Glasnik Matematički

SERIJA III

www.math.hr/glasnik

Tobias Hilgart and Volker Ziegler

On a conjecture of Levesque and Waldschmidt

Manuscript accepted
February 20, 2025.

ON A CONJECTURE OF LEVESQUE AND WALDSCHMIDT

Tobias Hilgart and Volker Ziegler
University of Salzburg, Austria

Abstract. One of the first parametrised Thue equations,

$$\left|X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3\right| = 1,$$

over the integers was solved by E. Thomas in 1990. If we interpret this as a norm-form equation, we can write this as

$$\left|N_{K/\mathbb{Q}}(X - \lambda_0 Y)\right| = \left|(X - \lambda_0 Y)(X - \lambda_1 Y)(X - \lambda_2 Y)\right| = 1$$

if $\lambda_0, \lambda_1, \lambda_2$ are the roots of the defining irreducible polynomial, and K the corresponding number field.

Levesque and Waldschmidt twisted this norm-form equation by an exponential parameter s and looked, among other things, at the equation

$$\left|N_{K/\mathbb{Q}}(X - \lambda_0^s Y)\right| = 1.$$

They solved this effectively and conjectured that introducing a second exponential parameter t and looking at $\left|N_{K/\mathbb{Q}}\left(X - \lambda_0^s \lambda_1^t Y\right)\right| = 1$ does not change the effective solvability.

We want to partially confirm this if

$$\min\{|2s-t|, |2t-s|, |s+t|\} > \varepsilon \cdot \max\{|s|, |t|\} > 2,$$

i.e. the two exponents do not almost cancel in specific cases.

1. Motivation and Statement

Ever since the work of Baker [1], we know how to effectively solve any Thue equation, i.e. a diophantine equation $F(X, Y) = m$ over the integers, where F is an irreducible homogenous polynomial of degree at least 3. And if we can solve one, we can solve a finite number of them.

Thomas [9] was amongst the first to study infinitely many (non-binary) Thue equations by looking at parametrised equations $F_n(X, Y) = \pm 1$, where

---

2020 *Mathematics Subject Classification.* 11D25, 11D57.
Key words and phrases. Parametrised Thue Equations, Exponential Diophantine Equations.

$F_n \in \mathbb{Z}[n][X,Y]$ gives a Thue equation for each $n \in \mathbb{N}$. He proved that

$$F_n(X,Y) = X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3 = \pm 1 \tag{1.1}$$

has only a few trivial solutions for large parameters n while giving a complete list of all solutions for small parameters n. Because of the symmetry $F_{-n-1}(X,Y) = F_n(-Y,-X)$, only positive integers n need to be considered.

Levesque and Waldschmidt [6] took this equation one step further and *twisted* it by an exponential parameter t in the following way: We can write Thue equations as norm-form equations, e.g. we first factorise the polynomial $F_n(X,1)$ of Equation (1.1) into its complex roots $\lambda_0, \lambda_1, \lambda_2$ (and forego to denote their dependency on the parameter n). Then Equation (1.1) can be written as

$$N_{K/\mathbb{Q}}(X - \lambda_0 Y) = \pm 1,$$

where $K = \mathbb{Q}(\lambda_0)$ and $N_{K/\mathbb{Q}}$ denotes the norm relative to $K/\mathbb{Q}$. Levesque and Waldschmidt *twisted* this equation by an integer t and looked, among other things, at the equation

$$N_{K/\mathbb{Q}}\left(X - \lambda_0^t Y\right) = \pm 1,$$

for which they managed to prove that there exists an effectively computable number κ such that any solution (x,y,n,t) to the above equation with either $|x| \geq 2$ or $|y| \geq 2$ satisfies $\max\{|x|,|y|,|n|,|t|\} < \kappa$, i.e. there are but finitely many solutions and we, at least in theory, could find them all.

One conjecture Levesque and Waldschmidt posed in their paper is whether a *twist* by multiple exponential parameters, i.e. the norm-form equation

$$N_{K/\mathbb{Q}}\left(X - \lambda_0^s \lambda_1^t Y\right) = \pm 1$$

can be tackled analogously, working with the existing tools used to study Thue equations—a conjecture we want to give a partial positive answer for in this paper. Our theorem is as follows:

Theorem 1.1. *Let $\lambda_0 = \lambda_0(n), \lambda_1 = \lambda_1(n), \lambda_2 = \lambda_2(n)$ be the roots of the integer polynomial*

$$f(X) = f(X;\, n) = X^3 - (n-1)X^2 - (n+2)X - 1$$

and

$$F(X,Y) = F(X,Y;\, n,s,t) = \left(X - \lambda_0^s\lambda_1^t Y\right)\left(X - \lambda_1^s\lambda_2^t Y\right)\left(X - \lambda_2^s\lambda_0^t Y\right).$$

Let $(x,y;n,s,t) \in \mathbb{Z}^5$ be a solution to the Thue-Equation

$$|F(X,Y)| = 1, \tag{1.2}$$

with $|y| \geq 2, n \geq 3$, $st \neq 0$ and let us assume that for a given $\varepsilon > 0$ we have

$$\min\left(|2s-t|, |2t-s|, |s+t|\right) > \varepsilon \cdot \max\left(|s|,|t|\right) > 2. \tag{1.3}$$

Then there exists an effectively computable constant $\kappa > 0$, depending only on ε, such that

$$\max\{|x|,|y|,n,|s|,|t|\} < \kappa.$$

2. Outline of the Proof

We strive to improve the readability of our paper by separating the conceptual proof from the necessary detail work. Thus we first give an outline of the proof of Theorem 1.1, which follows the general strategy to solve Thue equations via Baker bounds on linear forms in logarithms. At the intermediate steps that require careful technical consideration, we refer to the corresponding lemmas in the subsequent section, where we also collect other auxiliary results that we use in the process.

Let us start by briefly mentioning the degree of restriction we placed on the theoretical solution $(x, y; n, s, t)$ of Equation (1.2). There are infinitely many solutions where $|y| \le 2$, e.g. $(1, 0; n, s, t)$ and $(0, 1; n, s, t)$, hence the condition $|y| \ge 2$ is necessary. Due to the symmetry $F_{-n-1}(X, Y) = F_n(-Y, -X)$, we only have to consider $n \ge 3$, while we drop the n smaller than 3 for $\log(n)$ and $\log(\log(n))$ to be positive (and indeed defined at all over the real numbers). But we could alternatively take $n \ge 3$ to be the absolute value of the integer parameter and conduct the same proof, where we just have to be mindful of taking λ_0 in absolute value as well. Finally, Condition (1.3) is the degree to which we could solve Levesque and Waldschmidt's conjecture and can be read thus: No quotient of two conjugates of $\lambda_0^s\lambda_1^t$ is allowed to be too close to 1, for the whole argument falls apart otherwise.

We want to improve the readability of the terms and equations and thus introduce several short hands. First, we want to hide constants in our expressions that do not affect their asymptotic without immediately resorting to O-notations. So when we write $z \ll_\delta z'$, we mean $z = O(z')$, i.e. that there exists an effectively computable constant $c(\delta)$ which does not depend on z or z', but which may depend on δ such that $z \le c(\delta)\, z'$. If both $z \ll z'$ and $z' \ll z$, we write $z = \theta(z')$.

