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Iva Kodrnja and Helena Koncul

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POLYNOMIALS VANISHING ON A BASIS OF $S_m(\Gamma_0(N))$

Iva Kodrnja, Helena Koncul
University of Zagreb, Croatia

Abstract. In this paper we compute the bases of homogeneous polynomials of degree d such that they vanish on cuspidal modular forms of even weight $m \geq 4$ that form a basis for $S_m(\Gamma_0(N))$. Among them we find irreducibile ones.

1. Introduction

Let $N > 1$, $m \geq 4$ be an even number and $f_0, ..., f_{t-1}$ be elements of the basis of the space of cuspidal modular forms $S_m(\Gamma_0(N))$ of weight m with $\dim S_m(\Gamma_0(N)) = t$. Let $X_0(N)$ be the modular curve for $\Gamma_0(N)$. We look at the holomorphic map $X_0(N) \to \mathbb{P}^{t-1}$ defined by

$$\mathfrak{a}_z \mapsto (f_0(z) : \cdots : f_{t-1}(z)) \tag{1.1}$$

and we denote the image curve of this map by

$$\mathcal{C}(N, m) \subseteq \mathbb{P}^{t-1}.$$

Let us set $g = f_{t-1}$. Then the map (1.1) can be written as

$$\mathfrak{a}_z \mapsto (f_0(z)/g(z) : \cdots : f_{t-1}(z)/g(z)) \tag{1.2}$$

and is a rational map of algebraic curves. Here, we are following the work in [18], where is shown that the complete linear system attached to this map, consisting of integral divisors of degree $t + g - 1$ attached to modular forms f_i, obtained from usual $\mathrm{div}(f_i)$ by subtracting contributions at elliptic points and cusps, satisfy the conditions for the map to be an embedding, namely the linear system is base point free when $t \geq g+1$ and very ample when $t \geq g+2$.

We repeat the following facts about the image curve $\mathcal{C}(N, m)$ from [18]:

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LEMMA 1.1. *Assume that* $m \geq 4$ *even. Let* $t = \dim S_m(\Gamma_0(N))$, $f_0, \dots, f_{t-1}$ *be a basis of* $S_m(\Gamma_0(N))$, g *be the genus of* $\Gamma_0(N)$ *and we denote by*

$$\mathcal{C}(N,m) = C(f_0, \cdots, f_{t-1})$$

*the image of the map (*1.1*). Then*

i) $\mathcal{C}(N,m)$ *is an irreducible smooth projective curve in* $\mathbb{P}^{t-1}$.

ii) *If* $m \geq 4$*, then* $t \geq g+2$ *and the degree of the curve is* $t+g-1$.

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PROOF. The i) part of Lemma 1.1 follows from Chow's theorem, ii) and its proof can be found in [18], Corollary 3-4a. □

In our previous work we have used map (1.1) for $t = 3$ to map the modular curve $X_0(N)$ to projective plane, find its irreducible equation and check conditions for birationality, ([12], [18], [19]).

Maps to higher dimensional projective spaces generate a projective curve in $\mathbb{P}^{t-1}$, $t \geq 3$. Here curves are no longer defined by just one equation. Our goal is to adapt the algorithms used in [12] and compute all linearly independent homogeneous polynomials of a certain given degree that vanish on the curve $\mathcal{C}(N,m)$. Geometrically, these polynomials define hypersurfaces in $\mathbb{P}^{t-1}$ lying over the curve.

In weight 2, space of cusp forms $S_2(\Gamma_0(N))$ is isomorphic to holomorphic 1-forms, we have $\dim S_2(\Gamma_0(N)) = g$, divisors of cusp forms defining the map (1.1) make the canonical linear system of the map and (1.1) is a canonical embedding ([11] Ch IV.5). In [9] the bases of $S_2(\Gamma_0(N))$ are used to obtain canonical models for modular curves.

Canonical curves and their ideals are well studied ([4] Ch III,[8], [23]), their ideal is generated by quadrics except when the curve is trigonal or isomorphic to a smooth plane quintic and then at least one cubic generator appears in the minimal generating system of the ideal. In [10] one can find the complete list of trigonal modular curves $X_0(N)$.

For $m > 2$ the space of cusp forms of weight m is bigger then the set of differentials of degree $m/2$, ([15]). But the complete linear system of integral divisors attached to cusp forms consists of special divisors ([18]), so by results from [3] the ideal of our image curve is generated by quadrics.

