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Summability of solutions to some degenerate elliptic equations

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SUMMABILITY OF SOLUTIONS TO SOME DEGENERATE ELLIPTIC EQUATIONS

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Abstract. This paper deals with boundary value problems for elliptic equations with degenerate coercivity whose prototype is

$$\begin{cases} -\operatorname{div}\left(a(x)|\nabla u(x)|^{p-2}\nabla u(x)\right) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

with $0 < a(x) \leqslant \beta$. Some summability properties of solutions are given.

§1 Introduction and Statement of Results

The purpose of this paper is to study the boundary value problem

$$\begin{cases} -\operatorname{div}\mathcal{A}(x, u(x), \nabla u(x)) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

here Ω stands for a bounded open subset of $\mathbb{R}^n$, $n \geqslant 2$, $\partial\Omega$ is the boundary of Ω, $\mathcal{A}(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a Carathéodory vector (that is, measurable with respect to x for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and continuous with respect to (s, ξ) for almost every $x \in \Omega$) satisfying the following assumptions: there exist $1 < p \leqslant n$, a function $a(x)$ and a constant β, $0 < a(x) \leqslant \beta < \infty$, a.e. Ω, such that

$$\mathcal{A}(x, s, \xi)\xi \geqslant a(x)|\xi|^p, \tag{1.2}$$

and

$$|\mathcal{A}(x, s, \xi)| \leqslant \beta|\xi|^{p-1}. \tag{1.3}$$

As far as the datum f in (1.1) is concerned, we assume that it belongs to the Lebesgue space $L^m(\Omega)$, or the Marcinkiewicz space $M^m(\Omega)$, respectively.

A prototype of $\mathcal{A}(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfying (1.2) and (1.3) is

$$\mathcal{A}(x, s, \xi) = a(x)|\xi|^{p-2}\xi, \quad 0 < a(x) \leqslant \beta.$$

Let us first recall the definition of Marcinkiewicz space, also called weak Lebesgue space, which is defined as follows: if $m > 1$, then the space $M^m(\Omega)$ consists of all measurable functions g on Ω such that

$$\sup_{t>0} t\,|\{x \in \Omega : |g(x)| > t\}|^{\frac{1}{m}} < +\infty. \tag{1.4}$$

This condition is equivalently stated as

$$|||g|||_m = \sup_{\substack{E \subset \Omega \\ |E|>0}} \frac{1}{|E|^{\frac{1}{m'}}} \int_E |g|dx < \infty.$$

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It is well-known that $M^m(\Omega)$ is a Banach space under $|||\cdot|||_m$ and, moreover, if the supremum in (1.4) is denoted by $A_m(g)$, then

$$A_m(g) \leqslant |||g|||_m \leqslant m' A_m(g). \tag{1.5}$$

A useful result is

$$\left.\begin{matrix} g \in M^m(\Omega) \\ 1 \leqslant \sigma < m \end{matrix}\right\} \Longrightarrow \begin{cases} |g|^\sigma \in M^{\frac{m}{\sigma}}(\Omega), \\ A_{\frac{m}{\sigma}}(|g|^\sigma) = A_m^\sigma(g), \\ |||\,|g|^\sigma|||_{\frac{m}{\sigma}} \leqslant \frac{m}{m-\sigma}|||g|||_m^\sigma. \end{cases} \tag{1.6}$$

In fact, by (1.4),

$$\begin{aligned} A_{\frac{m}{\sigma}}(|g|^\sigma) &= \sup_{t>0} t|\{|g|^\sigma > t\}|^{\frac{\sigma}{m}} = \left(\sup_{t>0} t^{\frac{1}{\sigma}}|\{|g| > t^{\frac{1}{\sigma}}\}|^{\frac{1}{m}}\right)^\sigma \\ &= \left(\sup_{t>0} t|\{|g| > t\}|^{\frac{1}{m}}\right)^\sigma = A_m^\sigma(g), \end{aligned}$$

which together with (1.5) implies

$$|||\,|g|^\sigma|||_{\frac{m}{\sigma}} \leqslant \left(\frac{m}{\sigma}\right)' A_{\frac{m}{\sigma}}(|g|^\sigma) = \frac{m}{m-\sigma}A_m^\sigma(g) \leqslant \frac{m}{m-\sigma}|||g|||_m^\sigma.$$

Another useful result is, see Proposition 3.13 in [3], if $f \in M^m(\Omega)$, $m > 1$, then there exists a positive constant $B = B(|||f|||_m, m)$, such that for every measurable set $E \subset \Omega$,

$$\int_E |f|dx \leqslant B|E|^{1-\frac{1}{m}}. \tag{1.7}$$

The alternate name, the weak Lebesgue space, of $M^m(\Omega)$ is due to the fact that, if Ω has finite measure, then

$$L^m(\Omega) \subset M^m(\Omega) \subset L^{m-\varepsilon}(\Omega), \tag{1.8}$$

for every $m > 1$ and every $0 < \varepsilon \leqslant m-1$. For a detailed analysis of Marcinkiewicz spaces we refer to [8].

red In the following, for $1 < p \leqslant n$, we shall use the symbol p^* which is defined as:

$$p^* = \begin{cases} \frac{np}{n-p}, & p < n, \\ \text{any constant} > p, & p = n. \end{cases}$$

Definition 1.1. *Let* $f \in L^m(\Omega)$, $m \geqslant (p^*)'$. *A function* $u \in W_0^{1,p}(\Omega)$ *is called a solution to (1.1) if*

$$\int_\Omega \mathcal{A}(x, u(x), \nabla u(x))\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx, \quad \forall\varphi(x) \in W_0^{1,p}(\Omega). \tag{1.9}$$

We note that in the above definition, we restrict ourselves to the case $f \in L^m(\Omega)$, $m \geqslant (p^*)'$. Sobolev embedding ensures $\varphi \in L^{p^*}(\Omega)$ for $\varphi(x) \in W_0^{1,p}(\Omega)$, thus the right hand side integral of (1.9) is well-defined. We note that there is a function $a(x)$ in condition (1.2). If $a(x) \geqslant \alpha > 0$, a.e. Ω, then we are in the usual coercivity sense. The results of this equation are very rich, we refer, among others, to the classical monographs by Ladyženskaya-Ural'ceva [16], Gilbarg-Trudinger [14], Heinonen-Kilpeläinen-Martio [9] and Boccardo-Croce [3]. But if $a(x)$ is not bounded from below by a positive constant, then the coercivity is degenerate, as the following example shows.

