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Abstract

Denoting by B_{r_0} the open ball with radius r_0, centered at the origin, we consider the so called "limiting case" in the Sobolev embedding theorem, $W^{j+m,p}(B_{r_0}) \to W^{j,q}(B_{r_0})$, namely the case $mp = n$, $1 < p \leq q$, where the embedding for $q = \infty$ does *not* hold. We show that in the case $j = 1$, contrary to the case $j = 0$, radial-symmetric counterexamples, that is radial-symmetric functions in $W^{m+1,p}(B_{r_0}) \setminus W^{1,\infty}(B_{r_0})$ do not exist, if one assumes C^2-regularity away from the origin. Moreover, we characterize in dimension $n = 2$ the set $W^{m+1,p}(B_{r_0}) \setminus W^{1,\infty}(B_{r_0})$, i.e. $W^{2,2}(B_{r_0}) \setminus W^{1,\infty}(B_{r_0})$ within a reasonable large class of functions.

1 Introduction

The famous Soboloev embedding theorem describes the continuous embedding of Sobolev spaces into spaces of (Hölder) continuous functions on the one hand side, and the embedding of certain Sobolev spaces into other Sobolev spaces with different indices, see, for example, [1], Theorem 4.12. More precisely, we have in the latter case, if $mp > n$ or $m = n$ and $p = 1$, the continuous embedding

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega),$$

for $p \leq q \leq \infty$, where Ω is a domain satisfying a cone condition. Moreover, there is the so called "limiting case", $mp = n$, $p > 1$, where we only have

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega), \tag{1}$$

for $p \leq q < \infty$, which means, for example, that functions in $W^{m,p}(\Omega)$, with $mp = n$, $p > 1$, are not necessarily bounded.

This limiting case found some interest in the literature. For example, Trudinger showed in [9] that $W^{m,p}(\Omega)$ can be embedded in an Orlicz space with a defining function of exponential type, see also [1], Theorem 8.27.

Moreover, both, the classical Sobolev embedding and the "Trudinger embedding", can be generalized, using more complicated spaces like Lorentz spaces, resp. even more complicated spaces, described in [4]; see also [7].

Now, the failure of the embedding $W^{m,p}(\Omega) \to L^\infty(\Omega)$, with $mp = n$, $p > 1$ is illustrated by a radial symmetric function in Example 4.43 of [1]. Moreover, it is claimed that this example multiplied

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by the distance to the origin r can serve as radial-symmetric counterexample for $q = \infty$ in the case $j = 1$ in (1) above, see Example 4.44 there. Unfortunately, this seems to be not correct. Roughly speaking, the reason, why it doesn't work is the following: The final "deciding radial integral" for membership in $W^{m,p}$ of Example 4.43 has an integrand of the form $\frac{1}{r}(-\ln r)^{-p}$, such that the radial integral in the vicinity of zero is clearly finite. On the other hand, the corresponding integrand of Example 4.44 has the form $\frac{1}{r}(\ln(-\ln r))^p$, which leads clearly to an infinite integral, see section 2 below.

For convenience, we shall restrict our considerations to the case $\Omega = B_{r_0}$. Motivated by the failure of the mentioned example, one can pose the question, whether there are radial-symmetric counterexamples in the case $j = 1$ at all. It turns out that this is note the case, if we assume C^2-regularity away from the origin.

Furthermore, we can characterize the set of possible counterexamples within a reasonable large set of functions in the two-dimensional case.

The schedule of our paper is the following. We give details to the calculation of Example 4.44 of [1] in section 2. In section 3 we solely consider the two-dimensional situation. In the first subsection we show that radial-symmetric counterexamples are not possible at all (under the mentioned regularity assumption), whereas in the second one we give the characterization of counterexamples mentioned above. Finally, we show in section 4 that, also for general dimension n, radial-symmetric counterexamples are not possible for functions C^2 away from the origin.

It would be an interesting question, whether similar characterization as in subsection 3.2 can be given for general dimension n and/or values of $j \geq 2$.

2 A radial-symmetric example

We have already mentioned Example 4.44 of [1] in the Introduction. It is claimed in [1] that the function $v(x) := |x| \ln(\ln(4R/|x|))$, with $x = (x_1, x_2, ..., x_n) \in \mathbf{R}^n$, is an element of $W^{m,p}(B_R)$, but not an element of $W^{1,\infty}(B_R) = C^{0,1}(\overline{B_R})$. Here $(m-1)p = n$ and $p > 1$ have to hold, B_R is a ball with radius R, centered at the origin. The aim of this section is, to show that this assertion is not true. As this example was the starting point for our consideration in the subsequent sections, we provide details of the calculations.

More precisely, we claim

Lemma 2.1 *Let* $v(x) := r \ln(\ln(1/r))$, $r = \sqrt{x_1^1 + x_2^2 + ... + x_n^2}$. *Then we have for* $(m-1)p = n, p > 1$, *that* $v \notin W^{m,p}(B_{r_0})$, *for arbitrary small radius* r_0.

Proof. Set $\tilde{v} = \ln(\ln(1/r))$. Leibniz' rule implies, using the notation $\tilde{v}^{(k)} := \frac{\partial^k \tilde{v}}{\partial x_1^k}$,

$$v^{(m)} = \sum_{k=0}^{m} \binom{m}{k} r^{(k)} \tilde{v}^{(m-k)}, \quad \text{for } r > 0. \tag{2}$$

Note that, if v would be an element of $W^{m,p}(B_{r_0})$, by the definition of Sobolev spaces, see for example [1], Definition 3.2b, its distributional derivatives up to order m have to be in $L^p(B_{r_0})$. Since our candidate function v is smooth away from the origin (which has Lebesgue measure zero), the classical derivatives of v, away from zero, would have to be in $L^p(B_{r_0})$. Hence, we need Leibniz' rule only for $r \neq 0$, and we shall prove finally, that these classical derivatives are not in $L^p(B_{r_0})$.

By induction one finds

$$r^{(k)} = \frac{Q_k(x_1, x_2, ... x_n)}{r^{2k-1}},$$

where Q_k denotes a homogeneous polynomial of order k in x, that is $Q_k(\lambda x) = \lambda^k Q_k(x)$, for all $x \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$. For example $Q_3(x_1, x_2, ..., x_n) = -3x_1(x_2^2 + ... + x_n^2)$.