In Section 2 we present the algorithm to compute homogeneous polynomials vanishing on $\mathcal{C}(N,m)$ and in Section 3 we present the results of computations and some examples.

2. COMPUTING HOMOGENEOUS POLYNOMIALS VANISHING ON CUSP FORMS

Let $\mathcal{P} = \mathbb{Q}\left[X_0, \dots, X_{t-1}\right]$ be the ring of polynomials in t variables and $\mathcal{P}_d = \mathbb{Q}\left[X_0, \dots, X_{t-1}\right]_d$ subring of homogeneous polynomials of degree d. We regard $\mathcal{P}$ as the graded ring $\mathcal{P} = \bigoplus_{d \geq 0} \mathcal{P}_d$.

Let $I(\mathcal{C}(N,m)) \subseteq \mathcal{P}$ be the homogenous ideal of the curve $\mathcal{C}(N,m)$ consisting of all homogenous polynomials that vanish on $\mathcal{C}(N,m)$. Then $f \in I(\mathcal{C}(N,m))$ defines a hypersurface $\mathcal{C}(N,m) \subset V(f)$ in $\mathbb{P}^{t-1}$. There is a graded structure on the ideal $I(\mathcal{C}(N,m))$

$$I(\mathcal{C}(N,m)) = \bigoplus_{d\geq 0} I(\mathcal{C}(N,m))_d$$

if we set

$$I(\mathcal{C}(N,m))_d = \mathcal{P}_d \cap I(\mathcal{C}(N,m)) \tag{2.3}$$

we get the vector space $I(\mathcal{C}(N,m))_d$ of all homogenous polynomials of degree d which vanish on $\mathcal{C}(N,m)$. Product of two homogeneous polynomials of degrees d_1 and d_2 is again a homogeneous polynomial of degree $d_1 + d_2$. We can see this graded structure as vector spaces or modules,

$$\mathcal{P}_j I(\mathcal{C}(N,m))_d \subseteq I(\mathcal{C}(N,m))_{jd}.$$

Let $f_0, \ldots, f_{t-1} \in S_m(\Gamma_0(N))$ be a basis of the space of cuspidal modular forms for the congruence subgroup $\Gamma_0(N)$ of weight $m \geq 4$.

Let $P \in \mathbb{Q}[x_0, \ldots, x_{t-1}]$ be a homogeneous polynomial of degree d

$$P(x_0, \ldots, x_{t-1}) = \sum_{\substack{0\leq i_0,\ldots,i_{t-1}\leq d \\ i_0+\cdots+i_{t-1}=d}} a_{i_0,\ldots,i_{t-1}} x_0^{i_0} \cdots x_{t-1}^{i_{t-1}}.$$

For a given degree $d \geq 0$, we are interested in those polynomials which vanish on the elements of the basis $f_0, \ldots, f_{t-1}$,

$$P(f_0(z), \cdots, f_{t-1}(z)) = \sum_{\substack{0\leq i_0,\ldots,i_{t-1}\leq d \\ i_0+\cdots+i_{t-1}=d}} a_{i_0,\ldots,i_{t-1}} f_0^{i_0} \cdots f_{t-1}^{i_{t-1}} = 0 \tag{2.4}$$

for all $\mathfrak{a}_z \in X_0(N)$.

Vector space $\mathcal{P}_d$ of all homogeneous polynomials of degree d is generated with monomials and its dimension can be viewed as the number of coefficients $a_{i_0,\ldots,i_{t-1}}$ with respect to the indexing set of the set of monomials of degree d

$$I = \{(i_0, \ldots, i_{t-1}) : 0 \leq i_0, \ldots, i_{t-1} \leq d, i_0 + \cdots + i_{t-1} = d\}.$$

The cardinality of $|I|$ is known as the weak composition problem in combinatorics and the solution is

$$d' = \dim \mathcal{P}_d = |I| = \binom{d+t-1}{d}. \tag{2.5}$$

We will order I using the lexicographical ordering ([7]), so that we consider a polynomial P as a finite linear array of its coefficients

$$P \longrightarrow (a_0, \ldots, a_{d'-1}) \tag{2.6}$$

satisfying the order of corresponding monomials, as basis representation of P.

We are interested in subspaces $I(\mathcal{C}(N,m))_d \subseteq \mathcal{P}_d$ containing polynomials that vanish on the basis $f_0,\dots,f_{t-1}$ of $S_m(\Gamma_0(N))$ for certain choices of d,N,m and their dimensions,

$$I(\mathcal{C}(N,m))_d = \{P \in \mathcal{P}_d :\ P(f_0,\cdots,f_{t-1}) = 0.\} \tag{2.7}$$

Each modular form is in practical computations given by a finitely many coefficients of its integral Fourier expansion in the cusp ∞.

