, by Lemma 7.2, (i), and by Proposition 6.4, (ii), it follows that for every measurable set $A \subseteq \Omega$ we have that $g(A)$ is measurable. Since $\lambda(A) < +\infty$, by the Luzin theorem, for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq A$ such that $\lambda(A\backslash K_\varepsilon) < \varepsilon$ and such that $g \in \mathrm{C}(K_\varepsilon)$. By Proposition 7.4 we get $\lim_{\varepsilon\longrightarrow 0} \lambda(g(A\backslash K_\varepsilon)) = 0$. On the other hand, since $K_\varepsilon$ is compact, $g(K_\varepsilon)$ is also compact. Hence, for sufficiently small $\varepsilon_0 > 0$ we get

$$\lambda(g(K_{\varepsilon_0})) < +\infty \ , \quad \lambda(g(A\backslash K_{\varepsilon_0})) < +\infty \ , \tag{7.15}$$

which proves the assertion (ii). □

Remark 7.7. We note that, if two measurable functions $g_1, g_2 : \Omega \longrightarrow \mathbf{R}$ satisfy $g_1(s) = g_2(s)$ (a.e. $s \in \Omega$), and if $g_1$ satisfies the Banach (S) property

((F) property, resp.) on $\Omega$, it does not follows that $g_2$ satisfies the Banach (S) ((F) property, resp.) property on $\Omega$. However, we have the following:

LEMMA 7.8. *Consider a measurable set $\Omega \subseteq \mathbf{R}$ and two measurable functions $g_1, g_2 : \Omega \longrightarrow \mathbf{R}$. Assume that the following assumptions are fulfilled:*

(i) *$g_1(s) = g_2(s)$ (a.e. $s \in \Omega$),*

(ii) *$g_1$ satisfies the Banach (S) property ((F) property, resp.) on $\Omega$,*

(iii) *$g_2$ satisfies the Luzin (N) property on $\Omega$.*

*Then $g_2$ satisfies the Banach (S) property ((F) property, resp.) on $\Omega$.*

PROOF. By Lemma 7.2 and Lemma 6.6 it follows that for every measurable set $E \subseteq \Omega$ we have $\lambda(g_1(E)) = \lambda(g_2(E))$, and the claim follows (The assertion follows from Proposition 7.6, (i), and from Lemma 6.6, resp.). □

The topic of the next corollary is a result regarding the preservation of Banach (S) property and (F) property under composition.

COROLLARY 7.9. *Consider two non-empty measurable set $\Omega \subseteq \mathbf{R}$ and $D \subseteq \mathbf{R}$, a measurable function $g : \Omega \longrightarrow \mathbf{R}$, and a bijection $\psi : D \longrightarrow \Omega$. Then we have the following conclusion: If $\psi$ satisfies the Banach (S) property ((F) property, resp.) on $D$, if $\psi^{-1}$ is a measurable function which satisfies the Luzin (N) property on $\Omega$, and if $g$ satisfies the Banach (S) property ((F) property, resp.) on $\Omega$, then $g \circ \psi$ is a measurable function which satisfies the Banach (S) property ((F) property, resp.) on $D$.*

PROOF. By Corollary 6.9, (i), it follows that $g \circ \psi$ is a measurable function on $D$. To show that $g \circ \psi$ satisfies the Banach (S) property ((F) property, resp.) on $D$, we consider a sequence of measurable sets $(D_n)$ such that $D_n \subseteq D$ and such that $\lim_{n \longrightarrow +\infty} \lambda(D_n) = 0$ (we consider a measurable set $A \subseteq D$ such that $\lambda(A) < +\infty$, resp.). Then we have $(g \circ \psi)(D_n) = g(\psi(D_n))$ $((g \circ \psi)(A) = g(\psi(A))$, resp.), and the claim follows by Proposition 7.4, since $\psi$ satisfies the Banach (S) property on $D$ and $g$ satisfies the Banach (S) property on $\Omega$ (the claim follows since $\psi$ satisfies (F) property on $D$ and $g$ satisfies (F) property on $\Omega$, resp.). □

LEMMA 7.10. *If $\psi \in \mathrm{AC}(\mathbf{R})$ is monotonic, then $\psi$ satisfies the Banach (S) property on $\mathbf{R}$.*