In Example 4.43 of [1] one finds

$$\tilde{v}^{(m-k)} = \sum_{j=1}^{m-k} P_{m-k,j}(x) r^{-2(m-k)} \left(\ln(\frac{1}{r})\right)^{-j}, \quad k < m,$$

where the $P_{m-k,j}$ are homogeneous polynomials of degree $m-k$ in x. Hence, for $r < r_0$, $r_0 > 0$ small, we have, denoting a generic constant depending on m, which may vary from place to place by C_m

$$\begin{aligned}
\left|\sum_{k=0}^{m-1} \binom{m}{k} r^{(k)} \tilde{v}^{(m-k)}\right| &\leq C_m \sum_{k=0}^{m-1} \left|r^{(k)}\right| \left|\tilde{v}^{(m-k)}\right| \\
&= C_m \sum_{k=0}^{m-1} \frac{|Q_k(x)|}{r^{2k-1}} \sum_{j=1}^{m-k} |P_{m-k,j}(x)| \, r^{-2(m-k)} \left(\ln(\frac{1}{r})\right)^{-j} \\
&\leq C_m \left(\ln(\frac{1}{r})\right)^{-1} \sum_{k=0}^{m-1} \frac{|Q_k(x)|}{r^{2k-1}} r^{-2(m-k)} \sum_{j=1}^{m-k} |P_{m-k,j}(x)| \\
&\leq C_m \left(\ln(\frac{1}{r})\right)^{-1} \sum_{k=0}^{m-1} \frac{r^k}{r^{2k-1}} r^{-2(m-k)} r^{m-k} = C_m \left(\ln\left(\frac{1}{r}\right)\right)^{-1} \frac{1}{r^{m-1}}. \quad (3)
\end{aligned}$$

On the other hand, one has

$$r^{(m)} \tilde{v} = \frac{Q_m(x_1, x_2, ..., x_n)}{r^{2m-1}} \ln\left(\ln\left(\frac{1}{r}\right)\right). \tag{4}$$

As Q_m is clearly not identically zero, we choose a vector $\bar{x} := (\bar{x}_1, ..., \bar{x}_n)$, such that $\frac{|Q_m(\bar{x})|}{\bar{r}^m} \geq c > 0$ holds. Here c denotes some positive constant, and $\bar{r}$ denotes the length of the vector $\bar{x}$. In the following c will denote some generic positive constant, which may vary from place to place.

Since the fraction $\frac{|Q_m(x_1, x_2, ..., x_n)|}{r^m}$ depends only on the direction of the vector $(x_1, x_2, ..., x_n)$, for example,

$$\left|\frac{Q_3(x_1, x_2, ..., x_n)}{r^3}\right| = \left|-3\frac{x_1}{r}\left(\frac{x_2^2}{r^2} + ... + \frac{x_n^2}{r^2}\right)\right|,$$

we can write $\frac{|Q_m(x)|}{r^m} = f(e_1, e_2, ..., e_n)$, where $e_i = \frac{x_i}{r}$, for all $i = 1, 2, ..., n$ with a clearly continuous function f. Hence, we can find a vicinity of $\bar{e} := (\bar{e}_1, \bar{e}_2, ..., \bar{e}_n)$, say $U(\bar{e})$ of unit vectors, say $\tilde{e}$, such that we have

$$f(\tilde{e}) \geq c > 0,$$

for all $\tilde{e} \in U(\bar{e})$. Therefore we can find a spherical cone, say S, with center vector $\bar{x}$, such that $\frac{|Q_m(x)|}{r^m} \geq c > 0$ holds, for $x \in S$. By a spherical cone we understand the intersection of an infinite cone and a ball with radius r_0. This implies

$$\left|r^{(m)} \tilde{v}\right| \geq \frac{c}{r^{m-1}} \ln\left(\ln\left(\frac{1}{r}\right)\right), \qquad x \in S \tag{5}$$

By (2),(3),(5) and the triangle inequality, we get

$$\begin{aligned}
\left|v^{(m)}\right| &= \left|r^{(m)} \tilde{v} + \sum_{k=0}^{m-1} \binom{m}{k} r^{(k)} \tilde{v}^{(m-k)}\right| \\
&\geq \left|r^{(m)} \tilde{v}\right| - \left|\sum_{k=0}^{m-1} \binom{m}{k} r^{(k)} \tilde{v}^{(m-k)}\right| \\
&\geq \frac{c}{r^{m-1}} \ln\left(\ln\left(\frac{1}{r}\right)\right) - C_m \left(\ln\left(\frac{1}{r}\right)\right)^{-1} \frac{1}{r^{m-1}} \\
&\geq \frac{c}{r^{m-1}} \ln\left(\ln\left(\frac{1}{r}\right)\right), \quad (6)
\end{aligned}$$

on S for $r \le r_0$, with small r_0 and a $c > 0$.

Finally, we find

$$\begin{aligned}\int_S \left|v^{(m)}\right|^p dx &\ge \int_S \frac{c}{r^{(m-1)p}} \left(\ln\left(\ln\left(\frac{1}{r}\right)\right)\right)^p dx \\ &= c\lambda\left(\hat{S}\cap\partial B_1\right)\int_0^{r_0} \frac{r^{n-1}}{r^{(m-1)p}}\left(\ln\left(\ln\left(\frac{1}{r}\right)\right)\right)^p dr \\ &= c\lambda\left(\hat{S}\cap\partial B_1\right)\int_0^{r_0} \frac{1}{r}\left(\ln\left(\ln\left(\frac{1}{r}\right)\right)\right)^p dr = \infty,\end{aligned}$$

where λ denotes the surface measure, and $\hat{S}$ the infinite cone corresponding to S. Hence, $v^{(m)} \notin L^p(B_{r_0})$, and therefore $v \notin W^{m,p}(B_{r_0})$. □

3 The two dimensional case

In this section we shall only consider the two-dimensional case, that is $n = 2$. We shall prove two results. On the one hand side, we show that radial-symmetric functions in $W^{m,p}(B_{r_0}) \setminus W^{1,\infty}(B_{r_0})$, with $(m-1)p = n$, $r_0 > 0$, that is, since we have $n = 2$, radial-symmetric functions in $W^{2,2}(B_{r_0}) \setminus W^{1,\infty}(B_{r_0})$ are not possible at all, if we, motivated by the examples in [1], **assume that our functions are C^2 away from the origin**.

On the other hand, we shall give a characterization of such functions in a set of "well-behaved" functions.

The reader finds these results in the next two subsections, respectively.