The polynomial combination $P(f_0,\dots,f_{t-1})$ is again a modular form of weight md, where d is the degree of the polynomial P, since cuspidal forms on a given group also make a graded ring $S(\Gamma_0(N)) = \oplus_m S_m(\Gamma_0(N))$.

The condition of vanishing of the modular form $P(f_0,\dots,f_{t-1})$ is known as the Sturm bound saying that we only consider a finite number B of coefficients of the q-expansion of the form to distinguish forms,

$$B_m = \left\lfloor \frac{m\,[SL_2(\mathbb{Z}) : \Gamma_0(N)]}{12} \right\rfloor. \tag{2.8}$$

Similar to [9], [12], [15], [19], the algorithm for computing polynomials vanishing on a basis of $S_m(\Gamma_0(N))$ is based on the following linear algebra considerations: for fixed values of d,N,m we are solving a homogeneous system of equations, where the unknowns are coefficients $a_0,\dots,a_{d'-1}$ of a polynomial P, as in (2.6) and the coefficients of the system are values of q-expansions of evaluated monomials $f_0^{i_0}\dots f_t^{i_{t-1}}$ over the indexing set I,

$$\begin{aligned} P(f_0,\dots,f_{t-1}) &= \sum_{\substack{0\le i_0,\dots,i_{t-1}\le d\\ i_0+\cdots+i_{t-1}=d}} a_{i_0,\dots,i_{t-1}} f_0^{i_0}\cdots f_{t-1}^{i_{t-1}} \\ &= \sum_{\substack{0\le i_0,\dots,i_{t-1}\le d\\ i_0+\cdots+i_{t-1}=d}} a_{i_0,\dots,i_{t-1}} \left(a_0^{(i_0,\dots,i_{t-1})} + a_1^{(i_0,\dots,i_{t-1})} q + \dots\right) \\ &= p_0 + p_1 q + p_2 q^2 + \cdots . \end{aligned}$$

The homogeneous system is $p_0 = p_1 = \cdots = p_{B_{md}} = 0$ and its solutions are obtained as the basis of the right kernel of the transpose of $d' \times B_{md}$ matrix whose rows are made of coefficients of $f_0^{i_0}\dots f_{t-1}^{i_{t-1}}$, after ordering the index set I.

Here is the algorithm, for a given N and weight m, with the use of lexicographic ordering on the set of monomials of degree d:

Input: q-expansions of $f_0,\dots,f_{t-1}$ basis of $S_m(\Gamma_0(N))$

- for a degree $d \ge 0$:
  - for each monomial index $(i_0,\dots,i_{t-1}) \in I$ in the ordered set of monomials of degree d:

    compute $f_0^{i_0}\dots f_{t-1}^{i_{t-1}}$

– create a $d' \times B_{md}$ matrix A, whose rows are first B_{md} coefficients of q-expansion of $f_0^{i_0} \dots f_{t-1}^{i_{t-1}}$
– return elements of the right kernel of A

Output: linearly independent homogeneous polynomials of degree $d \geq 0$ vanishing on all forms, i.e. such that $P(f_0, \dots, f_{t-1}) = 0$.

In our computations we are using the SAGE software system [22] and the cusp form basis we are using is generated by command

```
CuspForms(Gamma_0(N),m).q_integral_basis(prec).
```

3. Results

Let $t = \dim S_m(\Gamma_0(N))$, g be the genus of $\Gamma_0(N)$. The formula for t is derived from Riemann-Roch theorem, ([14] Prop 6.1)

$$t = \dim S_m(\Gamma_0(N)) = (m-1)(g-1) + \left(\frac{m}{2} - 1\right) c_0 + \mu_{0,2} \left\lfloor \frac{m}{4} \right\rfloor \mu_{0,3} \left\lfloor \frac{m}{3} \right\rfloor \tag{3.9}$$

for even $m \geq 4$ where c_0 is the number of inequivalent cusps and $\mu_{0,i}$ is the number of inequivalent elliptic points of order i of $\Gamma_0(N)$.