Let furthermore $\tau := \max(|s|, |t|)$ and

$$\begin{aligned} &\alpha_0 := \lambda_0^s\lambda_1^t, \qquad \alpha_1 := \lambda_1^s\lambda_2^t, \qquad \alpha_2 := \lambda_2^s\lambda_0^t,\\ &\beta_0 := x - \alpha_0 y,\ \ \beta_1 := x - \alpha_1 y,\ \ \beta_2 := x - \alpha_2 y. \end{aligned} \tag{2.4}$$

In this notation, α_i, β_i resp., is the image of α_0, β_0 resp., under the automorphism that maps λ_0 to λ_i. The numeration of the roots $\lambda_0, \lambda_1, \lambda_2$ is chosen such that λ_0 is close to n, λ_1 is close to 0 and λ_2 is close to -1. The exact inequalities used are given here in Lemma 3.2, from Lemma 10 of [5].

One particular β_i stands out as being very small, or $\frac{x}{y}$ is a good rational approximation to one particular α_i respectively. We denote this as β_j, i.e. $|\beta_j| = \min\{|\beta_0|, |\beta_1|, |\beta_2|\}$ and the remaining two, in no particular order, as

β_k and β_l. For the corresponding α_i, we define

$$\alpha_{j,k} = \begin{cases} \alpha_j, & \text{if } |\alpha_j| > |\alpha_k| \\ \alpha_k, & \text{if } |\alpha_k| > |\alpha_j| \end{cases} \tag{2.5}$$

and $\alpha_{j,l}$ analogously.

By the triangle inequality, we have that

$$2\,|\beta_k| \geq |\beta_k - \beta_j| = |y\,(\alpha_j - \alpha_k)|\,, \tag{2.6}$$

same for β_l. We have to be mindful of the difference between two of the algebraic numbers, e.g. α_j, α_k, not becoming too small, which we do in Lemma 3.5. Having done that, we can use the above inequality in combination with $|\beta_j\beta_k\beta_l| = |F(x,y)| = 1$ to derive the upper bound

$$|\beta_j| \leq \frac{1}{|\beta_k\beta_l|} \ll \frac{1}{|y|^2\,|\alpha_j - \alpha_k|\,|\alpha_j - \alpha_l|} \overset{\text{Lemma 3.5}}{\ll} \frac{1}{|y|^2\,|\alpha_{j,k}|\,|\alpha_{j,l}|}.$$

Using Lemma 3.6, the simple property that the product of pairwise maxima of positive real numbers multiplying to 1 is bounded from below by an appropriate root of the global maximum, we get

$$\max\{|\alpha_j|\,,|\alpha_k|\}\max\{|\alpha_j|\,,|\alpha_l|\} \geq \sqrt{\max\{|\alpha_0|\,,|\alpha_1|\,,|\alpha_2|\}}.$$

We ascertain that at least the largest of the algebraic numbers α_i behaves as expected, and indeed, by Lemma 3.7, the above square root is at least $n^{\frac{c_1}{2}\tau}$. We thus get that our bound for $|\beta_j|$ is indeed small, namely

$$|\beta_j| \ll \frac{1}{|y|^2\,|\alpha_{j,k}|\,|\alpha_{j,l}|} \ll \frac{1}{|y|^2\,n^{\frac{c_1}{2}\tau}}. \tag{2.7}$$

We then use this information to rewrite Siegel's Identity into a unit equation where we now know one addend to be very small. Stating the identity

$$\beta_j(\alpha_k - \alpha_l) + \beta_l(\alpha_j - \alpha_k) + \beta_k(\alpha_l - \alpha_j) = 0 \tag{2.8}$$

and dividing by the last addend with flipped sign gives

$$\frac{\beta_j}{\beta_k}\frac{\alpha_k - \alpha_l}{\alpha_j - \alpha_l} + \frac{\beta_l}{\beta_k}\frac{\alpha_j - \alpha_k}{\alpha_j - \alpha_l} = 1. \tag{2.9}$$

The first addend on the left-hand side is very small: We use the inequality from Equation (2.6) again, in combination with the bound from Inequality (2.7) derived above to get

$$\left|\frac{\beta_j}{\beta_k}\frac{\alpha_k - \alpha_l}{\alpha_j - \alpha_l}\right| \ll \frac{|\alpha_k - \alpha_l|}{|y|^3\,n^{\frac{c_1}{2}\tau}\,|\alpha_j - \alpha_k|\,|\alpha_j - \alpha_l|} \ll \frac{\max\{|\alpha_k|\,,|\alpha_l|\}}{|y|^3\,n^{\frac{c_1}{2}\tau}\,|\alpha_{j,k}|\,|\alpha_{j,l}|}.$$

We differentiate the cases for whether α_j has the maximal absolute value or not in Lemma 3.8 and resolve the quotients of the various maxima with an upper bound of $2|y|$. We thus have

$$\left|\frac{\beta_j}{\beta_k}\frac{\alpha_k-\alpha_l}{\alpha_j-\alpha_l}\right| \ll \frac{1}{|y|^2 n^{\frac{c_1}{2}\tau}},$$

and unless both n and τ are small, we can assume this to be smaller than $\frac{1}{2}$.

Note that neither of the two addends in Equation (2.9) is zero: None of the β_i is since each α_i is irrational, and any two α_p, α_q are distinct. The latter fact of which can be seen by looking at Equation (3.12) from the proof of Lemma 3.5, together with the fact that $\log \lambda_0, \log|\lambda_2|$ are $\mathbb{Q}$-linearly independent, since λ_0, λ_2 give a fundamental system of units for $\mathbb{Z}[\lambda_0]$ by Proposition 3.4, and $st \neq 0$.

This gives for Equation (2.9) that

$$\frac{\beta_l}{\beta_k}\frac{\alpha_j-\alpha_k}{\alpha_j-\alpha_l} = 1 - \frac{\beta_j}{\beta_k}\frac{\alpha_k-\alpha_l}{\alpha_j-\alpha_l} \neq 1,$$

and taking the logarithm, using the relation $|\log(1-v)| \leq 2|v|$ for $|v| \leq \frac{1}{2}$, yields

$$0 < \left|\log\left|\frac{\beta_l}{\beta_k}\right| + \log\left|\frac{\alpha_j-\alpha_k}{\alpha_j-\alpha_l}\right|\right| \leq 2\left|\frac{\beta_j}{\beta_k}\frac{\alpha_k-\alpha_l}{\alpha_j-\alpha_l}\right| \ll \frac{1}{|y|^2 n^{\frac{c_1}{2}\tau}}, \tag{2.10}$$

and we call the linear form $\log\left|\frac{\beta_l}{\beta_k}\right| + \log\left|\frac{\alpha_j-\alpha_k}{\alpha_j-\alpha_l}\right| = \Lambda$.

With some extra effort exerted in Lemma 4.1, this allows us to directly derive an upper bound for $\log|y|$, without troubling, say, Bugeaud and Győry's result [4], which holds much more generally. We get that

$$\log|y| \ll \tau(\log n)^3(\log\tau + \log\log n).$$

Returning to the linear form in logarithms Λ, we managed to derive the bound for $\log|y|$ by shifting the dependence on the exponents s, t from the logarithm of $\left|\frac{\beta_l}{\beta_k}\right|$ into its coefficient. If we manage to do the same for the second logarithm $\log\left|\frac{\alpha_j-\alpha_k}{\alpha_j-\alpha_l}\right|$ as well, then the tools we used before will give us, also using the bound for $\log|y|$, an absolute bound for the parameters n and τ, thus proving the theorem.

