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Manuscript accepted
July 15, 2024.

GENERIC IRREDUCIBILITY OF PARABOLIC INDUCTION FOR REAL REDUCTIVE GROUPS

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Abstract. Let G be a real reductive linear group in the Harish-Chandra class. Suppose that P is a parabolic subgroup of G with Langlands decomposition $P = MAN$. Let π be an irreducible representation of the Levi factor $L = MA$. We give sufficient conditions on the infinitesimal character of π for the induced representation $i_P^G(\pi)$ to be irreducible. In particular, we prove that if π_M is an irreducible representation of M, then for a generic character χ_ν of A, the induced representation $i_P^G(\pi_M \boxtimes \chi_\nu)$ is irreducible. Here the parameter ν is in $\mathfrak{a}^* = (\mathrm{Lie}(A) \otimes_{\mathbb{R}} \mathbb{C})^*$ and generic means outside a countable, locally finite union of hyperplanes which depends only on the infinitesimal character of π. Notice that there is no other assumption on π or π_M than being irreducible, so the result is not limited to generalised principal series or standard representations, for which the result is already well known.

1. Introduction

Let G be a real reductive group. We assume that there is a connected algebraic reductive group $\mathbb{G}$ defined over $\mathbb{R}$ and that G has finite index in $\mathbb{G}(\mathbb{R})$. Let P be a parabolic subgroup of G, with Langlands decomposition $P = MAN$, let π_M be an irreducible representation of M, and χ a character of A. Consider the induced representation $\tau = i_P^G(\pi_M \otimes \chi)$ where i_P^G is the functor of (normalized) parabolic induction.

Theorem 1.1. *For generic χ, the representation $\tau = i_P^G(\pi_M \otimes \chi)$ is irreducible.*

2020 *Mathematics Subject Classification.* 20G05, 22E45.
Key words and phrases. Representation of real reductive groups, generic irreducibility of parabolic induction, Kazhdan-Lusztig-Vogan algorithm.

More precisely, since $A = \exp \mathfrak{a}_0$ where $\mathfrak{a}_0$ is a real abelian Lie algebra, characters of A are of the form

$$\chi_\nu : \exp(X) \mapsto \exp(\nu(X)), \quad (X \in \mathfrak{a}_0) \tag{1.1}$$

for some $\nu \in \mathfrak{a}^*$. "Generic" in the Theorem means "for $\chi = \chi_\nu$ with ν outside a countable, locally finite union of hyperplanes in $\mathfrak{a}^*$". See Hypotheses 4.2 for the precise conditions when the infinitesimal character of π_M is non singular and Hypotheses 5.1 when infinitesimal character of π_M is singular. For a different perspective on potential applications, notice that Hypotheses 4.2 and 5.1 give in fact sufficient conditions on the infinitesimal character of an irreducible representation π of $L = MA$ for the induced representation $i_P^G(\pi)$ to be irreducible (Theorems 4.4 and 5.4).

The result may seem obvious to experts, and I was surprised not being able to find a reference in the literature. For p-adic groups, a proof of the analog result is given by F. Sauvageot in [10], and a totally different one by J-F. Dat in [2]. Both proofs seem difficult to adapt to the real case, however. I propose here a very simple argument, based on a very sophisticated theory, namely the Kazhdan-Lusztig-Vogan theory of character multiplicities that I will try to describe (partly) below. The motivation for writing this note came from a question of Nadir Matringe, who asked for a reference for the result in Theorem 1.1, since it is used in his work with O. Offen and C. Yang [6].

After the first version of the paper was written, David Vogan informed me that he knew the argument given here. This was not a surprise since the proof consists mainly in giving references to his work. He also sketched a more elementary one (in the sense that it uses less sophisticated results, on Lie algebra cohomology), but probably not shorter to expose from the published results. I also became aware that the idea of using the Kazhdan-Lusztig-Vogan (KLV for short from now on) algorithm to prove irreducibility of parabolically induced representations has been used before in published works, notably Matumoto [7] and Gan-Ichino [3]. I had also thought about using this argument in our work with Colette Moeglin on Arthur packets for real classical groups [8]. Indeed, the last step is the construction of arbitrary packets from packets of "good parity" on a maximal Levi subgroup, by parabolic induction. In *ibid*, Thm 4.4, it is stated that this induction preserves irreducibility, and I had the vague impression that it could be a consequence of the ideas explained in the paper. Eventually, we didn't use that strategy and another quite difficult and circumvoluted argument is given [9], Thm. 5.4. These works prompt me to phrase the main result of the present paper in Corollary 3.7, following the idea of [7] and [3]. In the final section, I go back to [9], Thm. 5.4, and explain how the results here provide some shortcuts in the proof (and even a complete argument in most cases, but not more than that, some difficulties remain in some degenerate cases).

Let me now describe more precisely the content of the paper. Let $\mathcal{L}$ be a real reductive group, with Lie algebra $\mathfrak{l}_0$ and let $\mathcal{K}$ be maximal compact subgroup. Let $\mathfrak{l}$ be the complexified Lie algebra of $\mathfrak{l}_0$. By "representations of $\mathcal{L}$", we mean finite length Harish-Chandra modules for the pair $(\mathfrak{l}, \mathcal{K})$. Fix an infinitesimal character χ for $\mathcal{L}$ and denote by $\mathcal{HC}(\mathfrak{l}, \mathcal{K})_\chi$, or simply $\mathcal{HC}_\chi$, the category of representations of $\mathcal{L}$ with infinitesimal character χ, and by $\mathbb{K}_\mathbb{Z}\mathcal{HC}(\mathfrak{l}, \mathcal{K})_\chi$, or simply $\mathbb{K}_\mathbb{Z}\mathcal{HC}_\chi$, its Grothendieck group with coefficients in $\mathbb{Z}$. If π is a representation in $\mathcal{HC}_\chi$, denote by $[\pi]$ its image in $\mathbb{K}_\mathbb{Z}\mathcal{HC}_\chi$.

A result of Harish-Chandra asserts that the number of equivalence classes of irreducible representations with fixed infinitesimal character is finite and Langlands classification for irreducible representations, as reformulated by Vogan (see [11], [13]), gives us a set $\mathcal{P}_\chi^\mathcal{L} = \mathcal{P}_\chi$ which parametrizes the equivalence classes of irreducible representations in $\mathcal{HC}_\chi$ (in fact, it is the set $\mathcal{P}_{\chi/\sim\mathcal{K}}$ of $\mathcal{K}$-conjugacy classes in $\mathcal{P}_\chi$ which is in one-to-one correspondence with equivalence classes of irreducible representations). I will be more precise later, but for the moment, I will just say that a parameter γ in $\mathcal{P}_\chi$ is roughly a character of a Cartan subgroup of $\mathcal{L}$ with some additional data, from which one can construct a "standard" representation $\mathbf{std}(\gamma)$ in $\mathcal{HC}_\chi$ (parabolically induced from a limit of discrete series modulo the center of the corresponding Levi subgroup). The standard representation $\mathbf{std}(\gamma)$ has a Langlands quotient $\mathbf{irr}(\gamma)$ which is irreducible, and appears with multiplicity one in $\mathbf{std}(\gamma)$. Thus, $([\mathbf{irr}(\gamma)])_{\gamma\in\mathcal{P}_{\chi/\sim\mathcal{K}}}$ is a basis of the Grothendieck group $\mathbb{K}_\mathbb{Z}\mathcal{HC}_\chi$. Therefore, in the Grothendieck group, one can write for all $\delta \in \mathcal{P}_\chi$,

$$[\mathbf{std}(\delta)] = \sum_{\gamma\in\mathcal{P}_{\chi/\sim\mathcal{K}}} m(\gamma,\delta)\,[\mathbf{irr}(\gamma)] \tag{1.2}$$

for some non negative integers $m(\gamma, \delta)$ (the multiplicity of $\mathbf{irr}(\gamma)$ in $\mathbf{std}(\delta)$).

It is known from properties of the Langlands classification relative to "exponents" that one can invert the linear system (1.2). Since we will not use exponents in this paper, we explain this using the length function l_I on $\mathcal{P}_\chi$ introduced by Vogan ([11], 8.1.4). Indeed, if $m(\gamma, \delta) \neq 0$, then $l_I(\gamma) < l_I(\delta)$, or $\gamma = \delta$, and furthermore $m(\gamma, \gamma) = 1$. Therefore we can write

$$[\mathbf{irr}(\delta)] = \sum_{\gamma\in\mathcal{P}_{\chi/\sim\mathcal{K}}} M(\gamma,\delta)\,[\mathbf{std}(\gamma)] \tag{1.3}$$

for some integers $M(\gamma, \delta)$. The Kazhdan-Lusztig-Vogan theory gives an algorithm to compute these integers $M(\gamma, \delta)$ (or equivalently the $m(\gamma, \delta)$). We give details about the KLV algorithm in Section 3.

Let us apply this to the problem of determining when a representation $\tau = i_P^G(\pi)$ is irreducible, for π an irreducible representation of $L = MA$, the Levi factor of P, as in the beginning of this introduction. Applying the statements in the previous paragraph to $\mathcal{L} = L$ and to the infinitesimal character χ of π, we can write $\pi = \mathbf{irr}(\delta)$ for some parameter $\delta \in \mathcal{P}^L_\chi$ and write

$$[\pi] = [\mathbf{irr}(\delta)] = \sum_{\gamma \in \mathcal{P}^L_{\chi\,/\sim K_L}} M^L(\gamma,\delta)\,[\mathbf{std}(\gamma)].$$

By the exactness of the functor i_P^G, we get,

$$[\tau] = [i_P^G(\pi)] = [i_P^G(\mathbf{irr}(\delta))] = \sum_{\gamma \in \mathcal{P}^L_{\chi\,/\sim K_L}} M^L(\gamma,\delta)\,[i_P^G(\mathbf{std}(\gamma))].$$

Now, the infinitesimal character χ for L determines an infinitesimal character for G that we can still denote by χ. Let us assume first that the infinitesimal character χ of π is non singular. Then, a parameter $\gamma \in \mathcal{P}^L_\chi$ can be extended to a parameter $\gamma^G \in \mathcal{P}^G_\chi$, giving a correspondence

$$\gamma \mapsto \gamma^G, \quad \mathcal{P}^L_\chi \longrightarrow \mathcal{P}^G_\chi, \tag{1.4}$$

so that

$$i_P^G(\mathbf{std}(\gamma)) = \mathbf{std}(\gamma^G). \tag{1.5}$$

Thus we get

$$[\tau] = \sum_{\gamma \in \mathcal{P}^L_{\chi\,/\sim K_L}} M^L(\gamma,\delta)\,[\mathbf{std}(\gamma^G)] \tag{1.6}$$

We can compare this to

$$[\mathbf{irr}(\delta^G)] = \sum_{\eta \in \mathcal{P}^G_{\chi\,/\sim K}} M^G(\eta,\delta^G)\,[\mathbf{std}(\eta)] \tag{1.7}$$

to conclude that $\tau = \mathbf{irr}(\delta^G)$ if the following conditions are satisfied :

a) The correspondence (1.4) is injective.
b) $M^L(\gamma,\delta) = M^G(\gamma^G,\delta^G)$ for any $\gamma,\delta \in \mathcal{P}^L_\chi$.
c) $M^G(\eta,\delta^G) = 0$ if η is not in the image of (1.4).