Table 1. (N, m) for $2 \leq \dim S_m(\Gamma_0(N)) \leq 8$

t	g	(N,m)
2	0	(2,12), (2,14), (3,10), (4,8)
	1	(11,4)
3	0	(2,16), (2,18), (3,12), (3,14), (4,10), (5,8), (5,10), (6,6), (7,6), (7,8), (8,6), (9,6), (10,4), (12,4), (13,4), (16,4)
4	0	(2,20), (2,22), (3,16), (4,12)
	1	(14,4), (15,4), $(17,4)^*$, $(19,4)^*$, (11,6)
5	0	(2,24), (2,26), (3,18), (3,20), (4,14), (5,12), $(5,14)^*$, (6,8), (7,10), (8,8), (9,8), $(10,6)^*$, (13,6), (18,4), (25,4)
	2	(23,4)
6	0	(2,28),(2,30), (3,22), (4,16)
	1	(11,8), (17,6), $(20,4)^{**}$, $(21,4)^{***}$, $(27,4)^*$
7	0	(2,32), (2,34), (3,24), (3,26), (4,18), (5,16), (5,18) (6,10), (7,12), $(7,14)^{**}$, (8,10), (9,10), (12,6), (13,8), $(16,6)^*$
	2	(22,4), $(29,4)^*$, (31,4)
8	0	(2,36), (2,38), (3,28), (4,20)
	1	(11,10), $(14,6)^*$, (15,6), (19,6), (24,4), $(32,4)^\dagger$

Remark 3.1. For the ordered pairs denoted with asterisk $(N, m)^*$, $(N, m)^{**}$, $(N, m)^{***}$ the number of irreducible polynomials differs from other in the group (Table 8), and for $(N, m)^\dagger$ no computation could be made.

Using the algorithm in Section 2 we were able to compute homogeneous polynomials that vanish on all elements of basis of $S_m(\Gamma_0(N))$, and the irreducible ones among them, for small degrees d up to 10 or at times lower due to the limitations of calculations on huge numbers. For $g \geq 0$ of the modular curve $X_0(N)$ we will denote possible cases for maps (1.1) defined by basis of $S_m(\Gamma_0(N))$ by listing ordered pairs (N, m) in Table 1.

TABLE 2. Number of polynomials for $2 \leq t \leq 8$ and $2 \leq d \leq 10$

t	g	degree d of P								
		2	3	4	5	6	7	8	9	10
2	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
3	0	1	3	6	10	15	21	28	36	45
4	0	3	10	22	40	65	98	140	192	255
	1	2	8	19	36	60	92	133	184	246
5	0	6	22	53	105	185	301	462	678	960
	2	4	18	47	97	175	289	448	662	942
6	0	10	40	105	226	431	756	1246	1956	2952
	1	9	38	102	222	426	750	1239	1948	2943
7	0	15	65	185	431	887	1673	2954	4950	7947
	2	13	61	179	423	877	1661	2940	4934	7929
8	0	21	98	301	756	1673	3382	6378	11376	19377
	1	20	96	298	752	1668	3376	6371	11368	19368

REMARK 3.2. The blue colored numbers present the assumed numbers but they are not calculated due to the computational limitation.

REMARK 3.3. There are no homogeneous polynomial of degree $d \in \{0, 1\}$ that vanish on all elements of the basis of $S_m(\Gamma_0(N))$, therefore they are omitted in the table.

PROPOSITION 3.4. *In Table 2 for $2 \leq t \leq 8$ are given the numbers of all linearly independent homogeneous polynomials of degree $2 \leq d \leq 10$ that vanish on the basis of $S_m(\Gamma_0(N))$, $P(f_0, f_1, \cdots, f_{t-1}) = 0$.*

3.1. *Case $t = 3$.* Since $X_0(N)$ is mapped by (1.1) to $\mathbb{P}^2$ its image is a planar curve, given by one irreducible equation. The degree of this equation is the degree of the curve and for all higher degrees we can find more than one polynomial vanishing on the curve. These higher degree polynomials are reducible, because they have the defining polynomial as a factor.

The numbers appearing in Proposition 3.4 for $t = 3$ are the initial part of the integer sequence called triangular numbers A000217, [20]. They also appear in the usual genus-degree formula for curves ([2] Thm 2.1). This happens because to raise the degree we multiply a polynomial with a monomial.