Example 1.2. Let us consider the case $p = 2$. We claim that the differential operator $-\text{div}\mathcal{A}(x, u(x), \nabla u(x))$ with $\mathcal{A}$ satisfying (1.2) and (1.3) is not coercive on $W_0^{1,2}(\Omega)$, even if it is well defined between $W_0^{1,2}(\Omega)$ and its dual. To see that it is sufficient to consider the sequence

$$u_m(x) = |x|^{\frac{m(1-n)}{2(m+1)}} - 1, \quad m = 1, 2, \cdots,$$

and

$$a(x) = |x|$$

defined in $B_1(0)$, the unit ball centered at 0 in $\mathbb{R}^n$. It satisfies

$$\int_{B_1(0)} |Du_m|^2 \mathrm{d}x = \left(\frac{m(n-1)}{2(m+1)}\right)^2 \int_{B_1(0)} \frac{1}{|x|^{\frac{m(n+1)+2}{m+1}}} \mathrm{d}x = +\infty, \quad \text{for every } m \geqslant n-2,$$

so

$$\|u_m\|_{W_0^{1,2}(\Omega)} = +\infty, \quad \text{for every } m \geqslant n-2. \tag{1.10}$$

At the same time, for all $m = 1, 2, \cdots$,

$$\int_{B_1(0)} a(x)|Du_m|^2 \mathrm{d}x = \left(\frac{m(n-1)}{2(m+1)}\right)^2 \int_{B_1(0)} \frac{1}{|x|^{\frac{nm+1}{m+1}}} \mathrm{d}x < +\infty. \tag{1.11}$$

(1.10) together with (1.11) implies

$$\frac{1}{\|u_m\|_{W_0^{1,2}(\Omega)}} \int_{B_1(0)} a(x)|Du_m|^2 \mathrm{d}x = 0, \quad \text{as } m \to +\infty.$$

For some recent developments related to elliptic equations with degenerate coefficients, we refer to Boccardo-Croce [3] and Bella and Schäffner [5, 6]. If there is no restriction on the function $a(x)$, then one can not expect any regularity results for the boundary value problem (1.1). We now assume

$$0 < \frac{1}{a(x)} \in L^\sigma(\Omega), \quad \sigma > \max\left\{\frac{n}{p}, \frac{1}{p-1}\right\}, \tag{1.12}$$

then we will have some summability results.

We first consider the case when

$$f \in M^m(\Omega), \quad m > \frac{np\sigma}{np\sigma - n - n\sigma + p\sigma}. \tag{1.13}$$

THEOREM 1.3. *Assume (1.2), (1.3), (1.12) and (1.13). Let* $u \in W_0^{1,p}(\Omega)$ *be a solution of problem (1.1).*

(i) *If* $m > \frac{n\sigma}{p\sigma-n}$, *then there exists a positive constant* c, *depending upon* $n, p, \sigma, |\Omega|, m$, $\|\frac{1}{a}\|_{L^\sigma(\Omega)}$ *and* $|||f|||_m$, *such that*

$$\|u\|_{L^\infty(\Omega)} \leqslant c;$$

(ii) *If* $m = \frac{n\sigma}{p\sigma-n}$, *then there exists a positive constant* λ, *depending upon* n, p, σ, m, $\|\frac{1}{a}\|_{L^\sigma(\Omega)}$ *and* $|||f|||_m$, *such that*

$$e^{\lambda|u|} \in L^1(\Omega);$$

(iii) *If* $\frac{np\sigma}{np\sigma-n-n\sigma+p\sigma} < m < \frac{n\sigma}{p\sigma-n}$, *then*

$$u \in M^\tau(\Omega), \quad \tau = \frac{nm(p-1)\sigma}{nm - mp\sigma + n\sigma}. \tag{1.14}$$

If $0 < a \leqslant a(x)$, that is, the function $a(x)$ is bounded from below by a positive constant a, then $\sigma = +\infty$ in (1.12). In this case, we have the following corollary of Theorem 1.3.

COROLLARY 1.4. *Assume (1.2) with* $a(x) \geqslant a > 0$, *(1.3) and* $f \in M^m(\Omega), m > (p^*)' = \frac{np}{np-n+p}$. *Let* $u \in W_0^{1,p}(\Omega)$ *be a solution of problem (1.1).*

(i) *If* $m > \frac{n}{p}$, *then there exists a positive constant* c, *depending upon* $n, p, |\Omega|, m, a$ *and* $|||f|||_m$, *such that*

$$\|u\|_{L^\infty(\Omega)} \leqslant c;$$

(ii) *If* $m = \frac{n}{p}$, *then there exists a positive constant* λ, *depending upon* n, p, σ, m, a *and* $|||f|||_m$, *such that*

$$e^{\lambda|u|} \in L^1(\Omega);$$

(iii) *If* $(p^*)' < m < \frac{n}{p}$, *then*

$$u \in M^\tau(\Omega), \quad \tau = \frac{nm(p-1)}{n - mp}. \tag{1.15}$$

In case of $p = 2$, the above results (i), (ii) and (iii) coincide with [3, Theorems 6.11, 6.13 and 6.12], respectively.

If we weaken the summability hypotheses on f, then the gradient of u (and even u itself) may no longer be in $L^1(\Omega)$. However, it is possible to give a meaning of solutions for problem (1.1), using the concept of entropy solutions which has been introduced in [1] by Bénilan et al. In order to give the definition of entropy solution, we define, for $k > 0$, the truncation function

$$T_k(s) = \max\{-k, \min\{s, k\}\} = \begin{cases} s, & |s| \leqslant k, \\ k \operatorname{sgn}(s), & |s| > k. \end{cases}$$

Definition 1.5. *Let $f \in L^1(\Omega)$. A measurable function u is called an entropy solution of (1.1) if $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$ and if*

$$\int_\Omega \mathcal{A}(x, u(x), \nabla u(x)) \nabla T_k(u(x) - \varphi(x)) dx \leqslant \int_\Omega f(x) T_k(u(x) - \varphi(x)) dx, \tag{1.16}$$

for every $k > 0$ and every $\varphi(x) \in W_0^{1,p}(\Omega) \bigcap L^\infty(\Omega)$.