We will give sufficient conditions for this to hold (Hypotheses 4.2). When the infinitesimal character χ of π is singular, we give conditions in Hypotheses 5.1 so that the correspondence (1.4) is still well-defined and a), b), c) still hold. The corresponding irreducibility results are Theorems 4.4 and 5.4, and Theorem 1.1 is obtained as a corollary.

The multiplicities $M(\gamma,\delta)$ are computed by the KLV algorithm, and this algorithm is determined by a set of data attached to the parameters. Under the conditions we give on the infinitesimal character, this set of data is preserved under the one-to-one correspondence $\gamma \mapsto \gamma^G$. To see this, and explain how the KLV algorithm works, we need to introduce a lot of structure theory and results taken from Vogan's papers (about integral root systems, Cayley transforms, cross-actions, etc, in[11], [12], [13]), which makes the paper a little bit heavy, but the proofs consist mostly in careful bookkeeping.

In Section 2 and 3 of the paper, we introduce the material to be able to describe the KLV algorithm (in case of non singular infinitesimal character). The algorithm itself (what the KLV polynomials are, how they give the multiplicities $M(\gamma, \delta)$ and how to compute them) is explained at the end of Section 3, and the main result for us here is that this algorithm is completely determined by the structural data introduced in Section 2.6. In section 4, we show that these data are "the same" for L and G, if the Hypotheses 4.2 on the infinitesimal character are satisfied. Indeed, we see first that the correspondence (1.4) between the Langlands-Vogan parameters sets $\mathcal{P}^{L^\flat}_{\chi\ /\sim M^\flat_K}$ and $\mathcal{P}^G_{\chi\ /\sim K}$ is injective and its image is a union of blocks (this term will be explained in §3.13), among which is the block containing δ^G. Then we see that the integral root systems attached to χ are the same in L and G, and likewise for all the data in Section 2.6. In the last section, we show how to extend the irreducibility result to the case of singular infinitesimal character, using the "translation data" in Chapter 16 of [1].

2. Notation, preliminaries and structural data

For any real Lie algebra $\mathfrak{b}_0$, we denote by $\mathfrak{b}$ its complexification. Let G be a real reductive group as in the introduction. Let $\mathfrak{g}_0$ be the Lie algebra of G. We also fix a Cartan involution θ of G with associated maximal compact subgroup K, and associated Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$. We denote by σ the complex conjugation in $\mathfrak{g}$ relative to the real form $\mathfrak{g}_0$. We fix a G-invariant non-degenerate symmetric bilinear form $\langle .\,,. \rangle$ on $\mathfrak{g}$ (and $\mathfrak{g}^*$), preserved by θ and which is positive definite on $\mathfrak{s}_0$ and definite negative on $\mathfrak{k}_0$.

If a group $\mathcal{G}$ acts on a set X and if Y is a subset of X, we denote by $\mathrm{Centr}_{\mathcal{G}}(Y)$ or simply $\mathcal{G}^Y$ the centraliser of Y in $\mathcal{G}$ and by $\mathrm{Norm}_{\mathcal{G}}(Y)$ its normaliser and we use analogous notation for a linear action of a Lie algebra.

2.1. *Cartan subalgebras, Cartan subgroups, roots.* We recall the following well-known facts about Cartan subgroups. A Cartan subgroup H is the centraliser in G of a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$. If the Cartan subalgebra $\mathfrak{h}_0$ is θ-stable and decomposes as $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, then the Cartan subgroup H decomposes as $H = TA$ (direct product) with $T = H \cap K$ and $A = \exp \mathfrak{a}_0$. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$. Let us denote by $R(\mathfrak{g}, \mathfrak{h})$ the root system of $\mathfrak{h}$ in $\mathfrak{g}$, by $W(\mathfrak{g}, \mathfrak{h})$ the corresponding complex Weyl group, and by $W(G, H) = \mathrm{Norm}_G(H)/H$ the real Weyl group. Depending on their values on $\mathfrak{h}_0$, roots are classified as real, complex or imaginary. One can furthermore distinguish between compact imaginary and non-compact imaginary roots (see [13] p. 150). Let us denote by

$$R_{\mathbb{R}}(\mathfrak{g}, \mathfrak{h}), \quad R_{i\mathbb{R}}(\mathfrak{g}, \mathfrak{h}), \quad R_{i\mathbb{R},c}(\mathfrak{g}, \mathfrak{h}), \quad R_{i\mathbb{R},nc}(\mathfrak{g}, \mathfrak{h}), \quad R_{\mathbb{C}}(\mathfrak{g}, \mathfrak{h})$$

the subsets of $R(\mathfrak{g},\mathfrak{h})$ consisting of the real, imaginary, imaginary compact, imaginary non compact, and complex roots respectively. Denote by $\check{\alpha} = 2\frac{\alpha}{\langle\alpha,\alpha\rangle} \in \mathfrak{h}^*$ the coroot associated to a root $\alpha \in R(\mathfrak{g},\mathfrak{h})$ and by s_α the reflection in $W(\mathfrak{g},\mathfrak{h})$ associated to α.

Via Harish-Chandra isomorphisms, any element λ in the dual $\mathfrak{h}^*$ of any Cartan subalgebra $\mathfrak{h}$ determines a character χ_λ of $\mathfrak{Z}(\mathfrak{g})$, the center of the enveloping algebra of $\mathfrak{g}$ (*i.e.* an infinitesimal character). Both λ and χ_λ are said to be non singular when $\langle\alpha,\lambda\rangle \neq 0$ for all $\alpha \in R(\mathfrak{g},\mathfrak{h})$. If a positive root system $R^+(\mathfrak{g},\mathfrak{h})$ is given, λ is said to be dominant if $-\langle\check{\alpha},\lambda\rangle \notin \mathbb{N}$ for all $\alpha \in R^+(\mathfrak{g},\mathfrak{h})$.

For $\lambda \in \mathfrak{h}^*$, set $R(\lambda) = \{\alpha \in R(\mathfrak{g},\mathfrak{h}) \,|\, \langle\check{\alpha},\lambda\rangle \in \mathbb{Z}\}$, the set of integral roots (for λ) and let $W(\lambda) = W(R(\lambda))$ be the Weyl group of the root system $R(\lambda)$. If λ is non singular, then put

$$R^+(\lambda) = \{\alpha \in R(\lambda) \,|\, \langle\check{\alpha},\lambda\rangle > 0\}$$

and let $\Pi(\lambda)$ and $S(\lambda)$ be respectively the set of simple roots in $R^+(\lambda)$ and the set of simple reflections in $W(\lambda)$.

In order to be able to compare roots and Weyl groups on different Cartan subalgebras, we will use the abstract Cartan subalgebra $\mathfrak{h}_a$ of $\mathfrak{g}$ (see [13], Section 2). We fix a positive root system $R^+(\mathfrak{g},\mathfrak{h}_a)$ in $R(\mathfrak{g},\mathfrak{h}_a)$ and a non singular dominant weight $\lambda_a \in \mathfrak{h}_a^*$. This defines an infinitesimal character χ_{λ_a}. We also define $R^a = R(\lambda_a)$, $R^{a,+}$, W^a, Π^a, S^a to be respectively the abstract integral root system, the abstract integral positive root system, the abstract integral Weyl group, the abstract set of simple roots, and the abstract set of simple reflections.

If $\mathfrak{h}$ is any Cartan subalgebra of $\mathfrak{g}$ and $\lambda \in \mathfrak{h}^*$ is such that $\chi_\lambda = \chi_{\lambda_a}$, there is an isomorphism $i_\lambda : \mathfrak{h}_a^* \to \mathfrak{h}^*$ sending λ_a to λ which induces isomorphisms $R^a \to R(\lambda)$, $W^a \to W(\lambda)$, and so forth.

2.2. *Parabolic subgroups.* Let P be a parabolic subgroup of G with Langlands decomposition $P = MAN$ and Levi factor $L = MA$ (direct product). Denote by $\mathfrak{p}_0$, $\mathfrak{m}_0$, $\mathfrak{a}_0$, $\mathfrak{n}_0$ and $\mathfrak{l}_0$ the respective Lie algebras of P, M, A, N and L. We also introduce the opposite parabolic P^- and its Lie algebra $\mathfrak{p}_0^-$. Conjugating with an element of G, we may assume that $\mathfrak{l}_0$ is θ-stable and $M_K := L \cap K = M \cap K$ is a maximal compact subgroup of L and M. Both L and M are in the class of groups defined in the introduction.

Let $\mathfrak{h}_0$ be a θ-stable Cartan subalgebra of $\mathfrak{l}_0$. It decomposes as

$$\mathfrak{h}_0 = \mathfrak{h}_{M,0} \oplus \mathfrak{a}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_{M,0} \oplus \mathfrak{a}_0.$$

Let $R(\mathfrak{n},\mathfrak{h})$ be the set of roots in $R(\mathfrak{g},\mathfrak{h})$ such that the corresponding root space is in $\mathfrak{n}$. Then

$$R(\mathfrak{g},\mathfrak{h}) = R(\mathfrak{l},\mathfrak{h}) \coprod R(\mathfrak{n},\mathfrak{h}) \coprod (-R(\mathfrak{n},\mathfrak{h})).$$

The roots $\alpha \in R(\mathfrak{n},\mathfrak{h})$ are either real, or complex with $\sigma(\alpha) = -\theta(\alpha)$ also in $R(\mathfrak{n},\mathfrak{h})$. Let us choose a positive root system $R^+(\mathfrak{l},\mathfrak{h})$ and set $R^+(\mathfrak{g},\mathfrak{h}) = R^+(\mathfrak{l},\mathfrak{h}) \coprod R(\mathfrak{n},\mathfrak{h})$. By [5], Lemma 11.13 and (11.12) there is an element $h_{\delta(\mathfrak{n})} \in \mathfrak{a}_0$ such that $\mathfrak{l}_0 = \mathfrak{g}_0^{h_{\delta(\mathfrak{n})}}$. Therefore, as $\mathfrak{a}_0$ is central in $\mathfrak{l}_0$, we have $\mathfrak{l}_0 = \mathfrak{g}_0^{\mathfrak{a}_0}$.