To that end, we extract $\alpha_{j,k}$ from the deonimantor and $\alpha_{j,l}$ from the numerator. We can then separate the factor $\log\left|\frac{\alpha_{j,k}}{\alpha_{j,l}}\right|$, and the remaining logarithm is at most $4n^{-c_1\tau}$ by Lemma 3.5. We shift this into the upper bound and get a new small linear form in logarithms Λ', where

$$|\Lambda'| = \left|\log\left|\frac{\beta_l}{\beta_k}\right| + \log\left|\frac{\alpha_{j,k}}{\alpha_{j,l}}\right|\right| \ll \frac{1}{n^{\frac{c_1}{2}\tau}}. \tag{2.11}$$

Note that we can no longer argue that Λ' is non-zero by construction. We must argue that it is indeed non-zero, which we do at length in Lemma 4.2.

We end up either way with a (possibly different) linear form Λ'' in logarithms of $\beta_k, \beta_l, \alpha_j, \alpha_k, \alpha_l$ and 2, for which inequalities of the form

$$0 < |\Lambda''| < \frac{1}{n^{\frac{c_1}{2}\tau}}$$

hold again. We can now again explicitly shift the dependency on s, t into the coefficients and write $\Lambda'' = A \log|\lambda_0| + B \log|\lambda_2| - C \log 2$, where A, B are linear-combinations of s, t and C is 0 or 1. We can use the bounds for the coefficients, heights and $\log|y|$ we already established in Lemmas 3.9-4.1 to then use bounds for linear forms in logarithms again and derive in Lemma 4.3 that

$$\tau < c_3 \log n \log\log n.$$

With this information, we can say that in all our decompositions of α_i and β_i into powers of λ_0, λ_2, the second power is of no consequence unless n is already sufficiently "small", i.e. $n < \kappa$, since $\log|\lambda_2| = \theta\left(\frac{1}{n}\right)$ goes to zero, even pitted against the logarithmic bound for the exponents s, t.

Carrying out a rigorous case-differentiation for the type j and the order of the $|\alpha_i|$ in Lemma 4.4 gives a contradiction to $|y| \geq 2$ in each case under the assumption that $n \geq \kappa$ is sufficiently large for the powers of λ_2 to be of no consequence.

This gives $n < \kappa$, which gives a bound first for τ by Lemma 4.3 and then $\log|y|$ by Lemma 4.1 and concludes the proof of Theorem 1.1.

3. Auxilary Results

We first define the absolute Weil height and Mahler's measure for the sake of completeness—see, e.g., [3] or [7]. If K is a number field of degree $d = [K : \mathbb{Q}]$, and for every place ν, write $d_\nu = [K_\nu : \mathbb{Q}_\nu]$ for the completions $K_\nu, \mathbb{Q}_\nu$ with respect to ν, we normalise the absolute value $|\cdot|_\nu$ so that

1. if $\nu|p$ for a prime number p, then $|p|_\nu = p^{-d_\nu/d}$,
2. if $\nu|\infty$ and ν is real, then $|x|_\nu = |x|^{1/d}$,
3. if $\nu|\infty$ and ν is complex, then $|x|_\nu = |x|^{2/d}$,

where $|x|$ denotes the Euclidian absolute value in $\mathbb{R}$ or $\mathbb{C}$. Given this normalisation, the product formula

$$\prod_\nu |\alpha|_\nu = 1$$

holds for every $\alpha \in K^*$. The absolute height of $\alpha \in K$ is then defined as

$$H(\alpha) = \prod_\nu \max\{1, |\alpha|_\nu\},$$

and the absolute logarithmic height as $h(\alpha) = \log H(\alpha)$. The absolute height is then equal to the Mahler measure $M(m_\alpha)$ of its minimal polynomial m_α, i.e. if $m_\alpha(X) = a_d \prod_{i=1}^{d}(X - \alpha_i) \in \mathbb{Z}[X]$ is the minimal polynomial of $\alpha \in K$, then

$$h(\alpha) = \frac{1}{d}\log M(m_\alpha) = \frac{1}{d}\left(\log|a_d| + \sum_{i=1}^{d}\log\max\{1, |\alpha_i|\}\right).$$

After the notion of height, we next give the fundamental tool for solving Thue equations effectively, lower bounds for linear forms in logarithms. We give the version of Wüstholz and Baker himself [2] in

PROPOSITION 3.1. *Let $\gamma_1, \ldots, \gamma_k$ be algebraic numbers not 0 or 1 in $K = \mathbb{Q}(\gamma_1, \ldots, \gamma_k)$, which is of degree D. Let $b_1, \ldots, b_k \in \mathbb{Z}$ and*

$$\Lambda = b_1 \log\gamma_1 + \cdots + b_k \log\gamma_k \neq 0.$$

Then

$$\log|\Lambda| \geq -C \cdot h_1 \cdots h_k \cdot \log B,$$

where $C = 18(k+1)!k^{k+1}(32D)^{k+2}\log(2kD)$, $B \geq \max\{3, |b_1|, \cdots, |b_k|\}$ and

$$h_i \geq \max\left\{h(\gamma_i), \log|\gamma_i|\; D^{-1}, 0.16\, D^{-1}\right\}$$

for $i \in \{1, \ldots, k\}$.

Next we make some observations about the roots $\lambda_0, \lambda_1, \lambda_2$ of the polynomial f, starting with

LEMMA 3.2. *The roots $\lambda_0, \lambda_1, \lambda_2$ of the polynomial $f(X) = X^3 - (n-1)X^2 - (n+2)X - 1$ satisfy the following inequalities:*

$$\begin{aligned} n + \frac{1}{n} &< \lambda_0 && < n + \frac{2}{n} \\ -\frac{1}{n+1} &< \lambda_1 = -\frac{1}{\lambda_0 + 1} && < -\frac{1}{n+2} \\ -1 - \frac{1}{n} &< \lambda_2 = -\frac{\lambda_0 + 1}{\lambda_0} && < -1 - \frac{1}{n+1}. \end{aligned}$$

PROOF. See, for instance, Lemma 10 in [5]. The asymptotics are also easy to check directly, e.g. $f(n + \frac{2}{n})$ is positive and $f(n + \frac{2}{n} - \frac{3}{n^2})$ is negative; by the Intermediate Value Theorem, the root lies between $n + \frac{2}{n}$ and $n + \frac{2}{n} - \frac{3}{n^2} > n + \frac{1}{n}$. For λ_1 and λ_2, closer bounds are $-\frac{1}{n} + \frac{1}{n^2} \pm \frac{1}{n^3}$ and $-1 - \frac{1}{n} \pm \frac{3}{n^3}$, but the bounds from [5] given above suffice for our subsequent proofs. ∎

COROLLARY 3.3. *The logarithms of the roots of f satisfy the following inequalities:*

$$\begin{aligned} \log n < \log\lambda_0 &< \log n + \frac{2}{n^2} \\ -\log n - \frac{2}{n} < \log|\lambda_1| &< -\log n - \frac{1}{2n} \\ \frac{1}{n} - \frac{2}{n^2} < \log|\lambda_2| &< \frac{1}{n} + \frac{1}{n^2}. \end{aligned}$$

PROOF. The proof follows from the inequalities of Lemma 3.2 and the Taylor expansion of $\log(1+\xi)$; for example, if $(1<)\,\lambda_0 < n+\frac{2}{n}$, then

$$\log\lambda_0 < \log\left(n+\frac{2}{n}\right) = \log n + \log\left(1+\frac{2}{n^2}\right)$$
$$= \log n + \frac{2}{n^2} - \frac{1}{n^4} + \cdots < \log n + \frac{2}{n^2}.$$

∎

The following result of Thomas [8] allows us to decompose the various units in our proof into products of powers of λ_0 and λ_2.