We have the following

Theorem 1.6. *Suppose (1.2), (1.3), (1.12), and*

$$f \in M^m(\Omega),\ 1 < m < \frac{np\sigma}{np\sigma - n - n\sigma + p\sigma},$$

then for any entropy solution u of problem (1.1), one has $u \in M^\tau(\Omega)$ with τ be as in (1.14) and

$$|\nabla u| \in M^\nu(\Omega),\ \nu = \frac{nm(p-1)\sigma}{nm - m\sigma + n\sigma}.$$

In case of $a(x) \geqslant a > 0$, we have the following corollary.

Corollary 1.7. *Suppose (1.2) with $a(x) \geqslant a > 0$, (1.3) and $f \in M^m(\Omega), 1 < m < (p^*)'$, then for any entropy solution u of problem (1.1), one has $u \in M^\tau(\Omega)$ with τ be as in (1.15) and*

$$|\nabla u| \in M^\nu(\Omega),\ \nu = \frac{nm(p-1)}{n-m}.$$

The above corollary coincides with [15, Theorem 1.7, i), ii)].

In Theorems 1.3 and 1.6, we deal with the case when f lies in Marcinkiewicz space. We now assume that f belongs to Lebesgue space, that is,

$$f \in L^m(\Omega),\ m > \frac{np\sigma}{np\sigma - n - n\sigma + p\sigma}. \tag{1.17}$$

We have the following

Theorem 1.8. *Suppose (1.2), (1.3), (1.12) and (1.17). Let $u \in W_0^{1,p}(\Omega)$ be a solution of problem (1.1).*

(i) *If $m > \frac{n\sigma}{p\sigma - n}$, then $u \in L^\infty(\Omega)$;*
(ii) *If $m = \frac{n\sigma}{p\sigma - n}$, then $e^{\bar\lambda|u|} \in L^1(\Omega)$ for every $\bar\lambda > 0$;*
(iii) *If $\frac{np\sigma}{np\sigma - n - n\sigma + p\sigma} \leqslant m < \frac{n\sigma}{p\sigma - n}$, then $u \in L^\tau(\Omega)$ with τ be as in (1.14).*

In case of $a(x) \geqslant a > 0$, we have the following corollary.

Corollary 1.9. *Suppose (1.2) with $a(x) \geqslant a > 0$, (1.3) and $f \in L^m(\Omega), m > (p^*)'$. Let $u \in W_0^{1,p}(\Omega)$ be a solution of problem (1.1).*

(i) *If $m > \frac{n}{p}$, then $u \in L^\infty(\Omega)$;*
(ii) *If $m = \frac{n}{p}$, then $e^{\bar\lambda|u|} \in L^1(\Omega)$ for every $\bar\lambda > 0$;*
(iii) *If $(p^*)' \leqslant m < \frac{n}{p}$, then $u \in L^\tau(\Omega)$ with τ be as in (1.15).*

In case of $p = 2$, the above results (i), (ii) and (iii) coincide with [3, Theorems 6.6, 6.10 and 6.9], respectively.

We end this section by the following remarks: we note that Theorem 1.3 (i) is a particular case of [7]; we note also that, the present paper deals with elliptic equations with variable coefficients, the original regularity results related to variable coefficients go back to results due to Trudinger [18] and Marthy-Stapaccha [17], and in the linear case $p = 2$, Theorem 1.8 is essentially contained in [18, Theorem 4.1]; we refer to [4] for some similar results related to elliptic equations with degenerate coercivity, and to [2] for some Marcinkiewicz estimates for solutions of some elliptic problems with nonregular data; we point out that the monograph [3] by Boccardo and Croce provides fruitful ideas.

§2 Proof of the Main Theorems

In order to prove Theorems 1.3 and 1.6, we need the following Stampacchia Lemma, which can be found, for example, in [19, Lemma 4.1].

LEMMA 2.1. *Let c, α, β be positive constants and $k_0 \in \mathbb{R}$. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be nonincreasing and such that*

$$\varphi(h) \leqslant \frac{c}{(h-k)^\alpha}[\varphi(k)]^\beta \tag{2.1}$$

for every h, k with $h > k \geqslant k_0$. It results that:

(i) *if $\beta > 1$ then*

$$\varphi(k_0 + d) = 0,$$

where

$$d^\alpha = c[\varphi(k_0)]^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}.$$

(ii) *if $\beta = 1$ then for any $k \geqslant k_0$,*

$$\varphi(k) \leqslant \varphi(k_0) e^{1-(ce)^{-\frac{1}{\alpha}}(k-k_0)}.$$

(iii) *if $0 < \beta < 1$ and $k_0 > 0$ then for any $k \geqslant k_0$,*

$$\varphi(k) \leqslant 2^{\frac{\alpha}{(1-\beta)^2}} \left\{ c^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} \varphi(k_0) \right\} \left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}}.$$

For some remarks on the classical Stampacchia Lemma we refer to [13]. For some generalizations we refer to [10–12].

Proof of Theorem 1.3. Suppose (1.2), (1.3), (1.12), (1.13) and let $u \in W_0^{1,p}(\Omega)$ be a solution to problem (1.1) in the sense of (1.9). Define, for $s \in \mathbb{R}$ and $k \geqslant 0$,

$$G_k(s) = s - T_k(s).$$

If we take $G_k(u)$ as test function in (1.9) and use hypothesis (1.2), we then obtain

$$\begin{aligned} &\int_{A_k} a(x)|\nabla u|^p dx \leqslant \int_\Omega \mathcal{A}(x, u, \nabla u)\nabla G_k(u) dx \\ &= \int_\Omega f G_k(u) dx \leqslant \int_{A_k} |f||G_k(u)| dx, \end{aligned} \tag{2.2}$$

where $A_k = \{x \in \Omega : |u| > k\}$ is the superlevel set of u. Let us denote $q = \frac{p\sigma}{1+\sigma}$ with σ the number in (1.12). It is obvious that $1 < q < p \leqslant n$ and $\frac{q}{p-q} = \sigma$. (1.12), (2.2) and Hölder inequality give

$$
\begin{aligned}
&\int_{A_k}|\nabla u|^q dx\\
=\ &\int_{A_k} a(x)^{\frac{q}{p}}|\nabla u|^q\Big(\frac{1}{a(x)}\Big)^{\frac{q}{p}}dx\\
\leqslant\ &\Big(\int_{A_k} a(x)|\nabla u|^p dx\Big)^{\frac{q}{p}}\Big(\int_{A_k}\Big(\frac{1}{a(x)}\Big)^{\frac{q}{p-q}}dx\Big)^{\frac{p-q}{p}}\\
\leqslant\ &\Big(\int_{A_k}|f||G_k(u)|dx\Big)^{\frac{q}{p}}\Big(\int_{A_k}\Big(\frac{1}{a(x)}\Big)^{\sigma}dx\Big)^{\frac{q}{p\sigma}}\\
\leqslant\ &\Big(\int_{A_k}|f||G_k(u)|dx\Big)^{\frac{q}{p}}\Big\|\frac{1}{a}\Big\|^{\frac{q}{p}}_{L^\sigma(\Omega)}\\
\leqslant\ &\Big(\int_{A_k}|f|^{(q^*)'}dx\Big)^{\frac{q}{(q^*)'p}}\Big(\int_{\Omega}|G_k(u)|^{q^*}dx\Big)^{\frac{q}{q^*p}}\Big\|\frac{1}{a}\Big\|^{\frac{q}{p}}_{L^\sigma(\Omega)}.
\end{aligned}
\tag{2.3}
$$