Since $L = MA$ it is clear that $L \subset G^{\mathfrak{a}_0} = G^A$. If $g \in G^{\mathfrak{a}_0} = G^A$, it preserves $\mathfrak{l}_0$, $\mathfrak{n}_0$ and $\mathfrak{n}_0^-$ which are stable under the adjoint action of $\mathfrak{a}_0$. Therefore, since $L = \mathrm{Norm}_G(\mathfrak{p}) \cap \mathrm{Norm}_G(\mathfrak{p}^-)$, we get

$$L = G^A = G^{\mathfrak{a}_0}. \tag{2.8}$$

Similar consideration apply to the complex connected group $\mathbb{G}(\mathbb{C})$, and there we have

$$\mathbb{L}(\mathbb{C}) := \mathrm{Norm}_{\mathbb{G}(\mathbb{C})}(\mathfrak{p}) \cap \mathrm{Norm}_{\mathbb{G}(\mathbb{C})}(\mathfrak{p}^-) = \mathbb{G}(\mathbb{C})^{\mathfrak{a}} = \mathbb{G}(\mathbb{C})^A. \tag{2.9}$$

Some parabolic subgroups called cuspidal are attached to Cartan subgroups : let $H = TA$ be a θ-stable Cartan subgroup. Set $L = G^A$ Then L is a Levi factor of parabolic subgroups $P = LN$ of G, with Langlands decomposition $L = MA$, and T is a compact Cartan subgroup of M.

2.3. *Cayley transforms.* For the results in this section, we refer to [11], §8.3. Suppose that $H = TA$ is a θ-stable Cartan subgroup of G, and let $\alpha \in R(\mathfrak{g},\mathfrak{h})$ be a real root. Then the root vectors for α generate a subalgebra of $\mathfrak{g}_0$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and we get a Lie algebra morphism

$$\phi_\alpha : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{g}_0$$

satisfying $\phi_\alpha(-{}^tX) = \theta(X)$. We choose ϕ_α so that

$$Z_\alpha := \phi_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}_0 \subset \mathfrak{h}_0 \quad \text{and} \quad \phi_\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_0^\alpha.$$

Set

$$\tilde{Z}_\alpha := \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k}_0.$$

Since G is linear, this map exponentiate to a group morphism :

$$\Phi_\alpha : \mathbf{SL}(2,\mathbb{R}) \longrightarrow G. \tag{2.10}$$

Put

$$\sigma_\alpha = \Phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad m_\alpha = \sigma_\alpha^2.$$

Then $m_\alpha \in T$ and $\sigma_\alpha \in K$. If α is real, then σ_α normalizes H and represents $s_\alpha \in W(G,H)$.

Define $\mathfrak{h}_0^\alpha = \mathfrak{t}_0^\alpha \oplus \mathfrak{a}_0^\alpha$ by setting $\mathfrak{a}_0^\alpha = \{X \in \mathfrak{a}_0 \,|\, \alpha(X) = 0\}$ and $\mathfrak{t}_0^\alpha = \mathfrak{t}_0 \oplus \mathbb{R}\widetilde{Z}_\alpha$. The corresponding Cartan subgroup will be denoted $c_\alpha(H) = H^\alpha = T^\alpha A^\alpha$. Notice that $\sigma_\alpha \in T^\alpha$ and $m_\alpha \in T^\alpha \cap T$.

Let $\tilde{\alpha} = c_\alpha(\alpha)$ be the non compact imaginary root of $\mathfrak{h}^\alpha$ in $\mathfrak{g}$ supported on $\widetilde{Z}_\alpha$ (the Cayley transform of α).

If $H_1 = T_1 A_1$ is a θ-stable Cartan subgroup of G, and $\beta \in R(\mathfrak{g}, \mathfrak{h})$ is a non-compact imaginary root, one can also define a Lie algebra morphism

$$\phi_\beta : \mathfrak{sl}(2, \mathbb{R}) \longrightarrow \mathfrak{g}_0$$

which exponentiate to a group morphism

$$\Phi_\beta : \mathbf{SL}(2, \mathbb{R}) \longrightarrow G, \tag{2.11}$$

another Cartan subgroup $c^\beta(H_1) = H_1^\beta = T_1^\beta A_1^\beta$ and a real root $\tilde{\beta} = c^\beta(\beta) \in R(\mathfrak{g}, \mathfrak{h}_1^\beta)$. The two constructions are inverse of each other : if $H = TA$ is a θ-stable Cartan subgroup of G, and $\alpha \in R(\mathfrak{g}, \mathfrak{h})$ is a real root then $\Phi_\alpha = \Phi_{\tilde{\alpha}}$, $c^{\tilde{\alpha}}(c_\alpha(H)) = H$ and $c^{\tilde{\alpha}}(\tilde{\alpha}) = \alpha$. If $H_1 = T_1 A_1$ is a θ-stable Cartan subgroup of G, and $\beta \in R(\mathfrak{g}, \mathfrak{h}_1)$ is a non compact imaginary root then $\Phi_\beta = \Phi_{\tilde{\beta}}$, $c_{\tilde{\beta}}(c^\beta(H_1)) = H_1$ and $c_{\tilde{\beta}}(\tilde{\beta}) = \beta$.

The following equivalent conditions define type I roots (for a real root α or the corresponding non compact imaginary root $\tilde{\alpha}$) :

a) the reflection $s_{\tilde{\alpha}}$ does not belong to $W(G, H^\alpha)$,
b) $T^\alpha \cap T = T$,
c) $\alpha : T \to \{\pm 1\}$ is not onto.

The following equivalent conditions define type II roots :

a) the reflection $s_{\tilde{\alpha}}$ belongs to $W(G, H^\alpha)$,
b) $T^\alpha \cap T$ has index 2 in T and $s_{\tilde{\alpha}}$ has a representative in $T \setminus T^\alpha$,
c) $\alpha : T \to \{\pm 1\}$ is onto.

2.4. *Parameters for Langlands classification.* We start by fixing an infinitesimal character $\chi = \chi_{\xi_a}$ by picking an element $\xi_a \in \mathfrak{a}_a^*$. We assume that ξ_a is non singular. Abstract integral roots, etc, defined in §2.1 with respect to an element $\lambda_a \in \mathfrak{h}_a^*$ are now defined with respect to this element ξ_a.

We recall the set of parameters $\mathcal{P}^G_{\chi\ /\sim K}$ for the Langlands classification of irreducible representations of G with infinitesimal character ξ_a (see [11] , [13]).

Definition 2.1. *A parameter γ is a multiplet*

$$\gamma = (H = TA, \Gamma, \bar{\gamma})$$

where $H = TA$ is a θ-stable Cartan subgroup of G, Γ is a character of H, and $\bar{\gamma} \in \mathfrak{h}^*$, satisfying the following conditions a), b), c):

a) $\bar{\gamma}_{|\mathfrak{t}} \in i\mathfrak{t}_0^*$*, and* $\langle \alpha, \bar{\gamma} \rangle \neq 0$, $(\forall \alpha \in R_{i\mathbb{R}}(\mathfrak{g}, \mathfrak{h}))$.

Set

$$R_{i\mathbb{R}}^+ = R_{i\mathbb{R}}^+(\mathfrak{g},\mathfrak{h}) = \{\alpha \in R_{i\mathbb{R}}(\mathfrak{g},\mathfrak{h}) |\, \langle \alpha, \bar{\gamma}\rangle > 0\}, \quad R_{i\mathbb{R},c}^+ = R_{i\mathbb{R}}^+ \cap R_{i\mathbb{R},c}(\mathfrak{g},\mathfrak{h}),$$
$$\rho(R_{i\mathbb{R}}^+) = \frac{1}{2}\sum_{\alpha\in R_{i\mathbb{R}}^+} \alpha, \qquad \rho(R_{i\mathbb{R},c}^+) = \frac{1}{2}\sum_{\alpha\in R_{i\mathbb{R},c}^+} \alpha.$$

b) $d\Gamma = \bar{\gamma} + \rho(R_{i\mathbb{R}}^+) - 2\rho(R_{i\mathbb{R},c}^+)$.

c) *The infinitesimal character* $\chi_{\bar{\gamma}}$ *equals* χ.

Attached to a parameter $\gamma = (H = TA, \Gamma, \bar{\gamma})$ as above, there is a standard representation $\mathbf{std}(\gamma)$ (see [11] or [13]); it may be defined by parabolic induction from a discrete series representation on a cuspidal parabolic subgroup $P = MAN$ attached to the Cartan subgroup $H = TA$. The group N is chosen so that the Langlands subquotients appear as quotients of $\mathbf{std}(\gamma)$. For non singular infinitesimal character, this quotient is irreducible and is denoted by $\mathbf{irr}(\gamma)$.

Let us denote by $\mathcal{P}_\chi^G$ the set of parameter γ as above, and by $\mathcal{P}_{\chi\,/\sim K}^G$ the set of K-conjugacy classes in $\mathcal{P}_\chi^G$. The Langlands classification for non singular infinitesimal character χ is the following theorem (see [13], Thm. 2.13 and the references given there).

Theorem 2.2. *Suppose that* π *is an irreducible representation of* G *with non singular infinitesimal character* χ. *Then there is a parameter* $\gamma \in \mathcal{P}_\chi^G$ *such that* $\pi = \mathbf{irr}(\gamma)$. *If two parameters* γ_1 *and* γ_2 *satisfy* $\pi = \mathbf{irr}(\gamma_1) = \mathbf{irr}(\gamma_2)$, *then* γ_1 *and* γ_2 *are* K*-conjugate.*

We will constantly abuse notation by denoting the K-conjugacy class of an element $\gamma \in \mathcal{P}_\chi^G$ also by γ, and conversely, for a conjugacy class $\gamma \in \mathcal{P}_{\chi\,/\sim K}^G$, we denote again by γ the choice of a representative in $\mathcal{P}_\chi^G$. Usually, this should not lead to any confusion.

2.5. *Cayley transforms and cross-action on parameters.* For any $\gamma = (H = TA, \Gamma, \bar{\gamma}) \in \mathcal{P}_\chi^G$ and for any $w \in W(\bar{\gamma})$, a new element $w \times \gamma$ in $\mathcal{P}_\chi^G$, with first component $H = TA$ is defined in [11], §8.3. When $\alpha \in R^+(\bar{\gamma})$ is a simple root, the other components of $s_\alpha \times \gamma$ are given explicitly in *ibib.* Lemma 8.3.2. One can use the isomorphisms $i_{\bar{\gamma}}$ of §2.1 to transport this to an action of W^a on $\mathcal{P}_\chi^G$ and $\mathcal{P}_{\chi\,/\sim K}^G$ (see [13], Section 2).

In Section 2.3, the Cayley transform of a θ-stable Cartan subgroup $H = TA$ with respect to a real root α has been recalled. In *ibib.* §8.3 this definition is extended to Langlands parameters.

We recall first the parity conditions on real integral roots. If $\gamma = (H = TA, \Gamma, \bar{\gamma}) \in \mathcal{P}_\chi^G$ and if $\alpha \in R(\bar{\gamma})$ is a real root, we say that α satisfies the parity condition with respect to γ if

$$\Gamma(m_\alpha) = \epsilon_\alpha^G (-1)^{\langle\check{\alpha},\bar{\gamma}\rangle}. \tag{2.12}$$

Here $\epsilon_\alpha^G \in \{\pm\}$ is defined in [11], Def. 8.3.11.