TABLE 3. Basis of $I(\mathcal{C}(N,m))_d$ for $X_0(N)$ with $g=0$ and $t=3$

$(16,4)$	**d=2** : $p_2 = xz - y^2$,
	d=3 : xp_2, yp_2, zp_2,
	d=4 : $x^2p_2, z^2p_2, xyp_2, xzp_2, yzp_2, (xz+y^2)p_2$
$(13,4)$	**d=2** : $q_2 = xz - y^2 + yz - 3z^2$,
	d=3 : $xq_2, yq_2, (x+y+3z)q_2$
	d=4 : x^2q_2, y^2q_2, xyq_2, $x(x+y+3z)q_2$, $y(x+y+3z)q_2$, $(x^2+(2y+3z)x-2y^2+3yz+9z^2)q_2$

The formula relating the degree d of the image curve $\mathcal{C}(N,m)$ and the degree $d(f_0,f_1,f_2)$ of the map (1.1) is ([12], [19])

$$d\cdot d(f_0,f_1,f_2) = \dim S_m(\Gamma_0(N)) + g(\Gamma_0(N)) - 1 - \epsilon_m, \tag{3.10}$$

where $\epsilon_2 = 1$ and $\epsilon_m = 0$ for $m \geq 4$ is the number of possible common zeroes of the basis cusp forms. Given $t=3$, the right-hand side of (3.10) can attain values $3+0-1-0=2$ for $g=0$ and even $m\geq 4$. Since we have computed irreducible equation for $\mathcal{C}(N,m)$ of that exact degree we can conclude that the map is birational.

COROLLARY 3.5. *Assume that* $\dim S_m(\Gamma_0(N)) = 3$ *and let* $\{f_0,f_1,f_2\}$ *be the basis of* $S_m(\Gamma_0(N))$. *Then the map* $X_0(N)\to\mathbb{P}^2$ *given by*

$$\mathfrak{a}_z \mapsto (f_0(z) : f_1(z) : f_2(z))$$

is birational equivalence of $X_0(N)$ *and the image curve* $C(f_0,f_1,f_2)$ *is a conic if* $g(X_0(N))=0$.

3.2. *Number of computed polynomials.* The numbers for $g=0$, $3\leq t\leq 8$ in Table 2 appear to be diagonals of the number sequence A124326 from OEIS database [20] written as a triangle of numbers. This sequence of numbers can be obtained as a difference of Pascal triangle A00731 and rascal triangle A077028 omitting zeros and satisfies the formula

$$T(m,n) = \binom{m}{n} - (1+n(m-n)). \tag{3.11}$$

The table 2 is filled with the assumed blue numbers $T(m,n)$ which could not be computed by the algorithm.

We can deduce:

LEMMA 3.6. *The numbers in Table 2 of all linearly independent homogeneous polynomials that vanish on the basis of* $S_m(\Gamma_0(N))$, *for* $3 \leq t \leq 8$ *and* $3 \leq d \leq 10$ *can be obtained as:*

i) *first six diagonals of the number sequence A124326 written as a triangle, for* $g = 0$,
ii) *number of polynomials of same degree as genus 0 subtracted by* $g(d-1)$, *for* $g = 1, 2$.

This is in accordance with what is known for the dimensions of ideals of projective curves. For $d \geq 0$ the Hilbert function ([13] Ch 5) of the curve $\mathcal{C}(N,m)$ is the Hilbert function of its coordinate ring:

$$HF_{\mathcal{C}(N,m)}(d) = HF_{\mathcal{P}/I(\mathcal{C}(N,m))}(d) = \dim \mathcal{P}_d - \dim I_d. \tag{3.12}$$

For the polynomial ring $\mathcal{P}$ we have

$$HF_{\mathcal{P}}(d) = \dim \mathcal{P}_d = \binom{t+d-1}{d}. \tag{3.13}$$

By Hilbert-Serre theorem ([11] Thm 7.5) for a projective curve there is a unique linear polynomial such that for $d \gg 0$

$$HP_{\mathcal{C}(N,m)}(d) = HF_{\mathcal{C}(N,m)}(d).$$

But the condition $\gg$ here is excessive. Bounds for the regularity index of the Hilbert function, minimal index from which it coincides with this linear polynomial are known ([6] Prop 4.2.12, [21], [5]) and they show that the two functions coincide for d close to zero. We have used CoCoA System ([1]) to compute the Hilbert polynomial of ideals generated by polynomials of degree 2 and 3 we have computed and the numbers coincide for $d \geq 2$. This linear polynomial has known form

$$HP_{\mathcal{C}(N,m)}(d) = \deg \mathcal{C}(N,m) \cdot d + 1 - g.$$

THEOREM 3.7.

$$\dim(I(\mathcal{C}(N,m))_d) = \binom{t+d-1}{d} - (t+g-1)d - 1 + g. \tag{3.14}$$

PROOF. From (3.11) for $n = d$ and $m - n = t - 1$ and (3.13) we obtain the formula (3.14) for the case $g = 0$ in which the linear polynomial $HP_{\mathcal{C}(N,m)}(d)$ appears. For $g = 1, 2$ we use Lemma 3.6[ii] to get (3.14). ∎

We give examples of computed polynomials in Tables 4, 5, 6 and 7.