PROOF. Let $\varepsilon > 0$ be given. If $\psi$ is increasing (decreasing, resp.) and continuous, we have $\psi((a, b)) = (\psi(a), \psi(b))$ $(\psi((a, b)) = (\psi(b), \psi(a))$, resp.) and $\lambda(\psi((a, b))) = \psi(b) - \psi(a)$ $(\lambda(\psi((a, b))) = \psi(a) - \psi(b)$, resp.), where $a, b \in \mathbf{R}$ satisfy $a < b$. On the other hand, since $\psi$ is absolutely continuous, by Remark 3.2. in [21], there exists $\delta > 0$ such that for every sequence of pairwise disjoint open intervals $(I_n)$ we have that $\sum_{n=1}^{+\infty} \lambda(I_n) \leq \delta$ implies $\sum_{n=1}^{+\infty} \lambda(\psi(I_n)) \leq \varepsilon$. We recall that, by the definition of the outer Lebesgue measure $\lambda^*$, we can find a sequence $(J_n)$ of open intervals such that $E \subset$

$\cup_{n=1}^{+\infty} J_n$ and $\sum_{n=1}^{+\infty} \lambda(J_n) \leq \lambda(E) + \frac{\delta}{2}$. Since $\cup_{n=1}^{+\infty} J_n$ is an open set, it can be rewritten as a union of disjoint open intervals $(I_n)$, getting $E \subseteq \cup_{n=1}^{+\infty} J_n = \cup_{n=1}^{+\infty} I_n$ and

$$\sum_{n=1}^{+\infty} \lambda(I_n) = \lambda(\cup_{n=1}^{+\infty} I_n) = \lambda(\cup_{n=1}^{+\infty} J_n) \leq \sum_{n=1}^{+\infty} \lambda(J_n) \leq \lambda(E) + \frac{\delta}{2} \, .$$

Therefore, if $\lambda(E) \leq \frac{\delta}{2}$, it results $\psi(E) \subset \cup_{n=1}^{+\infty} \psi(I_n)$ and $\sum_{n=1}^{+\infty} \lambda(\psi(I_n)) \leq \varepsilon$, and so we get $\lambda(\psi(E)) \leq \varepsilon$. □

Proposition 7.11. *Consider two disjoint non-empty intervals $U$ and $V$ such that $\lambda(U) > 0$ and $\lambda(V) > 0$, and such that $U \cup V$ is an interval.*

(i) *If it holds that $\psi \in \mathrm{C}(U \cup V)$, $\psi \in \mathrm{AC}(U)$ and $\psi \in \mathrm{AC}(V)$, then we have $\psi \in \mathrm{AC}(U \cup V)$.*
(ii) *If $\psi$ satisfies the Banach (S) property on $U$ and on $V$, then $\psi$ satisfies the Banach (S) property on $U \cup V$.*

Proof. The assertion (i) follows directly from Theorem 6.14. Indeed, since $\psi$ is differentiable at almost every point in $U \cup V$, it suffices to verify the following three properties of $\psi$: (I) $\psi' \in \mathcal{L}^1_{loc}(U \cup V)$; (II) $\psi$ is equi-integrable on $U \cup V$; (III) $\psi$ satisfies the Luzin (N) property on $U \cup V$. We set $\overline{U} \cap \overline{V} = \{a\}$. By Exercise 3.8 in [21] we get $\psi \in \mathrm{AC}([a-\delta, a])$ and $\psi \in \mathrm{AC}([a, a+\delta])$, where $\delta > 0$ is chosen sufficiently small, while by Corollary 3.10 in [21] we observe that we have $\psi' \in \mathcal{L}^1([a-\delta, a])$ and $\psi' \in \mathcal{L}^1([a, a+\delta])$. Thus, (I) follows since it holds that $\psi' \in \mathcal{L}^1_{loc}(\overline{U})$ and $\psi' \in \mathcal{L}^1_{loc}(\overline{V})$. Furthermore, (II) ((III), resp.) follows since $\psi$ is equi-integrable on $U$ and on $V$ (since $\psi$ satisfies the Luzin (N) property on $U$ and on $V$, resp.). To prove the assertion (ii), we consider an arbitrary $\varepsilon > 0$ and we choose an arbitrary measurable set $E \subseteq U \cup V$. We set $E_1 := E \cap U$ and $E_2 := E \cap V$. By the assumption, there exists $\delta_U > 0$ ($\delta_V > 0$, resp.) such that for every measurable set $F \subseteq U$ ($F \subseteq V$, resp.) which satisfies $\lambda(F) \leq \delta_U$ ($\lambda(F) \leq \delta_V$, resp.) we have $\lambda(\psi(F)) \leq \frac{\varepsilon}{2}$. We set $\delta := \min\{\delta_U, \delta_V\}$. If $E \subseteq U \cup V$ satisfies $\lambda(E) \leq \delta$, we conclude that $\lambda(\psi(E_i)) \leq \frac{\varepsilon}{2}$, where $i = 1, 2$. In effect, we get $\lambda(\psi(E)) \leq \lambda(\psi(E_1)) + \lambda(\psi(E_2)) \leq \varepsilon$, which completes the proof. □

Proposition 7.12. *If $\psi \in \mathrm{AC}(\mathbf{R})$, then $\psi$ satisfies the Banach (S) property on $\mathbf{R}$.*

Proof. Let $\varepsilon > 0$ be given. The proof is divided in several steps.