If α is a real integral root satisfying the parity condition, then the Cayley transform $c_\alpha(\gamma)$ is defined as a subset of $\mathcal{P}_\chi^G$. It is a singleton if α is type II, and we set $c_\alpha(\gamma) = \{\gamma_\alpha\}$, and if α is type I, then $c_\alpha(\gamma) = \{\gamma_\alpha^+, \gamma_\alpha^-\}$, with $s_{\tilde{\alpha}} \times \gamma_\alpha^\pm = \gamma_\alpha^\mp$. The first component of γ_α or $\gamma_\alpha^\pm$ is $H^\alpha = c_\alpha(H)$.

In the other direction, one can define also Cayley transform of a parameter $\gamma = (H = TA, \Gamma, \bar{\gamma}) \in \mathcal{P}_\chi^G$ with respect to a non compact imaginary integral root β. The Cayley transform $c^\beta(\gamma)$ is a singleton if α is type I, and we set $c^\beta(\gamma) = \{\gamma^\beta\}$, and if β is type II, then $c^\beta(\gamma) = \{\gamma_+^\beta, \gamma_-^\beta\}$, with $s_{\tilde{\beta}} \times \gamma_\pm^\beta = \gamma_\mp^\beta$. The first component of γ^β or $\gamma_\pm^\beta$ is $H^\beta = c^\beta(H)$.

2.6. *Data associated to a Langlands parameter.* We associate to any $\gamma = (H = TA, \Gamma, \bar{\gamma}) \in \mathcal{P}_\chi^G$ the following set of data :

(1) $R(\bar{\gamma})$, the integral root system defined by $\bar{\gamma}$ and $R^+(\bar{\gamma})$, the set of integral positive roots.
(2) $R_{i\mathbb{R}}^+(\bar{\gamma}) = R^+(\bar{\gamma}) \cap R_{i\mathbb{R}}^+(\mathfrak{g}, \mathfrak{h})$, the set of integral imaginary positive roots.
(3) $R_{i\mathbb{R},c}^+(\bar{\gamma}) = R^+(\bar{\gamma}) \cap R_{i\mathbb{R},c}(\mathfrak{g}, \mathfrak{h})$, the set of integral imaginary compact positive roots and $R_{i\mathbb{R},nc}^+(\bar{\gamma}) = R^+(\bar{\gamma}) \cap R_{i\mathbb{R},nc}(\mathfrak{g}, \mathfrak{h})$, the set of integral imaginary non compact positive roots.
(4) $R_{i\mathbb{R},nc}^{+,I}(\bar{\gamma})$ the set of integral imaginary non compact positive roots of type I and $R_{i\mathbb{R},nc}^{+,II}(\bar{\gamma})$ the set of integral imaginary non compact positive roots of type II.
(5) $R_{\mathbb{R}}(\bar{\gamma}) = R(\bar{\gamma}) \cap R_{\mathbb{R}}(\mathfrak{g}, \mathfrak{h})$, the set of integral real roots and $R_{\mathbb{R}}^+(\bar{\gamma}) = R^+(\bar{\gamma}) \cap R_{\mathbb{R}}(\mathfrak{g}, \mathfrak{h})$, the set of integral real positive roots.
(6) $R_{\mathbb{R},0}(\bar{\gamma})$, the set of integral real roots not satisfying the parity condition for γ and $R_{\mathbb{R},1}(\bar{\gamma})$ is the set of integral real roots satisfying the parity condition for γ.
(7) $R_{\mathbb{R},1}^I(\bar{\gamma})$, the set of integral real roots satisfying the parity condition for γ of type I and $R_{\mathbb{R},1}^{II}(\bar{\gamma})$ the set of integral real roots satisfying the parity condition for γ of type II.
(8) $R_{\mathbb{C},1}^+(\bar{\gamma})$, the set of integral complex positive roots such that $\theta(\alpha) \in R_{\mathbb{C}}^+(\bar{\gamma})$ and $R_{\mathbb{C},0}^+(\bar{\gamma})$, the set of integral positive roots such that $\theta(\alpha) \notin R_{\mathbb{C}}^+(\bar{\gamma})$.

Remarks 2.3. a) The integral root system $R(\bar{\gamma})$ is θ-stable ([11], Lemma 8.2.5).

b) A parameter $\gamma \in \mathcal{P}_\chi^G$ is said to be minimal if $R_{\mathbb{R},1}(\bar{\gamma})$ and $R_{\mathbb{C},0}^+(\bar{\gamma})$ are empty. It is equivalent to the fact that for any non real simple root in $R^+(\bar{\gamma})$, $\theta(\alpha) \in R^+(\bar{\gamma})$ and no real simple root in $R^+(\bar{\gamma})$ satisfies the parity condition (see [11], Def. 8.6.5). The standard representation $\mathbf{std}(\gamma)$ is irreducible if and only if γ is minimal ([11], Thm. 8.6.6).

c) As we will see in the next section, the data above for all $\gamma \in \mathcal{P}^G_{\chi\ /\sim K}$ (a finite set) is sufficient to determine the multiplicities $M(\gamma,\delta)$ or $m(\gamma,\delta)$ for any $\gamma,\delta \in \mathcal{P}^G_{\chi\ /\sim K}$ via the KLV algorithm.

3. The KLV-algorithm

We follow here [13], Section 12.

3.1. *Blocks.* To describe (part of) the KLV-algorithm, we first introduce the notion of block in $\mathcal{P}^G_{\chi\ /\sim K}$. Blocks are equivalence classes on $\mathcal{P}^G_{\chi\ /\sim K}$ for the equivalence relation generated by

$$\gamma_1 \sim \gamma_2 \text{ if } m(\gamma_1,\gamma_2) \neq 0,$$

i.e. $\mathbf{irr}(\gamma_1)$ occurs as a subquotient in $\mathbf{std}(\gamma_2)$. This is also the equivalence relation generated by the weaker condition that $\gamma_1 \sim \gamma_2$ if $\mathbf{irr}(\gamma_1)$ and $\mathbf{irr}(\gamma_2)$ both occur as some subquotient of the same standard representation $\mathbf{std}(\delta)$. Another characterisation is given in [11], Thm 9.2.11. Block equivalence is generated by the following relations : if $\gamma \in \mathcal{P}^G_\chi$ and β is a simple non-compact imaginary root in $R^+(\bar{\gamma})$, then $\gamma \sim \gamma'$ for any $\gamma' \in c^\beta(\gamma)$, and if α is a simple complex root in $R^+(\bar{\gamma})$, then $\gamma \sim s_\alpha \times \gamma$.

Via the Langlands-Vogan parametrisation (Thm. 2.2), block equivalence gives an equivalence relation on equivalence classes of irreducible representations which can be characterized in terms of Ext groups, namely, it is the equivalence relation generated by $\pi_1 \sim \pi_2$ if $\mathrm{Ext}^1(\pi_1,\pi_2) \neq \{0\}$ (see [11], Def. 9.2.1 and Prop. 9.2.10). A result of Casselman (see [11], Cor. 9.2.24) states that if two irreducible representations π_1 and π_2 of G are in different blocks, then $\mathrm{Ext}^*(\pi_1,\pi_2) = 0$. Therefore the set of parameters $\mathcal{P}^G_\chi$ admits a partition in blocks

$$\mathcal{P}^G_{\chi\ /\sim K} = \coprod_i \mathfrak{B}_i. \tag{3.13}$$

If two parameters $\gamma,\delta \in \mathcal{P}^G_\chi$ are in different blocks, then $M(\gamma,\delta) = 0$.

Let us fix a block $\mathfrak{B}$ in the partition above.

Definition 3.1. *The integral length of a parameter* $\gamma = (H = TA, \Gamma, \bar{\gamma}) \in \mathcal{P}^G_\chi$ *is*

$$l_I(\gamma) = \frac{1}{2}\left|\{\alpha \in R^+(\bar{\gamma})|\, \theta(\alpha) \notin R^+(\bar{\gamma})\}\right| + \frac{1}{2}\dim A - c_0^G$$

Remark 3.2. The constant c_0^G may be chosen so that $l_I(\gamma) \in \mathbb{N}$, for all $\gamma \in \mathfrak{B}$, but the choice of c_0^G is irrelevant for the KLV algorithm since it is always the difference $l_I(\gamma_1) - l_I(\gamma_2)$ between the integral length of two parameters γ_1, $\gamma_2 \in \mathcal{P}^G_\chi$ which matters.

The Hecke algebra $\mathcal{H} = \mathcal{H}(W^a) = \mathcal{H}(W(\xi_a))$ of the abstract Weyl group W^a is defined in [13], Def. 12.4. This is an algebra over $\mathbb{Z}[u^{\frac{1}{2}}, u^{-\frac{1}{2}}]$ generated by elements T_w, $w \in W^a$ with the relations given in *ibid.*

The Hecke module of $\mathfrak{B}$ is the free module over $\mathbb{Z}[u^{\frac{1}{2}}, u^{-\frac{1}{2}}]$ with basis $\{\gamma \in \mathfrak{B}\}$. Let us denote this module by $\mathfrak{M}(\mathfrak{B})$. The action of $\mathcal{H}$ on $\mathfrak{M}(\mathfrak{B})$ is given also in *ibid.* More precisely, what is given are formulas for $T_s\gamma$ when $\gamma \in \mathfrak{B}$ and $s \in S^a$ is a simple reflection. This simple reflection corresponds to a simple root $\alpha \in R^+(\bar{\gamma})$ via the isomorphisms $i_{\bar{\gamma}}$ and the formula depends on α. For instance, if α is type II real satisfying the parity condition, then

$$T_s\gamma = (u-1)\gamma - s_\alpha \times \gamma + (u-1)c_\alpha(\gamma).$$

Let s be a simple reflection in S^a. For any γ_1, $\gamma_2 \in \mathfrak{B}$, write $\gamma_1 \xrightarrow{s} \gamma_2$ if α_1, the corresponding simple root in $R^+(\bar{\gamma}_1)$ is

• either complex with $\theta\alpha_1 \notin R^+(\bar{\gamma}_1)$ and $\gamma_2 = s \times \gamma_1$

• or real, satisfying the parity condition with respect to γ_1 and $\gamma_2 \in c_{\alpha_1}(\gamma_1)$.

Equivalently, if α_2 is the corresponding simple root in $R^+(\bar{\gamma}_2)$, then α_2 is complex and $\theta\alpha_2 \notin R^+(\bar{\gamma}_2)$ or α_2 is non compact imaginary with respect to γ_2 and $\gamma_1 \in c^{\alpha_2}(\gamma_2)$.

If $\gamma_1 \xrightarrow{s} \gamma_2$, then $l_I(\gamma_2) = l_I(\gamma_1) - 1$, and we have also

$$s \times \gamma_1 \xrightarrow{s} s \times \gamma_2, \quad \gamma_1 \xrightarrow{s} s \times \gamma_2, \quad s \times \gamma_1 \xrightarrow{s} \gamma_2$$

In [13], Def. 12.12 an order relation is defined on $\mathcal{B}$ and denoted $\gamma_1 \leq_r \gamma_2$. Let us recall some properties of this partial order relation.

a) If $\gamma_1 \leq_r \gamma_2$, then $l_I(\gamma_1) \leq l_I(\gamma_2)$ and if $\gamma_1 \leq_r \gamma_2$, and $l_I(\gamma_1) = l_I(\gamma_2)$ then $\gamma_1 = \gamma_2$.

b) If $m(\gamma_1, \gamma_2) \neq 0$, or $M(\gamma_1, \gamma_2) \neq 0$, then $\gamma_1 \leq_r \gamma_2$.