Sobolev inequality yields

$$
\int_{A_k}|\nabla u|^q dx=\int_{\Omega}|\nabla G_k(u)|^q dx\geqslant C_*^q\left(\int_{\Omega}|G_k(u)|^{q^*}dx\right)^{\frac{q}{q^*}},
\tag{2.4}
$$

where q^* is Sobolev exponent for q and C_* is a positive constant depending upon n and q.

(2.3) and (2.4) merge into

$$
\left(\int_{\Omega}|G_k(u)|^{q^*}dx\right)^{\frac{q}{q^*p'}}\leqslant\frac{1}{C_*^q}\Big\|\frac{1}{a}\Big\|^{\frac{q}{p}}_{L^\sigma(\Omega)}\Big(\int_{A_k}|f|^{(q^*)'}dx\Big)^{\frac{q}{(q^*)'p}}.
\tag{2.5}
$$

The condition $\frac{np\sigma}{np\sigma-n-n\sigma+p\sigma}<m$ is equivalent to $(q^*)'<m$. We use (1.6) and (1.7) to get

$$
\begin{aligned}
&\left(\int_{\Omega}|G_k(u)|^{q^*}dx\right)^{\frac{q}{q^*p'}}\\
\leqslant&\frac{1}{C_*^q}\Big\|\frac{1}{a}\Big\|^{\frac{q}{p}}_{L^\sigma(\Omega)}\left[B|A_k|^{1-\frac{(q^*)'}{m}}\right]^{\frac{q}{(q^*)'p}}\\
=&\frac{1}{C_*^q}\Big\|\frac{1}{a}\Big\|^{\frac{q}{p}}_{L^\sigma(\Omega)}B^{\frac{q}{(q^*)'p}}|A_k|^{\left(1-\frac{(q^*)'}{m}\right)\frac{q}{(q^*)'p}},
\end{aligned}
\tag{2.6}
$$

where B is a constant depending upon $|||f|||_m, n, p, \sigma$. Let $h>k\geqslant 0$, then

$$
\begin{aligned}
&(h-k)^{\frac{q}{p'}}|A_h|^{\frac{q}{q^*p'}}\\
\leqslant\ &\left(\int_{A_h}(u-k)^{q^*}dx\right)^{\frac{q}{q^*p'}}\\
=\ &\left(\int_{A_h}|G_k(u)|^{q^*}dx\right)^{\frac{q}{q^*p'}}\\
\leqslant\ &\left(\int_{\Omega}|G_k(u)|^{q^*}dx\right)^{\frac{q}{q^*p'}}\\
\leqslant\ &\frac{1}{C_*^q}\Big\|\frac{1}{a}\Big\|^{\frac{q}{p}}_{L^\sigma(\Omega)}B^{\frac{q}{(q^*)'p}}|A_k|^{\left(1-\frac{(q^*)'}{m}\right)\frac{q}{(q^*)'p}},
\end{aligned}
\tag{2.7}
$$

from which we derive

$$
|A_h|\leqslant\frac{\left(\frac{1}{C_*}\right)^{q^*p'}\left\|\frac{1}{a}\right\|^{\frac{q^*}{p-1}}_{L^\sigma(\Omega)}B^{\frac{q^*p'}{(q^*)'p}}}{(h-k)^{q^*}}|A_k|^{\left(1-\frac{(q^*)'}{m}\right)\frac{q^*p'}{(q^*)'p}}.
\tag{2.8}
$$

The assumption (2.1) of Lemma 2.1 holds with

$$
\varphi(k)=|A_k|,
$$

$$
c=\left(\frac{1}{C_*}\right)^{q^*p'}\Big\|\frac{1}{a}\Big\|^{\frac{q^*}{p-1}}_{L^\sigma(\Omega)}B^{\frac{q^*p'}{(q^*)'p}},
$$

$$\alpha = q^*,$$

$$\beta = \Big(1 - \frac{(q^*)'}{m}\Big)\frac{q^* p'}{(q^*)' p},$$

and

$$k_0 = 0.$$

We now divide the following proof into three cases.

Case 1: $m > \frac{n\sigma}{p\sigma - n}$. In this case $\beta > 1$. Lemma 2.1 (i) tells us that there exists a constant $d = d\left(n, p, \sigma, |\Omega|, m, \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}, |||f|||_m\right) > 0$, such that

$$|\{|u| > d\}| = 0,$$

thus $|u| \leqslant d$, a.e. Ω.

Case 2: $m = \frac{n\sigma}{p\sigma - n}$. In this case $\beta = 1$. We use lemma 2.1 (ii) to get, for any $k \geqslant 0$,

$$|\{|u| > k\}| \leqslant |\{|u| > 0\}| e^{1-(ce)^{-\frac{1}{\alpha}} k} \leqslant |\Omega| e e^{-(ce)^{-\frac{1}{\alpha}} k}, \tag{2.9}$$

thus

$$\sum_{k=0}^{\infty} e^{\frac{(ce)^{-\frac{1}{\alpha}} k}{2}} |\{|u| > k\}| \leqslant \sum_{k=0}^{\infty} e^{\frac{(ce)^{-\frac{1}{\alpha}} k}{2}} |\Omega| e e^{-(ce)^{-\frac{1}{\alpha}} k} = |\Omega| e \sum_{k=0}^{\infty} e^{-\frac{(ce)^{\frac{1}{\alpha}} k}{2}} < \infty. \tag{2.10}$$

Proposition 6.4 in [3] states that for $\lambda > 0$,

$$\int_\Omega e^{\lambda|u|} dx < \infty \Longleftrightarrow \sum_{k=0}^{\infty} e^{\lambda k} |\{|u| > k\}| < \infty.$$