PROPOSITION 3.4 (Thomas; [8]). *Let $\lambda_0, \lambda_1, \lambda_2$ be the roots of the polynomial $f(X) = X^3 - (n-1)X^2 - (n+2)X - 1$ and K be the number field generated by them. Then $\{\lambda_0, \lambda_2\}$ is a fundamental system of units for the order $\mathbb{Z}[\lambda_0]$.*

In the next lemma we show that any two of the algebraic numbers α_0, α_1 and α_2 as defined in Equation (2.4) are separated.

LEMMA 3.5. *Let $\alpha_p, \alpha_q \in \{\alpha_0, \alpha_1, \alpha_2\}$ such that $|\alpha_p| < |\alpha_q|$. Then there exists an effectively computable constant $c_1 = c_1(\varepsilon) > 0$ which, for sufficiently large n, can be chosen arbitrarily close to ε, with the following properties:*

1. $|\alpha_p/\alpha_q| < n^{-c_1\tau} \leq 1/2$.
2. $|\alpha_p - \alpha_q| > |\alpha_q|/2$.

PROOF. The second statement follows directly from the first. To prove $|\alpha_p/\alpha_q| < n^{-c_1\tau} \leq 1/2$, we first use Lemma 3.2 to express the quotient α_p/α_q via powers of λ_0 and λ_2. For example, if $|\alpha_0| = |\lambda_0^s\lambda_1^t| < |\alpha_1| = \left|\lambda_0^{-s}\lambda_2^{-s+t}\right|$, then the quotient $\alpha_0/\alpha_1 = \pm\lambda_0^{2s-t}\lambda_2^{-s+2t}$. We prove in this case $|\alpha_0/\alpha_1| < n^{-c_1\tau} < 1/2$ by proving $-\log|\alpha_0/\alpha_1| > c_1\tau\log n \geq \log 2$. In the other cases, the logarithm has the form

$$\log\left|\frac{\alpha_p}{\alpha_q}\right| = \begin{cases} (2s-t)\log\lambda_0 - (-s+2t)\log|\lambda_2| & \text{if } (p,q) = (0,1) \\ (s-2t)\log\lambda_0 \quad -(s+t)\log|\lambda_2| & \text{if } (p,q) = (0,2) \\ (s+t)\log\lambda_0 - (-2s+t)\log|\lambda_2| & \text{if } (p,q) = (1,2) \end{cases} \tag{3.12}$$

or with flipped signs for the coefficients of $\log\lambda_0$ and $\log|\lambda_2|$ if (p,q) is $(1,0)$ or $(2,0)$ or $(2,1)$, respectively. In any case, the absolute value of the coefficient of $\log\lambda_0$ is greater than $\varepsilon\tau$ by Condition (1.3). Furthermore, the absolute value of the coefficient of $\log|\lambda_2|$ is at most 3τ in each case. Applying the reverse triangle inequality thus yields

$$-\log\left|\frac{\alpha_p}{\alpha_q}\right| > \varepsilon\tau\log\lambda_0 - 3\tau\log|\lambda_2|.$$

Using Corollary 3.3 to estimate the logarithms, we get

$$-\log\left|\frac{\alpha_p}{\alpha_q}\right| > \varepsilon\tau\log\lambda_0 - 3\tau\log|\lambda_2| > \varepsilon\tau\log n - \frac{3\tau}{n} - \frac{3\tau}{n^2},$$

and unless both τ and n are already small, there exists an effectively computable constant $c_1 > 0$, dependent only on ε, and for sufficiently large n arbitrarily close to ε, such that

$$\varepsilon\tau\log n - \frac{3\tau}{n} - \frac{3\tau}{n^2} > c_1\tau\log n \geq \log 2.$$

□

LEMMA 3.6. *Let a, b, c be positive real numbers with $abc = 1$, then*

$$\max\{a,b\}\max\{a,c\} \geq \sqrt{\max\{a,b,c\}}.$$

PROOF. Let $a = \max\{a,b,c\}$, then $abc = 1$ implies that $a \geq 1$ and thus $\max\{a,b\}\max\{a,c\} = a^2 \geq \sqrt{a}$ holds. If instead, after possibly renaming b and c, the maximum is $b = \max\{a,b,c\}$ then $a \geq \frac{1}{\sqrt{b}}$ or $c \geq \frac{1}{\sqrt{b}}$, with equality in the case that $a = c$ for $abc = 1$ to hold. This in turn means that $\max\{a,b\}\max\{a,c\} \geq \frac{b}{\sqrt{b}} = \sqrt{b}$. □

LEMMA 3.7. *We have*

$$n^{3\tau} \geq \max\{|\alpha_0|, |\alpha_1|, |\alpha_2|\} \geq n^{c_1\tau}.$$

PROOF. Analogously to the proof of Lemma 3.5, we have for the maximum α that

$$\log|\alpha| = \begin{cases} (s-t)\log\lambda_0 \quad -t\log|\lambda_2| & \text{if } \alpha = \alpha_0 \\ -s\log\lambda_0 + (-s+t)\log|\lambda_2| & \text{if } \alpha = \alpha_1 \\ t\log\lambda_0 \quad +s\log|\lambda_2| & \text{if } \alpha = \alpha_1 \end{cases}$$

Since $|\alpha|$ is the maximum of $|\alpha_0|, |\alpha_1|, |\alpha_2|$, its logarithm is the largest of the three expressions given above. Since it is either $|-s| = \tau$ or $|t| = \tau$, we can infer

$$3\tau\log n > \log|\alpha| \geq \tau\log\lambda_0 - 2\tau\log|\lambda_2| > \tau\log n - 2\tau\left(\frac{1}{n} + \frac{1}{n^2}\right)$$

in each of the three cases for α. We can certainly bound the right-hand side by $c_1\tau\log n$, the same expression as used in Lemma 3.5 but could also use a constant that does not depend on ε, such as 0.19: Since $\tau\log n - 2\tau\left(\frac{1}{n} + \frac{1}{n^2}\right) > 0.19\,\tau\log n$ if and only if $1 - \left(\frac{2}{n\log n} + \frac{2}{n^2\log n}\right) > 0.19$ which is true for $n \geq 3$.

□

LEMMA 3.8. *We have*

$$\frac{\max\{|\alpha_k|, |\alpha_l|\}}{\max(|\alpha_j|, |\alpha_k|)\max\{|\alpha_j|, |\alpha_l|\}} < 2|y|.$$

Proof. Irrespective of whether α_j has the largest absolute value of the three or not, we can bound the quotient by its inverse, i.e.

$$\frac{\max\left(|\alpha_k|, |\alpha_l|\right)}{\max\left(|\alpha_j|, |\alpha_k|\right)\max\{|\alpha_j|, |\alpha_l|\}} < \frac{1}{|\alpha_j|}.$$

If $|\alpha_j| = \max\{|\alpha_0|, |\alpha_1|, |\alpha_2|\}$, then the right-hand side is at most $n^{-c_1\tau}$, by Lemma 3.7. And unless n and τ are already absolutely bounded, we can assume this to be smaller than any constant, say, $4 \le 2|y|$.