The next lemma is [13], Lemma 12.18. It is used to set up the induction step for computing KLV polynomials.

Lemma 3.3. *Suppose that $\gamma, \delta \in \mathfrak{B}$ and $m(\gamma, \delta) \neq 0$. Then we can find $\delta' \in \mathcal{B}$ and a simple reflection $s \in W^a$ such that $\delta \xrightarrow{s} \delta'$, and for any such s, one of the following conditions is satisfied*

(i) $m(\gamma, \delta') \neq 0$.

(ii) *There exists $\gamma' \in \mathfrak{B}$ such that $\gamma \xrightarrow{s} \gamma'$ and $m(\gamma', \delta') \neq 0$.*

(iii) *Let α be the simple root in $R^+(\bar{\delta})$ corresponding to s. Then α is real, satisfying the parity condition with respect to δ and (i) or (ii) holds with $s \times \delta'$ replacing δ'.*

The next ingredient in the KLV algorithm is the duality map D on $\mathfrak{M}(\mathfrak{B})$ ([13], Lemma 12.14).

LEMMA 3.4. *There is a unique $\mathbb{Z}$ linear map $D : \mathfrak{M}(\mathfrak{B}) \to \mathfrak{M}(\mathfrak{B})$ with the following properties. For any $\gamma \in \mathfrak{B}$, write*

$$D\gamma = u^{-l_I(\gamma)} \sum_{\phi \in \mathfrak{B}} (-1)^{l_I(\gamma)-l_I(\phi)} R_{\phi\gamma}\, \phi$$

for some polynomials $R_{\phi\gamma} \in \mathbb{Z}[u^{\frac{1}{2}}, u^{-\frac{1}{2}}]$. Then

a) $D(um) = u^{-1}D(m)$, $(\forall m \in \mathfrak{M}(\mathfrak{B}))$.

b) $D((T_s+1)m) = u^{-1}(T_s+1)D(m)$, $(\forall m \in \mathfrak{M}(\mathfrak{B}), \forall s \in S^a)$.

c) $R_{\gamma\gamma} = 1$, $(\forall \gamma \in \mathfrak{B})$.

d) $R_{\phi\gamma} \neq 0 \implies \phi \leq_r \gamma$, $(\forall \gamma \in \mathfrak{B})$.

The map D has the following extra properties

e) *$R_{\phi\gamma}$ is a polynomial in u of degree $\leq l_I(\gamma) - l_I(\phi)$.*

f) $D^2 = \mathrm{Id}_{\mathfrak{M}(\mathfrak{B})}$.

g) *The specialisation of D at $u = 1$ is the identity.*

We can finally define the KLV polynomials. This is [13], Lemma 12.15.

LEMMA 3.5. *For any $\gamma \in \mathfrak{B}$, there is a unique element $C_\gamma = \sum_{\phi\in\mathfrak{B}} P_{\phi\gamma}\phi \in \mathfrak{M}(\mathfrak{B})$ (with coefficients $P_{\phi\gamma}$ in $\mathbb{Z}[u^{\frac{1}{2}}, u^{-\frac{1}{2}}]$) such that $D(C_\gamma) = u^{-l_I(\gamma)}C_\gamma$, $P_{\gamma\gamma} = 1$, $P_{\phi\gamma} \neq 0$ only if $\phi \leq_r \gamma$ and if $\phi \neq \gamma$, then $P_{\phi\gamma}$ is a polynomial in u of degree $\leq \frac{1}{2}\,(l_I(\gamma) - l_I(\phi) - 1)$.*

The next result proves the Kazhdan-Lusztig-Vogan conjecture on multiplicities.

THEOREM 3.6. *The integers $M(\gamma,\delta)$, $\gamma,\delta \in \mathfrak{B}$ are given by*

$$M(\gamma,\delta) = (-1)^{l_I(\delta)-l_I(\gamma)} P_{\gamma\delta}(1).$$

It is proved by Vogan, Lusztig-Vogan if the infinitesimal character is integral, and an argument by Bernstein settle the non integral case. See [13] and [1] for a discussion of this fundamental result and references to the original papers.

Finally, there is an algorithm which computes the KLV polynomials $P_{\gamma,\delta}$. It is described in Prop. 6.14 of [12]. It starts with the fact that $P_{\delta,\delta} = 1$ for any $\delta \in \mathcal{P}^G_{\xi\ /\sim K}$ and that $P_{\gamma\delta} = 0$ if $\gamma \not\leq_r \delta$ in $\mathcal{P}^G_{\xi\ /\sim K}$. If $P_{\gamma',\delta'}$ is known whenever $l_I(\delta') < l_I(\delta)$ or $l_I(\delta') = l_I(\delta)$ and $l_I(\gamma') > l_I(\gamma)$, then there are formulae for computing $P_{\gamma\delta}$.

To summarise, the KLV polynomials (and therefore the multiplicities $M(\gamma,\delta)$) are completely determined by the $\mathcal{H}(W(\xi_a))$-module structure of $\mathfrak{M}(\mathfrak{B})$, and this structure is in turn completely determined by the data associated to all $\gamma \in \mathfrak{B}$ in Section 2.6.

The following corollary was stated and used in [7] and [3].

Corollary 3.7. *Suppose we have two reductive groups G and G' in the class of groups we consider, two blocks $\mathfrak{B}$ and $\mathfrak{B}'$ of Langlands-Vogan parameters with non singular infinitesimal characters, respectively for G and G', and a bijection*

$$\iota : \mathfrak{B} \longrightarrow \mathfrak{B}'$$

which respects the data associated to any $\gamma \in \mathfrak{B}$ (resp. $\gamma' \in \mathfrak{B}'$) in Section 2.6. Then

$$M^G(\gamma,\delta) = M^{G'}(\iota(\gamma),\iota(\delta)), \quad (\gamma,\delta \in \mathfrak{B}).$$

4. Data in G versus data in $L^\flat$

Let us fix now a parabolic subgroup $P^\flat = M^\flat A^\flat N^\flat$ of G with θ-stable Levi factor $L^\flat = M^\flat A^\flat$. We also fix a fundamental θ-stable Catan subgroup $H^\flat$ of L. Such a Cartan subgroup has a decomposition $H^\flat = T^\flat A_{M^\flat} A^\flat$. Of course there are similar decompositions for the Cartan subalgebras.

All the notation and results in Section 2 apply to the group $L^\flat$ instead of G. When needed, we will add a superscript G or $L^\flat$ to distinguish between objects defined with respect to G or $L^\flat$.

Cartan subalgebras of $\mathfrak{l}_0^\flat$ are Cartan subalgebras of $\mathfrak{g}_0$ and Cartan subgroups of $L^\flat$ are Cartan subgroups of G. This is particular the case for $\mathfrak{h}_0^\flat$ and $H^\flat = T^\flat A^\flat_{M^\flat} A^\flat$.

In general, a Cartan subgroup H in $L^\flat$ decomposes as $H = H_{M^\flat} A^\flat$ with $H_{M^\flat}$ a Cartan subgroup of $M^\flat$. If H is θ-stable, one can further decompose $H_{M^\flat}$ as $H_{M^\flat} = T A_{M^\flat}$ and H as $H = T A_{M^\flat} A^\flat$. For such a Cartan subgroup, we have as in Section 2.2

$$R(\mathfrak{g},\mathfrak{h}) = R(\mathfrak{l}^\flat,\mathfrak{h}) \coprod R(\mathfrak{n}^\flat,\mathfrak{h}) \coprod (-R(\mathfrak{n}^\flat,\mathfrak{h})) \tag{4.14}$$

The roots $\alpha \in R(\mathfrak{n}^\flat,\mathfrak{h})$ are either real, or complex with $\sigma(\alpha) = -\theta(\alpha)$ also in $R(\mathfrak{n}^\flat,\mathfrak{h})$. Therefore

$$R_{i\mathbb{R}}(\mathfrak{g},\mathfrak{h}) = R_{i\mathbb{R}}(\mathfrak{l}^\flat,\mathfrak{h}), \quad R_{i\mathbb{R},c}(\mathfrak{g},\mathfrak{h}) = R_{i\mathbb{R},c}(\mathfrak{l}^\flat,\mathfrak{h})$$

We can therefore simply write $R_{i\mathbb{R}}$ and $R_{i\mathbb{R},c}$ for these systems of imaginary roots.

For the definition of the Hirai order used in the next proposition, see [4].

Proposition 4.1. *The following conjugacy classes of Cartan subgroups are in natural one-to-one correspondences.*

a) *G-conjugacy classes of Cartan subgroups of G containing a G-conjugate of $A^\flat$.*
b) *K-conjugacy classes of θ-stable Cartan subgroups of G containing a G-conjugate of $A^\flat$.*

The following conjugacy classes of Cartan subgroups are in natural one-to-one correspondences.

c) *$L^\flat$-conjugacy classes of Cartan subgroups of $L^\flat$.*
d) *$M_K^\flat$-conjugacy classes of θ-stable Cartan subgroups of $L^\flat$.*
e) *$M^\flat$-conjugacy classes of Cartan subgroups of $M^\flat$.*
f) *$M_K^\flat$-conjugacy classes of θ-stable Cartan subgroups of $M^\flat$.*

Furthermore, the natural map from the set of conjugacy classes in c) to the set of conjugacy classes in a) is surjective. The G-conjugacy classes in a) are exactly the ones which are greater than $H^\flat$ in the Hirai order for G. In particular, it contains the maximally split G-conjugacy class of Cartan subgroups of G.

Proof. The equivalence of *a*) and *b*) is in [11], Lemma 0.1.6. Of course, it gives also the equivalence between *e*) and *f*) and *c*) and *d*). The equivalence of *c*) and *e*) is clear since $A^\flat$ is central in $L^\flat$. A Cartan subgroup containing $A^\flat$ is contained in $G^{A^\flat} = L^\flat$ since Cartan subgroups are abelian for linear groups, proving that the natural map from the set of conjugacy classes in *c*) to the set of conjugacy classes in *a*) is surjective. This map respects the Hirai order (for $L^\flat$ and G respectively), and from this we get that the set of conjugacy classes in *a*) are greater than the one of $H^\flat$ in the Hirai order for G. Conversely, a Cartan subgroup of G with G-conjugacy class greater than the one of $H^\flat$ in the Hirai order for G has a G-conjugate containing $A^\flat$. If two Cartan subgroups of $L^\flat$ are $L^\flat$-conjugate, they are G-conjugate. In general, two G-conjugate Cartan subgroups of $L^\flat$ are not $L^\flat$-conjugate, unless they are maximally compact or split in $L^\flat$. ∎

We now fix an infinitesimal character $\chi = \chi_\xi$, both for G and $L^\flat$, by choosing $\xi \in (\mathfrak{h}^\flat)^*$. We decompose ξ as $\xi = \xi_{M^\flat} + \nu$, according to the decomposition $(\mathfrak{h}^\flat)^* = (\mathfrak{h}^\flat_{M^\flat})^* \oplus (\mathfrak{a}^\flat)^*$.