We use this fact for $\lambda = \frac{(ce)^{-\frac{1}{\alpha}}}{2}$. Note that λ is a constant depending on $n, p, \sigma, \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}, |||f|||_m$. We use the above proposition and (2.10) and we derive that

$$\int_\Omega e^{\lambda|u|} dx < \infty.$$

Case 3: $\frac{np\sigma}{np\sigma - n - n\sigma + p\sigma} < m < \frac{n\sigma}{p\sigma - n}$. In this case $0 < \beta < 1$. Since the assumption (2.1) of Lemma 2.1 holds with $k_0 = 0$, and Lemma 2.1 (iii) requires $k_0 > 0$, then one can use the fact that the assumption (2.1) of Lemma 2.1 holds with $k_0 = 1$ and we have

$$|\{|u| > k\}| \leqslant c\left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}} = c\left(\frac{1}{k}\right)^{\tau}, \ \forall k \geqslant 1,$$

where

$$\tau = \frac{nm(p-1)\sigma}{mn - mp\sigma + n\sigma},$$

the desired result $u \in M^\tau(\Omega)$ follows from the fact

$$|\{|u| > k\}| \leqslant |\Omega| \left(\frac{1}{k}\right)^{\tau} + c\left(\frac{1}{k}\right)^{\tau} = (|\Omega| + c)\left(\frac{1}{k}\right)^{\tau}, \ \forall k > 0.$$

Proof of Theorem 1.6. For any $h > k \geqslant 0$, we take $h - k$ in place of k in (1.16), and we use $\varphi = T_k(u)$ as a test function. Note that

$$T_{h-k}(u - T_k(u)) = 0 \quad \text{for } x \in \{|u| \leqslant k\},$$

$$|T_{h-k}(u - T_k(u))| \leqslant h - k$$

and

$$\nabla T_{h-k}(u - T_k(u)) = \begin{cases} 0, & |u| \leqslant k, \\ \nabla u, & k < |u| \leqslant h, \\ 0, & |u| > h, \end{cases}$$

then (1.2) and (1.16) yield

$$\begin{aligned}
&\int_{B_{k,h}} a(x)|\nabla u|^p dx \\
\leqslant &\int_\Omega \mathcal{A}(x,u,\nabla u)\nabla T_{h-k}\left(u-T_k(u)\right)dx \\
\leqslant &\int_\Omega f T_{h-k}(u-T_k(u))dx \\
\leqslant &(h-k)\int_{A_k}|f|dx,
\end{aligned}$$

where

$$B_{k,h}=\{x\in\Omega : k<|u|\leqslant h\}.$$

As in the proof of Theorem 1.3 we take $1<q=\frac{p\sigma}{1+\sigma}<p$. Hölder inequality gives

$$\begin{aligned}
&\int_{B_{k,h}}|\nabla u|^q dx \\
= &\int_{B_{k,h}} a(x)^{\frac{q}{p}}|\nabla u|^q\Big(\frac{1}{a(x)}\Big)^{\frac{q}{p}}dx \\
\leqslant &\Big(\int_{B_{k,h}} a(x)|\nabla u|^p dx\Big)^{\frac{q}{p}}\Big(\int_{B_{k,h}}\left(\frac{1}{a(x)}\right)^{\sigma}dx\Big)^{\frac{q}{p\sigma}} \\
\leqslant &\Big((h-k)\int_{A_k}|f|dx\Big)^{\frac{q}{p}}\Big\|\frac{1}{a}\Big\|^{\frac{q}{p}}_{L^\sigma(\Omega)} \\
\leqslant &(h-k)^{\frac{q}{p}}B^{\frac{q}{p}}\Big\|\frac{1}{a}\Big\|^{\frac{q}{p}}_{L^\sigma(\Omega)}|A_k|^{\frac{q}{pm'}},
\end{aligned}\tag{2.11}$$

where we used again (1.7). Sobolev inequality yields

$$\begin{aligned}
&\int_{B_{k,h}}|\nabla u|^q dx \\
= &\int_\Omega|\nabla T_{h-k}(G_k(u))|^q dx \\
\geqslant &C_*^q\left(\int_\Omega|T_{h-k}(G_k(u))|^{q^*}dx\right)^{\frac{q}{q^*}} \\
\geqslant &C_*^q\left(\int_{A_h}|T_{h-k}(G_k(u))|^{q^*}dx\right)^{\frac{q}{q^*}} \\
\geqslant &C_*^q(h-k)^q|A_h|^{\frac{q}{q^*}},
\end{aligned}\tag{2.12}$$

where q^* is the Sobolev exponent for q and C_* is a constant depending upon n,q. Combining (2.11) and (2.12) we arrive at

$$|A_h|\leqslant\frac{B^{\frac{q^*}{p}}C_*^{-q^*}\left\|\frac{1}{a}\right\|^{\frac{q^*}{p}}_{L^\sigma(\Omega)}}{(h-k)^{\frac{q^*}{p'}}}|A_k|^{\frac{q^*}{pm'}}.$$

The assumption (2.1) of Lemma 2.1 holds with

$$\varphi(k)=|A_k|,$$

$$c=B^{\frac{q^*}{p}}C_*^{-q^*}\Big\|\frac{1}{a}\Big\|^{\frac{q^*}{p}}_{L^\sigma(\Omega)},$$

$$\alpha=\frac{q^*}{p'},$$

$$\beta=\frac{q^*}{pm'},$$

and

$$k_0=0.$$

(2.1) holds true for $k_0 = 1$ as well. Since $1 < m < \frac{np\sigma}{np\sigma - n - n\sigma + p\sigma}$, then $0 < \frac{q^*}{pm'} < 1$. We use Lemma 2.1 (iii), and note that

$$\frac{\alpha}{1-\beta} = \frac{\frac{q^*}{p'}}{1-\frac{q^*}{pm'}} = \frac{nm(p-1)\sigma}{mn - mp\sigma + n\sigma} = \tau,$$

we derive that

$$|\{|u| > k\}| \leqslant (|\Omega| + c)\left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}} = (|\Omega| + c)\left(\frac{1}{k}\right)^{\tau}, \ \forall k > 0,$$

where c is a constant depending upon $n, p, \sigma, |\Omega|, |||f|||_m$ and $\left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}$. This shows that $u \in M^\tau(\Omega)$.