Otherwise, we use Inequality (2.7), which by the same argument is smaller still than, say $\frac{1}{2}$, and get

$$\frac{1}{2} > |x - \alpha_j y| \ge |x| - |\alpha_j|\,|y|.$$

Since $x \ne 0$—the only solutions with $x = 0$ have $y = \pm 1$—we have $|x| \ge 1$ and thus the above inequality can be read as

$$|\alpha_j| > \frac{1}{2|y|}, \tag{3.13}$$

which proves the assertion. □

Lemma 3.9. *Let $\beta_0 = \pm\lambda_0^a\lambda_2^b$ be the decomposition of the unit $\beta_0 \in \mathbb{Z}[\lambda_0]$ into powers of λ_0, λ_2. Then we have*

$$\max\{|a|, |b|\} = \theta\left(\frac{\log|y|}{\log n} + \tau\right).$$

Proof. If $\beta_0 = \pm\lambda_0^a\lambda_2^b$ then, by conjugation, $\beta_1 = \pm\lambda_1^a\lambda_0^b$, and $\beta_2 = \pm\lambda_2^a\lambda_1^b$. Depending on the type j, we take the decompositions of the other two units β_k and β_l and take the logarithm of their absolute values; for example, if $j = 0$, we look at the system of linear equations

$$a\log|\lambda_1| + b\log|\lambda_0| = \log|\beta_1|$$
$$a\log|\lambda_2| + b\log|\lambda_1| = \log|\beta_2|$$

or $M\begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix}\log|\beta_k| \\ \log|\beta_l|\end{pmatrix}$ in matrix notation. Depending on whether j is $0, 1$ or 2, the matrix M is, up to switching rows,

$$M = \begin{pmatrix}\log|\lambda_1| & \log|\lambda_0| \\ \log|\lambda_2| & \log|\lambda_1|\end{pmatrix}, \begin{pmatrix}\log|\lambda_0| & \log|\lambda_2| \\ \log|\lambda_2| & \log|\lambda_1|\end{pmatrix}, \begin{pmatrix}\log|\lambda_0| & \log|\lambda_2| \\ \log|\lambda_1| & \log|\lambda_0|\end{pmatrix}.$$

Since $\log\lambda_0 = \theta(\log n), \log|\lambda_1| = \theta(-\log n)$ and $\log|\lambda_2| = \theta(\frac{1}{n})$ by Corollary 3.3, we have that $|\det M| = \theta((\log n)^2)$ in either case. In particular, the determinant is non-zero, the matrix thus invertible.

We then multiply the system $M\begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix}\log|\beta_k| \\ \log|\beta_l|\end{pmatrix}$ with the inverse M^{-1} and take the column-wise maximum norm. We can calculate it by taking the row-wise maximum norm of M and dividing by $|\det M|$, which gives a $\theta\left(\frac{1}{\log n}\right)$. We thus get, with the consistency

$$\left\| M^{-1} \begin{pmatrix} \log|\beta_k| \\ \log|\beta_l| \end{pmatrix} \right\| \leq \left\| M^{-1} \right\| \cdot \left\| \begin{pmatrix} \log|\beta_k| \\ \log|\beta_l| \end{pmatrix} \right\|,$$

that

$$\max\{|a|, |b|\} = \theta \left\{ \frac{\max\{|\log|\beta_k||, |\log|\beta_l||\}}{\log n} \right\}.$$

Furthermore, we have

$$\begin{aligned} \log|\beta_k| &= \log|x - \alpha_k y + \alpha_j y - \alpha_j y| \\ &= \log|y| + \log|\alpha_j - \alpha_k| + \log\left|1 + \left|\frac{\beta_j}{y\,|\alpha_j - \alpha_k|}\right|\right|, \end{aligned} \tag{3.14}$$

and since $|\alpha_j - \alpha_k| \geq \frac{1}{2}|\alpha_j| \geq \frac{1}{4}|y|^{-1}$ by Equation (3.13), while $|\beta_j| \ll |y|^{-2} n^{-\frac{c_1}{2}\tau}$ by Inequality (2.7), the last logarithm is very small and can certainly be effectively bounded by $n^{-\frac{c_1}{2}\tau}$. Thus,

$$\max\{|\log|\beta_k||, |\log|\beta_l||\} = \theta(\log|y| + \log\max\{|\alpha_j - \alpha_k|, |\alpha_j - \alpha_l|\}).$$

Finally, we have by Lemma 3.5 that $|\alpha_j - \alpha_k|$ is at least $\frac{1}{2}\max(|\alpha_j|, |\alpha_k|)$, and it is at most 2 times the maximum. That is to say,

$$\log\max\{|\alpha_j - \alpha_k|, |\alpha_j - \alpha_l|\} = \theta(\log\max\{|\alpha_j|, |\alpha_k|, |\alpha_l|\}),$$

which by Lemma 3.7 is a $\theta(\tau\log n)$. This gives us that

$$\max\{|\log|\beta_k||, |\log|\beta_l||\} = \theta(\log|y| + \tau\log n)$$

and thus

$$\max\{|a|, |b|\} = \theta\left(\frac{\log|y|}{\log n} + \tau\right),$$

which proves the assertion. ∎

4. Proof of Theorem 1.1

With most of the technical considerations done in the previous section, we can formally finish the proof of Theorem 1.1, as outlined in Section 2. That is to say, we now use bounds for linear forms in logarithms, namely Proposition 3.1, to deduce bounds for the size of the parameters of our Thue equations.

Lemma 4.1. *We have, for some effectively computable constant $c_2 > 0$,*

$$\log|y| < c_2\,\tau\,(\log n)^3(\log\tau + \log\log n).$$

Proof. We take the decomposition $\beta_0 = \pm\lambda_0^a\lambda_2^b$ and conjugate the expression to get decompositions for β_1 and β_2. Only this time, we express everything in powers of λ_0 and λ_2 using the connections from Lemma 3.2. This gives that $\beta_0 = \pm\lambda_0^a\lambda_2^b$, $\beta_1 = \pm\lambda_0^{-a+b}\lambda_2^{-a}$, and $\beta_2 = \pm\lambda_0^{-b}\lambda_2^{a-b}$.

We can thus write for the logarithm of the absolute value of the quotient, $\log\left|\frac{\beta_l}{\beta_k}\right| = a'\log\lambda_0 + b'\log|\lambda_2|$, where both a', b' are the respective linear combination of a, b and can write the linear form in logarithms Λ from Equation (2.10) as

$$0 < \left|a'\log\lambda_0 + b'\log|\lambda_2| + \log\left|\frac{\alpha_j - \alpha_k}{\alpha_j - \alpha_l}\right|\right| \ll \frac{1}{|y|^2\, n^{\frac{c_1}{2}\tau}}.$$

We now use lower bounds for this linear form in logarithms, Proposition 3.1.