Let us note that we shall prove in section 4 an analogue to the result in section 3.1 for general dimension n. But as the proof in the two-dimensional setting is much easier, we found it worthwhile, to present it.

3.1 Radial-symmetric examples don't work

Let us define the set of radial-symmetric functions by

$$\mathcal{H} := \{v(x_1,x_2)|v(x_1,x_2) = f(r) \in C^2((0,r_0])\}, \;\; r_0 > 0,$$

with $r = \sqrt{x_1^2 + x_2^2}$, as usual. Then we have

Proposition 3.1 *For $n = 2$, one has*

$$W^{2,2}(B_{r_0}) \cap \mathcal{H} \subset W^{1,\infty}(B_{r_0}) \cap \mathcal{H}.$$

Proof. Let $v(x_1,x_2) = f(r) \in W^{2,2}(B_{r_0}) \cap \mathcal{H}$. We have to show that $v \in W^{1,\infty}(B_{r_0})$, that is $f' \in L^\infty((0,r_0))$.

A simple calculation gives for the partial derivatives

$$v_{x_1} = f'(r)\frac{x_1}{r} =: f'(r)\cos(\phi),$$

as well as

$$v_{x_1x_1} = f''(r)\cos^2(\phi) + \frac{f'(r)}{r}\sin^2(\phi).$$

Using our assumption that $v \in W^{2,2}(B_{r_0})$, one finds for arbitrary l_1, l_2, with $0 < l_1 < l_2 \le r_0$, and some positive finite C

$$C \;\; > \;\; \int_{B_{l_2}\setminus B_{l_1}} v_{x_1x_1}^2 \, dx_1 \, dx_2$$

$$
\begin{aligned}
&= \int_0^{2\pi} d\phi \int_{l_1}^{l_2} dr\, r \left(f''(r)^2 \cos^4(\phi) + \frac{f'(r)^2}{r^2} \sin^4(\phi) + \frac{2f'(r)f''(r)}{r} \sin^2(\phi)\cos^2(\phi) \right) \\
&= \frac{3\pi}{4} \int_{l_1}^{l_2} dr\, r \left(f''(r)^2 + \frac{f'(r)^2}{r^2} + \frac{2f'(r)f''(r)}{3r} \right) \\
&= \frac{3\pi}{4} \int_{l_1}^{l_2} dr\, r \left(f''(r) + \frac{f'(r)}{r} \right)^2 - \pi \int_{l_1}^{l_2} dr\, r \frac{f'(r)f''(r)}{r} \\
&\geq -\pi \int_{l_1}^{l_2} dr\, r \frac{f'(r)f''(r)}{r} = \frac{-\pi}{2} \left(f'(l_2)^2 - f'(l_1)^2 \right).
\end{aligned}
$$

Note that we have used the absolute continuity of the function f' in the last step, which we shall show in Remark 4.1 of section 4, for general n.

As l_1 and l_2 are arbitrary, this proves the proposition. □

3.2 Characterization of functions in $W^{2,2}(B_{r_0}) \setminus W^{1,\infty}(B_{r_0})$

We have seen that radial-symmetric functions in $\mathcal{G} := W^{2,2}(B_{r_0}) \setminus W^{1,\infty}(B_{r_0})$ do not exist, under the regularity assumption that we are considering only functions, which are C^2 away from the origin. In this section we shall show that, within a certain class of "well-behaved" functions, we can characterize functions in $\mathcal{G}$ very precisely. We look for functions V with an isolated singularity of ∇V, situated at the origin, and which are C^2 elsewhere (as we have assumed already in section 3.1). Moreover, we assume here that V can be written as a product of a regularly varying function, a product of a power and a slowly varying function (like logarithms, iterated logarithms and powers of logarithms) in the radius r and a C^2-function in the angle ϕ. Without loss of generality we assume also $V(0) = 0$. More precisely, we consider

$$
\mathcal{M} = \{V(r,\phi) | V(r,\phi) = f(r)\Phi(\phi),\ f'' \in \mathcal{R}([0,r_0]) \cap C((0,r_0]), |f'(0+)| = \infty, f(0) = 0, \Phi \in C^2[0,2\pi]\}, \tag{7}
$$

where $\mathcal{R}$ are the regularly varying functions, which we define now (for more information on these kind of functions, see for example [3]).

Definition 3.1 *A positive or negative measurable function $L(r)$ on $[0, r_0]$ is called* slowly varying*, if we have*

$$
\lim_{r \to 0} L(\lambda r)/L(r) = 1, \ \ \textit{for all } \lambda > 0.
$$

We write $L \in \mathcal{L}$.

Furthermore, we call a function $f(r)$ of regular variation*, if we can represent it as $f(r) = r^\rho L(r)$, for some $\rho \neq 0$ and some slowly varying function $L(r)$.*

Note that we allow also negative slowly varying functions, hence negative regularly varying function. This is to avoid cumbersome notation as $\pm L$. Negative representatives of these classes can therefore written as a standard representative times (-1).

Our next theorem shows that for functions in $\mathcal{G} \cap \mathcal{M}$, the function Φ has to be a linear combination of the cos and the sin function, whereas the slowly varying ingredient of f'' has to fulfill a certain equality and a certain inequality. We have

Theorem 3.1

$$
\mathcal{G} \cap \mathcal{M} = \mathcal{K},
$$

with

$$
\mathcal{K} := \{V(r,\phi) | V = f(r)\Phi(\phi), \Phi(\phi) = A\cos(\phi) + B\sin(\phi), f''(r) = \frac{L(r)}{r}\},
$$

where L is a slowly varying function in $C((0, r_0])$, fulfilling $\left| \int_0^{r_0} \frac{L(r)}{r}\, dr \right| = \infty$ and $\int_0^{r_0} \frac{L^2(r)}{r}\, dr < \infty$, whereas A and B are real constants.