Let us take $h = 2k$ in (2.11), use (1.4) and the fact $u \in M^\tau(\Omega)$, then

$$\begin{aligned}
&\int_{B_{k,2k}} |\nabla u|^q dx \\
\leqslant\ & k^{\frac{q}{p}} B^{\frac{q}{p}} \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}} |A_k|^{\frac{q}{pm'}} \\
\leqslant\ & k^{\frac{q}{p}} B^{\frac{q}{p}} \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}} \left(A_\tau(u)^\tau k^{-\tau}\right)^{\frac{q}{pm'}} \\
=\ & k^{\frac{q}{p}(1-\frac{\tau}{m'})} B^{\frac{q}{p}} \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}} A_\tau(u)^{\frac{q\tau}{pm'}},
\end{aligned}$$

which yields, for any $k > 0$,

$$\begin{aligned}
&\int_{\{|u|\leqslant k\}} |\nabla u|^q dx \\
\leqslant\ & \sum_{j=0}^{\infty} \int_{\{2^{-j-1}k < |u| \leqslant 2^{-j}k\}} |\nabla u|^q dx \\
\leqslant\ & \sum_{j=0}^{\infty} (2^{-j-1}k)^{\frac{q}{p}(1-\frac{\tau}{m'})} B^{\frac{q}{p}} \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}} A_\tau(u)^{\frac{q\tau}{pm'}} \qquad (2.13)\\
=\ & \sum_{j=0}^{\infty} (2^{-j-1})^{\frac{q}{p}(1-\frac{\tau}{m'})} k^{\frac{q}{p}(1-\frac{\tau}{m'})} B^{\frac{q}{p}} \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}} A_\tau(u)^{\frac{q\tau}{pm'}}.
\end{aligned}$$

Since $m < \frac{np\sigma}{np\sigma - n\sigma - n + p\sigma}$, then $\frac{q}{p}(1 - \frac{\tau}{m'}) > 0$, so

$$\sum_{j=0}^{\infty} (2^{-j-1})^{\frac{q}{p} - \frac{q\tau}{pm'}} < \infty,$$

from (2.13) we obtain

$$\int_{\{|u|\leqslant k\}} |\nabla u|^q dx \leqslant c k^{\frac{q}{p}(1-\frac{\tau}{m'})} B^{\frac{q}{p}} \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}} A_\tau(u)^{\frac{q\tau}{pm'}}. \qquad (2.14)$$

Thus, for any $k > 0$ and $t > 0$,

$$\begin{aligned}
&\Big|\{|\nabla u| > t\}\Big| \\
=\ & \Big|\{|\nabla u| > t\} \cap \{|u| > k\}\Big| + \Big|\{|\nabla u| > t\} \cap \{|u| \leqslant k\}\Big| \\
\leqslant\ & \Big|\{|u| > k\}\Big| + t^{-q} \int_{\{|u|\leqslant k\}} |\nabla u|^q dx \\
\leqslant\ & A_\tau(u)^\tau k^{-\tau} + t^{-q} c k^{\frac{q}{p}(1-\frac{\tau}{m'})} B^{\frac{q}{p}} \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}} A_\tau(u)^{\frac{q\tau}{pm'}} \\
=\ & c_1 k^{-\tau} + c_2 t^{-q} k^{\frac{q}{p} - \frac{q\tau}{pm'}},
\end{aligned}$$

where

$$c_1 = A_\tau(u)^\tau, \quad c_2 = cB^{\frac{q}{p}} \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}} A_\tau(u)^{\frac{q\tau}{pm'}}.$$

Next we minimize this in k, i.e., choose

$$k = \left(\frac{c_1 \tau t^q}{c_2(\frac{q}{p} - \frac{q\tau}{pm'})} \right)^{\frac{1}{\frac{q}{p} - \frac{q\tau}{pm'} + \tau}},$$

and we arrive at

$$\Big| \{ |\nabla u| > t \} \Big| \leqslant c t^{\frac{-q\tau}{\tau + \frac{q}{p} - \frac{q\tau}{pm'}}},$$

where c is a constant depending upon $n, p, m, \sigma, |||f|||_m, \left\| \frac{1}{a} \right\|_{L^\sigma(\Omega)}$ and $A_\tau(u)$. Now we observe that

$$\nu = \frac{q\tau}{\tau + \frac{q}{p} - \frac{q\tau}{pm'}} = \frac{nm(p-1)\sigma}{nm - m\sigma + n\sigma},$$

then $|\nabla u| \in M^\nu(\Omega)$, as desired.

Proof of Theorem 1.8. Suppose (1.2), (1.3), (1.12), (1.17) and let $u \in W_0^{1,p}(\Omega)$ be a solution to problem (1.1) in the sense of (1.9).

(i) For the case $m > \frac{n\sigma}{p\sigma - n}$, we use the fact (1.8) and we have $f \in M^m(\Omega)$. Theorem 1.3 (i) gives the result.

(ii) For the case $m = \frac{n\sigma}{p\sigma - n}$, for every $\lambda > 0$, $k > 0$, $\ell > 0$ let us take

$$\varphi = \left[e^{p\lambda T_\ell |(G_k(u)|} - 1 \right] \mathrm{sgn}(u) \in W_0^{1,p}(\Omega)$$

as a test function in the weak formulation (1.9). Since

$$\nabla \varphi = p\lambda e^{p\lambda T_\ell |G_k(u)|} \nabla u \cdot 1_{B_{k,k+\ell}},$$

where

$$B_{k,k+\ell} = \{ x \in \Omega : k \leqslant |u| < k + \ell \},$$

and 1_E is the characteristic function for the set E, that is, $1_E(x) = 1$ for $x \in E$ and $1_E(x) = 0$ otherwise, then (1.9) gives

$$p\lambda \int_{B_{k,k+\ell}} \mathcal{A}(x, u, \nabla u) e^{p\lambda T_\ell |G_k(u)|} \nabla u dx = \int_{A_k} f \left[e^{p\lambda T_\ell |G_k(u)|} - 1 \right] \mathrm{sgn}(u) dx. \tag{2.15}$$

We study the two sides separately. The left hand side of (2.15) can be estimated from below by using (1.2),

$$\begin{aligned} & p\lambda \int_{B_{k,k+\ell}} \mathcal{A}(x, u, \nabla u) e^{p\lambda T_\ell |G_k(u)|} \nabla u dx \\ \geqslant \ & p\lambda \int_{B_{k,k+\ell}} a(x) e^{p\lambda T_\ell |G_k(u)|} |\nabla u|^p dx \\ = \ & \frac{p\lambda}{\lambda^p} \int_{B_{k,k+\ell}} a(x) \left| \nabla (e^{\lambda T_\ell |G_k(u)|} - 1) \right|^p dx. \end{aligned} \tag{2.16}$$