In the first step, we observe that WLG we can assume that $\psi(0) = 0$ (otherwise, we consider the function $\psi(s) - \psi(0)$).

In the second step, we set $U := [0, +\infty)$ and $V := (-\infty, 0)$, and we note that $\psi_+ := \psi \llcorner U$ ($\psi_- := \psi \llcorner V$, resp.) can be written as the difference between two monotonic and absolutely continuous functions. Indeed, by Theorem 6.14, there exists an equi-integrable function $f \in \mathcal{L}^1_{loc}(\mathbf{R})$ such that

$\psi(s) = \int_0^s f(\sigma)d\sigma$, where $0 \leq s$ ($\psi(s) = \int_s^0 f(\sigma)d\sigma$, where $s < 0$, resp.). Since $f^+(\sigma) := \max\{f(\sigma), 0\} \geq 0$ and $f^-(\sigma) := -\min\{f(\sigma), 0\} \geq 0$ belong to $\mathcal{L}^1_{loc}(\mathbf{R})$ and are equi-integrable on $\mathbf{R}$, by Theorem 6.14, we conclude that $\psi_1^+(s) := \int_0^s f^+(\sigma)d\sigma$ and $\psi_2^+(s) := \int_0^s f^-(\sigma)d\sigma$, where $s \geq 0$ ($\psi_1^-(s) := \int_s^0 f^+(\sigma)d\sigma$ and $\psi_2^-(s) := \int_s^0 f^-(\sigma)d\sigma$, where $s < 0$, resp.), are absolutely continuous on $[0, +\infty)$ ($(-\infty, 0)$, resp.) and increasing on $[0, +\infty)$ (decreasing on $(-\infty, 0)$, resp.), whereby $\psi_+ = \psi_1^+ - \psi_2^+$ on $[0, +\infty)$ ($\psi_- = \psi_1^- - \psi_2^-$ on $(-\infty, 0)$, resp.). At this point we note that we can extend $\psi_1^+$ and $\psi_2^+$ ($\psi_1^-$ and $\psi_2^-$, resp.) on $\mathbf{R}$ by zero, whereby the extensions (which we do not relabel) are monotonic on $\mathbf{R}$ and, by Proposition 7.11, (i), are absolutely continuous on $\mathbf{R}$.

In the next step, we recall that, by the same argument as in the proof of Lemma 7.10, if $E \subseteq \mathbf{R}$ is a measurable set, then for every $\delta > 0$, there is a sequence $(I_n)$ of pairwise disjoint open intervals such that $E \subseteq \cup_{n=1}^{+\infty} I_n$ and $\sum_{n=1}^{+\infty} \lambda(I_n) \leq \lambda(E) + \frac{\delta}{2}$.

In the last step, we show that $\psi^+$ ($\psi^-$, resp.) satisfies the Banach (S) property on $[0, +\infty)$ ($(-\infty, 0)$, resp.). To this end, we set $I_n := (a_n, b_n)$, and for every $s, t \in [a_n, b_n]$ we estimate $\psi_+(s) = \psi_1^+(s) - \psi_2^+(s) \leq \psi_1^+(b_n) - \psi_2^+(s) \leq \psi_1^+(b_n) - \psi_2^+(a_n)$, and so $|\psi_+(s) - \psi_+(t)| \leq \psi_1^+(b_n) - \psi_2^+(a_n) + \psi_2^+(b_n) - \psi_1^+(a_n) = \psi_1^+(b_n) - \psi_1^+(a_n) + \psi_2^+(b_n) - \psi_2^+(a_n)$. Since it holds that $\psi_+([a_n, b_n]) = [c_n, d_n]$, there exists $s_n \in [a_n, b_n]$ ($t_n \in [a_n, b_n]$, resp.) such that $\psi_+(s_n) = c_n$ ($\psi_+(t_n) = d_n$, resp.). It results that we have $\lambda(\psi_+(I_n)) \leq \psi_1^+(b_n) - \psi_1^+(a_n) + \psi_2^+(b_n) - \psi_2^+(a_n) = \lambda(\psi_1^+(I_n)) + \lambda(\psi_2^+(I_n))$. In a quite similar way we infer that $\lambda(\psi_-(I_n)) \leq \lambda(\psi_1^-(I_n)) + \lambda(\psi_2^-(I_n))$. By taking $\varepsilon_i > 0$, where $i = 1, 2$, from the proof of Lemma 7.10 we infer that there exists $\delta_i^\pm > 0$, such that if $\sum \lambda(I_n) \leq \delta_i^\pm$, then $\sum_{n=1}^{+\infty} \lambda(\psi_i^\pm(I_n)) \leq \varepsilon_i$. It follows that $\sum \lambda(I_n) \leq \delta$ (where $\delta := \min\{\delta_1^-, \delta_1^+, \delta_2^-, \delta_2^+\}$), implies $\sum_{n=1}^{+\infty} \lambda(\psi_i^\pm(I_n)) \leq \varepsilon_i$, where $i = 1, 2$. If we choose $\varepsilon_1 := \frac{\varepsilon}{2}$ and $\varepsilon_2 := \frac{\varepsilon}{2}$, we get $\sum_{n=1}^{+\infty} \lambda(\psi_\pm(I_n)) \leq \sum_{n=1}^{+\infty} \lambda(\psi_1^\pm(I_n)) + \sum_{n=1}^{+\infty} \lambda(\psi_2^\pm(I_n)) \leq \varepsilon$. Finally, we conclude that $\lambda(E) \leq \frac{\delta}{2}$ implies $\lambda(\psi_\pm(E)) \leq \varepsilon$ and the claim follows by Proposition 7.11, (ii). ◻