TABLE 4. Basis of $I(\mathcal{C}(4,12))_d$

d=2 : $p_2 = y^2 + 40wy - xz - 20z^2,\ q_2 = xw - zy + 20zw,$ $r_2 = xz - y^2 - 20wy + 800w^2$
d=3 : $xp_2,\ xq_2,\ xr_2,\ yp_2,\ yq_2,\ yr_2,$ $p_3 = (x - 60z)y^2 + 80xwy - x^2z + 400z^3,$ $q_3 = y^3 + 40wy^2 - 2xzy + x^2w - 400z^2w$ $r_3 = (x - 40z)y^2 + 60xwy - x^2z + 16000zw^2,$ $s_3 = y^3 + 60wy^2 - 3xzy + 2x^2w - 32000w^3$
d=4, $x^2p_2,\ x^2q_2,\ x^2r_2,\ xyp_2,\ xyq_2,\ xyr_2,\ y^2p_2,\ y^2q_2,\ y^2r_2,$ $xp_3,\ xq_3,\ xr_3,\ xs_3,\ yp_3,\ yq_3,\ yr_3,\ ys_3,$ $60y^4 + 2400wy^3 + (x^2 - 160xz)y^2 + 120x^2wy - x^3z - 8000z^4,$ $(2x - 60z)y^3 + 120xwy^2 - 3x^2zy + x^3w + 8000z^3w,$ $40y^4 + 1600wy^3 - (120xz - x^2)y^2 + 100x^2wy - x^3z - 320000z^2w^2,$ $(3x - 80z)y^3 + 180xwy^2 - 5x^2zy + 2x^3w + 640000zw^3,$ $30y^4 + 1000wy^3 - (90xz - x^2)y^2 + 80x^2wy - x^3z - 12800000w^4$

TABLE 5. Basis of $I(\mathcal{C}(15,4))_d$

d=2 : $p_2 = 2y^2 - (z + 3w)y - 2xz + 4z^2 + xw - 4zw,$ $q_2 = y^2 - (z + w)y - xz + 2z^2 + xw - w^2$
d=3 : $xp_2,\ xq_2,\ yp_2,\ yq_2,$ $p_3 = 12y^3 + (6x - 4z - 20w)y^2 + (12z^2 - 13xz + xw)y - 6x^2z$ $+4xz^2 + 16z^3 + x^2w,$ $q_3 = 18y^3 + (4x - 15z - 29w)y^2 + (28z^2 - 18xz + 7xw)y - 4x^2z$ $+8xz^2 + 16z^2w,$ $r_3 = 16y^3 + (2x - 14z - 26w)y^2 + (28z^2 - 15xz + 9xw)y - 2x^2z$ $+4xz^2 - x^2w + 8zw^2,$ $s_3 = 30y^3 - (25z + 51w)y^2 + (52z^2 - 24xz + 19xw)y - 6x^2w + 8w^3$
d=4 : $x^2p_2,\ x^2q_2,\ y^2p_2,\ y^2q_2,\ xyp_2,\ xyq_2,\ xp_3,\ xq_3,\ xr_3,\ xs_3,$ $yp_3,\ yq_3,\ yr_3,\ ys_3,$ $38y^4 + (14x - 40z - 60w)y^3 + (2x^2 - 53xz + 66z^2 + 3xw)y^2+$ $(36xz^2 - 14x^2z + 3x^2w)y - 2x^3z + 12x^2z^2 - 32z^4,$ $142y^4 + (52x - 129z - 227w)y^3 + (12x^2 - 166xz + 260z^2 + 17xw)y^2$ $-(54x^2z - 96xz^2 - 2x^2w)y - 12x^3z + 24x^2z^2 + 2x^3w - 64z^3w,$ $68y^4 + (18x - 59z - 109w)y^3 + (4x^2 - 73xz + 122z^2 + 20xw)y^2$ $-(19x^2z - 32xz^2 + 2x^2w)y - 4x^3z + 8x^2z^2 + x^3w - 16z^2w^2,$ $258y^4 + (40x - 223z - 413w)y^3 + (12x^2 - 252xz + 460z^2 + 121xw)y^2$ $-(46x^2z - 72xz^2 + 24x^2w)y - 12x^3z + 24x^2z^2 + 6x^3w - 32zw^3,$ $122y^4 + (6x - 106z - 194w)y^3 + (6x^2 - 109xz + 218z^2 + 81xw)y^2$ $-(12x^2z - 12xz^2 + 19x^2w)y - 6x^3z + 12x^2z^2 + 6x^3w - 8w^4$