Hypotheses 4.2. Consider the following conditions on $\xi = \xi_{M^\flat} + \nu \in (\mathfrak{h}^\flat)^*$,

A. ξ is non-singular for $L^\flat$, or equivalently, $\xi_{M^\flat}$ is non-singular for $M^\flat$ *i.e.*

$$\langle \xi_{M^\flat}, \check{\alpha} \rangle \neq 0 \text{ for all } \alpha \in R(\mathfrak{l}^\flat, \mathfrak{h}^\flat).$$

B. For all $\alpha \in R(\mathfrak{n}^\flat, \mathfrak{h}^\flat)$,

$$\langle \check{\alpha}, \xi \rangle = \langle \check{\alpha}, \xi_{M^\flat} + \nu \rangle \notin \mathbb{Z}.$$

Remark 4.3. If ξ satisfies Hypothesis 4.2, B., then

$$R^G(\xi) = R^G(\xi_{M^\flat} + \nu) = R^{L^\flat}(\xi_{M^\flat} + \nu) = R^{M^\flat}(\xi). \tag{4.15}$$

Furthermore if ξ also satisfies Hypothesis 4.2, A., then ξ is non singular also for G.

Our main result is

THEOREM 4.4. *Suppose $\xi = \xi_{M^\flat} + \nu \in (\mathfrak{h}^\flat)^*$ satisfies Hypotheses 4.2, A. and B. Let π be an irreducible representation of $L^\flat$ of the form $\pi = \pi_{M^\flat} \boxtimes \chi_\nu$ with infinitesimal character χ_ξ . Then the induced representation $i_{P^\flat}^G(\pi) = i_{P^\flat}^G(\pi_{M^\flat} \boxtimes \chi_\nu)$ is irreducible.*

REMARK 4.5. We see that under hypothesis A., Theorem 1.1 is a corollary of the result above, since condition B. is generic in ν, *i.e* it holds for $\nu \in (\mathfrak{a}^\flat)^*$ outside a locally finite, countable number of affine hyperplanes.

We will prove this theorem following the ideas given in the introduction (see Corollary 3.7). We start by comparing the parameters for irreducible representations with infinitesimal character $\chi = \chi_\xi$, for G and $L^\flat$. To be coherent with our preceding notation, we also fix a dominant ξ_a in the dual of the abstract Cartan subalgebra $\mathfrak{h}_a$ such that $\chi_{\xi_a} = \chi_\xi = \chi$.

Consider a parameter $\gamma^{L^\flat} = (H = TA, \Gamma, \bar{\gamma}) \in \mathcal{P}_\chi^{L^\flat}$ as in Section 2.4 (but for $L^\flat$). Since imaginary roots are the same for $\mathfrak{l}^\flat$ and $\mathfrak{g}$, it is clear that it is also a parameter in $\mathcal{P}_\chi^G$ (see (2) and (3) below) and conversely if H is greater than $H^\flat$ in the Hirai order. So the identity map

$$\mathcal{P}_\chi^{L^\flat} \longrightarrow \mathcal{P}_\chi^G, \quad \gamma^L \mapsto \gamma^G,$$

induces a map

$$\mathcal{P}_{\chi\ /\sim M_K^\flat}^{L^\flat} \longrightarrow \mathcal{P}_{\chi\ /\sim K}^G, \quad \gamma^L \mapsto \gamma^G, \tag{4.16}$$

with image the set of parameters $\eta = (H = TA, \Gamma, \bar{\gamma}) \in \mathcal{P}_\chi^G$ with H greater than $H^\flat$ in the Hirai order.

PROPOSITION 4.6. *Under Hypotheses 4.2, A. and B. the map (4.16) is injective.*

Proof. We have to show that if two parameters $\gamma_1 = (H_1 = T_1A_1, \Gamma_1, \bar{\gamma}_1)$ and $\gamma_2 = (H_2 = T_2A_2, \Gamma_2, \bar{\gamma}_2) \in \mathcal{P}_\chi^{L^\flat}$ are G-conjugate, then they are $L^\flat$-conjugate. Since $\bar{\gamma}_1$ and $\bar{\gamma}_2$ define the same infinitesimal character as ξ, there are elements l_1 and l_2 in the complex group $L^\flat(\mathbb{C})$ such that $l_1 \cdot \bar{\gamma}_1 = \xi = l_2 \cdot \bar{\gamma}_2$ and $l_1 \cdot \mathfrak{h}_1 = \mathfrak{h}^\flat = l_2 \cdot \mathfrak{h}_2$. Since γ_1 and γ_2 are G-conjugate, there is an element $g \in G$ such that $g \cdot \mathfrak{h}_1 = \mathfrak{h}_2$ and $g \cdot \bar{\gamma}_1 = \bar{\gamma}_2$. Therefore, setting $n = l_2 g l_1^{-1} \in G(\mathbb{C})$, we get $n \cdot \xi = \xi$ with $n \in \mathrm{Norm}_{G(\mathbb{C})}(\mathfrak{h}^\flat)$. Since ξ is non-singular, we must have $n \in \mathrm{Centr}_{G(\mathbb{C})}(\mathfrak{h}^\flat) = H^\flat \subset L^\flat(\mathbb{C})$ and so $g \in L^\flat(\mathbb{C})$. Thus $g \in G \cap L^\flat(\mathbb{C}) = L^\flat$. ∎

We now check that the data (1) to (8) in Section 2.6 associated to parameters are preserved by the correspondence $\gamma^{L^\flat} \mapsto \gamma^G$ in (4.16). So, let us fix $\gamma^{L^\flat} = (H = TA, \Gamma, \bar{\gamma}) \in \mathcal{P}_\chi^{L^\flat}$. We add superscript G or $L^\flat$ to the various objects defined in Section 2.4 to distinguished the ones defined with respect to G from the ones defined with respect to $L^\flat$.

(1) Because of hypothesis 4.2.B on $\nu \in (\mathfrak{h}^\flat)^*$, we have $R^G(\xi_{M^\flat} + \nu) = R^{L^\flat}(\xi_{M^\flat} + \nu)$ and therefore $R^{L^\flat}(\bar{\gamma}) = R^G(\bar{\gamma})$. Thus the integral root systems for $\bar{\gamma}$ are the same for $L^\flat$ and G.

(2) and (3) We have seen that roots in $R(\mathfrak{n}^\flat, \mathfrak{h})$ are either real, or complex with $\sigma(\alpha) = -\theta(\alpha)$. Therefore, the imaginary roots for $\mathfrak{h}$ are the same in $\mathfrak{l}^\flat$ and $\mathfrak{g}$, and such an imaginary root is compact in G if and only if it is compact in $L^\flat$, and so we have

$$R^G_{i\mathbb{R}} = R^{L^\flat}_{i\mathbb{R}}, \quad R^G_{i\mathbb{R},c} = R^{L^\flat}_{i\mathbb{R},c} \quad R^G_{i\mathbb{R},nc} = R^{L^\flat}_{i\mathbb{R},nc}$$

$$R^{G,+}_{i\mathbb{R}} = R^{L^\flat,+}_{i\mathbb{R}}, \quad R^{G,+}_{i\mathbb{R},c} = R^{L^\flat,+}_{i\mathbb{R},c} \quad R^{G,+}_{i\mathbb{R},nc} = R^{L^\flat,+}_{i\mathbb{R},nc}$$

$$\rho(R^{G,+}_{i\mathbb{R}}) = \rho(R^{L^\flat,+}_{i\mathbb{R}}), \quad \rho(R^{G,+}_{i\mathbb{R},c}) = \rho(R^{L^\flat,+}_{i\mathbb{R},c})$$

(4) The fact that a non compact imaginary root $\tilde{\alpha}$ is type I or type II in G depends only on the map $\Phi_\alpha : \mathbf{SL}(2,\mathbb{R}) \longrightarrow G$ in (2.10) as it is clear from the equivalent conditions defining type I or type II. Since in our context we can choose the same map $\Phi_\alpha : \mathbf{SL}(2,\mathbb{R}) \longrightarrow L^\flat \subset G$ for G and $L^\flat$, we see that a non compact imaginary root is type I in $L^\flat$ if and only if it is type I in G.

(5) Since $R^G(\bar{\gamma}) = R^{L^\flat}(\bar{\gamma})$, we have also $R^G_{\mathbb{R}}(\bar{\gamma}) = R^{L^\flat}_{\mathbb{R}}(\bar{\gamma})$ and likewise $R^{G,+}_{\mathbb{R}}(\bar{\gamma}) = R^{L^\flat,+}_{\mathbb{R}}(\bar{\gamma})$.

(6) Let $\alpha \in R^G(\bar{\gamma}) = R^{L^\flat}(\bar{\gamma})$ be an integral real root. We want to compare the parity condition for $L^\flat$ and G. Since m_α is defined via the same map Φ_α for $L^\flat$ and G, we have only to check that $\epsilon^{L^\flat}_\alpha = \epsilon^G_\alpha$. We do that in the lemma below. Therefore the parity condition is the same in $L^\flat$ and G:

$$R^{L^\flat}_{\mathbb{R},0}(\bar{\gamma}) = R^{L^\flat}_{\mathbb{R},0}(\bar{\gamma}) \text{ and } R^{L^\flat}_{\mathbb{R},1}(\bar{\gamma}) = R^{L^\flat}_{\mathbb{R},1}(\bar{\gamma}).$$

(7) As for non compact imaginary roots, real roots are of the same type (I or II) with respect to $L^\flat$ and G.

(8) Since $R^G(\bar{\gamma}) = R^{L^\flat}(\bar{\gamma})$, we have also $R^G_{\mathbb{C}}(\bar{\gamma}) = R^{L^\flat}_{\mathbb{C}}(\bar{\gamma})$ and likewise $R^{G,+}_{\mathbb{C}}(\bar{\gamma}) = R^{L^\flat,+}_{\mathbb{C}}(\bar{\gamma})$, $R^{G,+}_{\mathbb{C},0}(\bar{\gamma}) = R^{L^\flat,+}_{\mathbb{C},0}(\bar{\gamma})$ and $R^{G,+}_{\mathbb{C},1}(\bar{\gamma}) = R^{L^\flat,+}_{\mathbb{C},1}(\bar{\gamma})$.