By Lemma 3.9, we have that $\max\{|a'|, |b'|\} \ll \frac{\log|y|}{\log n} + \tau$. Due to the sub-additivity and sub-multiplicativity of the absolute logarithmic height, we have $h\left(\frac{\alpha_j-\alpha_k}{\alpha_j-\alpha_l}\right) \ll \tau\, h(\lambda_0)$. And switching to the Mahler measure of the minimal polynomial f of λ_0 gives

$$h(\lambda_0) = M(f) = \frac{1}{3}\left(\log(n+2) + \log|\lambda_0| + \log|\lambda_2|\right) = \theta(\log n). \tag{4.15}$$

We plug everything into Proposition 3.1 and get that for some large effective constant $c_2 > 0$, since the constant in Proposition 3.1 is already larger than 10^{14}, that

$$-c_2\,\tau(\log n)^3\log\left(\frac{\log|y|}{\log n} + \tau\right) < -\log|y|$$

holds, which implies the assertion. □

Lemma 4.2. *There exists a non-zero linear form Λ'' in logarithms of $\beta_k, \beta_l, \alpha_0, \alpha_1, \alpha_2, 2$ with coefficients 0 or ±1, such that*

$$0 < |\Lambda''| < \frac{1}{n^{\frac{c_1}{2}\tau}}$$

holds.

Proof. If the linear form Λ' from Equation (2.11) is already non-zero, then the statement follows immediately for $\Lambda'' = \Lambda'$.

Assume instead that $\Lambda' = 0$, which holds if and only if $|\beta_l\alpha_{j,k}| = |\beta_k\alpha_{j,l}|$. We now differentiate the cases for $\alpha_{j,k}$ and $\alpha_{j,l}$ and the sign. For simplicity we assume $|\alpha_k| > |\alpha_l|$, otherwise we swap the roles of k and l.

Case $\alpha_{j,k} = \alpha_{j,l} = \alpha_j$: In this case, $\Lambda' = 0$ if and only if $|\beta_l| = |\beta_k|$. They cannot be equal, since $\beta_l = \beta_k$ implies $\alpha_l = \alpha_k$, which gives $s = t = 0$ due to the multiplicative independence of λ_0, λ_2. Thus, $\beta_l = -\beta_k$.

We substitute β_l for $-\beta_k$ in Siegel's Identity (2.8) and obtain

$$\beta_j(\alpha_k - \alpha_l) - \beta_k(\alpha_j - \alpha_k) + \beta_k(\alpha_l - \alpha_j) = 0;$$

moving the term $\beta_k\alpha_k$ to the other side and dividing by $-2\beta_k\alpha_j$ then yields

$$-\frac{\beta_j(\alpha_k - \alpha_l)}{2\beta_k\alpha_j} - \frac{\alpha_l}{2\alpha_j} + 1 = \frac{\alpha_k}{2\alpha_j}. \tag{4.16}$$

Since $|\alpha_j| > |\alpha_k|$, the right-hand side is not 1. Equivalently, the two fractions on the left do not cancel. Applying Lemma 3.5 to the fraction $\dfrac{\alpha_l}{2\alpha_j}$, since we also have $|\alpha_j| > |\alpha_l|$, gives $\left|\dfrac{\alpha_l}{2\alpha_j}\right| \ll n^{-c_1\tau}$. By a combination of inequalities (2.6) and (2.7), and by Lemma 3.5 and 3.7, we can similarly bound

$$\left|\frac{\beta_j\left(\alpha_k - \alpha_l\right)}{2\beta_k\alpha_j}\right| \ll \frac{\max\left\{|\alpha_k|, |\alpha_l|\right\}}{|y|^3 n^{\frac{c_1}{2}\tau} |\alpha_j|^2} \ll \frac{1}{|y|^3 n^{\frac{3c_1}{2}\tau}}.$$

We then apply the logarithm to Equation (4.16). The right-hand side is not 1, its logarithm non-zero. We can bound both fractions on the left-hand side by $n^{-c_1\tau}$ and thus

$$0 < |\log|\alpha_k| - \log|\alpha_j| - \log 2| \ll \frac{1}{n^{c_1\tau}},$$

so the statement holds for $\Lambda'' = \log|\alpha_k| - \log|\alpha_j| - \log 2$.

Case $\alpha_{j,k} = \alpha_k, \alpha_{j,l} = \alpha_l$*:* In this case, we have $\Lambda' = 0$ if and only if $|\beta_l\alpha_k| = |\beta_k\alpha_l|$. If the absolute value cancels without a sign change, the term $\beta_k\alpha_l - \beta_l\alpha_k$ vanishes in Siegel's Identity (2.8). What remains is

$$\beta_j(\alpha_k - \alpha_l) + \beta_l\alpha_j - \beta_k\alpha_j = 0,$$

or

$$-\frac{\beta_j(\alpha_k - \alpha_l)}{\beta_k\alpha_j} + 1 = \frac{\beta_l}{\beta_k}.$$

The right-hand side is again not 1, since $\alpha_l \neq \alpha_k$. Analogously to the above case and then using Inequality (3.13), we have that

$$\left|\frac{\beta_j(\alpha_k - \alpha_j)}{\beta_k\alpha_j}\right| \ll \frac{|\alpha_k|}{|y|^3 n^{\frac{c_1}{2}\tau} |\alpha_k - \alpha_j| |\alpha_j|} \ll \frac{1}{|y|^2 n^{\frac{c_1}{2}\tau}},$$

and thus

$$0 < |\log|\beta_l| - \log|\beta_k|| < \frac{1}{n^{\frac{c_1}{2}\tau}}.$$

If the absolute value cancels with a sign change, then $\beta_k\alpha_l - \beta_l\alpha_k = -2\beta_l\alpha_k$. Plugging this into Siegel's identity yields

$$-\frac{\beta_j(\alpha_k - \alpha_l)}{2\beta_l\alpha_k} - \frac{\alpha_j}{2\alpha_k} + 1 = -\frac{\beta_k\alpha_j}{2\beta_l\alpha_k}.$$

The right-hand side is not 1: On the one hand, we have that

$$|\beta_k \alpha_j| \ll |y| \, |\alpha_k - \alpha_j| \, |\alpha_j| \ll |y| \, |\alpha_k| \, |\alpha_j| \, ,$$

and on the other hand that

$$|y| \, |\alpha_l| \, |\alpha_k| \ll |y| \, |\alpha_l - \alpha_j| \, |\alpha_k| \ll |2\beta_l \alpha_k| \, .$$

Thus, the right-hand side is effectively bounded by $\left|\frac{\alpha_j}{\alpha_l}\right|$, which in this case is smaller than 1, even taking into account the implied constants. We can similarly effectively bound the fractions on the left-hand side by $n^{-\frac{c_1}{2}\tau}$, from which it follows that

$$0 < |\log|\beta_k| - \log|\beta_l| + \log|\alpha_j| - \log|\alpha_k| - \log 2| < \frac{1}{n^{\frac{c_1}{2}\tau}}.$$

The case where $\alpha_{j,k} = \alpha_k, \alpha_{j,l} = \alpha_j$ follows analogously. □

LEMMA 4.3. *There exists an effectively computable constant* $c_3 > 0$ *such that*

$$\tau < c_3 \log n \log\log n.$$

PROOF. By Lemma 4.2, we have a linear form Λ'' in logarithms of β_k, β_l, α_j,α_k,α_l and 2 with coefficients in $\{-1, 0, 1\}$, for which

$$0 < |\Lambda''| < \frac{1}{n^{\frac{c_1}{2}\tau}}$$

holds. We proceed analogously to the proof of Lemma 4.1 and write $\beta_0 = \pm\lambda_0^a \lambda_2^b$ and so on. We do the same for the α_i, i.e. $\alpha_0 = \lambda_0^s \lambda_1^t = \lambda_0^{s-t}\lambda_2^{-t}$. We can then write Λ'' alternatively as a linear form in logarithms of $\lambda_0, \lambda_2, 2$.