TABLE 6. Basis of $I(\mathcal{C}(25,4))_d$

d=2 : $p_2 = y^2 - wy - xz + z^2$, $q_2 = (2u - z)y + xw - zw$, $r_2 = wy - xu + zu$, $s_2 = y^2 - 2wy - xz - w^2 + 2xu$, $t_2 = (u - z)y + xw + wu$, $u_2 = y^2 - wy - xz + xu + 2u^2$
d=3 : $xp_2,\ xq_2,\ xr_2,\ xs_2,\ xt_2,\ xu_2,\ yp_2,\ yq_2,\ yr_2,\ ys_2,\ yt_2,\ yu_2$, $(2u - x - 2z)y^2 + 2xwy + x^2z - z^3$, $y^3 - 3wy^2 + (4xu - 2xz)y + x^2w - z^2w$, $(2u - z)y^2 + 2xwy - x^2u + z^2u$, $(x + 3z - 4u)y^2 - 5xwy - x^2z - zw^2 + 2x^2u$, $y^3 - 2wy^2 + (3xu - 2xz)y + x^2w + zwu$, $(x + 2z - 2u)y^2 - 3xwy - x^2z + x^2u + 2zu^2$, $2y^3 - 5wy^2 + (8xu - 5xz)y + 3x^2w + w^3$, $(x + 2z - 3u)y^2 - 4xwy - x^2z + 2x^2u + w^2u$, $3wy^2 - y^3 + (3xz - 5xu)y - 2x^2w + 2wu^2$, $(x + 2z - 4u)y^2 - 5xwy - x^2z + 3x^2u - 4u^3$

TABLE 7. Basis of $I(\mathcal{C}(23,4))_d$

d=2 : $p_2 = y^2 - 2uy - xz - z^2 + 3zw + xu + 2zu$, $q_2 = y^2 + (4z - 3w)y - xz + 4z^2 - 4xw - 9zw + 6w^2 + 3xu$, $r_2 = y^2 + (4z - 2w - 2u)y - xz + 3z^2 - 4xw - 5zw + 3xu + 4wu$, $s_2 = 5y^2 + (8z - 14u)y - 5xz + 5z^2 - 8xw - zw + 7xu + 8u^2$
d=3 : $xp_2,\ xq_2,\ xr_2,\ xs_2,\ yp_2,\ yq_2,\ yr_2,\ ys_2$, $(23x - 20z + 72u)y^2 - 36y^3 + (32xz + 4z^2 - 30xw - 162zw - 78xu)y - 23x^2z - 11xz^2 - 38z^3 + 4x^2w + 185xzw + 33x^2u$, $(11x - 36z + 84u)y^2 - 42y^3 + (50xz + 30z^2 - 16xw - 170zw - 72xu)y - 11x^2z + 3xz^2 - 8x^2w + 105xzw - 76z^2w + 29x^2u$, $22y^3 + (8z - 51x - 44u)y^2 + (10xz + 6z^2 + 12xw + 42zw + 92xu)y + 51x^2z - 7xz^2 - 32x^2w - 169xzw - 17x^2u + 152z^2u$, $(29x - 240z + 180u)y^2 - 90y^3 + (194xz - 66z^2 + 96xw - 6zw - 252xu)y - 29x^2z + 77xz^2 - 104x^2w + 111xzw - 456zw^2 + 111x^2u$, $70y^3 + (136z - 31x - 140u)y^2 + (102z^2 - 134xz - 100xw - 46zw + 196xu)y + 31x^2z - 43xz^2 + 64x^2w - 61xzw - 61x^2u + 304zwu$, $214y^3 + (216z - 47x - 428u)y^2 + (86z^2 - 262xz - 284xw + 146zw + 356xu)y + 47x^2z - 227xz^2 + 48x^2w + 339xzw + 35x^2u + 608zu^2$, $(35x - 2208z + 228w - 396u)y^2 + (926xz - 2226z^2 + 2388xw + 4254zw - 1908xu)y - 30y^3 - 35x^2z + 431xz^2 - 896x^2w - 39xzw - 2736w^3 + 417x^2u$, $(15x + 2072z - 456w + 156u)y^2 + (2162z^2 - 602xz - 2516xw - 3714zw + 1788xu)y$ $74y^3 - 15x^2z - 141xz^2 + 528x^2w - 147xzw - 147x^2u + 1824w^2u$, $(37x + 1848z - 304w - 284u)y^2 + (-350xz + 1918z^2 - 2396xw - 3142zw + 1492xu)y$ $142y^3 - 37x^2z + 17xz^2 + 208x^2w - 241xzw + 63x^2u + 1216wu^2$, $(517x + 4312z - 3196u)y^2 + (3918z^2 - 1374xz + 3918z^2 - 5692xw - 4646zw + 2772xu)y + 1294y^3 - 517x^2z - 239xz^2 + 80x^2w + 1135xzw + 831x^2u + 2432u^3$