LEMMA 4.7. *For any integral real root $\alpha \in R_{\mathbb{R}}(\bar{\gamma})$, $\epsilon^{L^\flat}_\alpha = \epsilon^G_\alpha$.*

Proof. The sign ϵ^G_α is defined as $(-1)^{d+1}$ where d is an integer given by one of the definitions in [11], Lemma 8.3.9. For us, the most convenient is the first one, *i.e.* we take $d = d_1$ in *ibid*. We fix $\Phi_\alpha : \mathbf{SL}(2,\mathbb{R}) \to L^\flat$ as in Section 2.3. Thus we get m_α and $H^\alpha = T^\alpha A^\alpha$. Consider a cuspidal parabolic subgroup $P_\alpha = M_\alpha A_\alpha N_\alpha$ attached to H^α, *i.e.* $L^\alpha = M_\alpha A_\alpha = G^{A_\alpha}$. Up to conjugacy in $L^\flat$, we may assume that $A^\flat \subset A^\alpha$ since H^α is a Cartan subgroup of $L^\flat$, and therefore greater than $H^\flat$ in the Hirai order. Thus

$M_\alpha \subset M^\flat$. Therefore the integer d_1 defined in [11], Lemma 8.3.9, which is $d_1 = \frac{1}{2} \dim ((-1) -$ eigenspace of m_α in $\mathfrak{m}_\alpha \cap \mathfrak{k})$ equals

$$\frac{1}{2} \dim ((-1) - \text{ eigenspace of } m_\alpha \text{ in } \mathfrak{m}_\alpha \cap \mathfrak{k}_{\mathfrak{l}^\flat}),$$

since $\mathfrak{m}_\alpha \cap \mathfrak{k} = \mathfrak{m}_\alpha \cap \mathfrak{m}^\flat \cap \mathfrak{k} = \mathfrak{m}_\alpha \cap \mathfrak{k}_{\mathfrak{l}^\flat}$. Therefore $\epsilon_\alpha^{L^\flat} = (-1)^{d_1+1} = \epsilon_\alpha^G$. □

Let us now consider the decomposition of the parameter sets $\mathcal{P}^{L^\flat}_{\chi\ /\sim M_K^\flat}$ and $\mathcal{P}^G_{\chi\ /\sim K}$ into blocks as in (3.13). From the characterization of blocks in terms of Cayley transforms and cross-action, we see that the injective correspondence $\gamma^{L^\flat} \mapsto \gamma^G$ respects blocks.

Lemma 4.8. *Let us consider a block $\mathfrak{B}^G$ in $\mathcal{P}^G_{\chi\ /\sim K}$. Then all elements in $\mathfrak{B}^G$ are in the image of (4.16), or none of them are. Therefore, (4.16) induces a bijection between corresponding blocks.*

Proof. This is clear from the characterisation of blocks given in §3.13. Indeed, suppose that in $\mathfrak{B}^G$, there is a parameter $\eta = (H = TA, ...)$ which is in the image of (4.16) and one $\eta' = (H' = T'A', ...)$ which is not. Then H is greater or equal to $H^\flat$ in the Hirai order, and H' is not. Furthermore, there would be a sequence of parameters $\eta_0 = \eta$, η_1, ..., $\eta_r = \eta'$ in $\mathfrak{B}^G$ such that η_{i+1} is obtained from η_i either by the cross-action with respect to a Cayley transform associated to a real or non compact imaginary simple integral root or by the cross-action of a complex simple integral root. Since cross-action doesn't change the conjugacy class of the associated Cartan subgroup, there is an index i such that η_i is in the image of (4.16), η_{i+1} is not, and furthermore η_i and η_{i+1} are related by a Cayley transform associated to a real or non compact imaginary simple integral root. Our problem is reduced to the case $\eta = \eta_i$ and $\eta' = \eta_{i+1}$. But then η' would also be in the image of (4.16). □

Given a block $\mathfrak{B}^{L^\flat}$ in $\mathcal{P}^{L^\flat}_{\chi\ /\sim M_K^\flat}$, we see that the integral length (Def. 3.1) is the same for $\mathfrak{B}^{L^\flat}$ and the corresponding block $\mathfrak{B}^G$, if we choose the constants c_0^G and $c_0^{L^\flat}$ to be equal.

5. The case of singular infinitesimal character

We turn now to the case of possibly singular infinitesimal character $\chi = \chi_\xi$ with $\xi \in {\mathfrak{h}^\flat}^*$. The relevant discussion may be found in [1], Chapter 11 and [14]. First, the parameters have to be enriched by an extra piece of data, so a parameter is now a multiplet

$$\gamma = (H = TA, \Gamma, \bar{\gamma}, R^+_{i\mathbb{R}}),$$

where $(H = TA, \Gamma, \bar{\gamma})$ is as before and $R^+_{i\mathbb{R}}$ is a system of positive imaginary roots of $\mathfrak{h}$ in $\mathfrak{g}$. The conditions imposed on γ are the following $a)$, $b)$, $c)$, $d)$ and $e)$:

a) $\langle \alpha, \bar{\gamma} \rangle \geq 0$, $(\forall \alpha \in R^+_{i\mathbb{R}})$.

With $R^+_{i\mathbb{R},c}$, $\rho(R^+_{i\mathbb{R}})$ and $\rho(R^+_{i\mathbb{R},c})$ as above, conditions *b*) and *c*) are the same as in Definition 2.1.

d) Suppose α is a simple root in $R^+_{i\mathbb{R}}$ such that $\langle \alpha, \bar{\gamma} \rangle = 0$. Then α is non compact.

e) Suppose α is a real root in $R(\mathfrak{g}, \mathfrak{h})$ such that $\langle \alpha, \bar{\gamma} \rangle = 0$. Then α does not satisfy the parity condition, *i.e.* $\Gamma(m_\alpha) = -\epsilon^G_\alpha$.

Hypotheses 5.1. Consider the following conditions on $\xi = \xi_{M^\flat} + \nu \in (\mathfrak{h}^\flat)^*$,

B. For all $\alpha \in R(\mathfrak{n}^\flat, \mathfrak{h}^\flat)$,

$$\langle \check{\alpha}, \xi \rangle = \langle \check{\alpha}, \xi_{M^\flat} + \nu \rangle \notin \mathbb{Z}.$$

C1. For any $w \in W(\mathfrak{g}, \mathfrak{h}^\flat)$ such that $\xi_{M^\flat} - w \cdot \xi_{M^\flat}$ is non-zero, ν is not in the strict affine subspace in $(\mathfrak{a}^\flat)^*$ of solutions of $w \cdot \nu - \nu = \xi_{M^\flat} - w \cdot \xi_{M^\flat}$.

C2. For all $\alpha \in R(\mathfrak{n}^\flat, \mathfrak{h}^\flat)$, $\langle \check{\alpha}, \nu \rangle \neq 0$.

D. For any $w \in W(\mathfrak{g}, \mathfrak{h}^\flat)$, $w \cdot \xi = \xi$ implies $w \in W(\mathfrak{l}^\flat, \mathfrak{h}^\flat)$.

Remark 5.2. Condition B. is the same as in Hypotheses 4.2, and we replace condition A. there, which is the assumption of non singular infinitesimal character, by either condition C. (meaning C1. and C2.) or condition D.

We start with the analog of Prop 4.6.

Proposition 5.3. *Under Hypotheses 5.1, C1. and C2., or Hypothesis 5.1, D. the map (4.16) is well-defined and injective.*

Proof. We first have to check that the extra conditions d) and e) in the definition of the parameters are preserved, but this is straightforward. (See Lemma 4.7 for condition e)). Starting the proof for injectivity as in the proof of Proposition 4.6, with $l_1 \cdot \mathfrak{h}_1 = \mathfrak{h}^\flat$, $l_1 \cdot \bar{\gamma}_1 = \xi$, $l_2 \cdot \mathfrak{h}_2 = \mathfrak{h}^\flat$, $l_2 \cdot \bar{\gamma}_2 = \xi$, and $g \cdot \mathfrak{h}_1 = \mathfrak{h}_2$, $g \cdot \bar{\gamma}_1 = \bar{\gamma}_2$, we get $n \cdot \xi = \xi$ with $n \in \mathrm{Norm}_{G(\mathbb{C})}(\mathfrak{h}^\flat)$ and we conclude under Hypothesis 5.1, D. as in the proof of Prop 4.6. If we assume instead Hypothesis 5.1, C. we rewrite $n \cdot \xi = \xi$ as $\xi_{M^\flat} - n \cdot \xi_{M^\flat} = n \cdot \nu - \nu$. So if the linear map

$$\phi_n : (\mathfrak{a}^\flat)^* \longrightarrow (\mathfrak{h}^\flat)^*, \quad \nu \mapsto n \cdot \nu - \nu$$

is non zero, its kernel is a strict subspace of $(\mathfrak{a}^\flat)^*$ and the set of solutions in $(\mathfrak{a}^\flat)^*$ of $n \cdot \nu - \nu = \xi_{M^\flat} - n \cdot \xi_{M^\flat}$ is strict affine subspace. Since $\xi_{M^\flat} - n \cdot \xi_{M^\flat}$ takes only a finite number of non-zero values for $n \in \mathrm{Norm}_{\mathbb{G}(\mathbb{C})}(\mathfrak{h}^\flat)$, we see that for ν outside a finite number of strict affine subspaces in $(\mathfrak{a}^\flat)^*$, $\xi_{M^\flat} - n \cdot \xi_{M^\flat} = n \cdot \nu - \nu$ implies $n \cdot \nu = \nu$. Since Hypothesis 5.1, C2. implies that $\mathfrak{g}^\nu = \mathfrak{l}$ and since $L^\flat(\mathbb{C})$ is connected $G(\mathbb{C})^\nu = L^\flat(\mathbb{C})$. We can then conclude as in the proof of Prop 4.6 that $g \in L^\flat$. □

We now use the results of [1], Chapter 16, using what is called there a translation datum to reduce the problem to the case of non singular infinitesimal character. The translation datum consists of our singular infinitesimal character ξ, a weight μ for $H^\flat \cap M^\flat = T^\flat A_{M^\flat}$ satisfying $\xi' = \xi + \mu$ such that

a) ξ' is non-singular for $\mathfrak{l}^\flat$
b) If $\langle \check{\alpha}, \xi \rangle$ is a positive integer for $\alpha \in R(\mathfrak{g}, \mathfrak{h}^\flat)$, then $\langle \check{\alpha}, \xi' \rangle$ is a positive integer.