The coefficients are then linear combinations of a, b, s, t and thus effectively bounded by $\frac{\log|y|}{\log n} + \tau$ by Lemma 3.9. Applying the bound for $\log|y|$ from Lemma 4.1 gives, up to effective constants, the bound

$$\tau(\log n)^2(\log\tau + \log\log n)$$

for the coefficients of the linear form. We plug this bounds, and the bound for the logarithmic heights of λ_0, λ_2 from Equation (4.15) into Proposition 3.1 and get

$$\tau \log n \ll (\log n)^2 \log(\tau(\log n)^2(\log\tau + \log\log n)).$$

If $\tau \ll \log n$, then the proposed bound holds in particular. If $\log n \ll \tau$ instead, then $\log\log n \ll \log\tau$ as well and the right-hand side of the above inequality becomes

$$\tau \ll \log n \log(\tau^3 \log\tau) \ll \log n \log\tau,$$

let us call the implied constant c. Now we do the same thing again with a slight variation; it is either the case that $\tau < 2c \log n \log\log n$ (from which follows the statement), or it is the case that $\tau \geq 2c \log n \log\log n$. But combining the latter inequality with $\tau \leq c \log n \log \tau$ yields

$$2c \log n \log\log n \leq \tau \leq c \log n \log \tau$$

and thus $2 \log\log n \leq \log \tau$ or $\log n \leq \sqrt{\tau}$. If we go back to $\tau \leq c \log n \log \tau$ and insert this last inequality, we get $\tau \leq c\sqrt{\tau} \log \tau$, and so $\tau < c_3$; this again implies the statement in particular. □

If we use the bound $\tau < c_3 \log n \log\log n$ and plug it into Lemma 4.1, we get that

$$\log |y| < c_4 (\log n)^4 (\log\log n)^2, \tag{4.17}$$

and plugging both bounds into Lemma 3.9, we get

$$\max\{|a|, |b|\} < c_5 (\log n)^3 (\log\log n)^2. \tag{4.18}$$

All we have left to do is to bound n by an absolute constant, which we do in the following

LEMMA 4.4. *There exists an effectively computable constant κ such that $n < \kappa$.*

PROOF. We assume $n \geq \kappa$ is sufficiently large, such that the powers of λ_2 do not influence the following arguments, and will derive a contradiction to $|y| \geq 2$.

We first recall the forms of the β_i, α_i in terms of λ_0, λ_2, i.e. $\beta_0 = \pm\lambda_0^a \lambda_2^b$, $\beta_1 = \pm\lambda_0^{-a+b}\lambda_2^{-a}$ and $\beta_2 = \pm\lambda_0^{-b}\lambda_2^{a-b}$, while $\alpha_0 = \lambda_0^{s-t}\lambda_2^{-t}$, $\alpha_1 = \lambda_0^{-s}\lambda_2^{-s+t}$ and $\alpha_2 = \lambda_0^t \lambda_2^s$. We differentiate the cases for j. For each j, we denote according to Equation (2.5) by $\alpha_{j,k}$ the greater of α_j and α_k in terms of absolute value, and by $\alpha_{j,l}$ the greater of α_j and α_l. We start with the case $j = 0$ and choose $(k, l) = (2, 1)$.

The linear form Λ' of Equation (2.11) is

$$\Lambda' = \log\left|\frac{\beta_l}{\beta_k}\right| + \log\left|\frac{\alpha_{j,k}}{\alpha_{j,l}}\right| = \log\left|\frac{\beta_1}{\beta_2}\right| + \log\left|\frac{\alpha_{0,2}}{\alpha_{0,1}}\right|,$$

and we can write this as a linear form in the logarithms $\log|\lambda_0|$ and $\log|\lambda_2|$, whose coefficients we call ξ and η. From Lemma 4.3 and Equation (4.18), we have that $|\xi|, |\eta| < c_5 (\log n)^3 (\log\log n)^2$. We now show that via $\xi = \eta = 0$ the linear form Λ' vanishes and then use the information about a, b, s, t derived from this to proof the lemma.

In a first step, we assume that the coefficient ξ of $\log|\lambda_0|$ is non-zero, and thus $|\xi| \geq 1$. We can then rewrite the linear form Λ' and its upper bound in

$$|\xi| \log|\lambda_0| - |\eta| \left|\log|\lambda_2|\right| < n^{-\frac{c_1}{2}\tau},$$

or, with Corollary 3.3,

$$\log n < |\xi| \log|\lambda_0| < |\eta| \left|\log|\lambda_2|\right| + n^{-\frac{c_1}{2}\tau} < |\eta| \left(\frac{1}{n} + \frac{1}{n^2}\right) + n^{-\frac{c_1}{2}\tau}.$$

But this gives a contradiction for sufficiently large $n \geq \kappa$, since $|\eta| < c_5(\log n)^3(\log\log n)^2$, and thus our assumption that $\xi \neq 0$ was false.

Next, we do the same for the coefficient η of $\log|\lambda_2|$. We assume $\eta \neq 0$ and therefore $|\eta| \geq 1$. Together with $\xi = 0$ and Corollary 3.3 this gives

$$\frac{1}{n} - \frac{1}{n^2} < |\eta| \, |\log|\lambda_2|| < n^{-\frac{c_1}{2}\tau}.$$

This again gives a contradiction: For sufficiently large n, the constant c_1 from Lemma 3.5 can be chosen arbitrarily close to ε. In particularly, such that $\varepsilon\tau > 2$ from Condition 1.3 implies $c_1\tau > 2$, which makes the above inequality contradictory for sufficiently large n.

So we have that both $\xi = 0$ and $\eta = 0$, and in the following steps we differentiate their exact forms depending on the cases for $\alpha_{0,2}$ and $\alpha_{0,1}$.

Case $\alpha_{0,2} = \alpha_{0,1} = \alpha_0$*:* In this case, we have $|\alpha_0| = \max\{|\alpha_0|, |\alpha_1|, |\alpha_2|\}$. We recall that $|\alpha_0\alpha_1\alpha_2| = 1$ and that they are all distinct, so the largest - $|\alpha_0|$ in this case - is strictly greater than 1. Plugging in $\lambda_0^{s-t}\lambda_2^{-t}$ for α_0 yields $s - t > 0$ for sufficiently large $n \geq \kappa$ - so the logarithm of λ_2^{-t} does not influence the sign of $\log|\alpha_0|$.

In the linear form Λ', $\log\left|\frac{\alpha_{0,2}}{\alpha_{0,1}}\right| = \log\left|\frac{\alpha_0}{\alpha_0}\right| = 0$ vanishes in this case. If we substitute for β_1 and β_2, we get that the coefficients of $\log|\lambda_0|$ and $\log|\lambda_2|$ are $\xi = -a + 2b$ and $\eta = -2a + b$, respectively. Thus, $\xi = \eta = 0$ implies that $a = b = 0$.

But if we plug this into $x - \alpha_0 y = \beta_0 = \pm\lambda_0^a\lambda_2^b$, and since α_0 is irrational, we derive $x = \pm 1$ and $y = 0$, which contradicts $|y| \geq 2$.