Proof. Let $V(r,\phi) = f(r)\Phi(\phi) \in \mathcal{G} \cap \mathcal{M}$. By definition of the set $\mathcal{M}$, we conclude that f'' is a regularly varying function, hence, it can be written as $f''(r) = \frac{L(r)}{r^\alpha}$, for some slowly varying function L and some real constant α.
Case I: $\alpha = 1$, that is $f''(r) = \frac{L(r)}{r}$, for some slowly varying function L.
A lengthy but elementary calculation provides

$$\begin{aligned}
V_{xx}(r,\phi) &= \cos^2\phi\, V_{rr} + \frac{\sin^2\phi}{r^2} V_{\phi\phi} - \frac{\sin 2\phi}{r} V_{r\phi} + \frac{\sin^2\phi}{r} V_r + \frac{\sin 2\phi}{r^2} V_\phi \\
V_{yy}(r,\phi) &= \sin^2\phi\, V_{rr} + \frac{\cos^2\phi}{r^2} V_{\phi\phi} + \frac{\sin 2\phi}{r} V_{r\phi} + \frac{\cos^2\phi}{r} V_r - \frac{\sin 2\phi}{r^2} V_\phi \\
2V_{xy}(r,\phi) &= \sin 2\phi\, V_{rr} - \frac{\sin 2\phi}{r^2} V_{\phi\phi} + \frac{2\cos 2\phi}{r} V_{r\phi} - \frac{\sin 2\phi}{r} V_r - \frac{2\cos 2\phi}{r^2} V_\phi,
\end{aligned}$$

or, using the fact that $V \in \mathcal{M}$,

$$\begin{aligned}
V_{xx}(r,\phi) &= \cos^2\phi\, f''\Phi + \frac{\sin^2\phi}{r^2} f\Phi'' - \frac{\sin 2\phi}{r} f'\Phi' + \frac{\sin^2\phi}{r} f'\Phi + \frac{\sin 2\phi}{r^2} f\Phi' \\
V_{yy}(r,\phi) &= \sin^2\phi\, f''\Phi + \frac{\cos^2\phi}{r^2} f\Phi'' + \frac{\sin 2\phi}{r} f'\Phi' + \frac{\cos^2\phi}{r} f'\Phi - \frac{\sin 2\phi}{r^2} f\Phi' \\
2V_{xy}(r,\phi) &= \sin 2\phi\, f''\Phi - \frac{\sin 2\phi}{r^2} f\Phi'' + \frac{2\cos 2\phi}{r} f'\Phi' - \frac{\sin 2\phi}{r} f'\Phi - \frac{2\cos 2\phi}{r^2} f\Phi'.
\end{aligned}$$

Integrating $f''(r) = \frac{L(r)}{r}$, we find for $0 \le r \le r_0$, using the notation $C_1 := f'(r_0)$,

$$\begin{aligned}
f'(r) &= C_1 - \int_r^{r_0} \frac{L(s)}{s}\, ds =: C_1 - \tilde{L}(r), \\
f(r) &= C_1 r - \int_0^r \tilde{L}(s)\, ds =: C_1 r - r\hat{L}(r),
\end{aligned}$$

where we have employed $f(0) = 0$. Plugging this into the formulas for the second order derivatives above, gives, after some calculations,

$$\begin{aligned}
V_{xx}(r,\phi) &= \frac{L(r)}{r}\left(\cos^2\phi\,\Phi\right) + \frac{\hat{L}(r)}{r}\left(-\sin^2\phi\,\Phi'' - \sin 2\phi\,\Phi'\right) + \frac{\tilde{L}(r)}{r}\left(\sin 2\phi\,\Phi' - \sin^2\phi\,\Phi\right) \\
&+ \frac{C_1}{r}\left(\sin^2\phi\,\Phi'' + \sin^2\phi\,\Phi\right) \\
V_{yy}(r,\phi) &= \frac{L(r)}{r}\left(\sin^2\phi\,\Phi\right) + \frac{\hat{L}(r)}{r}\left(-\cos^2\phi\,\Phi'' + \sin 2\phi\,\Phi'\right) + \frac{\tilde{L}(r)}{r}\left(-\sin 2\phi\,\Phi' - \cos^2\phi\,\Phi\right) \\
&+ \frac{C_1}{r}\left(\cos^2\phi\,\Phi'' + \cos^2\phi\,\Phi\right) \\
2V_{xy}(r,\phi) &= \frac{L(r)}{r}\left(\sin 2\phi\,\Phi\right) + \frac{\hat{L}(r)}{r}\left(\sin 2\phi\,\Phi'' + 2\cos 2\phi\,\Phi'\right) + \frac{\tilde{L}(r)}{r}\left(-2\cos 2\phi\,\Phi' + \sin 2\phi\,\Phi\right) \\
&+ \frac{C_1}{r}\left(-\sin 2\phi\,\Phi'' - \sin 2\phi\,\Phi\right).
\end{aligned} \tag{8}$$

Now we want to express the fact that $V \notin W^{1,\infty}$ in terms of the slowly varying functions $\hat{L}$ and $\tilde{L}$. For the gradient of V we find

$$\nabla V = \begin{pmatrix} \cos\phi f'\Phi - \frac{\sin\phi}{r} f\Phi' \\ \sin\phi f'\Phi + \frac{\cos\phi}{r} f\Phi' \end{pmatrix} = \begin{pmatrix} \cos\phi(C_1 - \tilde{L})\Phi - \sin\phi(C_1 - \hat{L})\Phi' \\ \sin\phi(C_1 - \tilde{L})\Phi + \cos\phi(C_1 - \hat{L})\Phi' \end{pmatrix}. \tag{9}$$

Now,

$$\nabla V \notin L^\infty \Leftrightarrow \begin{pmatrix} -\cos\phi\tilde{L}\Phi + \sin\phi\hat{L}\Phi' \\ -\sin\phi\tilde{L}\Phi - \cos\phi\hat{L}\Phi' \end{pmatrix} \notin L^\infty,$$

or using the Euclidean norm of the vector above

$$\nabla V \notin L^\infty \Leftrightarrow \tilde{L}^2\Phi^2 + \hat{L}^2\left(\Phi'\right)^2 \notin L^\infty.$$

Now, since L and hence $\tilde{L}$, as well as $\hat{L}$, are continuous away from the origin, the above condition is equivalent to $\tilde{L}^2(0+)\Phi^2+\hat{L}^2(0+)(\Phi')^2 \notin L^\infty$, with $\tilde{L}(0+) = \lim_{r\to 0}\tilde{L}(r)$. As $\Phi \in C^2([0,2\pi])$ and non trivial, either $|\tilde{L}(0+)|$ or $|\hat{L}(0+)|$ has to be equal to infinity, which is the same as $\tilde{L}^2(0+)+\hat{L}^2(0+) = \infty$, and we end up with

$$V \notin W^{1,\infty} \Leftrightarrow \tilde{L}^2 + \hat{L}^2 \notin L^\infty. \tag{10}$$

Next, we analyse the asymptotic behaviour of the involved slowly varying functions $L, \tilde{L}, \hat{L}$: Since L is slowly varying, we conclude by [3], formula (1.5.8) that $\tilde{L}$, defined by $\tilde{L}(r) = \int_r^{r_0} \frac{L(s)}{s}\,ds$, is also slowly varying and satisfies

$$\lim_{r\to 0}\left|\frac{\tilde{L}(r)}{L(r)}\right| = \infty. \tag{11}$$

Note that the result is formulated there for the limit $r \to \infty$, but a simple inversion of the independent variable reveals, that it holds for $r \to 0$ too.