3.3. *Irreducibility.* For the computed polynomials we check irreducibility by standard argument:

LEMMA 3.8. *If $P(X_0, \ldots, X_n) \in \mathbb{C}[X_0, \cdots, X_n]$ is irreducible as an univariate polynomial $P(X_i) \in \mathbb{C}[X_0, \cdots X_{i-1}, X_{i+1}, \cdots, X_n][X_i]$ then P is irreducible.*

TABLE 8. Number of irreducible polynomials for $3 \leq t \leq 8$

t	g	degree d of P								
		2	3	4	5	6	7	8	9	10
3	0	1	0	0	0	0	0	0	0	0
4	0	3	4	5	6	7	8	9	10	11
	1	2	4	5	6	7	8	9	10	11
	1^*	2	3	5	6	7	8	9	10	11
5	0	6	10	15	21	28	36	45	55	66
	0^*	6	12	23	39	61	90	127	173	229
	2	4	10	15	21	28	36	45	55	66
6	0	10	20	35	56	84	120	165	220	286
	1	9	20	35	56	84	120	165	220	286
	1^*	9	17	35	56	84	120	165	220	286
	1^{**}	9	20	44	82	139	214	324	454	
	1^{***}	9	25	55	107	187	303	464	680	
7	0	15	35	70	126	210	330	495	715	1001
	0^*	15	38	82	182	322	552	877		
	0^{**}	15	39	89	180	334				
	2	13	35	70	126	210	330	495	715	1001
	2^*	13	39	96	205	394	699			
8	0	21	56	126	252	462	792	1287	2002	3003
	1	20	56	126	252	462	792	1287	2002	3003
	1^*	20	56	131						

PROPOSITION 3.9. *In Table 8 for $3 \leq t \leq 8$ we give the numbers of computed irreducible polynomials of degree $2 \leq d \leq 10$ among all linearly independent homogeneous polynomials that vanish on the basis of $S_m(\Gamma_0(N))$, $P(f_0, f_1, \cdots, f_{t-1}) = 0$.*

PROPOSITION 3.10. *Let $2 \leq t \leq 8$.*

i) *There are $\dfrac{t(t-3)}{2} - g + 1$ homogeneous polynomials of degree 2 vanishing on the basis of $S_m(\Gamma_0(N))$ and all are irreducible.*
ii) *For $d \geq 3$ the number of linearly independent homogeneous polynomials of degree d vanishing on the basis of $S_m(\Gamma_0(N))$ is greater than the number of irreducible polynomials of degree d.*

If the ordered pairs denoted with asterisk (see Table 1) are omitted from the Table 8 then we can deduce the following conjecture

CONJECTURE 3.11. *For $t \geq 4$ and $d \geq 3$ the number of irreducible polynomials of degree d is $\binom{d+1}{t-3}$.*

Specially for $t = 5$ we have triangular numbers A00217, for $t = 6$ tetrahedral numbers A000292, for $t = 7$ binomial coefficient C(n,4) A000332, $t = 8$ binomial coefficient C(n,5) A000389, [20].

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I. Kodrnja
Faculty of Geodesy
University of Zagreb
10 000 Zagreb
Croatia
E-mail: `iva.kodrnja@geof.unizg.hr`

H. Koncul
Faculty of Civil Engineering
University of Zagreb
10 000 Zagreb
Croatia
E-mail: `helena.koncul@grad.unizg.hr`