We use the following inequality, satisfied by every $t \geqslant 1$, $p > 1$ and $Q > 1$:

$$t^p - 1 \leqslant Q(t-1)^p + \left(1 - Q^{-\frac{1}{p-1}}\right)^{1-p} - 1,$$

then the right hand side of (2.15) can be estimated as

$$\begin{aligned} & \int_{A_k} f \left[e^{p\lambda T_\ell |G_k(u)|} - 1 \right] \mathrm{sgn}(u) dx \\ \leqslant \ & \int_{A_k} |f| \left[e^{p\lambda T_\ell |G_k(u)|} - 1 \right] dx \\ \leqslant \ & Q \int_{A_k} |f| (e^{\lambda T_\ell |G_k(u)|} - 1)^p dx + C(Q, p) \int_{A_k} |f| dx \\ \leqslant \ & Q \left(\int_{A_k} |f|^m dx \right)^{\frac{1}{m}} \left(\int_{A_k} \left[e^{\lambda T_\ell |G_k(u)|} - 1 \right]^{pm'} dx \right)^{\frac{1}{m'}} + C(Q, p) \|f\|_{L^1(\Omega)}, \end{aligned} \tag{2.17}$$

where

$$m=\frac{n\sigma}{p\sigma-n},\quad C(Q,p)=\Big[\big(1-Q^{-\frac{1}{p-1}}\big)^{1-p}-1\Big],$$

and we have used Hölder inequality.

Substituting (2.16) and (2.17) into (2.15),

$$\begin{aligned}&\int_{B_{k,k+\ell}}a(x)\left|\nabla(e^{\lambda T_\ell|G_k(u)|}-1)\right|^p dx\\ \leqslant\ &\frac{Q\lambda^{p-1}\|f\|_{L^m(A_k)}}{p}\left(\int_{A_k}\left[e^{\lambda T_\ell|G_k(u)|}-1\right]^{pm'}dx\right)^{\frac{1}{m'}}+\frac{\lambda^{p-1}C(Q,p)\|f\|_{L^1(\Omega)}}{p}.\end{aligned}\tag{2.18}$$

As in the proof of Theorems 1.3 and 1.6, we take $1<q=\frac{p\sigma}{1+\sigma}<p$. Hölder inequality together with (2.18) gives

$$\begin{aligned}&\int_{B_{k,k+\ell}}\left|\nabla(e^{\lambda T_\ell|G_k(u)|}-1)\right|^q dx\\ =\ &\int_{B_{k,k+\ell}}a(x)^{\frac{q}{p}}\left|\nabla(e^{\lambda T_\ell|G_k(u)|}-1)\right|^q\left(\frac{1}{a(x)}\right)^{\frac{q}{p}}dx\\ \leqslant\ &\left(\int_{B_{k,k+\ell}}a(x)\left|\nabla(e^{\lambda T_\ell|G_k(u)|}-1)\right|^p dx\right)^{\frac{q}{p}}\left(\int_{B_{k,k+\ell}}\left(\frac{1}{a(x)}\right)^{\frac{q}{p-q}}dx\right)^{\frac{p-q}{p}}\\ \leqslant\ &2^{\frac{q}{p}}\left[\left(\frac{Q\lambda^{p-1}\|f\|_{L^m(A_k)}}{p}\right)^{\frac{q}{p}}\left(\int_{A_k}\left[e^{\lambda T_\ell|G_k(u)|}-1\right]^{pm'}dx\right)^{\frac{q}{pm'}}\right.\\ &\left.+\left(\frac{\lambda^{p-1}C(Q,p)\|f\|_{L^1(\Omega)}}{p}\right)^{\frac{q}{p}}\right]\left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}}.\end{aligned}\tag{2.19}$$

Sobolev inequality gives

$$\begin{aligned}&\int_{B_{k,k+\ell}}\left|\nabla(e^{\lambda T_\ell|G_k(u)|}-1)\right|^q dx\\ =\ &\int_\Omega\left|\nabla(e^{\lambda T_\ell|G_k(u)|}-1)\right|^q dx\\ \geqslant\ &C_*^q\left(\int_\Omega\left|e^{\lambda T_\ell|G_k(u)|}-1\right|^{q^*}dx\right)^{\frac{q}{q^*}}\\ =\ &C_*^q\left(\int_{A_k}\left|e^{\lambda T_\ell|G_k(u)|}-1\right|^{q^*}dx\right)^{\frac{q}{q^*}}.\end{aligned}\tag{2.20}$$

where C_* is a constant depending upon n and q. (2.19) and (2.20) merge into

$$\begin{aligned}&\left(\int_{A_k}\left|e^{\lambda T_\ell|G_k(u)|}-1\right|^{q^*}dx\right)^{\frac{q}{q^*}}\\ \leqslant\ &\frac{2^{\frac{q}{p}}}{C_*^q}\left[\left(\frac{Q\lambda^{p-1}\|f\|_{L^m(A_k)}}{p}\right)^{\frac{q}{p}}\left(\int_{A_k}\left[e^{\lambda T_\ell|G_k(u)|}-1\right]^{pm'}dx\right)^{\frac{q}{pm'}}\right.\\ &\left.+\left(\frac{\lambda^{p-1}C(Q,p)\|f\|_{L^1(\Omega)}}{p}\right)^{\frac{q}{p}}\right]\left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}}.\end{aligned}\tag{2.21}$$

Recall $q=\frac{p\sigma}{1+\sigma}$, $m=\frac{n\sigma}{p\sigma-n}$, which imply $q^*=pm'$ and $\frac{q}{q^*}=\frac{q}{pm'}$. Since $\|f\|_{L^m(A_k)}\to 0$ as $k\to+\infty$, then there exists $k_\lambda>0$ such that

$$\frac{2^{\frac{q}{p}}}{C_*^q}\left(\frac{Q\lambda^{p-1}\|f\|_{L^m(A_k)}}{p}\right)^{\frac{q}{p}}\left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}}\leqslant\frac{1}{2},\quad\forall k\geqslant k_\lambda.$$