Moreover, Karamatas Theorem gives that $\hat{L}$, defined by $\hat{L}(r) = \frac{1}{r}\int_0^r \tilde{L}(s)\,ds$, fulfills

$$\lim_{r\to 0}\frac{\hat{L}(r)}{\tilde{L}(r)} = 1. \tag{12}$$

We rearrange now V_{xx} from (8) to get

$$\begin{aligned} V_{xx}(r,\phi) &= \frac{\hat{L}(r)}{r}\left(-\sin^2\phi\,\Phi'' - \sin 2\phi\,\Phi'\right) + \frac{\hat{L}(r)}{r}\left(\sin 2\phi\,\Phi' - \sin^2\phi\,\Phi\right) \\ &- \frac{\hat{L}(r)}{r}\left(\sin 2\phi\,\Phi' - \sin^2\phi\,\Phi\right) + \frac{\tilde{L}(r)}{r}\left(\sin 2\phi\,\Phi' - \sin^2\phi\,\Phi\right) + o(\frac{\hat{L}(r)}{r}) \\ &= \frac{\hat{L}(r)}{r}\left(-\sin^2\phi(\Phi''+\Phi)\right) + o(\frac{\hat{L}(r)}{r}), \end{aligned}$$

where we have used (10)-(12). Hence, necessary for $\int_{B_{r_0}} V_{xx}^2\,dx\,dy < \infty$ is, $\Phi''(\phi) + \Phi(\phi) = 0$, or

$$\Phi(\phi) = A\cos\phi + B\sin\phi, \tag{13}$$

for some real constants A and B. This implies

$$V_{xx}(r,\phi) = \frac{L(r)}{r}\left(\cos^2\phi\ \Phi\right) + \frac{\hat{L}(r)-\tilde{L}(r)}{r}\left(-\sin^2\phi\ \Phi'' - \sin 2\phi\ \Phi'\right). \tag{14}$$

Our next step is, to determine the asymptotic behaviour of $\tilde{L}(r) - \hat{L}(r)$.

Integration by parts yields

$$\hat{L}(r) = \frac{1}{r}\int_0^r \tilde{L}(s)\,ds = \frac{1}{r}\left(\tilde{L}(r)r - \int_0^r \tilde{L}'(s)s\,ds\right) = \tilde{L}(r) - \frac{1}{r}\int_0^r \tilde{L}'(s)s\,ds.$$

This provides

$$\tilde{L}(r) - \hat{L}(r) = \frac{1}{r}\int_0^r \tilde{L}'(s)s\,ds = \frac{1}{r}\int_0^r s\left(-\frac{L(s)}{s}\right)ds = -\frac{1}{r}\int_0^r L(s)\,ds \sim -L(r),$$

where the last asymptotic relation holds for $r \to 0$, and is true by Karamatas Theorem. As the ϕ-dependent coefficient functions of $\frac{L(r)}{r}$, respectively $\frac{\hat{L}(r)-\tilde{L}(r)}{r}$ are obviously linear independent, we get as necessary and sufficient condition for $\int_{B_{r_0}} V_{xx}^2\,dx\,dy < \infty$ the validity of (13) and

$$\int_0^{r_0}\frac{L(r)^2}{r}\,dr < \infty. \tag{15}$$

Because of (12), (10) is equivalent to $|\tilde{L}(0+)| = \infty$, which by definition of $\tilde{L}$ is equivalent to

$$\left|\int_0^{r_0}\frac{L(r)}{r}\,dr\right| = \infty. \tag{16}$$

(13),(15) and (16) prove the Theorem in Case I, since the argument for V_{yy} and $2V_{xy}$ works out in an identical way.

Case II: $\alpha \in (1,2)$, that is $f''(r) = \frac{L(r)}{r^\alpha}$, for some slowly varying function L.

Karamatas theorem gives

$$\frac{f'(r)}{r} \sim \frac{C_1(\alpha)}{r^\alpha} L(r),$$
$$\frac{f(r)}{r^2} \sim \frac{C_2(\alpha)}{r^\alpha} L(r),$$

for $r \to 0$, with known constants C_i, depending on α. Using the expressions for the second order derivatives above, gives, after some elementary calculations,

$$\Delta V \sim \frac{L(r)}{r^\alpha} \left\{ \frac{\alpha - 2}{\alpha - 1} \Phi + \frac{1}{(\alpha-1)(\alpha-2)} \Phi'' \right\}.$$

If we want the Laplacean of V to be square integrable, the curly bracket has to be identically zero, which gives

$$\Phi'' + (2-\alpha)^2 \Phi = 0, \tag{17}$$

or

$$\Phi(\phi) = A \cos{(2-\alpha)\phi} + B \sin{(2-\alpha)\phi}, \tag{18}$$

for some real constants A and B.

For $2V_{xy}$ we get analogously

$$2V_{xy} \sim \frac{L(r)}{r^\alpha} \left\{ \frac{\alpha}{\alpha-1} \Phi \sin 2\phi + \frac{2}{(2-\alpha)} \Phi' \cos 2\phi + \frac{1}{(\alpha-1)(2-\alpha)} \Phi'' \sin 2\phi \right\}.$$

Again, the curly bracket has to vanish identically, and one shows, using (17), that this is equivalent to $(2-\alpha)\Phi + \cot 2\phi \Phi' = 0$. Solving this ODE, gives

$$\Phi(\phi) = C(\cos 2\phi)^{(2-\alpha)/2},$$

with some constant C, obviously contradicting (18). Hence, Case II is impossible.

Case III: $\alpha = 2$, that is $f''(r) = \frac{L(r)}{r^2}$, for some slowly varying function L.