Case $\alpha_{0,2} = \alpha_0, \alpha_{0,1} = \alpha_1$*:* Here, the relation between the sizes of the algebraic numbers α_i is

$$|\alpha_1| > |\alpha_0| > |\alpha_2| \text{ and } |\alpha_2| > 1.$$

Taking the logarithm gives $-s > s - t > t$ and $-s > 0$, and in particular $t < 0$. If we replace $\log|\alpha_{0,2}/\alpha_{0,1}|$ for

$$\log\left|\frac{\alpha_0}{\alpha_1}\right| = (2s - t)\log|\lambda_0| + (s - 2t)\log|\lambda_2|,$$

we get $\xi = -a + 2b + 2s - t = 0$ and $\eta = -2a + b + s - 2t = 0$. From this we can deduce $a = -t$ and $b = -s$.

Similar to Equation (3.14) we use $\log|\beta_2|$ and $\log|\alpha_0 - \alpha_2|$ to approximate $\log|y|$, and make an error of at most $O\left(n^{-\frac{c_1}{2}\tau}\right)$. Using Lemma 3.5 to bound the error going from $\log|\alpha_0 - \alpha_2|$ to $\log|\alpha_{0,2}| = \log|\alpha_0|$, we have

$$\begin{aligned}\log|y| &= \log|\beta_2| - \log|\alpha_0 - \alpha_2| + \log\left|1 + \frac{\beta_0}{y\,|\alpha_0 - \alpha_2|}\right| \\ &= \log|\beta_2| - \log|\alpha_0| + O\left(n^{-\frac{c_1}{2}\tau}\right).\end{aligned}$$

We first replace β_2 and α_0 by the respective powers of λ_0 and λ_2 and then substitute a and b for the derived linear combinations of s and t, which gives

$$\begin{aligned}\log|y| &= (-b-s+t)\log|\lambda_0| + (a-b+t)\log|\lambda_2| + O\left(n^{-\frac{c_1}{2}\tau}\right)\\ &= t\log|\lambda_0| + s\log|\lambda_2| + O\left(n^{-\frac{c_1}{2}\tau}\right).\end{aligned}$$

But in the case $\alpha_{0,2} = \alpha_0, \alpha_{0,1} = \alpha_1$ we have $t < 0$, and so $\log|y|$ is negative for sufficiently large $n \geq \kappa$, which contradicts $|y| \geq 2$.

Case $\alpha_{0,2} = \alpha_2, \alpha_{0,1} = \alpha_0$*:* Here the relation is

$$|\alpha_2| > |\alpha_0| > |\alpha_1| \text{ and } |\alpha_2| > 1,$$

from which we deduce $-s < 0$ in particular. If we calculate $\log\left|\frac{\alpha_2}{\alpha_0}\right|$, we get $\xi = -a + 2b - s + 2t = 0$ and $\eta = -2a + b + s + t = 0$, from which $a = s$ and $b = s - t$ follows.

Putting this into our approximation of $\log|y|$ by $\log|\beta_2|$ and $\log|\alpha_{0,2}| = \log|\alpha_2|$ gives

$$\begin{aligned}\log|y| &= \log|\beta_2| - \log|\alpha_2| + O\left(n^{-\frac{c_1}{2}\tau}\right)\\ &= (-b-t)\log|\lambda_0| + (a-b-s)\log|\lambda_2| + O\left(n^{-\frac{c_1}{2}\tau}\right)\\ &= -s\log|\log|\lambda_0|| + (-s+t)\log|\lambda_2| + O\left(n^{-\frac{c_1}{2}\tau}\right).\end{aligned}$$

But $-s < 0$, so this is negative for sufficiently large $n \geq \kappa$, which contradicts $|y| \geq 2$.

Case $\alpha_{0,2} = \alpha_2, \alpha_{0,1} = \alpha_1$*:* In this case, the coefficients of $\log|\lambda_0|$ and $\log|\lambda_2|$ in Λ' are $\xi = -a + 2b + s + t$ and $\eta = -2a + b + 2s - t$, respectively. $\xi = \eta = 0$ then implies that $a = s - t$ and $b = -t$.

But if we approximate $\log|y|$ again by $\log|\beta_2|$ and $\log|\alpha_{0,2}| = \log|\alpha_2|$, we get

$$\begin{aligned}\log|y| &= \log|\beta_2| - \log|\alpha_2| + O\left(n^{-\frac{c_1}{2}\tau}\right)\\ &= (-b-t)\log|\lambda_0| + (a-b-s)\log|\lambda_2| + O\left(n^{-\frac{c_1}{2}\tau}\right)\\ &= O\left(n^{-\frac{c_1}{2}\tau}\right),\end{aligned}$$

which is less than $\log 2$ for sufficiently large $n \geq \kappa$ and contradicts $|y| \geq 2$.

In all cases for $\alpha_{0,2}$ and $\alpha_{0,1}$, we have derived a contradiction of the assumption $n \geq \kappa$, and thus conclude our proof for the type $j = 0$. The proof for the type $j = 1$ and $j = 2$ follows analogously. □

By this Lemma 4.4, we have deduced an effectively computable constant κ, which depends implicitly only on ε (since the constant c_1 in Lemma 3.5 does), such that $n < \kappa$. Plugging this into Lemma 4.3 gives a fixed upper bound for τ, and into Inequality (4.17) a bound for $\log|y|$, and thus $|y|$. And if everything else is bounded, so must be $|x|$. By abuse of notation, we shall call κ the bound that holds for all these parameters, thus concluding the proof of Theorem 1.1.

Acknowledgements.

The second author was supported by the Austrian Science Fund (FWF) under the project I4406. We would also like to thank the referee for his careful reading of the manuscript and his detailed comments and feedback.

References

[1] A. Baker, *Contributions to the theory of diophantine equations. I: On the representation of integers by binary forms,* Philos. Trans. R. Soc. Lond., Ser. A **263** (1968), 173–191.

[2] A. Baker and G. Wüstholz, *Logarithmic forms and group varieties,* J. Reine Angew. Math., **442** (1993), 19–62.

[3] E. Bombieri, *On the Thue-Siegel-Dyson theorem,* Acta Math **148** (1982), 255–296.

[4] Y. Bugeaud and K. Győry, *Bounds for the solutions of Thue-Mahler equations and norm form equations,* Acta Arith. **74(3)** (1996), 273–292.

[5] G. Lettl, A. Pethő and P. Voutier, *Simple Families of Thue Inequalities,* Trans Am Math Soc. **351(5)** (1999), 1871–1894.

[6] C. Levesque and M. Waldschmidt, *A family of Thue equations involving powers of units of the simplest cubic fields,* J. Théor. Nombres Bordx. **27(2)** (2015), 537–563.

[7] C. Smyth, *The Mahler measure of algebraic numbers: a survey,* London Mathematical Society Lecture Note Series (2008), 322–349.

[8] E. Thomas, *Fundamental units for orders in certain cubic number fields,* J. Reine Angew. Math. **310** (1979), 33–55.

[9] E. Thomas, *Complete solutions to a family of cubic Diophantine equations,* J. Number Theory **34(2)** (1990), 235–250.

T. Hilgart
Department of Mathematics
University of Salzburg
5020 Salzburg
Austria
E-mail: tobias.hilgart@plus.ac.at

V. Ziegler
Department of Mathematics
University of Salzburg
5020 Salzburg
Austria
E-mail: volker.ziegler@plus.ac.at