By the Banach-Zaretsky theorem (cf. Corollary 3.49 in [21]), if $I \subseteq \mathbf{R}$ is an interval, and if $\psi \in \mathrm{AC}_{loc}(I)$, then $\psi$ satisfies the Luzin (N) property on $I$. In the last theorem of the paper, we recover a more precise result.

THEOREM 7.13. *Let $I \subseteq \mathbf{R}$ be an interval such that $\lambda(I) > 0$. If $\psi \in$ AC$(I)$, then we have the following:*

(i) *$\psi$ satisfies the Banach (S) property on $I$,*
(ii) *$\psi$ preserves measurability of sets in $I$,*
(iii) *$\psi$ satisfies (F) property on $I$.*

PROOF. We note that we can extend $\psi$ from $I$ to $\mathbf{R}$ in such a way that the extension $\overline{\Psi} : \mathbf{R} \longrightarrow \mathbf{R}$ satisfies $\overline{\Psi} \in \mathrm{AC}(\mathbf{R})$. Indeed, in the first step we extend $\psi$ from $I$ to its closure $\overline{I}$ by continuity (such an extension $\Psi : \overline{I} \longrightarrow \mathbf{R}$ is absolutely continuous on $\overline{I}$, compare Exercise 3.8 in [21]). In the second step we extend $\Psi$ from $\overline{I}$ to $\mathbf{R}$ by a constant function and by continuity. More precisely, if $\overline{I} = (-\infty, b]$ or $\overline{I} = [a, +\infty)$ (if $\overline{I} = [a, b]$ is a compact interval, resp.), we define $\overline{\Psi}(s) := \Psi(b)$, where $s \geq b$, or $\overline{\Psi}(s) := \Psi(a)$, where $s \leq a$ ($\overline{\Psi}(s) := \Psi(b)$, where $s \geq b$, and $\overline{\Psi}(s) := \Psi(a)$, where $s \leq a$, resp.). Hence, by Theorem 6.14, we get $\overline{\Psi} \in \mathrm{AC}(\mathbf{R})$. Finally, we apply Proposition 7.12, getting (i). Assertion (ii) ((iii), resp.) follows from Lemma 7.2 and from Proposition 6.4, (iii) (from assertion (i) and from Proposition 7.6, (ii), resp.). □

As a corollary, we derive a result which connects continuity properties of an injection and measurability of its inverse.

COROLLARY 7.14. *Let $I \subseteq \mathbf{R}$ be an interval such that $\lambda(I) > 0$. If $\psi \in \mathrm{AC}_{loc}(I)$ is an injection, then $\psi^{-1}$ is measurable.*

PROOF. In the first step we observe that, if $\psi \in \mathrm{AC}(I)$ is an injection, by Theorem 7.13 and Proposition 6.7, (i), it results that $\psi^{-1}$ is measurable. In the second step we assume that $\psi \in \mathrm{AC}_{loc}(I)$ is an injection. WLG we can assume that $I$ is an open (a semi-open, resp.) interval. Then there exists a sequence of compact intervals $(I_m)$ such that $I = \cup_{m=1}^{+\infty} I_m$, and such that $I_m \subseteq I_{m+1} \subseteq I$, whereby we get $\psi(I) = \cup_{m=1}^{+\infty} \psi(I_m)$ and $\psi(I_m) \subseteq \psi(I_{m+1})$. By the first step, it follows that $(\psi \llcorner I_m)^{-1} : \psi(I_m) \longrightarrow I_m$ is measurable for every $m \in \mathbf{N}$. Hence, $\psi^{-1}$ is measurable. □

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