We have the following expression for the Laplace operator $\Delta V = f''(r)\Phi(\phi) + \frac{f'(r)}{r}\Phi(\phi) + \frac{f(r)}{r^2}\Phi''(\phi)$. As $f''(r) = \frac{L(r)}{r^2}$, we have by Karamatas theorem $f'(r) = -\frac{\bar{L}(r)}{r}$, with $\bar{L}(r) \sim L(r)$, for $r \to 0$. Moreover, $f(r_0) - f(r) = -\int_r^{r_0} \frac{\bar{L}(s)}{s}\, ds$ holds. Hence, we can conclude, as for (11), that $\lim_{r\to 0} |f(r)/\bar{L}(r)| = \infty$ holds. Finally, we find

$$\lim_{r\to 0} \left| \frac{f(r)/r^2}{f'(r)/r} \right| = \lim_{r\to 0} \left| \frac{f(r)}{r f'(r)} \right| = \lim_{r\to 0} \left| \frac{f(r)}{\bar{L}(r)} \right| = \infty,$$

as well as

$$\lim_{r\to 0} \left| \frac{f'(r)/r}{f''(r)} \right| = \lim_{r\to 0} \left| \frac{\bar{L}(r)/r^2}{L(r)/r^2} \right| = 1.$$

The last two relations imply

$$\Delta V \sim \frac{f}{r^2} \Phi'',$$

which gives easily that ΔV is not square integrable. Therefore, Case III is impossible too.

Case IV: $\alpha > 2$, that is $f''(r) = \frac{L(r)}{r^\alpha}$, for some slowly varying function L.

Here we have

$$|\nabla V|^2 = (f')^2 \Phi^2 + \frac{f^2}{r^2} (\Phi')^2 > (f')^2 \Phi^2,$$

which shows $V \notin W^{1,2}$, hence $V \notin W^{2,2}$. Case IV is not possible too.

Case V: $\alpha < 1$, that is $f''(r) = \frac{L(r)}{r^\alpha}$, for some slowly varying function L.

This gives immediately that $\nabla V \in L^\infty(B_{r_0})$ holds. Summing up, the only interesting case is case I, which we have handled above.

Up to now we have proved that the functions in $\mathcal{K}$ are exactly those, fulfilling the proper integrability conditions. Finally, it remains to show that the classical derivatives, which exist away from the origin, are the correct distributional derivatives (see [1], Definition 3.2b). This means, that we have to show, see for example [8], 6.13 (2),

$$\int_{B_{r_0}} V_{xx}\rho\, dx\, dy = (-1)^2 \int_{B_{r_0}} V\rho_{xx}\, dx\, dy, \tag{19}$$

for all $\rho \in C_0^\infty(B_{r_0})$, the other second derivatives working out analogously. Hence, we have to show

$$\lim_{\epsilon\to 0+} \int_{B_{(r_0,\epsilon)}} V_{xx}\rho\, dx\, dy = \int_{B_{r_0}} V\rho_{xx}\, dx\, dy,$$

where $B_{(r_0,\epsilon)} := B_{r_0} \setminus B_\epsilon$ holds. We use now the divergence theorem, see for example [5], chapter 0, equation (1.1),

$$\int_\Omega \operatorname{div} \vec{F} = \int_{\partial\Omega} \vec{F}\,\vec{n}\, dS,$$

with $\Omega := B_{(r_0,\epsilon)}$, $\vec{F} := (\rho V_x, 0)$ and $\vec{n}$ denoting the exterior normal unit vector to the boundary $\partial\Omega$. This gives

$$\lim_{\epsilon\to 0+} \int_{B_{(r_0,\epsilon)}} V_{xx}\rho\, dx\, dy = -\lim_{\epsilon\to 0+} \int_{B_{(r_0,\epsilon)}} V_x\rho_x\, dx\, dy + \lim_{\epsilon\to 0} \int_{\partial\Omega} \rho V_x\, n_1\, dS,$$

where n_1 denotes the first component of the vector $\vec{n}$. Using on the one hand side the explicit form of V_x, given in (9), and on the other hand side that $\rho \in C_0^\infty(B_{r_0})$ holds, shows that the last limit vanishes. Indeed,

$$\begin{aligned}
\left|\int_{\partial\Omega} \rho V_x n_1\, dS\right| &\leq \int_{\partial B_{r_0}} |\rho V_x n_1|\, dS + \int_{\partial B_\epsilon} |\rho V_x n_1|\, dS = \int_{\partial B_\epsilon} |\rho V_x n_1|\, dS \\
&\leq ||\rho||_{L^\infty} \int_{\partial B_\epsilon} \left(D_1 + D_2|\tilde{L}| + D_3|\hat{L}|\right) dS \\
&= ||\rho||_{L^\infty} \left(D_1 + D_2|\tilde{L(\epsilon)}| + D_3|\hat{L(\epsilon)}|\right) \lambda(\epsilon),
\end{aligned}$$

where the D_i depend on the constant C_1 of (9) and the C^1-norm of Φ, and λ denotes the surface measure of ∂B_ϵ. As $\lambda(\epsilon) = 2\epsilon\pi$, and $\tilde{L}(\epsilon)$, respectively $\hat{L}(\epsilon)$, are slowly varying, which means that they grow slower than any power, the last expression tends to zero for $\epsilon \to 0+$.

This provides

$$\lim_{\epsilon\to 0+} \int_{B_{(r_0,\epsilon)}} V_{xx}\rho\, dx\, dy = -\lim_{\epsilon\to 0+} \int_{B_{(r_0,\epsilon)}} V_x\rho_x\, dx\, dy.$$

Using the divergence theorem once again, shows

$$\lim_{\epsilon\to 0+} \int_{B_{(r_0,\epsilon)}} V_{xx}\rho\, dx\, dy = \lim_{\epsilon\to 0+} \int_{B_{(r_0,\epsilon)}} V\rho_{xx}\, dx\, dy = \int_{B_{r_0}} V\rho_{xx}\, dx\, dy,$$

concluding our proof. □

We construct now an example of a function in $W^{2,2}(B_{r_0}) \setminus W^{1,\infty}(B_{r_0})$.