Set $\chi = \chi_\xi$, $\chi' = \chi_{\xi'}$. Then by *ibid*, (16.5)(a) and (16.5)(d), there is a injection $\iota^{L^\flat}_{\xi,\xi'} : \mathcal{P}^{L^\flat}_\chi \to \mathcal{P}^{L^\flat}_{\chi'}$ respecting K_M-conjugacy classes such that for $\gamma^{L^\flat}$, $\delta^{L^\flat} \in \mathcal{P}^{L^\flat}_\chi$:

$$M^{L^\flat}(\gamma^{L^\flat}, \delta^{L^\flat}) = M^{L^\flat}(\iota^{l^\flat}_{\xi,\xi'}(\gamma^{L^\flat}), \iota^{l^\flat}_{\xi,\xi'}(\delta^{L^\flat})).$$

We can use the same translation datum, but this time for G, and we get a injection $\iota^G_{\xi,\xi'} : \mathcal{P}^G_\xi \to \mathcal{P}^G_{\xi'}$ respecting K-conjugacy classes such that for γ^G, $\delta^G \in \mathcal{P}^G_\xi$:

$$M^G(\gamma^G, \delta^G) = M^G(\iota^G_{\xi,\xi'}(\gamma^G), \iota_{\xi,\xi'}(\delta^G)).$$

By the result for non singular infinitesimal character proved in the previous section, we have that

$$M^{L^\flat}(\iota^{L^\flat}_{\xi,\xi'}(\gamma^{L^\flat}), \iota^{l^\flat}_{\xi,\xi'}(\delta^{L^\flat})) = M^G(\iota^G_{\xi,\xi'}(\gamma^G), \iota_{\xi,\xi'}(\delta^G)).$$

Since

$$\begin{tikzcd}
\mathcal{P}^{L^\flat}_{\chi\ /\sim K_{M^\flat}} \arrow[r, "\iota^{L^\flat}_{\xi,\xi'}"] \arrow[d, "\mathcal{I}_\xi"] & \mathcal{P}^{L^\flat}_{\chi'\ /\sim K_{M^\flat}} \arrow[d, "\mathcal{I}_{\xi'}"] \\
\mathcal{P}^G_{\chi\ /\sim K} \arrow[r, "\iota^G_{\xi,\xi'}"] & \mathcal{P}^G_{\chi'\ /\sim K}
\end{tikzcd}$$

is a commutative diagram, where the vertical maps are the injective maps previously defined in (4.16) and denoted here $\mathcal{I}_\xi$ and $\mathcal{I}_{\xi'}$, we get :

$$M^{L^\flat}(\gamma^{L^\flat}, \delta^{L^\flat}) = M^G(\gamma^G, \delta^G). \tag{5.17}$$

We also need to prove that $M^G(\eta, \delta^G) = 0$ if η is not in the image of $\mathcal{I}_\xi$. Since we don't have the results on blocks we need readily available when the infinitesimal character is singular, we cannot apply Lemma 4.8 directly. Assume $M^G(\eta, \delta^G) \neq 0$ and write $\eta' = \iota^G_{\xi,\xi'}(\eta)$, $\delta'^G = \iota^G_{\xi,\xi'}(\delta^G)$. Therefore $M^G(\eta', \delta'^G)) \neq 0$, and η', δ'^G are in the same block. The map $\mathcal{I}_{\xi'}$ is surjective on the block which contains both η' and δ'^G by Lemma 4.8, thus there exists $\omega', \delta' \in \mathcal{P}^{L^\flat}_{\chi'\ /\sim K_{M^\flat}}$ with, $\mathcal{I}_{\xi'}(\omega') = \eta'$, $\mathcal{I}_{\xi'}(\delta') = \delta'^G$, and $M^{L^\flat}(\omega', \delta') \neq 0$ by the results of the previous section. In [1], Chapter 16, the maps $\iota_{\xi,\xi'}$ (called $\phi_\mathcal{T}$ there) are defined as the inverse of partially defined bijective maps $\psi_\mathcal{T}$, the domain of this map being given by the condition that $\psi_{\mathcal{T}}(\mathbf{std}(\sigma)) \neq 0$ where $\psi_{\mathcal{T}}$ is here the Zuckerman translation functor from ξ' to ξ. From [1], Propositions 11.16 and 11.18, we see that this condition can be checked on the data associated to the parameter σ in Section 2.6. Therefore, the domain of $\psi_{\mathcal{T}}^{L^\flat}$ is the inverse image of the domain of $\psi_{\mathcal{T}}^{G}$ by $\mathcal{I}_{\xi'}$ since the map $\mathcal{I}_{\xi'}$ preserves these data. We deduce that ω', and ${\delta'}^G$ are in the image of $\iota_{\xi,\xi'}^{L^\flat}$, let's say from ω and δ respectively. Now, by the commutativity of the diagram and the injectivity of the maps, we must have $\delta = \delta^{L^\flat}$ and $\mathcal{I}_\xi(\omega) = \eta$. Thus η is in the image of $\mathcal{I}_\xi$. $\square$

From this we deduce as before the following

Theorem 5.4. *Suppose $\xi = \xi_{M^\flat} + \nu \in (\mathfrak{h}^\flat)^*$ satisfies either Hypotheses 5.1, B. and C., or Hypotheses 5.1, B. and D. Let π be an irreducible representation of $L^\flat$ of the form $\pi = \pi_{M^\flat} \boxtimes \chi_\nu$ with infinitesimal character χ_ξ . Then the induced representation $i_{P^\flat}^G(\pi) = i_{P^\flat}^G(\pi_{M^\flat} \boxtimes \chi_\nu)$ is irreducible.*

Remark 5.5. We see that under hypothesis B and C., Theorem 1.1 is a corollary of the result above, since conditions C1. and C2. are generic in ν, *i.e* it holds for $\nu \in (\mathfrak{a}^\flat)^*$ outside a locally finite, countable number of strict affine subspaces.

6. An application

We explain how the results above lead to simplifications in the proof of [9], Theorems 5.3 and 5.4. For background on Arthur packets, we refer to [8], specially in the context of classical real groups.

Suppose that $\mathbb{G}$ is a classical group (symplectic or special orthogonal, the case of unitary groups is similar but requires some adaptation in the formulation of some definitions and statements below) over $\mathbb{R}$, of rank N. Let us denote by $\mathbf{Std}_{\mathbb{G}}$ the standard representation of the L-group of $\mathbb{G}$ in $\mathbf{GL}_N(\mathbb{C})$ (see [8], §3.1), for instance if $\mathbb{G} = \mathbf{Sp}_{2n}(\mathbb{R})$, ${}^LG = \mathbf{SO}_{2n+1}(\mathbb{C}) \times W_{\mathbb{R}}$ and $\mathbf{Std}_{\mathbb{G}}$ is given by the inclusion of $\mathbf{SO}_{2n+1}(\mathbb{C})$ in $\mathbf{GL}_{2n+1}(\mathbb{C})$.

Let $\psi_G : W_{\mathbb{R}} \times \mathbf{SL}_2(\mathbb{C}) \to {}^LG$ be an Arthur parameter for G, and set $\psi = \mathbf{Std}_G \circ \psi_G$, that we see ψ as a completely reducible representation of $W_{\mathbb{R}} \times \mathbf{SL}_2(\mathbb{C})$. In [8], §4.1, we give an explicit decomposition of ψ into a direct sum of irreducible representations and we define good (and bad) parity for these. The parameter ψ is then written as $\psi = \psi_{gp} \oplus \psi_{bp}$ where ψ_{gp} (resp. ψ_{bp}) is the part of good (resp. bad) parity of ψ.[1] The bad parity part ψ_{bp} can be further decomposed as $\psi_{bp} = \rho \oplus \rho^*$ for some representation ρ of $W_{\mathbb{R}} \times \mathbf{SL}_2(\mathbb{C})$ in $\mathbf{GL}_{N_\rho}(\mathbb{C})$, and ρ^* is the contragredient of ρ. By the Arthur-Langlands correspondence for $\mathbf{GL}_{N_\rho}$, ρ is the Arthur parameter of a representation, denoted again by ρ, of $\mathbf{GL}_{N_\rho}(\mathbb{R})$. The good parity part ψ_{gp}

[1] In [8] and [9], written in french, ψ_{bp} is the "bonne parité" part and ψ_{mp} is the "mauvaise parité" .

is an Arthur parameter for a group G' of the same type as G, but of rank $N - N_\rho$. Furthermore $G' \times \mathbf{GL}_{N_\rho}(\mathbb{R})$ is the Levi factor of a maximal parabolic subgroup P of G.

Representations in the Arthur packet for G with parameter ψ are obtained from irreducible representations $\pi_{G'}$ in the Arthur packet for G' with parameter ψ_{gp}, as induced representations $i_P^G(\pi_{G'} \boxtimes \rho)$. Theorems 5.3 and 5.4. of [9] state that these representations are indeed irreducible. In fact, as the main results of this paper show, we get irreducibility of $i_P^G(\pi_{G'} \boxtimes \rho)$ for any representation $\pi_{G'}$ of G' and for any representation ρ of $\mathbf{GL}_{N_\rho}(\mathbb{R})$, if their infinitesimal characters are the ones determined by ψ, under some assumption on the infinitesimal character of ρ.

Let us explain this for $\mathbb{G} = \mathbf{Sp}_{2N}$, the other cases being similar. In the usual coordinates, the infinitesimal character for a parameter of good parity for $\mathbb{G} = \mathbf{Sp}_{2(N-N_\rho)}$ consists in $N - N_\rho$ integers (up to the Weyl group action by permutation and sign changes), while the infinitesimal character corresponding to ρ, which comes from the bad parity part, consists in N_ρ complex numbers which are not integers. It is then obvious that the Hypotheses 5.1 B. is satisfied, and for D., unfortunately, our hypothesis only implies that the element w is in the product of the Weyl groups for $\mathbf{Sp}_{2(N-N_\rho)}$ and $\mathbf{Sp}_{2N_\rho}$, rather than in the Weyl group of $\mathbf{Sp}_{2(N-N_\rho)} \times \mathbf{GL}_{2N_\rho}$. If the coordinates of the infinitesimal character of ρ don't contain pairs of the form $(a, -a)$ (which is a condition easy to check starting from ψ), then Hypothesis D is satisfied and we get the irreducibility of $i_P^G(\pi_{G'} \boxtimes \rho)$.

In general, we can do the following (see [7] and [3] for similar strategy): we apply Corollary 3.7 for the relevant blocks in the groups $\mathbf{Sp}_{2(N-N_\rho)}(\mathbb{R}) \times \mathbf{Sp}_{2N_\rho}(\mathbb{R})$ and $\mathbf{Sp}_{2N}(\mathbb{R})$ rather than $\mathbf{Sp}_{2(N-N_\rho)}(\mathbb{R}) \times \mathbf{GL}_{N_\rho}(\mathbb{R})$ and $\mathbf{Sp}_{2N}(\mathbb{R})$, so that this time Hypothesis D is satisfied. Then, the problem is to show that the representation parabolically induced from ρ to $\mathbf{Sp}_{2N_\rho}(\mathbb{R})$ (using the Siegel parabolic subgroup of $\mathbf{Sp}_{2N_\rho}(\mathbb{R})$) is irreducible. This is a particular case of our original problem, but the arguments in the proof in [9] are then technically easier. Other approaches to this problem may work, for instance ρ being unitary, one may start by using Tadic's classification of the unitary dual of general linear groups to write it in terms of Speh representations (which can be done directly from ψ_{bp}) and then try to use the independence of "polarization results" of [5], Chapter XI, to reduce further the problem.

Acknowledgements.

I would like to thank Nadir Matringe for initiating this work. I would also like to thank David Vogan for useful conversations about the problem considered here and the referees for correcting numerous either typographical or English grammar errors.

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