For such k we deduce from (2.21) that

$$\begin{aligned}
&\left(\int_\Omega \left|e^{\lambda T_\ell |G_k(u)|}-1\right|^{q^*} dx\right)^{\frac{q}{q^*}} \\
= & \left(\int_{A_k} \left|e^{\lambda T_\ell |G_k(u)|}-1\right|^{q^*} dx\right)^{\frac{q}{q^*}} \\
\leqslant & \frac{2^{1+\frac{q}{p}}}{C_*^q}\left(\frac{\lambda^{p-1}C(Q,p)\|f\|_{L^1(\Omega)}}{p}\right)^{\frac{q}{p}} \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}} \\
< & +\infty.
\end{aligned}$$

Let $\ell \to +\infty$, we use Fatou Lemma and we derive that

$$\int_\Omega \left|e^{\lambda |G_k(u)|}-1\right|^{q^*} dx < +\infty, \quad \forall k \geqslant k_\lambda.$$

Now

$$\begin{aligned}
& [e^{\lambda|u|}-1]^{q^*} = [e^{\lambda|T_k(u)+G_k(u)|}-1]^{q^*} \\
= & [e^{\lambda|T_k(u)+G_k(u)|}-e^{\lambda k}+e^{\lambda k}-1]^{q^*} \\
\leqslant & 2^{q^*-1}e^{\lambda k q^*}\left[e^{\lambda|G_k(u)|}-1\right]^{q^*}+2^{q^*-1}(e^{\lambda k}-1)^{q^*}.
\end{aligned}$$

Therefore, for every $k \geqslant k_\lambda$,

$$\int_\Omega [e^{\lambda|u|}-1]^{q^*} dx \leqslant 2^{q^*-1}e^{\lambda k q^*}\int_\Omega \left[e^{\lambda|G_k(u)|}-1\right]^{q^*} dx + 2^{q^*-1}(e^{\lambda k}-1)^{q^*}|\Omega| < +\infty.$$

That is, $e^{\lambda|u|}$ belongs to $L^{q^*}(\Omega)$ for every $\lambda > 0$. The result (ii) follows with $\bar\lambda = \lambda q^*$.

(iii) For the case $\frac{np\sigma}{np\sigma-n-n\sigma+p\sigma} \leqslant m < \frac{n\sigma}{p\sigma-n}$, let $t \geqslant 0$ be a number to be fixed and let us take

$$\varphi = |T_k(u)|^{pt}T_k(u)$$

as a test function in (1.9). We use hypothesis (1.2) and we have

$$\begin{aligned}
&\frac{pt+1}{(t+1)^p}\int_\Omega a(x)\left|\nabla|T_k(u)|^{t+1}\right|^p dx \\
=&(pt+1)\int_\Omega a(x)|T_k(u)|^{pt}|\nabla T_k(u)|^p dx \\
\leqslant&(pt+1)\int_\Omega \mathcal{A}(x,u,\nabla u)|T_k(u)|^{pt}\nabla T_k(u)dx \\
=&\int_\Omega \mathcal{A}(x,u,\nabla u)\nabla\varphi dx \\
=&\int_\Omega f|T_k(u)|^{pt}T_k(u)dx \\
\leqslant&\int_\Omega |f||T_k(u)|^{pt+1}dx.
\end{aligned}$$

As in the proof of Theorems 1.3 and 1.6, we take $1 < q = \frac{p\sigma}{1+\sigma} < p$, then

$$
\begin{aligned}
&\int_\Omega \left|\nabla |T_k(u)|^{t+1}\right|^q dx \\
&= \int_\Omega a(x)^{\frac{q}{p}} \left|\nabla |T_k(u)|^{t+1}\right|^q \left(\frac{1}{a(x)}\right)^{\frac{q}{p}} dx \\
&\leqslant \left(\int_\Omega a(x) \left|\nabla |T_k(u)|^{t+1}\right|^p dx\right)^{\frac{q}{p}} \left(\int_\Omega \left(\frac{1}{a(x)}\right)^{\frac{q}{p-q}} dx\right)^{\frac{p-q}{p}} \\
&\leqslant \left(\frac{(t+1)^p}{pt+1}\int_\Omega |f||T_k(u)|^{pt+1} dx\right)^{\frac{q}{p}} \left(\int_\Omega \left(\frac{1}{a(x)}\right)^{\sigma} dx\right)^{\frac{q}{p\sigma}} \\
&\leqslant \left(\frac{(t+1)^p}{pt+1}\right)^{\frac{q}{p}} \|f\|_{L^m(\Omega)}^{\frac{q}{p}} \left(\int_\Omega |T_k(u)|^{(pt+1)m'} dx\right)^{\frac{q}{pm'}} \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}}.
\end{aligned}
\tag{2.22}
$$

Sobolev inequality gives

$$
\int_\Omega \left|\nabla |T_k(u)|^{t+1}\right|^q dx \geqslant C_*^q \left(\int_\Omega |T_k(u)|^{q^*(t+1)} dx\right)^{\frac{q}{q^*}}, \tag{2.23}
$$

where C_* depends upon n, q.

Let us choose t in such a way that

$$q^*(t+1) = (pt+1)m',$$

this is equivalent to

$$t + 1 = \frac{(p-1)m'}{pm' - q^*} = \frac{nm(p-1)\sigma}{(nm + n\sigma - mp\sigma)q^*} = \frac{\tau}{q^*}.$$

The facts $\frac{np\sigma}{np\sigma - n - n\sigma + p\sigma} \leqslant m < \frac{n\sigma}{p\sigma - n}$ imply $t \geqslant 0$ and $\frac{q}{q^*} > \frac{q}{pm'}$. (2.22) and (2.23) merge into

$$C_*^q \left(\int_\Omega |T_k(u)|^{q^*(t+1)} dx\right)^{\frac{q}{q^*} - \frac{q}{pm'}} \leqslant \left(\frac{(t+1)^p}{pt+1}\right)^{\frac{q}{p}} \|f\|_{L^m(\Omega)}^{\frac{q}{p}} \left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}^{\frac{q}{p}}.$$

Since

$$q^*(t+1) = \tau,$$

then the above inequality implies, for any $k > 0$,

$$\int_\Omega |T_k(u)|^{q^*(t+1)} dx \leqslant c,$$

with c a constant depending upon $n, p, \sigma, m, \|f\|_{L^m(\Omega)}$ and $\left\|\frac{1}{a}\right\|_{L^\sigma(\Omega)}$. To be finished, we apply Fatou lemma (as k tends to infinity) to deduce that

$$\int_\Omega |u|^\tau dx \leqslant c,$$

as desired.

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