Example 3.1 *We set in Theorem 3.1, $B = 0$, $A = 1$, $L(r) = 1/(-\ln r)$ and $r_0 = 1/2$, that is $f''(r) = \frac{1}{r(-\ln r)}$. Integrating twice and using $f(0) = 0$, provides*

$$f(r) = Dr + r\ln(-\ln r) + Ei_1(-\ln r),$$

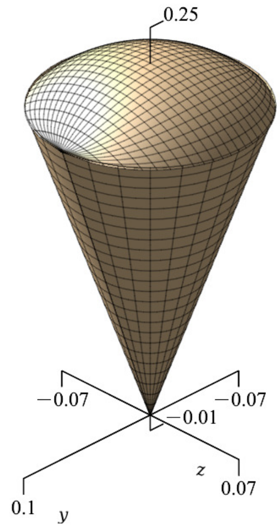

Figure 1: a spherical cone in 3 dimensions with central vector $(1, 0, 0)$

for some constant D, and Ei_1 denoting the exponential integral, defined by $Ei_1(z) := \int_1^\infty \frac{e^{-kz}}{k}\, dk$, see, for example, [2]. Hence,

$$V(r, \phi) = \cos\phi\left(Dr + r\ln(-\ln r) + Ei_1(-\ln r)\right).$$

4 The $n-$ dimensional case

In this section we show an analogue to Proposition 3.1 in n dimensions, that is, we claim, using

$$\mathcal{H} := \{V(x_1, x_2, ..., x_n) | V(x_1, x_2, ..., x_n) = f(r) \in C^2((0, r_0])\},$$

Theorem 4.1 *For $(m-1)p = n$, we have*

$$W^{m,p}(B_{r_0}) \cap \mathcal{H} \subset W^{1,\infty}(B_{r_0}) \cap \mathcal{H}.$$

Proof.
Case 1: $m = 2, p = n$**.** We start the proof in this case with the following

Remark 4.1 *For a function $V \in W^{2,p}(B_{r_0})$, one has $\nabla V \in W^{1,p}(B_{r_0})$. Hence, if V is radial-symmetric, one has $f'(r)\frac{x_i}{r} \in W^{1,p}(B_{r_0})$ for all $i = 1, 2, ...n$, hence $f'(r) \in W^{1,p}([l_1, l_2])$, or $f'(r)^p \in W^{1,1}([l_1, l_2])$ for arbitrary $0 < l_1 < l_2 \leq r_0$.*

We can now apply [10], Theorem 2.1.4, to get an absolutely continuous version of $f'(r)$ on $[l_1, l_2]$, which we shall use in the sequel.

Let $V(x_1, x_2, ..., x_n) \in W^{2,n}(B_{r_0}) \cap \mathcal{H}$ and we continue with the calculation of

$$V_{x_1x_1} = f''(r)\frac{x_1^2}{r^2} + \frac{f'(r)}{r}\frac{\overline{r}^2}{r^2} =: f''(r)\cos^2\phi_1 + \frac{f'(r)}{r}\sin^2\phi_1,$$

where we have used $\overline{r}^2 := x_2^2 + ... + x_n^2$.

Consider now $(n+1)$ disjunct, congruent, truncated $n-$dimensional spherical cones $S_i, i = 1, 2, ..., n+1$. By a truncated spherical cone we mean the difference of two spherical cones, with identical central vector and radius l_2, resp. l_1, with $0 < l_1 < l_2 \le r_0$. Moreover, we denote the first component of the unit central vectors by $0 \le a_{n+1} < a_n < ... < a_1 \le 1$. Finally, we denote the surface measure of the spherical part of our spherical cones with radius equal to one, by δ^{n-1}, where δ is small. More precisely: Let $\hat{S}_i$ be the spherical cone with radius equal to one, corresponding to S_i. δ is a small quantity, chosen in a way such that the surface measure of the spherical part of $\hat{S}_i$, that is $\hat{S}_i \cap B_1 =: F_i$, is equal to δ^{n-1}. The reader finds a picture of a spherical cone in 3 dimensions with central vector $(1,0,0)$ in Figure 1 above (a truncated spherical cone being just the difference of 2 such objects with identical central vector). Moreover, we provide in Figure 2 below a plot of the set F_1, for the same central vector. Note that we have used [6] to generate the figures. Let $C > 0$ be some positive constant. Then we calculate, using the well known volume element in spherical coordinates,

$$C > \int_{S_i} |V_{x_1x_1}|^n \, dx_1 \cdots dx_n \ge \int_{S_i} V_{x_1x_1}^n \, dx_1 \cdots dx_n =$$

$$\int_{l_1}^{l_2} r^{n-1}\, dr \int_{\rho_i} \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \cdot\cdot \sin\phi_{n-2} \left(f''(r)\cos^2\phi_1 + \frac{f'(r)}{r}\sin^2\phi_1\right)^n d\phi_1 \cdot\cdot d\phi_{n-1} > -C,$$

where $\phi_{n-1} \in [0, 2\pi]$, and $\phi_i \in [0,\pi]$, $i = 1, ..., n-2$ holds, and ρ_i denotes the area, where the angles vary to produce the sets F_i, defined above. We give a description of the set ρ_1 in spherical coordinates in 3 dimensions in the remark after the proof, assuming that the central vector is given by $(1,0,0)$ as above.

Now, the last but one term in the previous chain of inequalities, can be written as

$$\delta^{n-1} \sum_{k=0}^{n} \int_{l_1}^{l_2} dr\, r^{n-1} \binom{n}{k} f''(r)^k c_i^k \frac{f'(r)^{n-k}}{r^{n-k}} s_i^{n-k} =: \lambda_i(\delta). \tag{20}$$

Indeed, we have

$$\int_{l_1}^{l_2} r^{n-1}\, dr \int_{\rho_i} \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \cdot\cdot \sin\phi_{n-2} \left(f''(r)\cos^2\phi_1 + \frac{f'(r)}{r}\sin^2\phi_1\right)^n d\phi_1 \cdot\cdot d\phi_{n-1} =$$

$$\sum_{k=0}^{n} \int_{l_1}^{l_2} r^{n-1}\, dr \int_{\rho_i} \sin^{n-2}\phi_1 \cdot\cdot \sin\phi_{n-2} \binom{n}{k} f''(r)^k \cos^{2k}\phi_1 \frac{f'(r)^{n-k}}{r^{n-k}} \sin^{2(n-k)}\phi_1 \, d\phi_1 \cdot\cdot d\phi_{n-1} =$$

$$\sum_{k=0}^{n} \int_{l_1}^{l_2} r^{n-1} \binom{n}{k} f''(r)^k \frac{f'(r)^{n-k}}{r^{n-k}}\, dr \int_{\rho_i} \sin^{n-2}\phi_1 \cdot\cdot \sin\phi_{n-2} \cos^{2k}\phi_1 \sin^{2(n-k)}\phi_1 \, d\phi_1 \cdot\cdot d\phi_{n-1} =$$

$$\sum_{k=0}^{n} \int_{l_1}^{l_2} r^{n-1} \binom{n}{k} f''(r)^k \frac{f'(r)^{n-k}}{r^{n-k}} c_i^k s_i^{n-k}\, dr \int_{\rho_i} \sin^{n-2}\phi_1 \cdot\cdot \sin\phi_{n-2}\, d\phi_1 \cdot\cdot d\phi_{n-1} =$$

$$\sum_{k=0}^{n} \int_{l_1}^{l_2} r^{n-1} \binom{n}{k} f''(r)^k \frac{f'(r)^{n-k}}{r^{n-k}} c_i^k s_i^{n-k}\, dr\ \delta^{n-1},$$

where we have used in the last but one equality the mean value theorem. Since the cones become narrower and narrower with decreasing δ, c_i converges for $\delta \to 0$ to $a_i^2 =: b_i \in [0,1]$, which was defined above. Analogously, $s_i \to 1 - a_i^2 = 1 - b_i$ holds, for $\delta \to 0$.

Moreover, we have

$$|\lambda_i(\delta)| \le C,$$

uniformly in δ and the l_i. Now, (20) can be written as

$$\sum_{k=0}^{n} \binom{n}{k} c_i^k s_i^{n-k} \gamma_k = \frac{\lambda_i(\delta)}{\delta^{n-1}},\ i = 1, ..., n+1, \tag{21}$$

with $\gamma_k = \int_{l_1}^{l_2} r^{k-1} f''(r)^k f'(r)^{n-k}\, dr$. So this is a linear system for the γ_k, with a coefficient matrix, which converges for $\delta \to 0$ to

$$\begin{pmatrix} b_1^0(1-b_1)^n\binom{n}{0} & b_1^1(1-b_1)^{n-1}\binom{n}{1} & ... & b_1^n(1-b_1)^0\binom{n}{n} \\ b_2^0(1-b_2)^n\binom{n}{0} & b_2^1(1-b_2)^{n-1}\binom{n}{1} & ... & b_2^n(1-b_2)^0\binom{n}{n} \\ ... & ... & ... & ... \\ b_{n+1}^0(1-b_{n+1})^n\binom{n}{0} & b_{n+1}^1(1-b_{n+1})^{n-1}\binom{n}{1} & ... & b_{n+1}^n(1-b_{n+1})^0\binom{n}{n} \end{pmatrix}.$$

Elementary linear algebra shows that the determinant of this matrix is given by

$$D_n \begin{vmatrix} 1 & b_1 & ... & b_1^n \\ 1 & b_2 & ... & b_2^n \\ ... & ... & ... & ... \\ 1 & b_{n+1} & ... & b_{n+1}^n \end{vmatrix},$$

with $D_n := \binom{n}{0}\binom{n}{1}\cdots\binom{n}{n}$. Now, the last determinant is the Van der Monde determinant, which is well known. Summarizing, we get as value for the determinant of our coefficient matrix $D_n \Pi_{1\le i<j\le n+1}(b_j - b_i) \neq 0$.

Hence, for δ sufficiently small, say equal to δ_0, the determinant is still different from zero. So we can solve our system and get a finite solution for our γ_k, in particular we get, uniformly in the l_i, $|\gamma_1| \le C$, for some generic positive constant C(depending on δ_0), that is

$$\left| \int_{l_1}^{l_2} f''(r) f'(r)^{n-1}\, dr \right| < C,$$

uniformly in the l_i. By Remark 4.1, f' is absolutely continuous, hence, we get

$$|(f')^n(l_2) - (f')^n(l_2)| < C,$$

for arbitrary l_i. We conclude $\nabla V \in L^\infty(B_{r_0})$, hence $V \in W^{1,\infty}(B_{r_0})$, finishing our proof for case 1.
Case 2 - the general case: $(m-1)p = n, m > 2$**.** We apply [1], Theorem 4.12, Case C, (4): This yields, denoting the values there by j^A, and so on, and setting $j^A = m-1, m^A = 1, p^A = p$,

$$W^{m,p}(B_{r_0}) \subset W^{m-1, p^* = \frac{np}{n-p}}(B_{r_0}).$$

One easily calculates that $(m-2)p^* = n$ holds, such that the new indices $(m-1, p^*)$ also satisfy the defining relation for the critical case. By induction one gets

$$W^{m,p}(B_{r_0}) \subset W^{2,n}(B_{r_0}).$$

This finally shows

$$W^{m,p}(B_{r_0}) \cap \mathcal{H} \subset W^{2,n}(B_{r_0}) \cap \mathcal{H} \subset W^{1,\infty}(B_{r_0}) \cap \mathcal{H},$$

where the latter inclusion holds by Case 1, concluding our proof. □

Remark 4.2 *As announced above, we provide here a description of the set* ρ_1 *in 3 dimensions for the central vector* $(1,0,0)$ *in polar coordinates, which are defined as*

$$x_1 = r\cos\phi_1,$$
$$x_2 = r\sin\phi_1\cos\phi_2,$$
$$x_3 = r\sin\phi_1\sin\phi_2,$$

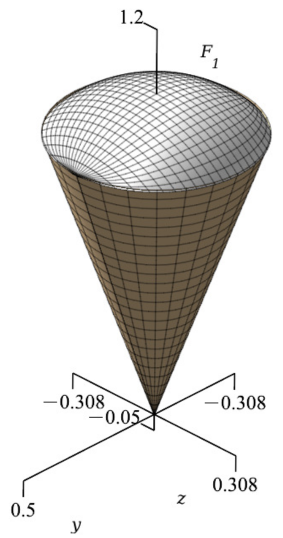

Figure 2: a plot of the set F_1 (white color), for the central vector $(1, 0, 0)$.

where $\phi_1 \in [0, \pi]$ *and* $\phi_2 \in [0, 2\pi]$ *hold. The set* ρ_1 *is defined as* $\rho_1 := \{(\phi_1, \phi_2) | \phi_1 \in [0, \nu], \phi_2 \in [0, 2\pi]\}$*, for a small quantity* ν*.*

Finally, the connection between ν*, defined here, and* δ*, defined in the proof above, can easily be calculated, using the equation*

$$\int_0^{\nu} \int_0^{2\pi} \sin\phi_1 \, d\phi_1 d\phi_2 = \delta^2,$$

providing $\delta = \sqrt{2\pi(1 - \cos\nu)} \sim \sqrt{\pi}\nu$*, where the last asymptotics holds for* $\nu \to 0+$*.*

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